0514 ①

$\mathrm{Im}(A) := \mathrm{Im}(\Phi_A)$

$\mathrm{Ker}(A) := \mathrm{Ker}(\Phi_A)$

Thm 2.24 ($\underbrace{\text{Rank}}_{\overset{\shortparallel}{\dim \mathrm{Im}(A)}}$ - $\underbrace{\text{Nullity}}_{\overset{\shortparallel}{\dim \mathrm{Ker} A}}$)

Let V, W be finite-dim vector spaces

Let $\Phi: V \to W$ be a linear transformation

$\dim V = \dim(\ker \Phi) + \dim(\mathrm{Im}\,\Phi)$

Rmk

① If $\dim(\mathrm{Im}(\Phi)) < \dim V$

$\Rightarrow \ker \Phi \neq \{0\} \Rightarrow \Phi$ is not 1-1

② 略

③ If $\dim V = \dim W$, then:

- Φ is 1-1
- Φ is onto
- Φ is bijective

3個会同時對/錯

2.8. Affine Subspaces

2.8.1. Affine Subspaces

Def'n (AS)

Let V be a vector space

$x_0 \in V$ & $U \leq V$

$\Rightarrow L = x_0 + U$ (平移) $= \{x_0 + u \mid u \in U\} \subseteq V$

L is called an affine subspaces/linear manifold

* Note that if $x_0 \notin U \Rightarrow 0 \notin x_0 + U \Rightarrow$ 他不是向量空間

$\Rightarrow$ AS可能不是向量空間

Rmk

Let $L = x_0 + U$ be an AS, 令 $B = \{b_1, \dots b_k\}$ be a basis of U

$\dim L := \dim U$

0514 ②

1. 一維AS 叫 line
2. 二維AS 叫 plane
3. hyperplane：$\mathbb{R}^n$裡的AS "L"，$\dim L = n-1$

2.8.2. Affine Mapping $\phi: V \to W$

Defn 在W裡存在一向量 a_1，及一個線性轉換 Φ
s.t. $\phi(x) = a_1 + \Phi(x)$

Rmk ① 每一個 AM 都是線性轉換再做平移
② 2個 AM 合完 still AM

CH3 Analytic Geometry 解析幾何

3.1 Norms

Defn' 3.1 (Norm)
現有一 sp V，A norm on V is a fcn
$\|\cdot\|: V \to \mathbb{R}$　　($\|x\|$ = 向量 x 的長度)
s.t. $\forall \lambda \in \mathbb{R}, x, y \in V$：
① Absolutely homogeneous
$\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
② Triangle inequality
$\|x+y\| \le \|x\| + \|y\|$
③ Positive definite 正定
(i) $\|x\| \ge 0$ (ii) $\|x\| = 0 \Leftrightarrow x = 0$

Manhattan Norm (One Norm)
$\|x\|_1 := \sum_{i=1}^{n} |x_i|$，$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Euclidean Norm (Two Norm)
$\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$

3.2. Inner Product 內積

3.2.1 Dot Product
(我們平常熟悉的 $x \cdot y$) $= \langle x, y \rangle$
$= x^T y$

3.2.2 General Inner Product
bilinear map：一個固定不動時是線性
$\Omega(x, y)$，$\forall x, y, z \in V$，$\lambda, \psi \in \mathbb{R}$
$\Omega(\lambda x + \psi y, \boxed{z}) = \lambda\Omega(x, \boxed{z}) + \psi\Omega(y, \boxed{z})$

0514 ③

Defn 3.2 Let $\Omega: V \times V$ be a bilinear map
① if $\Omega(x, y) = \Omega(y, x)$
$\Rightarrow \Omega$ is symmetric
② 所有不是0的向量，自己和自己內積要 >0
$\Rightarrow \Omega$ is 正定

Defn 3.3
① Ω is 正定 & symmetric on V
$\Rightarrow \Omega$ is inner product
通常內積寫作：$\langle x, y \rangle$
② 內積空間：$(V, \langle \cdot, \cdot \rangle)$

$\langle x, y \rangle = \underbrace{[x_1, x_2]}_{x^T} \underbrace{\begin{bmatrix} a & c \\ d & b \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y}$

- $A^T = A$ (symmetric)
- A is 正定
$\forall x \in \mathbb{R}^2$，$x^T A x \ge 0$，$x^T A x = 0 \Leftrightarrow x = 0$

Let $\langle x, y \rangle = [x]_B^T A [y]_B$
$a_{ij} = \langle b_i, b_j \rangle$