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Solution of the Matrix Riccati Equation for the Linear Quadratic Control Problems

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Abstract—This paper is concerned with the solution of the matrix Riccati differential equation with a terminal boundary condition. The solution of the matrix Riccati equation is given by using the solution of the algebraic form of the Riccati equation. An illustrative example for the proposed method is given.

Keywords—Matrix, Riccati, Solution, Control, Optimal.

1. INTRODUCTION

One of the most intensely studied nonlinear matrix equations arising in mathematics and engineering is the Riccati equation. This equation, in one form or another, has an important role in optimal control problems, multivariable and large scale systems, scattering theory, estimation, detection, transportation, and radiative transfer [1,2]. It is known that the boundedness of the solution of the matrix Riccati differential equation (MRDE) with a terminal boundary condition is equivalent to the "no-conjugate point to the final time" [3-5]. The solution of this equation is difficult to obtain from two points of view. One is that it is nonlinear, and the other is that it is in matrix form. Most general methods to solve (MRDE) with a terminal boundary condition are obtained on transforming (MRDE) into an equivalent linear differential Hamiltonian system [6]. By using this approach, the solution of (MRDE) is obtained by partitioning the transition matrix of the associated Hamiltonian system [2,7,8]. Another class of methods is based on transforming (MRDE) into a linear matrix differential equation and then solving (MRDE) analytically or computationally [9–11]. In [9,10], the problem is transformed into examining matrix differential equations of the type which have been studied in the fields of free vibration theory and aircraft

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX flutter analysis. These equations are solved analytically [9] or computationally [10]. However, the method in [9,10] is restricted for cases when certain coefficients of (MRDE) are nonsingular. In [11], an analytic procedure of solving the (MRDE) of the linear quadratic control problem for homing systems is presented. In [11], a solution $K(t)$ of the (MRDE) is obtained by using $K(t) = P(t)/f(t)$, where $f(t)$ and $P(t)$ are solutions of certain first-order ordinary linear differential equations. However, the given technique in [11] is restricted to single input.

The present paper presents a simple means of solving (MRDE) with a terminal boundary condition for the linear quadratic optimal control problems. This method transforms the problem into examining matrix differential equations of the Liapunov type which can be solved using Kronecker product [12]. An illustrative example is given to demonstrate the applicability of the proposed method.

2. STATEMENT OF THE PROBLEM

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \tag{1}$$

where $x(t)$ and $u(t)$ are $n \times 1$ and $m \times 1$ state and control vectors respectively, and A and B are constant matrices of appropriate dimensions. The optimal control problem would be to find an optimal control $u(t)$ satisfying (1) while minimizing the quadratic cost functional,

$$J = \frac{1}{2}x^{\top}(t_f)Sx(t_f) + \frac{1}{2}\int_0^{t_f} \left[x^{\top}(t)Qx(t) + u^{\top}(t)Ru(t)\right] dt, \tag{2}$$

where t_f is the final time which is given, S, Q, and R are constant matrices of appropriate dimensions, R is symmetric positive definite, while S and Q are symmetric positive semidefinite matrices and $\top$ denotes transposition. It is known that the optimal control is given by [13]

$$u(t) = -R^{-1}B^{\top}K(t)x(t), \tag{3}$$

where $K(t)$ is the solution of the (MRDE).

$$\dot{K}(t) = -K(t)A - A^{\top}K(t) + K(t)D_1K(t) - Q, \tag{4}$$

$$K(t_f) = S, \tag{5}$$

where

$$D_1 = BR^{-1}B^{\top}. \tag{6}$$

Usually, the infinite time problem is considered; in this case $\dot{K}(t) = 0$, and the Riccati differential equation is reduced to an algebraic equation:

$$0 = -K_sA - A^{\top}K_s + K_sD_1K_s - Q. \tag{7}$$

The solution of the algebraic form of Riccati equation has been considered in [1,14]. The aim of the present work is to obtain the solution of equation (4) with terminal condition given in equation (5).

3. SOLUTION OF THE MATRIX RICCATI DIFFERENTIAL EQUATION

To solve equation (4), subtracting equation (4) from equation (7) results in

$$-\dot{K}(t) = (K(t) - K_s)A + A^{\top}(K(t) - K_s) - K(t)D_1K(t) + K_sD_1K_s. \tag{8}$$

Equation (8) can be written as

$$-\dot{K}(t) = K_1(t)A_c + A_c^\top K_1(t) - K_1(t)D_1K_1(t), \tag{9}$$

where

$$A_c = A - D_1K_s, \qquad K_1(t) = K(t) - K_s. \tag{10}$$

Using equations (5) and (10), we obtain

$$K_1(t_f) = S - K_s. \tag{11}$$

To solve equation (9), we assume $K_1(t_f)$ is invertible and we let

$$P^{-1}(t) = K_1(t). \tag{12}$$

On differentiating equation (12), we obtain

$$\dot{K}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t). \tag{13}$$

Substituting equations (12) and (13) in equation (9), we get

$$\dot{P}(t) = P(t)A\top_c + A_cP(t) - D_1. \tag{14}$$

Using equations (11) and (12), we obtain

$$P(t_f) = (S - K_s)^{-1}. \tag{15}$$

By letting $t_1 = t_f - t$, equations (14) and (15) can be written as

$$\dot{P}(t_1) = -P(t_1)A\top_c - A_cP(t) + D_1, \tag{16}$$
$$P(0) = (S - K_s)^{-1}. \tag{17}$$

Equation (16), with the initial condition given in equation (17), is of the form of linear Liapunov differential equation, which can be solved using Kronecker product [12] as follows:

$$\dot{P}_v(t_1) = -[A_c \otimes I + I \otimes A_c]P_v(t_1) + (D_1)_v, \tag{18}$$

where $P_v(t_1)$ and $(D_1)_v$ are the vector form for the matrices $P(t_1)$ and D_1 and $\otimes$ denotes the Kronecker product. Equation (18) can be written as

$$\dot{P}_v(t_1) = -A_1P_v(t_1) + (D_1)_v, \tag{19}$$

where

$$A_1 = [A_c \otimes I + I \otimes A_c]. \tag{20}$$

To solve equation (19), since $P(t_1)$, A_1, and D_1 are symmetric matrices, we only need to solve $n(n+1)/2$ linear differential equations. Further, the transient time t_2 for equation (20) is given in [15] as

$$t_2 = \frac{2.5}{\min|\lambda_i(A_c)|}, \qquad i = 1, 2, \dots, n, \tag{21}$$

where $\lambda_i(A_c), i = 1, 2, \dots, n$ denote the eigenvalues for the matrix A_c. Furthermore, the transient time for the system in equation (1) is given by $2t_2$, thus, when we have [15]

$$t_f > \frac{7.5}{\min|\lambda_i(A_c)|}. \tag{22}$$

The optimal control in equation (3) can be obtained by solving the algebraic Riccati equation in equation (7) rather than the (MRDE) given in equation (4). The results obtained are illustrated by the following example.

5. NUMERICAL EXAMPLE

Consider minimizing the cost functional

$$J = \frac{1}{2}x^{\mathrm{T}}(t_f)\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}x(t_f) + \frac{1}{2}\int_0^{t_f}\left[x^{\mathrm{T}}(t)\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}x(t) + u^2(t)\right]dt, \tag{23}$$

subject to the conditions

$$\dot{x}(t) = \begin{bmatrix}9 & 1\\ -1 & 2\end{bmatrix}x(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \qquad x(0) = x_0. \tag{24}$$

In this problem, using equations (7) and (10), we get

$$K_s = \begin{bmatrix}5.6955 & 0.4142\\ 0.4142 & 4.6131\end{bmatrix}, \qquad A_c = \begin{bmatrix}0 & 1\\ -1.4142 & -2.6131\end{bmatrix}. \tag{25}$$

Using equation (19), we get

$$\dot{P}_v(t) = \begin{bmatrix}0 & -1 & -1 & 0\\ 1.4142 & 2.6131 & 0 & -1\\ 1.4142 & 0 & 2.6131 & -1\\ 0 & 1.4142 & 1.4142 & 5.2263\end{bmatrix}P_v(t) + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}. \tag{26}$$

Applying equation (17), the initial condition is given by

$$P_v(0) = [-1.5190 \quad 0.1364 \quad 01364 \quad -0.2290]^{\mathrm{T}}. \tag{27}$$

Using equation (26), it can be seen that we need to solve three linear differential equations. By solving these differential equations the solution for equation (4) is obtained from

$$K_{11}(t) = 5.6966 + \frac{1}{\det(K_1(t))}(-0.1913 - 2.1722E_1 + 3.9888E_2 - 1.8502E_3),$$

$$K_{12}(t) = K_{21}(t) = 0.4142 + \frac{1}{\det(K_1(t))}(-22.7663E_1 + 3.6311E_2 - 1.0012E_3),$$

$$K_{22}(t) = 4.6131 + \frac{1}{\det(K_1(t))}(-0.1353 - 3.6142E_1 + 2.7794E_2 - 0.5418E_3),$$

where $\det(K_1(t))$ is determinant of matrix $K_1(t)$ and

$$E_1 = \exp(1.5308(t_f - t)),$$
$$E_2 = \exp(2.6128(t_f - t)),$$
$$E_3 = \exp(3.6957(t_f - t)).$$

The eigenvalues of A_c in equation (25) are given by

$$\lambda_1(A_c) = -1.8478, \qquad \lambda_2(A_c) = -0.7654. \tag{28}$$

Using equations (22) and (28), if t_f is greater than 9.7992, then the optimal control can be calculated using K_s given in equation (25).

6. CONCLUSION

A method is proposed for the solution of the matrix Riccati equation which occurs in calculation of the optimal control for linear systems subject to quadratic cost criteria. This method transforms the problem into examining the linear Liapunov matrix differential equations. An example illustrating the concept involved is included.

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