Problem 1
To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

Solution
A: known to have Hansen's disease
B: known to have no Hansen's disease
H: Testing positive for Hansen's disease
$P(A) = 0.05$
$P(B) = 0.95$
$P(H \mid A) = 0.98$
$P(H \mid B) = 0.03$
The probability that someone testing positive for Hansen's disease under this new test actually has the disease is:

$$P(A|H) = \frac{P(H|A) \times P(A)}{P(H)}$$

$$= \frac{P(H|A) \times P(A)}{P(H|A) \times P(A) + P(H|B) \times P(B)}$$

Substituting the values, we have:

$$P(A|H) = \frac{0.98 \times 0.05}{0.98 \times 0.05 + 0.03 \times 0.95}$$

$$P(A|H) = \frac{0.049}{0.0775} \approx 0.6323$$

Problem 2
Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:
1. Univariate normal distribution.
2.(Optional) Multivariate normal distribution.

Solution
1.Univariate normal distribution
The area under the normal distribution curve should be equal to 1. Next is the proof:

$$Put\ I = \int e^{\frac{-x^2}{2}} dx, then\ we\ have\ I^2 = \left(\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-y^2+x^2}{2}} dxdy$$

$$Set\ x = rcos\theta,\ y = rsin\theta\ We\ have \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$Which\ J = [\frac{\partial(x,y)}{\partial(r,\theta)}]\ We\ have : dxdy = \begin{bmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{bmatrix} drd\theta = rdrd\theta$$

$$So : I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{x^2+y^2}{2}} dxdy = \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2}} rdrd\theta.$$

$$So : I^2 = \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2}} rdrd\theta = \int_0^{2\pi} [-e^{\frac{-r^2}{2}}]_0^{\infty} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi \Rightarrow I = \sqrt{2\pi}$$

Calculating mean:
Set:

$$Z = \frac{(X - \mu)}{\sigma}$$

We have:

$$E[Z] = \int_{-\infty}^{+\infty} x f_Z(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{\frac{-x^2}{2}} dx$$

$$= \frac{-1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} |_{-\infty}^{+\infty}$$

$$= 0$$

Because : $X = \mu + \sigma Z$

$$\Rightarrow E(X) = E(\mu) + E(\sigma Z)$$

$$\Leftrightarrow E(X) = \mu + E(Z)E(\sigma)$$

But: E(Z) = 0

$$\Rightarrow E(X) = \mu$$

Calculating variance:

$$Var(Z) = E(Z^2) - ((E(Z))^2 = E(Z^2)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-x^2}{2}} dx$$

Let:

$$u = x, \ dv = xe^{\frac{-x^2}{2}}$$

$$Var(Z) = \frac{1}{\sqrt{2\pi}} (-xe^{\frac{-x^2}{2}} |_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx = 1$$

But: $X = \mu + \sigma Z$

$$\Leftrightarrow Var(X) = Var(\mu) + \sigma^2 Var(Z)$$

$$\Rightarrow Var(X) = 0 + 1\sigma^2$$

$$\Rightarrow Var(X) = \sigma^2$$

Standard deviation: $S = \sigma$