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MATRIX RICCATI DIFFERENTIAL EQUATIONS*

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1. Introduction. We assume at the outset that coefficients of our equations are real-valued and continuous on the real line.

The Riccati equation

$$y' = p(t)y^2 + q(t)y + r(t) \tag{1}$$

has the following well-known properties:

(I) If one solution is known, (1) is reducible to a linear equation; the complete solution is obtainable by means of two quadratures.

(II) If two distinct solutions are known, say u and v, the complete solution is obtainable by means of one quadrature, and is given by

$$\frac{y-u}{y-v} = c \exp \int p(t)(u-v), \qquad c = \text{const.}$$

(III) If three distinct solutions are known, say u, v, and w, the complete solution is given by

$$\frac{(y-u)(w-v)}{(y-v)(w-u)} \equiv c, \qquad c \equiv \text{const};$$

i.e., the cross-ratio of y, u, w, and v is constant.

If $A(t)$ is a 2×2 matrix and $x(t)$ is a vector satisfying $x' = Ax$, then x_1/x_2 satisfies (1) with $p = -a_{21}$, $q = a_{11} - a_{22}$, and $r = a_{12}$. Thus consideration of the functions $y_i = x_i/x_n$, $1 \leqq i < n$, where $x' = A(t)x$ and $A(t)$ is $n \times n$, leads to the following generalization of (1):

$$y_i' = -y_i \sum_{k=1}^{n-1} a_{nk} y_k + \sum_{k=1}^{n-1} (a_{ik} - \delta_{ik} a_{nn}) y_k + a_{in}, \quad 1 \leqq i < n. \tag{2}$$

Chiellini [1] considered this system, and showed that knowledge of n solutions, not on the same $(n-2)$-flat, reduced the solution to quadratures (this generalizes (I)). In [2] it was shown that knowledge of k suitably independent solutions, $1 \leqq k < n$, reduces the solution to k quadratures and the solution of a matrix-vector linear homogeneous system of size $n - k$ (this generalizes (II) via Chiellini's result, and also generalizes (I)). In [2] it was also shown that the complete solution of (2) can be written in terms of $n + 1$ solutions, no n of which lie on the same $(n-2)$-flat; no quadratures are needed (this generalizes (III)).

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Another natural extension of (1) replaces the scalar functions by matrix functions of compatible dimensions:

$$(3) \qquad Y' = KY + L - YMY - YN.$$

Reid [5] considered (3) for $n \times n$ matrices; among his results was a generalization of (III). Levin [4] considered (3) for $n_1 \times n_2$ matrices Y; for $n_1 = n_2$ he obtained a generalization of (III) different from Reid's. Now (2) can be written in the form (3) by letting $Y = \text{col}\ (y_1 \cdots y_{n-1})$, $L = \text{col}\ (a_{1n} \cdots a_{n-1,n})$, $M = (a_{n1} \cdots a_{n,n-1})$, $N = a_{nn}$, and $K = A$ with row n and column n deleted. Since $n_1 = n - 1$ and $n_2 = 1$ in this case, the generalization of (III) in [2] differs from those of Reid and Levin, at least if $n > 2$.

Now let $n = \sum_{h=1}^{m} n_h$ and $A = \text{col}\ (A_1 \cdots A_m)$, where A_h is $n_h \times n$. The equation

$$(4) \qquad U' = AU - UA_hU$$

is a special case of the form (3), with $K = A$, $M = A_h$, and $L = N = 0$. In [3] it was shown that the form (4) is actually no less general than (3); indeed, if Y satisfies (3) then $U = \text{col}\ (Y\ E)$ satisfies (4) with

$$A = \begin{pmatrix} K & L \\ M & N \end{pmatrix}, \quad A_h = (M \quad N),$$

and so on. From this point of view, (2) can be written

$$(5) \qquad u' = Au - ua_nu, \qquad u_n(t_0) = 1, \qquad a_n = (a_{n1} \cdots a_{nn});$$

a solution is $u = \text{col}\ (y_1 \cdots y_{n-1}\ 1)$. Thus any system (3) is equivalent to a system (4), together with an initial condition $Y_h(t_0) = E$ (this restriction is nontrivial, and implies that $Y_h(t) \equiv E$).

It is clear from the preceding remarks that systems of the form

$$(6) \qquad Y' = AY - YBY, \qquad (Y \text{ is } n \times m),$$

and

$$(7) \qquad u' = Au - u\alpha u, \qquad (u \text{ is } n \times 1),$$

are of particular interest. This paper considers such systems, without any initial-value restrictions. §2 notes some relations between (6) and (7); in particular, it is established that each equation has an expression for the complete solution, and that the complete solution of (6) (for *any* B) can be obtained by quadratures from the complete solution of (7) (for *any* α). With the importance of (7) thus re-emphasized, §3 considers (7) in some detail, and establishes some new generalizations of (II) and (III).

2. Equivalence of Riccati systems.

THEOREM 1. (i) *Let* u^i, $i = 1, \cdots, m$, *be solutions* (*not necessarily distinct*) *of* (7), *and let* $U = (u^1 \cdots u^m)$. *Let* v^i, $i = 1, \cdots, m$, *satisfy*

$$v^{i\prime} = Bu^i - v^i \alpha u^i; \tag{8}$$

let $V = (v^1 \cdots v^m)$, *and suppose that* $V(t_0)$ *is nonsingular. Let* $Y = UV^{-1}$. *Then* Y *satisfies* (6), *and* rank Y = rank U.

(ii) *Let* Y *be a solution of* (6); *let* v *satisfy*

$$v' = BY \cdot v - v \cdot \alpha Y \cdot v. \tag{9}$$

Let $u = Yv$; *then* u *satisfies* (7).

Proof. (i) We have $YV = U$. Now $V' = BU - V \cdot \operatorname{diag}(\alpha u^i)$, and so

$$Y'V + YBU - YV \cdot \operatorname{diag}(\alpha u^i) = U' = AU - U \cdot \operatorname{diag}(\alpha u^i),$$

and

$$Y' = AY - YBY.$$

(ii)
$$u = Yv;$$
$$u' = Y'v + Yv' = AYv - YBYv + YBYv - Yv \cdot \alpha Yv;$$
$$u' = Au - u\alpha u.$$

The point in (i) is that (8) is solvable by quadratures. Thus, for any continuous α and B, knowledge of a solution of (7) leads immediately to a solution of (6), via a quadrature. On the other hand, a solution of (6) and a solution of (9) (which is of the same *form* as (7), but of different dimension) lead immediately to a solution of (7). The statement of *equivalence* of (6) and (7) is contained in the following corollary, whose proof is immediate from Theorem 1.

COROLLARY 1. (i) *If the complete solution of* (7) *is known, the complete solution of* (6) *is obtainable by quadratures.*

(ii) *If the complete solution of* $\{(6), (9)\}$ *is known, the complete solution of* (7) *is known.*

COROLLARY 2. *The systems* $u' = Au - u\alpha u$ *and* $v' = Av - v\beta v$ *are equivalent; solutions of one can be obtained from solutions of the other by multiplying by a scalar which can be got by quadrature.*

The preceding corollary can be stated in slightly more general terms: if $Y' = AY - YBY$ and $W' = CY - WBY$ and W^{-1} exists, then YW^{-1} satisfies $Z' = AZ - ZCZ$.

For completeness, we give the following theorem, known for (5).

THEOREM 2. *Equation* (7) *and the system* $X' = AX$ *are equivalent, except for quadratures.*

Proof. If $X' = AX$ and X is nonsingular,

$$y(t) = \frac{X(t)\cdot c}{1 + \int_{t_0}^{t} \alpha(s)X(s)c\,ds}, \qquad c = \text{col}\,(c_1 \cdots c_n),$$

satisfies (7). On the other hand, if $k = \text{col}\,(k_1 \cdots k_n)$ is given, then $x = y\lambda$ satisfies $x' = Ax$, $x(t_0) = k$, provided y satisfies (7), $y(t_0) = k$, and $\lambda(t) = \exp \int_{t_0}^{t} \alpha y$.

3. The equation $u' = Au - u\alpha u$. We first establish a property of type (II).

THEOREM 3. *Let u^i, $i = 1, \cdots, n$, be solutions of* (7) *such that the matrix $U = (u^1 \cdots u^n)$ is nonsingular at t_0; let*

$$\lambda_i(t) = \exp \int_{t_0}^{t} \alpha u^i, \qquad i = 1, \cdots, n. \tag{10}$$

Then the complete solution of (7) *can be written* (*near* t_0) *as*

$$y(t) = \frac{\sum_{i=1}^{n} c_i\,\lambda_i(t)u^i(t)}{c_0 + \sum_{i=1}^{n} c_i\,\lambda_i(t)}, \qquad \sum_{i=0}^{n} c_i = 1. \tag{11}$$

(*For* (5), $c_0 = 0$.)

Proof. First, the function given by (11) is a solution of (7). Indeed,

$$(\lambda_i u^i)' = u^i\alpha u^i\lambda_i + \lambda_i(Au^i - u^i\alpha u^i) = A\lambda_i u^i;$$

hence

$$y'\cdot\left(c_0 + \sum_{i=1}^{n} c_i\,\lambda_i\right) + y\cdot\left(\sum_{i=1}^{n} c_i\,\lambda_i\,\alpha u^i\right) = \sum_{i=1}^{n} c_i\,A\lambda_i\,u^i,$$

and so

$$y' = Ay - y\alpha y.$$

On the other hand, let $u(t)$ be a solution of (7) such that $u(t_0) = k = \text{col}\,(k_1 \cdots k_n)$. Let $c = \text{col}\,(c_1 \cdots c_n)$ and c_0, determined by

$$k = U(t_0)c, \qquad c_0 = 1 - \sum_{i=1}^{n} c_i,$$

define a solution $y(t)$ by (11). Since $y(t_0) = k$, uniqueness (well-known for such systems) implies that $y \equiv u$. This completes the proof.

To see that Theorem 3 does give a property (II), let us compare (11) with the formula in (II) for a solution y of (1). Letting $c = -c_2/c_1$ and solving for y, we obtain

$$y(t) = \frac{c_1\, u(t) \exp\left(-\int_{t_0}^{t} pu\right) + c_2\, v(t) \exp\left(-\int_{t_0}^{t} pv\right)}{c_1 \exp\left(-\int_{t_0}^{t} pu\right) + c_2 \exp\left(-\int_{t_0}^{t} pv\right)}.$$

In this case,

$$A = \begin{pmatrix} q & r \\ -p & 0 \end{pmatrix}, \quad \alpha = a_2 = (-p \quad 0), \quad u^1 = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad u^2 = \begin{pmatrix} v \\ 1 \end{pmatrix},$$

$$\lambda_1 = \exp\left(-\int_{t_0}^{t} pu\right), \quad \text{and} \quad \lambda_2 = \exp\left(-\int_{t_0}^{t} pv\right);$$

thus the above expression is exactly of the form (11) for the equation of type (5) which corresponds to (1).

The form (11) involves n quadratures, one for each $\lambda_i(t)$. At the expense of requiring another solution of (7), one can obtain a form requiring only one quadrature.

THEOREM 4. *Let* u^i, $i = 1, \cdots, n+1$, *be solutions of* (7) *such that, if* $U_0 = (u^1 \cdots u^n)$, $U_i = (u^1 \cdots u^{i-1}\, u^{n+1}\, u^{i+1} \cdots u^n)$, $i = 1, \cdots, n$, *and* $D_i = \det U_i$, $i = 0, \cdots, n$, *then* $D_i(t_0) \neq 0$. *Let*

$$\lambda(t) = \exp \int_{t_0}^{t} \left(\operatorname{tr} A - \sum_{i=1}^{n+1} \alpha u^i\right). \tag{12}$$

Then the complete solution of (7) *can be written* (*near* t_0) *as*

$$y(t) = \frac{\sum_{i=1}^{n} c_i\, D_i(t) u^i(t)}{c_0\, \lambda(t) + \sum_{i=1}^{n} c_i\, D_i(t)}, \quad 1 = c_0 + \sum_{i=1}^{n} c_i\, D_i(t_0). \tag{13}$$

(*For* (5), $c_0 = 0$.)

Proof. Let λ_i, $1 \leqq i \leqq n+1$, be defined by (10); let $\Lambda = \operatorname{diag}(\lambda_1 \cdots \lambda_n)$. Then

$$\det \Lambda = \prod_{i=1}^{n} \lambda_i, \quad (\det \Lambda)' = \left(\prod_{i=1}^{n} \lambda_i\right) \sum_{i=1}^{n} \alpha u^i.$$

Now, if $X = U_0 \Lambda$, it is easily verified that $X' = AX$, so that

$$\det(U_0\, \Lambda) = [\det(U_0(t_0)\Lambda(t_0))] \cdot \exp \int_{t_0}^{t} \operatorname{tr} A;$$

hence

$$D_0(t) \cdot \prod_{i=1}^{n} \lambda_i(t) = D_0(t_0) \cdot \exp \int_{t_0}^{t} \operatorname{tr} A,$$

and

$$D_0(t) \equiv D_0(t_0)\lambda(t)\lambda_{n+1}(t).$$

Similarly

$$D_i(t) \equiv D_i(t_0)\lambda(t)\lambda_i(t), \qquad i = 1, \cdots, n. \tag{14}$$

Multiplying the numerator and denominator of the right-hand side of (11) by $\lambda(t)$ gives

$$y(t) = \frac{\sum_{i=1}^{n} c_i \, \lambda(t)\lambda_i(t)u^i(t)}{c_0 \, \lambda(t) + \sum_{i=1}^{n} c_i \, \lambda(t)\lambda_i(t)}$$

$$= \frac{\sum_{i=1}^{n} (c_i/D_i(t_0))D_i(t)u^i(t)}{c_0 \, \lambda(t) + \sum_{i=1}^{n} (c_i/D_i(t_0))D_i(t)}$$

(permissible since $D_i(t_0) \neq 0$, $i = 1, \cdots, n$). Relabelling the arbitrary constants gives (13).

The following theorem gives a form for the complete solution of (7) requiring $n + 2$ solutions and no quadratures.

THEOREM 5. *Let $u^1, \cdots, u^{n+2}$ be solutions of* (7) *such that, if*

$$v^i = \begin{pmatrix} u^i \\ 1 \end{pmatrix}, \qquad i = 1, \cdots, n + 2,$$

$$D_0^+ = \det (v^1 \cdots v^{n+1}),$$

$$D_i^+ = \det (v^1 \cdots v^{i-1} \, v^{n+2} \, v^{i+1} \cdots v^{n+1}), \qquad i = 1, \cdots, n + 1,$$

then $D_i^+(t_0) \neq 0$, $i = 0, \cdots, n + 1$. Then the complete solution of (7) *can be written near t_0 in the form*

$$y(t) = \frac{\sum_{i=1}^{n+1} c_i \, D_i^+(t)u^i(t)}{\sum_{i=1}^{n+1} c_i \, D_i^+(t)}, \qquad 1 = \sum_{i=1}^{n+1} c_i \, D_i(t_0).$$

Proof. We know (see §1) that, if u satisfies (7) and $v = \begin{pmatrix} u \\ 1 \end{pmatrix}$, then

$$v' = Bv - v\beta v, \quad B = \begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}, \quad \beta = (\alpha \, 0). \tag{15}$$

The result then follows on application of Theorem 4 to (15); note that, since (15) is of type (5), $c_0 = 0$ in (13).

Note that (14) implies that n solutions of (7) are independent wherever they exist, if they are independent at a point.

Let us compare (13) with the expression in (III) for (1). In this case, $c_0 = 0$, so that $\lambda(t)$ does not appear. Solving for y in the formula in (III) gives

$$y = \frac{u(w - v) - cv(w - u)}{(w - v) - c(w - u)}.$$

Now if we let

$$u^1 = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad u^2 = \begin{pmatrix} v \\ 1 \end{pmatrix}, \quad u^3 = \begin{pmatrix} w \\ 1 \end{pmatrix}, \qquad c = -\frac{c_2}{c_1},$$

then

$$D_1 = \begin{vmatrix} w & v \\ 1 & 1 \end{vmatrix} = w - v, \quad D_2 = \begin{vmatrix} u & w \\ 1 & 1 \end{vmatrix} = u - w,$$

and so

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \frac{c_1 D_1 u^1 + c_2 D_2 u^2}{c_1 D_1 + c_2 D_2},$$

the form in (13).

The foregoing example shows that Theorem 4 is indeed a generalization of the cross-ratio property of (1). However, the *form* of the cross-ratio is not maintained, so that it is more natural to view (13) as a complete solution rather than an extended cross-ratio. A difficulty is that, if y, u, v, and w are matrix functions, they must be square in order that $(y - v)^{-1}$ and $(w - u)^{-1}$ exist, so that the coefficients in (3) must be square. Results in this case are given in [4] and [5].

Let us write the cross-ratio property as

$$\frac{\begin{vmatrix} y & u \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} w & v \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} w & u \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} y & v \\ 1 & 1 \end{vmatrix}} \equiv C.$$

The following theorem gives for (7) a direct generalization of this form.

THEOREM 6. *Let* $u^1, \cdots, u^{n+2}$ *be any solutions of* (7); *let*

$$D_i = (u^1 \cdots u^{i-1} u^{n+1} u^{i+1} \cdots u^n), \qquad D_i^* = (u^1 \cdots u^{i-1} u^{n+2} u^{i+1} \cdots u^n), \qquad 1 \leqq i \leqq n.$$

Then, for $1 \leqq j, k \leqq n$,

$$(16)\qquad D_k^*(t_0)D_j(t_0)D_k(t)D_j^*(t) \equiv D_k(t_0)D_j^*(t_0)D_k^*(t)D_j(t).$$

If, in addition, D_j , D_k , D_j^*, *and* D_k^* *are nonzero at* t_0 ,

$$(17)\qquad \frac{D_k(t)D_j^*(t)}{D_k^*(t)D_j(t)} \equiv \frac{D_k(t_0)D_j^*(t_0)}{D_k^*(t_0)D_j(t_0)} \equiv C.$$

Proof. Formula (14) does not depend on $D_i(t_0)$ being nonzero, and so is valid in either case. We have

$$\lambda_j(t)D_j(t_0)D_k(t) = \lambda_j(t)D_j(t_0)\lambda(t)D_k(t_0)\lambda_k(t) = \lambda_k(t)D_k(t_0)D_j(t).$$

Similarly, for the same λ_i's,

$$\lambda_j(t)D_j^*(t_0)D_k^*(t) = \lambda_k(t)D_k^*(t_0)D_j^*(t).$$

Eliminating λ_j and λ_k gives (16), which implies (17) under the additional hypotheses.

Note that (16) or (17) cannot be regarded as an expression for the complete solution of (7). However, if $u^1, \cdots, u^{n+1}$ are as in Theorem 4, the set of equations (16) obtained by taking different values for j and k will define u^{n+2}.

Finally, let us give an explicit expression for the complete solution of (17) in terms of the complete solution of (7). Such an expression is

$$(18)\qquad Y = U\Lambda CFZ,$$

where U and Λ are as previously defined, $C = (c_{ij})$ $(n \times m)$, $c_{0j} = 1 - \sum_{i=1}^n c_{ij}$, and $f_j = c_{0j} + \sum_{i=1}^n c_{ij}\lambda_i$, $F^{-1} = \operatorname{diag}(f_i)$, and Z, obtainable by quadratures, satisfies

$$Z' = BU\Lambda CF - Z\cdot\operatorname{diag}(a_nU\Lambda CF^i), \quad Z(t_0) = E$$

(F^i is column i of F). Further, (18) remains valid if λ_i is replaced by D_i , $i = 1, \cdots, n$, and c_{0j} by $c_{0j}\lambda$, $j = 1, \cdots, m$.

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