Problem 1
Prove: Multivariate Gaussian Distribution is normalize

Proof:
We have the PDF of the Gaussian distribution is:

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{D}{2}} \mid \Sigma \mid^{\frac{1}{2}}} \times e^{\frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

The Gaussian Distribution is normalize

$$\iff \int_{-\infty}^{+\infty} p(x \mid \mu, \sigma^2) = 1$$

where μ is a D-dimensional mean vector, Σ is a D x D covariance matrix, and $\mid \Sigma \mid$ denotes the determinant of Σ

Set $\Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) = \frac{-1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + \ constant$

Consider eigenvalues and eigenvvectors of Σ we have:

$$\Sigma u_i = \lambda_i u_i, i = 1, ..., D$$

Because Σ is a real, symmetric matrix, its eigenvalues will be real and its eigenvectors form an orthonormal set.
Proof:
1. its eigenvalues will be real
Example:

$$\begin{pmatrix} \sigma_1^2 & cov(\sigma_{1,2}) \\ cov(\sigma_{2,1}) & \sigma_2^2 \end{pmatrix}$$

$\iff$ The equation to find the eigenvalues is:

$$(\sigma_1^2 - \lambda) \times (\sigma_2^2 - \lambda) - (\sigma_{1,2})^2 = 0$$

$$\iff (\sigma_1^2 - \lambda) \times (\sigma_2^2 - \lambda) = (\sigma_{1,2})^2$$

$\implies \lambda$ must be a real number.
With $\lambda = \lambda_1$:
$\begin{pmatrix} \sigma_1^2 - \lambda_1 & cov(\sigma_{1,2}) \\ cov(\sigma_{1,2}) & \sigma_2^2 - \lambda_1 \end{pmatrix}$

$$(\sigma_1^2 - \lambda_1)x_1 + (\sigma_{1,2})x_2 = 0 \tag{1}$$
$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_1)x_2 = 0 \tag{2}$$

From (1) we have:
$x_1 = \frac{-y \times cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1} \ x_2 = x_2$

So the eigenvector in this case is: $\begin{pmatrix} \frac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1} \\ 1 \end{pmatrix}$

With $\lambda = \lambda_2$:

$$\begin{pmatrix} \sigma_1^2 - \lambda_2 & cov(\sigma_{1,2}) \\ cov(\sigma_{1,2}) & \sigma_2^2 - \lambda_2 \end{pmatrix}$$

$$(\sigma_1^2 - \lambda_2)x_1 + (\sigma_{1,2})x_2 = 0 \tag{3}$$

$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_2)x_2 = 0 \tag{4}$$

From (3) we have:

$$x_1 = \frac{-y \times cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}$$
$$x_2 = x_2$$

So the eigenvector in this case is: $\begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix}$

And: $\begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix}^T \times \begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix} = 1$

$So\ its\ eigenvectors\ form\ an\ orthonormal\ set. \Sigma = \Sigma_{i=1}^{D} \lambda_i u_i (u_i)^T \longrightarrow \Sigma^{-1} = \Sigma_{i=1}^{D} \frac{1}{\lambda i} u_i u_i^T$

$So\ that\ : \Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) \qquad = \sum_{i=1}^{D} \frac{1}{\lambda_i}(x-\mu)^T u_i u_i^T (x-\mu)$

$Let\ : y_i = u_i^T(x-\mu) \longrightarrow \Delta^2 = \sum_{i=1}^{D} \frac{y_i^2}{\lambda_i} \mid \Sigma \mid^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$

Now,we have:

$$p(x \mid \mu, \sigma^2) = p(y) = \prod_{j=1}^{D} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\dfrac{-(y_j)^2}{2\lambda_j}}$$

$$\Longleftrightarrow \int_{-\infty}^{+\infty} p(y)dy = \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\dfrac{-(y_j)^2}{2\lambda_j}}$$

But in the last homework we have proofed that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\dfrac{-y^2}{2}} dy = 1$$

So:

$$\int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\dfrac{-(y_j)^2}{2\lambda_j}} = 1$$

$$\Longleftrightarrow \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\dfrac{-(y_j)^2}{2\lambda_j}} = 1$$

Problem 2
Calculate conditional normal distribution

Solution

Problem 3
Calculate marginal normal distribution

Solution