

Problem 1

Re-transform the logistic regression algorithm

Solution

The model logistic regression:

$$p(C1 | \phi) = y(\phi) = \sigma(\mathbf{w}^T \phi)$$

$$p(C2 | \phi) = 1 - p(C1 | \phi)$$

For a data set ϕ_n, t_n , where $t_n \in 0, 1$ and $\phi_n = \phi(x_n)$, with $n = 1, \dots, N$, the likelihood function can be written:

$$p(t | \mathbf{w}) = \prod_{n=1}^N y_n^{t_n} (1 - y_n)^{1-t_n}$$

where $t_n = (t_1, \dots, t_N)^T$ and $y_n = p(C1 | \phi_n)$

Taking negative logarithm we have:

$$L = -\log p(t | \mathbf{w}) = -\sum_{n=1}^N t_n \log y_n + ((1 - t_n) \log(1 - y_n))$$

where $y_n = \sigma(a_n)$ and $a_n = \mathbf{w}^T \phi_n$

We have:

$$\frac{\partial L}{\partial w} = \sum_{n=1}^N \left(\frac{\partial L}{\partial y_n} \times \frac{\partial y_n}{\partial a_n} \times \frac{\partial a_n}{\partial w} \right)$$

$$\frac{\partial L}{\partial y_n} = -\frac{t_n}{y_n} + \frac{(1 - t_n)}{(1 - y_n)} = \frac{y_n - t_n}{y_n(1 - y_n)}$$

$$\frac{\partial y_n}{\partial a_n} = \sigma(a_n)(1 - \sigma(a_n)) = y_n(1 - y_n)$$

$$\frac{\partial a_n}{\partial w} = \phi_n$$

So :

$$\begin{aligned} \frac{\partial L}{\partial w} &= \sum_{n=1}^N \left(\frac{\partial L}{\partial y_n} \times \frac{\partial y_n}{\partial a_n} \times \frac{\partial a_n}{\partial w} \right) \\ &= \sum_{n=1}^N \frac{y_n - t_n}{y_n(1 - y_n)} \times y_n(1 - y_n) \times \phi_n \\ &= \sum_{n=1}^N (y_n - t_n) \phi_n \end{aligned}$$

Problem 2

Find the function: $f'(x) = f(x)(1 - f(x))$

Solution

$$f'(x) = f(x)(1 - f(x))$$

$$\Leftrightarrow \frac{d(f(x))}{dx} = f(x)(1 - f(x))$$

$$\Rightarrow \frac{d(f(x))}{f(x)(1 - f(x))} = dx$$

$$\Rightarrow \int \frac{d(f(x))}{f(x)(1 - f(x))} = \int dx$$

$$\Rightarrow \int \frac{1}{f(x)(1 - f(x))} d(f(x)) = \int dx$$

$$\Rightarrow \int \frac{1}{f(x)} + \frac{1}{1-f(x)} d(f(x)) = \int dx$$

$$\Leftrightarrow \ln(f(x)) - \ln(1-f(x)) = x$$

$$\Leftrightarrow \frac{f(x)}{1-f(x)} = e^x$$

$$\Rightarrow f(x) = e^x(1-f(x))$$

$$\Leftrightarrow f(x) = e^x - e^x f(x)$$

$$\Leftrightarrow f(x)(1+e^x) = e^x$$

$$\Leftrightarrow f(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}} = \sigma(x)$$