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## Finite Horizon Optimal Control of Singularly Perturbed Systems: A Differential Lyapunov Equation Approach

Thang Nguyen and Zoran Gajic

**Abstract**—In this note, we study the finite time (horizon) optimal control problem for singularly perturbed systems. The solution is obtained in terms of the corresponding solution of the algebraic Riccati equation and the decomposition of the singularly perturbed differential Lyapunov equation into reduced-order differential Lyapunov/Sylvester equations. An illustrative numerical example is provided to show the efficiency of the proposed approach.

**Index Terms**—Algebraic Riccati equation, differential Lyapunov/Sylvester equation, differential Riccati equation, optimal control, singularly perturbed systems.

### I. INTRODUCTION

Singularly perturbed systems contain slow and fast modes, which are characterized by a small positive parameter  $\epsilon$  that multiplies the derivatives of fast variables. The appearance of the small singular perturbation parameter causes ill-conditioning in computations. To overcome the stiffness of the problem, some useful techniques were proposed to simplify computations by separating pure slow and pure fast system variables [1], [2].

There are several approaches to study the finite time optimal control problem for singularly perturbed linear time-invariant systems, see for example [3]–[8] and references therein. The first approach is to exploit the asymptotic series expansion technique to overcome the computational difficulties [3]–[6]. When one considers a higher degree of accuracy, a considerable size of computations is required. Series expansion methods used in the early works possess disadvantages from the numerical point of view. In such cases, one can use the fixed-point approach as demonstrated in [2], [9]. Another approach that is numerically efficient for the finite time optimal control problem is based on the Hamiltonian form of the solution of the differential Riccati equation [7]. This approach uses the Chang nonsingular transformation to block diagonalize the Hamiltonian matrix so that the solution is obtained in terms of the reduced-order slow and fast subproblems. In a similar vein, the approach in [8] utilizes the Hamiltonian form to induce the exact slow and fast decomposition of the full-order singularly perturbed matrix differential Riccati equation.

In [10], a new method to solve the matrix differential Riccati equation was proposed. Specifically, the differential Lyapunov equation is utilized to numerically calculate the solution of the matrix differential Riccati equation. The method of [10] is shown to be robust and numerically efficient. Our objective is to apply this technique to the singularly

perturbed linear-quadratic optimal control problem. To that end, we decompose the full-order optimal linear-quadratic control problem into reduced-order subproblems where the reduced-order differential Lyapunov/Sylvester equations need to be addressed. A similar transformation idea can be found in the works of Glizer and Dmitriev [5], [6] in which a series expansion method is proposed. It will be observed that the computational cost of the proposed method is smaller than those of the other approaches reported in [7], [8].

The note is organized as follows. In Section II, we introduce the problem formulation. Section III reviews the differential Lyapunov equation approach. The technique to decompose the full-order problem into the reduced-order ones is presented in Section IV. A numerical example is given in Section V followed by the conclusion section.

### II. PROBLEM STATEMENT

Consider a linear time-invariant singularly perturbed system

$$\begin{bmatrix} \dot{x}(t) \\ \epsilon \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \quad (1)$$

where  $x(t)$  and  $z(t)$  are  $n_1$  and  $n_2$  dimensional slow and fast state vectors respectively,  $u(t)$  is an  $m$  dimensional vector control input and  $\epsilon$  is a small positive parameter.

The corresponding finite time optimal control problem is to find an optimal control  $u(t)$  that minimizes the following quadratic cost functional:

$$J = \frac{1}{2} \int_0^{t_f} \left\{ \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + u^T(t) R u(t) \right\} dt + \frac{1}{2} \begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix}^T F \begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix} \quad (2)$$

where

$$Q = Q^T = C C^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}^T \geq 0$$

$$R = R^T > 0, \quad F = F^T = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & \epsilon F_3 \end{bmatrix} \geq 0$$

are of appropriate dimensions. The optimal control law is achieved as [1], [2]

$$u(t) = -R^{-1} \left[ \left( B_1^T K_1(t) + B_2^T K_2^T(t) \right) x(t) + \left( \epsilon B_1^T K_2(t) + B_2^T K_3(t) \right) z(t) \right] \quad (3)$$

where

$$K(t) = \begin{bmatrix} K_1(t) & \epsilon K_2(t) \\ \epsilon K_2^T(t) & \epsilon K_3(t) \end{bmatrix} \quad (4)$$

is the solution of the differential Riccati equation

$$\dot{K}(t) = -K(t)A - A^T K(t) + K(t)S K(t) - Q$$

$$K(t_f) = F \quad (5)$$

where

$$S = B R^{-1} B^T = \begin{bmatrix} S_1 & \frac{1}{\epsilon} S_{12} \\ \frac{1}{\epsilon} S_{12}^T & \frac{1}{\epsilon^2} S_2 \end{bmatrix}$$

$$S_1 = B_1 R^{-1} B_1^T, \quad S_{12} = B_1 R^{-1} B_2^T$$

$$S_2 = B_2 R^{-1} B_2^T. \quad (6)$$

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The presence of the small positive parameter  $\epsilon$  makes the problem numerically ill-conditioned. We will utilize the method for solving the differential Riccati equation based on the differential Lyapunov equation [10], which requires the decomposition of the differential Lyapunov equations into the reduced-order differential Lyapunov and Sylvester equations.

### III. DIFFERENTIAL LYAPUNOV EQUATION APPROACH

Before presenting the main results, we will review the solution of the differential Riccati equation via the differential Lyapunov equation [10].

Let  $K^-$  be the negative definite solution of the algebraic Riccati equation

$$K^-A + A^TK^- - K^-SK^- + Q = 0. \quad (7)$$

One can compute the negative definite solution  $K^-$  by finding the positive definite solution of the algebraic equation

$$-K_nA - A^TK_n - K_nSK_n + Q = 0. \quad (8)$$

It is clear that  $K_n = -K^-$ . Note that the unique negative definite anti-stabilizing solution of (7) exists under the standard controllability observability conditions [11], which can be relaxed to stabilizability and observability conditions [7], [12]. As a result, we need the following assumption.

*Assumption 1:*  $(A, B)$  is controllable and  $(A, C)$  is observable.

Subtracting (5) from (7), we obtain

$$-\dot{K}(t) = (K(t) - K^-)A + A^T(K(t) - K^-) - K(t)SK(t) + K^-SK^-. \quad (9)$$

Introduce the change of variables as

$$P_0(t) = (K(t) - K^-)^{-1}. \quad (10)$$

The differential equation (9) becomes

$$\dot{P}_0(t) = A_0P_0(t) + P_0(t)A_0^T - S \quad (11)$$

with

$$P_0(t_f) = (F - K^-)^{-1} \quad (12)$$

and

$$A_0 = A - SK^-. \quad (13)$$

Let  $E$  be the solution of the following algebraic Lyapunov equation and

$$A_0E + EA_0^T - S = 0. \quad (14)$$

This equation can be efficiently solved by the method of Hammarling [13], [14]. Then the solution of the differential Lyapunov (11) can be obtained as

$$P_0(t) = e^{A_0(t-t_f)}(P_0(t_f) - E)e^{A_0^T(t-t_f)} + E. \quad (15)$$

The solution of the original differential Riccati (5) is given by

$$K(t) = K^- + P_0^{-1}(t). \quad (16)$$

It is shown that the solution is well-defined since  $P_0(t)$  is invertible for all  $0 \leq t \leq t_f$  [10].

### IV. APPLICATION TO SINGULARLY PERTURBED SYSTEMS

In this section, we will employ the differential Lyapunov equation approach to address the finite horizon linear-quadratic optimal control problem for the singularly perturbed linear time invariant system (1).

Partition the solution of the algebraic Riccati (7) as

$$K^- = \begin{bmatrix} K_1^- & \epsilon K_2^- \\ \epsilon K_2^{-T} & \epsilon K_3^- \end{bmatrix}. \quad (17)$$

The matrix  $K^-$  in (7) or  $K_n$  in (8) can be found by using either the fixed-point iterations [9], Hamiltonian method [15] or eigenvector approach [16] in terms of reduced-order pure-slow and pure-fast algebraic Riccati equations.

Matrix  $A_0$  from (13) has the form shown in (18), at the bottom of the page.

Due to the structures of  $A_0$  and  $S$ , the matrix differential equation (11) is the singularly perturbed differential Lyapunov equation that consists of coupled fast and slow dynamics. We will employ the Chang transformation [1], [2] to decouple the differential equation (11) into the reduced-order Lyapunov and Sylvester equations

$$T = \begin{bmatrix} I_{n_1} - \epsilon HL & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \quad (19)$$

and

$$T^{-1} = \begin{bmatrix} I_{n_1} & \epsilon H \\ -L & I_{n_2} - \epsilon LH \end{bmatrix} \quad (20)$$

where  $L$  and  $H$  are the solutions of the following algebraic equations

$$A_{04}L - A_{03} - \epsilon L(A_{01} - A_{02}L) = 0 \quad (21)$$

$$-H(A_{04} + \epsilon LA_{02}) + A_{02} + \epsilon(A_{01} - A_{02}L)H = 0. \quad (22)$$

$$\begin{aligned} A_0 &= \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix} - \begin{bmatrix} S_1 & \frac{1}{\epsilon}S_{12} \\ \frac{1}{\epsilon}S_{12}^T & \frac{1}{\epsilon^2}S_2 \end{bmatrix} \begin{bmatrix} K_1^- & \epsilon K_2^- \\ \epsilon K_2^{-T} & \epsilon K_3^- \end{bmatrix} \\ &= \begin{bmatrix} A_1 - (S_1K_1^- + S_{12}K_2^{-T}) & A_2 - (\epsilon S_1K_2^- + S_{12}K_3^-) \\ \frac{1}{\epsilon}(A_3 - S_{12}^TK_1^- - S_2K_2^{-T}) & \frac{1}{\epsilon}(A_4 - \epsilon S_{12}^TK_2^- - S_2K_3^-) \end{bmatrix} \\ &= \begin{bmatrix} A_{01} & A_{02} \\ \frac{1}{\epsilon}A_{03} & \frac{1}{\epsilon}A_{04} \end{bmatrix} \end{aligned} \quad (18)$$

*Remark 1:* These algebraic equations can be iteratively solved using methods such as the Newton method [7], the fixed point iteration [9] and the eigenvector approach [16]. Methods of [7], [9], [16] require that  $A_{04}$  is invertible. By a similar argument as in [3, Lemma 1], we can prove that the eigenvalues of  $A_4 - S_2 K_3^-$  have positive real parts with the assumption that  $(A_4, B_2)$  is controllable and  $(A_4, C_2)$  is observable. In other words,  $A_{04} = A_4 - S_2 K_3^- + O(\epsilon)$  is nonsingular for small enough  $\epsilon$ .

We introduce a new variable as

$$P(t) = T P_0(t) T^T. \quad (23)$$

As a result, the differential Lyapunov equation becomes

$$\dot{P}(t) = a P(t) + P(t) a^T - q \quad (24)$$

where

$$a = T^{-1} A_0 T = \begin{bmatrix} A_s & 0 \\ 0 & \frac{A_f}{\epsilon} \end{bmatrix} \quad (25)$$

$$A_s = A_{01} - A_{02} L \quad (26)$$

$$A_f = A_{04} + \epsilon L A_{02} \quad (27)$$

$$q = T S T^T = \begin{bmatrix} q_1 & \frac{1}{\epsilon} q_2 \\ \frac{1}{\epsilon} q_2^T & \frac{1}{\epsilon^2} q_3 \end{bmatrix} \quad (28)$$

$$q_1 = (I_{n_1} - \epsilon H L) (S_1 - \epsilon S_1 L^T H^T - S_{12} H^T) - H (S_{12}^T - \epsilon S_{12}^T L^T H^T - S_2 H^T) \quad (29)$$

$$q_2 = (I_{n_1} - \epsilon H L) (\epsilon S_1 L^T + S_{12}) - H (\epsilon S_{12}^T L^T + S_2) \quad (30)$$

$$q_3 = \epsilon^2 L S_1 L^T + \epsilon (L S_{12} + S_{12}^T L^T) + S_2. \quad (31)$$

The solution of (24) has the form

$$P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & \frac{1}{\epsilon} P_3(t) \end{bmatrix}.$$

Partitioning (24) according to (25)–(31) results in the following system of completely decoupled reduced-order differential Lyapunov/Sylvester equations

$$\dot{P}_1(t) = A_s P_1(t) + P_1(t) A_s^T - q_1 \quad (32)$$

$$\epsilon \dot{P}_2(t) = \epsilon A_s P_2(t) + P_2(t) A_f^T - q_2 \quad (33)$$

$$\epsilon \dot{P}_3(t) = A_f P_3(t) + P_3(t) A_f^T - q_3 \quad (34)$$

with the terminal condition

$$P(t_f) = T (F - K^-)^{-1} T^T = \begin{bmatrix} P_1(t_f) & P_2(t_f) \\ P_2^T(t_f) & \frac{1}{\epsilon} P_3(t_f) \end{bmatrix}. \quad (35)$$

To solve (32)–(34), we can use the method in [17], or any other method for solving the differential Lyapunov equation [13].

Let  $E_1$  be the solution of the algebraic Lyapunov equation

$$0 = A_s E_1 + E_1 A_s^T - q_1. \quad (36)$$

According the method in [17], the solution of (32) is given by

$$P_1(t) = e^{A_s(t-t_f)} (P_1(t_f) - E_1) e^{A_s^T(t-t_f)} + E_1. \quad (37)$$

Equations (33) and (34) can be solved using the same technique. Let  $E_2$  be the solution of the algebraic Sylvester equation

$$0 = \epsilon A_s E_2 + E_2 A_f^T - q_2. \quad (38)$$

Then, the solution of (33) is given by

$$P_2(t) = e^{A_s(t-t_f)} (P_2(t_f) - E_2) e^{A_f^T(t-t_f)/\epsilon} + E_2. \quad (39)$$

Let  $E_3$  be the solution of the algebraic equation

$$0 = A_f E_3 + E_3 A_f^T - q_3. \quad (40)$$

So, (34) has the solution

$$P_3(t) = e^{A_f(t-t_f)/\epsilon} (P_3(t_f) - E_3) e^{A_f^T(t-t_f)/\epsilon} + E_3. \quad (41)$$

Putting together all results obtained from (37), (39), and (41), we arrive at the solution of the original problem

$$\begin{aligned} K(t) &= K^- + P_0^{-1}(t) = K^- + T^T P^{-1}(t) T \\ &= K^- + T^T \begin{bmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & \frac{1}{\epsilon} P_3(t) \end{bmatrix}^{-1} T. \end{aligned} \quad (42)$$

It is seen that the inverse of  $P(t)$  in (42) may be ill-conditioned due to the presence of  $\epsilon$ . To overcome this, we can calculate it by the following formula:

$$P^{-1}(t) = V(t) = \begin{bmatrix} V_1(t) & \epsilon V_2(t) \\ \epsilon V_2^T(t) & \epsilon V_3(t) \end{bmatrix} \quad (43)$$

where

$$\begin{aligned} V_1(t) &= P_1^{-1}(t) [I_{n_1} + \epsilon P_2(t) (P_3(t) \\ &\quad - \epsilon P_2^T(t) P_1^{-1}(t) P_2(t))^{-1} P_2^T(t) P_1^{-1}(t)] \\ V_2(t) &= -P_1^{-1}(t) P_2(t) \left[ P_3(t) - \epsilon P_2^T(t) P_1^{-1}(t) P_2(t) \right]^{-1} \\ &\quad \text{and} \\ V_3(t) &= \left[ P_3(t) - \epsilon P_2^T(t) P_1^{-1}(t) P_2(t) \right]^{-1}. \end{aligned}$$

Hence, (42) can be rewritten as

$$K(t) = K^- + T^T \begin{bmatrix} V_1(t) & \epsilon V_2(t) \\ \epsilon V_2^T(t) & \epsilon V_3(t) \end{bmatrix} T. \quad (44)$$

One may be concerned with the invertibility of  $P_1(t)$  and  $P_3(t)$ . The following theorem will answer that issue.

*Theorem 1:*  $P_1(t)$  and  $P_3(t)$  are positive definite for all  $t \leq t_f$ .

*Proof:* We follow the proof of [10, Theorem 1]. In this way, the theorem will be proved if the eigenvalues of  $A_s$  and  $A_f$  have positive real parts, and  $q_1$  and  $q_3$  are positive semidefinite. Indeed, it is pointed out that  $A_0$  is anti-stable or all eigenvalues of  $A_0$  have positive real parts [10]. Hence, its equivalent matrix (25) implies that  $A_s$  and  $A_f$  are anti-stable. On the other hand, since  $S$  is positive semidefinite, so is  $q$ . For all nonzero vectors whose last  $n_2$  components are zero, we have

$$v^T q v = \begin{bmatrix} v_1^T & 0 \end{bmatrix} q \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = v_1^T q_1 v_1 \geq 0.$$

In other words,  $q_1$  is positive semidefinite. Similarly, the positive semidefinite property of  $q_3$  holds. ■

*Remark 2:* Since  $P_1(t)$  and  $P_3(t)$  are nonsingular,  $V_1(t)$ ,  $V_2(t)$ , and  $V_3(t)$  are well-defined for small  $\epsilon$ . As a consequence, the ill-conditioned issue of (42) is avoided by using (44). This demonstrates that our solution is stable with respect to  $\epsilon$ .

*Remark 3:* The numerical error of the proposed method is not straightforward to analyze. First, our solution depends on the computation of the solution of the algebraic Riccati equation and matrix exponentials.

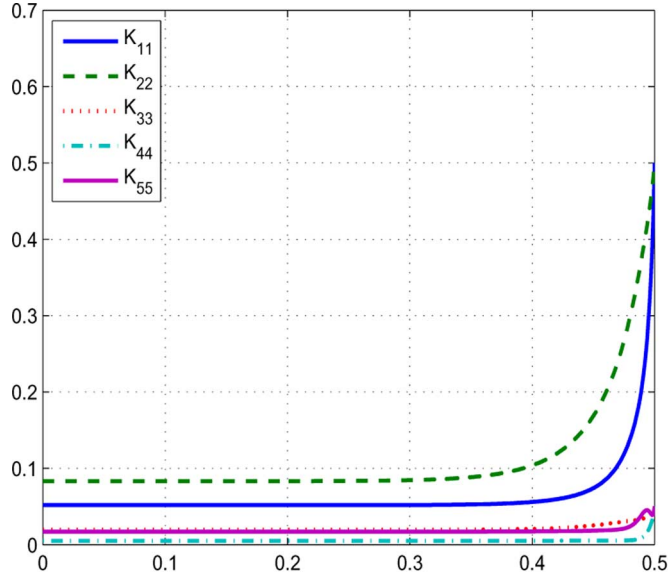
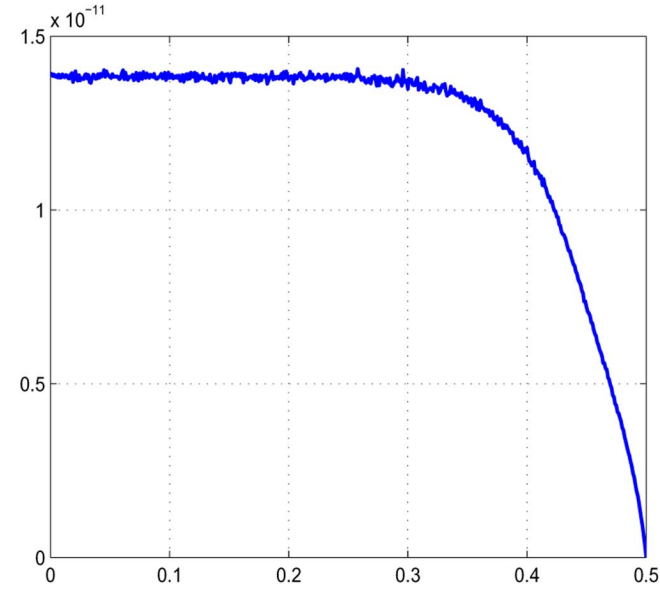
Fig. 1. Evolution of the diagonal elements of  $K(t)$  for the case of  $\epsilon = 0.1$ .

Fig. 2. Normwise relative errors for the new method.

Second, the solution of the Chang transformation equations also influence the accuracy of our solution. To make a rough estimate, assume that the errors for computing the solution of the algebraic Riccati (6) and matrix exponentials are small enough to neglect. Assume that the error of the solutions of the Chang transformation (20) and (21) is  $O(\epsilon^p)$  where  $p$  is a positive integer depending on the number of iterations used to calculate their solutions [7]. Then, the errors to compute  $A_s, A_f, q_1, q_2, q_3, P_1, P_2, P_3$  are  $O(\epsilon^p)$ . As a result, the final solution (44) has an error of  $O(\epsilon^p)$ .

The presented results are summarized in the form of the following algorithm.

- 1) Find the negative definite solution  $K^-$  of the algebraic Riccati (7) by calculating the positive solution  $K_n$  of the algebraic Riccati (8). One may use the reduced-order decomposition techniques of [9], [16].

- 2) Compute the solutions  $E_1, E_2$ , and  $E_3$  of the corresponding algebraic Lyapunov (37), (39), and (41).
- 3) Evaluate matrix exponentials

$$\Delta E_s = e^{-A_s \Delta t} \quad (45)$$

and

$$\Delta E_f = e^{-A_f \Delta t / \epsilon} \quad (46)$$

where  $\Delta t > 0$  is the step size.

- 4) For  $k = 1$  to  $(l - 1)$ , where  $l$  is the number of steps in the time interval  $[0, t_f]$ , with  ${}^l P = P(t_f) = T(F - K^-)^{-1}T^T$ , calculate

$${}^{l-k} P_1 = \Delta E_s \left( {}^{l-(k-1)} P_1 - E_1 \right) \Delta E_s^T + E_1 \quad (47)$$

$${}^{l-k} P_2 = \Delta E_s \left( {}^{l-(k-1)} P_2 - E_2 \right) \Delta E_f^T + E_2 \quad (48)$$

and

$${}^{l-k} P_3 = \Delta E_f \left( {}^{l-(k-1)} P_3 - E_3 \right) \Delta E_f^T + E_3. \quad (49)$$

- 5) Calculate the solution of the original equation according to (44)

$${}^{l-k} K = K^- + T^T \begin{bmatrix} {}^{l-k} V_1 & \epsilon \begin{pmatrix} {}^{l-k} V_2 \\ {}^{l-k} V_3 \end{pmatrix} \\ \epsilon \begin{pmatrix} {}^{l-k} V_2^T \\ {}^{l-k} V_3^T \end{pmatrix} & \epsilon \begin{pmatrix} {}^{l-k} V_2 \\ {}^{l-k} V_3 \end{pmatrix} \end{bmatrix} T. \quad (50)$$

where  ${}^k K = K(k \Delta t)$ .

*Remark 4:* The time step  $\Delta t$  is often chosen to be small. Assume that  $\Delta t \gg \epsilon$ . Then  $\Delta t / \epsilon$  is large. Since  $A_f$  is anti-stable,  $-A_f \Delta t / \epsilon$  is stable. This implies that  $e^{-A_f \Delta t / \epsilon}$  is very small. As a result, there is no ill-conditioned problem associated with  $\Delta t$  and  $\epsilon$  here.

The proposed method in this note provides a simpler way than the approaches in [7], [8]. All of them use the Chang transformation to decompose the full-order systems into reduced-order subsystems. However, the approaches in [7], [8] require solving two algebraic equations of dimensions  $(2n_2 \times 2n_1)$  and  $(2n_1 \times 2n_2)$ . Meanwhile, our approach only asks for the solutions of the two algebraic equations of dimensions  $(n_2 \times n_1)$  and  $(n_1 \times n_2)$ . Furthermore, the Hamiltonian approaches in [7], [8] requires solving a differential equation of dimensions  $2(n_1 + n_2) \times 2(n_1 + n_2)$ . In contrast, our method only requires solving the differential Lyapunov equation of dimension  $(n_1 + n_2) \times (n_1 + n_2)$ .

Both methods in [7], [8] exploit the Hamiltonian form of the solution of the Riccati equation. In addition to reduced-order pure-slow and pure-fast Riccati equations, the method in [8] requires solving two reduced-order initial value problems. Hence, its computational cost is larger than that of the method in [7]. The startup costs for the method of [7] consists of computational costs for calculating the matrix exponentials of order  $n_1$  and  $n_2$  and solving for the Chang transformation equations. A matrix exponential can be efficiently computed by the scaling and squaring method [18]. If the Hamiltonian approach in [7] is implemented in a recursive way, at each step, the solution is evaluated taking  $4n^3 + (4/3)n^3$  multiplications.

The proposed method's startup costs are due to forming the matrix exponentials of order  $n_1$  and  $n_2$  in (43), (44), and computing the Chang transformation in (21), (22). At each step, (47), (48), and (49) are evaluated taking  $n_1^3 + n_1^2(n_1 + 1)/2$ ,  $n_1^2 n_2 + n_1 n_2^2$  and  $n_2^3 + n_2^2(n_2 + 1)/2$  multiplications respectively. Furthermore, step (49) needs  $(2/3)n^3$  multiplications for the matrix inversion and  $n^3 + (n^2(n + 1))/2$  multiplications for matrix multiplications. Totally, we only use  $3/2(n_1^3 + n_2^3) + n_1^2 n_2 + n_1 n_2^2 + (n_1^2 + n_2^2)/2 + 2/3(n_1 + n_2)^3 + 3/2(n_1 + n_2)^3 + (n_1 + n_2)^2/2$  multiplications

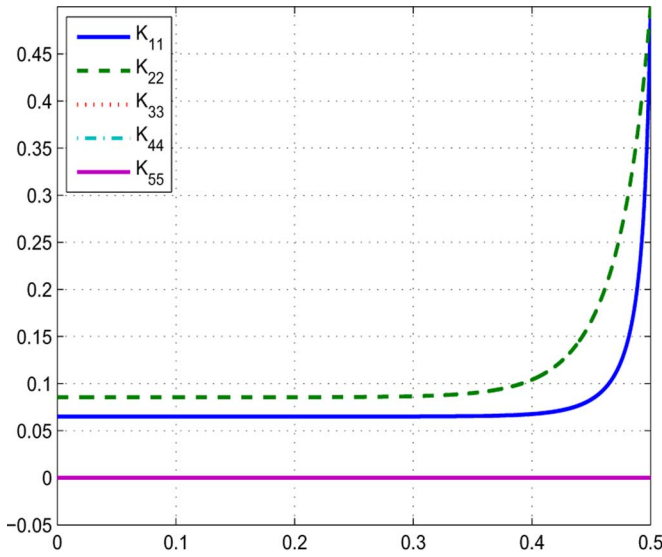


Fig. 3. Evolution of the diagonal elements of  $K(t)$  for the case of  $\epsilon = 10^{-7}$ .

for each step to compute the solution  $K(t)$ . This is smaller than the computational costs in the Hamiltonian approaches in [7], [8].

## V. NUMERICAL EXAMPLE

We use the example of a fluid catalytic reactor [8], [19] to illustrate the proposed approach. The fluid catalytic cracker is a fifth-order system with three fast variables and two slow variables.

Matrix  $A$  and matrix  $B$  are, respectively, given by

$$A_1 = \begin{bmatrix} -16.00 & -0.39 \\ 0.01 & -16.99 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 27.2 & 0 & 0 \\ 0 & 0 & 12.47 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1.511 & 0 \\ -5.336 & 0 \\ 0.227 & 6.91 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -5.36 & -1.657 & 7.178 \\ 0 & -10.72 & 23.211 \\ 0 & .2273 & -10.299 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 11.12 & -12.6 \\ -3.61 & 3.36 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2.191 & 0 \\ -5.36 & 0 \\ 6.91 & 0 \end{bmatrix}.$$

The cost weighting matrices are chosen as  $Q = I$  and  $R = I$ , where  $I$  is the identity matrix. The terminal time penalty matrix is taken as

$$F = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & \epsilon 0.5 & 0 & 0 \\ 0 & 0 & 0 & \epsilon 0.5 & 0 \\ 0 & 0 & 0 & 0 & \epsilon 0.5 \end{bmatrix}.$$

The small positive singular perturbation parameter is chosen as  $\epsilon = 0.1$ . The time step is  $\Delta t = 0.001$ . We plot the diagonal elements of the solution  $K(t)$  in Fig. 1. The Kalman-Englar method [12] is employed as a benchmark for comparison. The normwise relative errors which are calculated by  $\|K_{KM}(t) - K(t)\|_1 / \|K_{KM}(t)\|_1$ , and depicted in Fig. 2. It can be seen that the errors are  $O(10^{-11})$ .

Now, consider the case of  $\epsilon = 10^{-7}$ . Fig. 3 shows that our method is still effective in providing a good numerical solution  $K(t)$ . Meanwhile, the Kalman-Englar method fails due to the inversion of near-singular

matrices although a very small time step can be chosen. The method by Grodt and Gajic [7] is also unsuccessful to obtain a good numerical result. This case shows that the proposed method is very stable with respect to  $\epsilon$ .

## VI. CONCLUSION

A method has been proposed for the finite horizon optimal control problem for singularly perturbed linear time-invariant systems. Instead of directly looking for the solution of the singularly perturbed matrix differential Riccati equation, we exploit the structure of the differential Lyapunov equation and decompose it into the reduced-order differential Lyapunov and Sylvester equations, from which the final solution of the original problem is synthesized. It should be emphasized that our method is simpler and requires less computations than the previous methods reported in [7] and [8].

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