Solving the Matrix Differential Riccati Equation: A Lyapunov Equation Approach

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***Abstract*—In this technical note, we investigate a solution of the matrix differential Riccati equation that plays an important role in the linear quadratic optimal control problem. Unlike many methods in the literature, the approach that we propose employs the negative definite anti-stabilizing solution of the matrix algebraic Riccati equation and the solution of the matrix differential Lyapunov equation. An illustrative numerical example is provided to show the efficiency of our approach.**

***Index Terms*—Algebraic Riccati equation, differential Lyapunov equation, differential Riccati equation, optimal control.**

I. Introduction

The optimal control problem has been extensively studied since the work of [1]. Its solution is given in terms of the optimal feedback gain matrix that is defined by the solution of the matrix differential Riccati equation.

Several methods were proposed to numerically solve the matrix differential Riccati equation such as the direct integration method, Chandrasekhar algorithm, Davidson-Maki algorithm, Kalman-Englar method [2]–[4]. Some methods are based on the Hamiltonian approach to transform the nonlinear Riccati differential equation into a larger system of linear differential equations whose solution can be found by using the associated matrix exponential [2]. Unlike the aforementioned methods, the approach, that was proposed by Moore and Anderson [4], employs the positive definite solution of the algebraic Riccati equation. Several studies [5]–[7] also address the differential Riccati equation in similar ways. We call this approach the Anderson-Moore method. The advantage of this method is that it requires a smaller amount of computation than other methods [2]. However, its generality is limited due to the validity of the condition imposed. Therefore, its applicability is narrower than other methods. Furthermore, when the time interval of interest increases, there will be numerical difficulties due to the inverse of a near-singular matrix.

This technical note gives a new and efficient way to overcome these difficulties. Specifically, we employ the negative definite solution of the matrix algebraic Riccati equation and the solution of the matrix differential Lyapunov equation to solve the differential Riccati equation. In addition, the new method will be shown to be well-defined in the time interval of interest. The use of Lyapunov-type equations in solving optimal problems can be found in [8]. Because the solution is based on the differential Lyapunov equation, we call the new method the Lyapunov equation approach.

The rest of this technical note is organized as follows. Section II presents the problem formulation. The main result is exposed in Section III, which is followed in Section IV by a numerical example to illustrate the efficiency of the new method. The final section provides some conclusions.

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II. Problem Statement

Consider the linear time-invariant system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{1}$$

where $x(t)$ is an n dimensional state vector, $u(t)$ is an m dimensional control input vector.

The corresponding optimal control problem for this system is to find an optimal control $u(t)$ that minimizes the following quadratic cost functional

$$J = \frac{1}{2}\int_0^{t_f} \left\{x(t)^T Qx(t) + u^T(t)Ru(t)\right\} dt + \frac{1}{2}x^T(t_f)Fx(t_f) \tag{2}$$

where $Q = CC^T \geq 0, F \geq 0$ and $R > 0$ are constant matrices of appropriate dimensions.

The optimal control law is in the form $u(t) = -R^{-1}B^T K(t) \times t$ for $t \in [0, t_f]$, with the closed-loop system matrix $G = A - BR^{-1}B^T K(t)$, where $K(t)$ is the solution of the matrix differential Riccati equation

$$-\dot{K}(t) = K(t)A + A^T K(t) - K(t)SK(t) + Q \tag{3}$$

with $K(t_f) = F$ and $S = BR^{-1}B^T$.

Our purpose is to present a method for solving the matrix differential Riccati equation (3).

III. Main Results

Our method is similar to the approaches in [4]–[7]. Instead of directly looking for the solution of the differential Riccati equation, they sought way to find the inverse of the difference between the solution and the positive definite solution of the matrix algebraic Riccati equation. Thus, one numerical obstacle takes place when the difference is near singular. When t_f is large enough, $K(t = 0)$ tends to the positive semidefinite solution of the algebraic Riccati equation under certain conditions [9]. This also makes the inverse of the difference ill-conditioned.

Differing from these approaches, we make good use of the negative definite solution of the algebraic Riccati equation in the new method. Before presenting the new approach, we need the following assumption and lemma.

Assumption 1: (A, B) is completely controllable and (A, C) is completely observable.

Lemma 1: Under Assumption 1, the following algebraic Riccati equation has a unique positive definite solution K^+ and a unique negative definite solution K^-

$$0 = KA + A^T K - KSK + Q. \tag{4}$$

Proof: See [1] for a proof. ■

The positive definite solution of (4) can be numerically found using known algorithms such as Kleinman, matrix sign function, and eigenvector method and its variants (Schur method) [10]–[15]. Consider the $2n \times 2n$ Hamiltonian matrix

$$M = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}. \tag{5}$$

It is well known [9] that the eigenvalues of M are symmetrically distributed with respect to the imaginary axis.

According to [14], we can find an orthogonal transformation $W \in \mathbb{R}^{2n\times 2n}$ which transforms M into a quasi-upper-triangular structure:

$$W^T M W = D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}. \tag{6}$$

We can arrange that the eigenvalues of D_{11} have negative real parts while the real parts of the eigenvalues of D_{22} are positive [14]. Matrix W is partitioned into four $n \times n$ blocks

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \tag{7}$$

It was pointed out [14] that the positive definite solution of the algebraic Riccati equation can be written as

$$K^+ = W_{21} W_{11}^{-1}. \tag{8}$$

We can use the same technique as in [14] to show that the negative definite solution can be found in the form

$$K^- = W_{22} W_{12}^{-1}. \tag{9}$$

Alternatively, one can compute the negative definite solution K^- by finding the positive definite solution of the algebraic equation

$$0 = -K_n A - A^T K_n - K_n S K_n + Q. \tag{10}$$

It is clear that $K_n = -K^-$ is a positive definite solution of (10). We will show that the positive definite solution of (10) is unique. Because (A, B, C) is completely controllable and completely observable, so is $(-A, B, C)$. Indeed, (A, B) is completely controllable if and only if $\mathrm{rank}([B, AB, \ldots, A^{n-1}B]) = n$. Since the rank of a matrix does not change if we alter the sign of a row or a column, we have

$$\begin{aligned} &\mathrm{rank}([B, (-A)B, \ldots, (-A)^{n-1}B]) \\ &\qquad = \mathrm{rank}([B, AB, \ldots, A^{n-1}B]) = n. \end{aligned} \tag{11}$$

This means that $(-A, B)$ is completely controllable. Similarly, $(-A, C)$ is also completely observable. Therefore, from Lemma 1, there exists a unique positive definite solution K_n of the algebraic Riccati (10). As a consequence, we obtain $K^- = -K_n$.

Subtracting (4) from (3), with K replaced by K^- in (4), we have

$$\begin{aligned} -\dot{K}(t) = (K(t) - K^-)A + A^T(K(t) - K^-) \\ -K(t)SK(t) + K^- S K^-. \end{aligned} \tag{12}$$

By introducing the change of variable

$$P_0(t) = (K(t) - K^-)^{-1} \tag{13}$$

equation (3) becomes

$$\dot{P}_0(t) = A_0 P_0(t) + P_0(t) A_0^T - S \tag{14}$$

with

$$P_0(t_f) = (F - K^-)^{-1} \tag{15}$$

and

$$A_0 = A - SK^-. \tag{16}$$

Note that $\mathrm{Re}\{\lambda_i(A_0)\} > 0,\ \forall i$ [16]. It is seen that (14) is a differential Lyapunov equation, that is more easily solved than a differential Riccati one. To solve (14), we employ the methods of Serbin and Serbin [17], Gajic and Qureshi [18].

Let E be the solution of the algebraic Lyapunov equation

$$A_0 E + E A_0^T - S = 0. \tag{17}$$

The algebraic Lyapunov equation (17) can be easily solved using known methods [18], [19]. The solution of the differential Lyapunov equation (14) can be obtained by

$$P_0(t) = e^{A_0(t-t_f)}(P_0(t_f) - E)e^{A_0^T(t-t_f)} + E. \tag{18}$$

And the solution of the original problem (3) is

$$K(t) = K^- + P_0^{-1}(t). \tag{19}$$

One may ask for a question for the validity of (19), namely the invertibility of $P_0(t)$ for $t \in [0, t_f]$. The following theorem will give an answer to that concern.

Theorem 1: The solution $P_0(t)$ of (14) is positive definite for all $t \le t_f$.

Proof: The analytic solution of the differential Lyapunov equation (14) is described by the following expression [9]:

$$\begin{aligned} P_0(t) = e^{A_0(t-t_f)} P_0(t_f) e^{A_0^T(t-t_f)} \\ + \int_t^{t_f} e^{A_0(\tau-t_f)} S e^{A_0^T(\tau-t_f)} d\tau. \end{aligned} \tag{20}$$

Because $F \ge 0$ and $K^- < 0$ (from Lemma 1), $F - K^- > 0$. And hence, $P_0(t_f) = (F - K^-)^{-1} > 0$. As a result, $e^{A_0(t-t_f)} P_0(t_f) e^{A_0^T(t-t_f)} > 0$. In addition, the quantity under the integral is positive semidefinite since $S = BR^{-1}B^T$ is positive semidefinite. Therefore, $P_0(t)$ is positive definite for $t \le t_f$. ■

It is noted that when $t \longrightarrow -\infty$, $e^{A_0(t-t_f)} \longrightarrow 0$ since all eigenvalues of A_0 have positive real parts [6]. Because K^+ is a solution of (12), then from (14), the solution of the algebraic Lyapunov equation (17) is

$$E = (K^+ - K^-)^{-1}. \tag{21}$$

This agrees with the result provided in [16]. Thus from (18), we obtain

$$P_0(-\infty) = E = (K^+ - K^-)^{-1}.$$

As a result, from (19)

$$K(-\infty) = K^+.$$

This is exactly the positive definite solution of the algebraic Riccati equation, which was reported in [9] and references therein.

In short, we have a new algorithm to calculate the solution of the differential Riccati equation (3) as follows.

1) Find the negative definite solution K^- of the algebraic Riccati equation (4) by calculating the positive solution K_n of the algebraic Riccati equation (10).
2) Compute the solution E of the algebraic Lyapunov equation (17).
3) Calculate the solution of the algebraic differential Lyapunov equation (14) using formula (18), that is, first evaluate the matrix exponential

$$X = e^{-A_0 \Delta t} \tag{22}$$

where $\Delta t > 0$ is the step size.

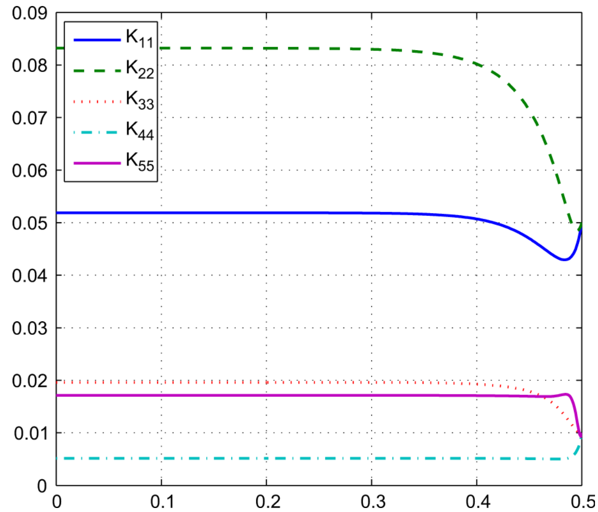

Fig. 1. Evolution of the diagonal elements of $K(t)$ for the Kalman-Englar method.

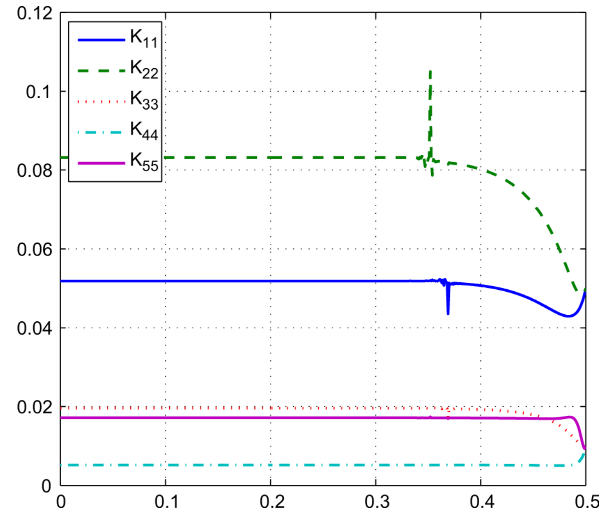

Fig. 2. Evolution of the diagonal elements of $K(t)$ for the Anderson-Moore method.

4) Obtain the solution of the differential Riccati equation (3) using formula (19). To that end, for $k = 1$ to $(l-1)$, where l is the number of steps in the time interval $[0, t_f]$, with $P_l = P_0(t_f) = (F - K^-)^{-1}$, calculate

$$P_{l-k} = X(P_{l-(k-1)} - E)X^T + E \tag{23}$$

and

$$K_{l-k} = K^- + P_{l-k}^{-1} \tag{24}$$

where $K_k = K(k\Delta t)$.

Remark 1: The new technique is similar to the Anderson and Moore method [4]. Unlike their approach, we obtain a solution that is well defined in the time region of interest. Specifically, it is not restricted to an assumption on the initial condition. In [4], the positive semidefinite matrix F is required to satisfy the condition that $F - K^+$ is invertible. Meanwhile, our requirement is only that $F \geq 0$. Instead of exploiting the positive definite solution of the algebraic Riccati equation, we utilize the negative definite one. This is one distinctive factor that leads to the validity of the solution.

Remark 2: The new method ensures the symmetry of the solution unlike the other approaches such as the Kalman-Englar method [9], the Davison-Maki algorithm [3]. Thus, we do not have to symmetrize the solution after each step.

Remark 3: The new method is more numerically efficient than other methods. Because of symmetry, it only requires $2n^3 + 1/6n^3 + 1/2n^2$ multiplications per time step which is the same as in the Anderson-Moore method while the Davidson-Maki method as well as other Hamiltonian ones require a total of $11n^3 + 1/3n^3$ multiplications per time step [2].

Remark 4: The startup cost of the proposed method includes the computational cost of the matrix exponential (22), the negative definite solution of the algebraic Riccati equation (4) and the solution of the Lyapunov equation (17). It is well known that the matrix exponential can be efficiently calculated by the scaling and squaring method [20].

IV. Numerical Example

We use the example of a fifth order fluid catalytic reactor [21] to illustrate our approach.

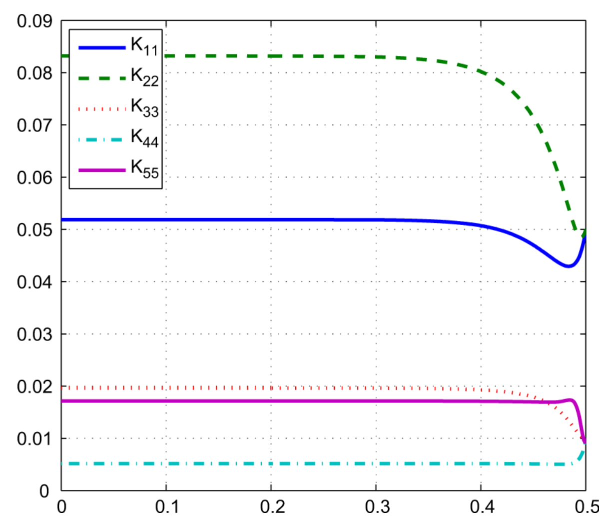

Fig. 3. Evolution of the diagonal elements of $K(t)$ for the new method.

Matrix A and matrix B are, respectively, given by

$$A = \begin{bmatrix} -16.00 & -0.39 & 27.20 & 0 & 0 \\ 0.01 & -16.99 & 0 & 0 & 12.47 \\ 15.11 & 0 & -53.60 & -16.57 & 71.78 \\ -53.36 & 0 & 0 & -107.20 & 232.11 \\ 2.27 & 69.10 & 0 & 2.273 & -102.99 \end{bmatrix},$$

$$B = \begin{bmatrix} 11.12 & -12.60 \\ -3.61 & 3.36 \\ -21.91 & 0 \\ -53.60 & 0 \\ 69.10 & 0 \end{bmatrix}.$$

The cost weighting matrices are chosen as $Q = I$ and $R = I$, where I is the identity matrix. The terminal time penalty matrix is taken as

$$F = \begin{bmatrix} 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix}.$$

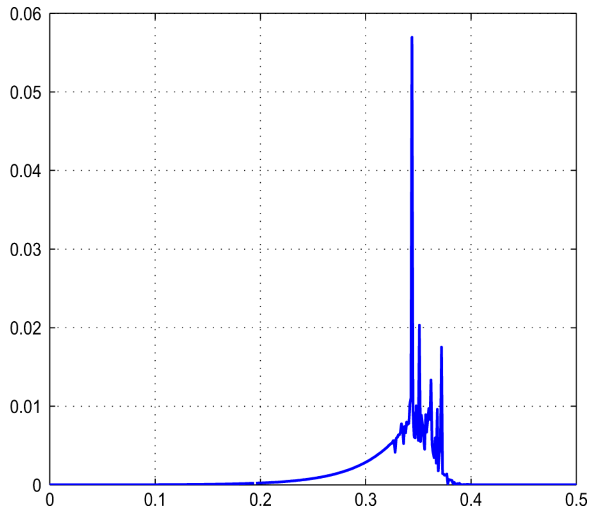

Fig. 4. Normwise relative errors for the Anderson-Moore method.

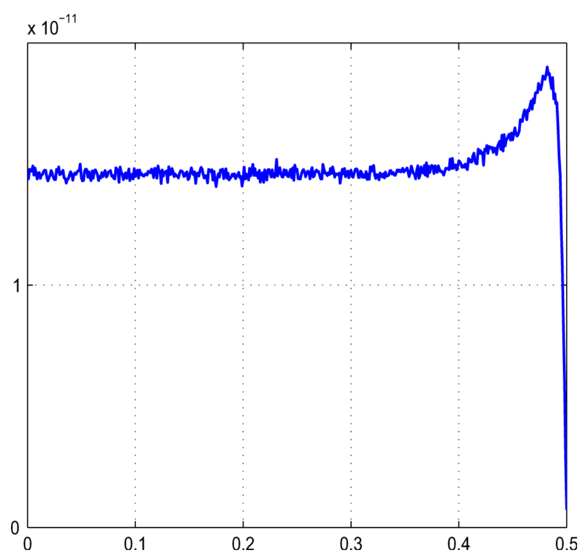

Fig. 5. Normwise relative errors for the new method.

For comparison, we use three methods: the Kalman-Englar method, the Anderson-Moore method, and the proposed one. Our computations are done using MATLAB 7. The diagonal elements of the solution $K(t)$ are plotted in Figs. 1–3. The time interval goes from $t = 0$ to $t = T = 0.5$. The Kalman–Englar method is regarded as a benchmark to compare the Anderson-Moore method and the new one. It is seen that the Anderson-Moore method exhibits numerical inaccuracies (Fig. 2). The normwise relative errors for the Anderson-Moore method and the new one which are calculated by $\|K_{exact}(t) - K(t)\|_1 / \|K_{exac}(t)\|_1$ are plotted in Figs. 4 and 5. These figures show that the proposed method provides better results than the Anderson-Moore method.

V. Conclusion

We provided a new way to solve the differential matrix Riccati equation. Its solution is given in terms of the negative definite solution of the algebraic Riccati equation and the solution of the differential matrix Lyapunov equation. Theoretical and numerical results showed that the new technique provides better performances than the Anderson-Moore method.

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