Probabilistic Reasoning Over Time

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Keeping Track

Notation



symbol	meaning		
$a, b, c, N \dots$	scalar number	symbol	meaning
$\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y} \dots$	column vector		
$\boldsymbol{X},\boldsymbol{Y}\dots$	matrix	<i>X</i> , <i>Y X</i> , <i>Y</i>	random variable multivariate random variable
\mathbb{R}	set of real numbers		
\mathbb{Z}	set of integer numbers	$x, y \dots$	value
\mathbb{N}	set of natural numbers	x, y	vector
\mathbb{R}^D	set of vectors	p, pr, P, Pr	probability
$\mathcal{D},\mathcal{X},\mathcal{Y},\dots$	set		
\mathcal{A}	algorithm		

Temporal Probabilistic Model

- Temporal Probabilistic Model
- Markov process



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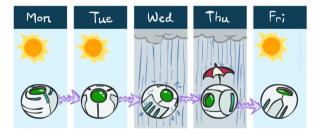
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Temporal Probabilistic Model



Why do we need temporal probabilistic model?

- The world changes over time (random process), and what happens now impacts what will happen in the future. We need to track and predict it
 - Stock market
 - Weather



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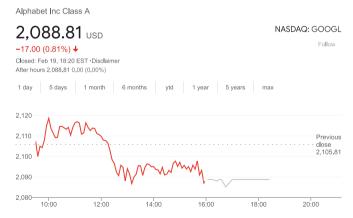
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Temporal Probabilistic Model (cont.)



How to model the time?

 View the world as time slices: discrete time steps; step size depends on problem



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Temporal Probabilistic Model (cont.)

Concept 1

- State variables X_t (often hidden): set of unobservable state variables at time t
 - State of the environment
 - Not directly observable but defines causal dynamics
- Evidence variables E_t: set of observable evidence variables at time t
 - Caused by the state of the environment
- Notation: $\boldsymbol{X}_{a:b} = \boldsymbol{X}_a, \boldsymbol{X}_{a+1}, \dots, \boldsymbol{X}_{b-1}, \boldsymbol{X}_b$

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Temporal Probabilistic Model (cont.)



Concept 2

• Transition model: How the world evolves

$$P(\boldsymbol{X}_t \mid \boldsymbol{X}_{0:t-1}) \tag{1}$$

• Sensor/observation model: How the evidence variables get their values

$$P(\boldsymbol{E}_t \mid \boldsymbol{X}_{0:t}, \boldsymbol{E}_{0:t-1}) \tag{2}$$

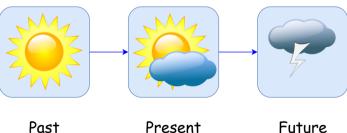
Markov process

Markov process



Concept 3

Markov process is a random process with Markov assumption " X_t depends on bounded subset of $X_{0:t-1}$ "



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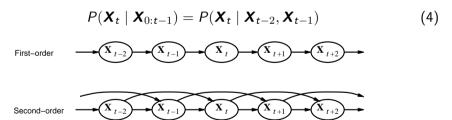
Markov process (cont.)



First-order Markov process:

$$P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1})$$
 (3)

Second-order Markov process:



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Markov process (cont.)



Sensor Markov assumption:

$$P(\boldsymbol{E}_t \mid \boldsymbol{X}_{0:t}, \boldsymbol{E}_{0:t-1}) = P(\boldsymbol{E}_t \mid \boldsymbol{X}_t)$$
 (5)

- Stationary process:
 - transition model $P(X_t \mid X_{t-1})$ fixed for all t
 - sensor model $P(\boldsymbol{E}_t \mid \boldsymbol{X}_t)$ fixed for all t

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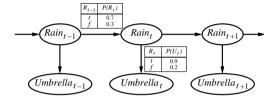
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Example





- First-order Markov assumption may not be exactly true in real world!
- Possible fixes:
 - 1. Increase order of Markov process
 - **2. Augment state**, e.g., add *Temp*_t, *Pressure*_t

Inference in Temporal Models

- Filtering
- Smoothing
- Most likely explanation

Inference tasks



• Filtering: computing the belief state, the posterior distribution over the most recent state given all evidence to date

$$P(X_t \mid e_{1:t})$$

• **Prediction**: computing the posterior distribution over the future state, given all evidence to date

$$P(X_{t+k} \mid e_{1:t}) \text{ for } k > 0$$

• Smoothing: computing the posterior distribution over a past state, given all evidence up to the present.

$$P(X_k \mid e_{1:t})$$
 for $0 \le k < t$

• Most likely explanation: given a sequence of observations, find the sequence of states that is most likely to have generated those observations

$$\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$$

Filtering



• Given the result of filtering up to time t, we needs to compute the result for t+1 from the new evidence e_{t+1} ,

$$P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$$

$$= \alpha P(e_{t+1} | X_{t+1}, e_{1:t}) P(X_{t+1} | e_{1:t})$$

$$= \alpha P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t})$$

 We obtain the one-step prediction for the next state by conditioning on the current state X_t :

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t, e_{1:t}) P(x_t \mid e_{1:t})$$

$$= \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})$$
(6)

The process is given by

$$f_{1\cdot t+1} = \text{FORWARD}(f_{1\cdot t}, e_{t+1}) \text{ where } f_{1\cdot t} = P(X_t \mid e_{1\cdot t})$$

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Filtering example

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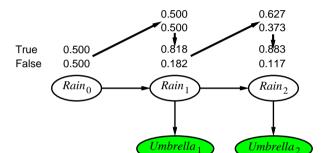
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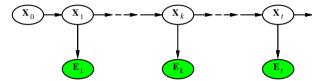
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Smoothing





• Divide evidence $e_{1:t}$ into $e_{1:k}$, $e_{k+1:t}$:

$$P(X_{k} \mid e_{1:t}) = P(X_{k} \mid e_{1:k}, e_{k+1:t})$$

$$= \alpha P(X_{k} \mid e_{1:k}) P(e_{k+1:t} \mid X_{k}, e_{1:k})$$

$$= \alpha P(X_{k} \mid e_{1:k}) P(e_{k+1:t} \mid X_{k})$$

$$= \alpha f_{1:k} b_{k+1:t}$$
(7)

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Smoothing (cont.)



Backward message computed by a backwards recursion:

$$P(e_{k+1:t} \mid X_k) = \sum_{x_{k+1}} P(e_{k+1:t} \mid X_k, x_{k+1}) P(x_{k+1} \mid X_k)$$

$$= \sum_{x_{k+1}} P(e_{k+1:t} \mid x_{k+1}) P(x_{k+1} \mid X_k)$$

$$= \sum_{x_{k+1}} P(e_{k+1:t} \mid x_{k+1}) P(e_{k+2:t} \mid x_{k+1}) P(x_{k+1} \mid X_k)$$
(8)

The process is given by

$$b_{k+1:t} = \text{BACKWARD}(b_{k+2:t}, e_{t+1}) \text{ where } b_{k+1:t} = P(e_{k+1:t} \mid X_k)$$

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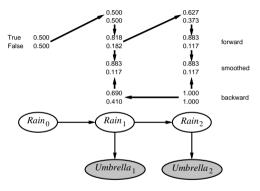
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Smoothing example





• Forward–backward algorithm: cache forward messages along the way Time linear in t (polytree inference), space O(t|f|)

Most likely explanation



- Most likely sequence ≠ sequence of most likely states!
- Most likely path to each $x_{t+1} = \text{most likely path to } some x_t$ plus one more step

$$\max_{x_t} P(x_1, \dots, x_t, X_{t+1} \mid e_{1:t+1})$$

$$= P(e_{t+1} \mid X_{t+1}) \max_{x_t} \left(P(X_{t+1} \mid x_t) \max_{x_1, \dots, x_{t-1}} P(x_1, \dots, x_{t-1}, x_t \mid e_{1:t}) \right)$$

• Identical to filtering, except $f_{1:t}$ replaced by

$$m_{1:t} = \max_{x_1...x_{t-1}} P(x_1,...,x_{t-1},X_t \mid e_{1:t}),$$

I.e., $m_{1:t}(i)$ gives the probability of the most likely path to state i.

• Update has sum replaced by max, giving the **Viterbi algorithm**:

$$m_{1:t+1} = P(e_{t+1} \mid X_{t+1}) \max_{x_t} (P(X_{t+1} \mid x_t) m_{1:t})$$

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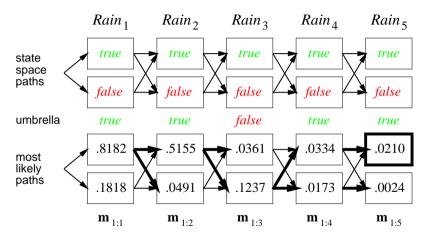
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Viterbi example





Hidden Markov Models

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Problem



Consider a temporal probabilistic model with two random variables

- Weather: *sunny* or *rainny*
- Baby: happy or sad

WEATHER



BABY



Example Hidden Markov Models

Data

Complete observations in 15 days

 D_{15} WEATHER

Partial observations in 15 days



Example

Markov Model

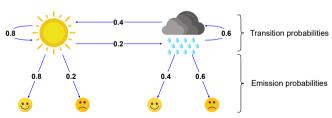


- Markov model with the following assumptions
 - Transition probabilities

$$P(weather_{t+1} \mid weather_t)$$

Emission probabilities

 $P(baby_t \mid weather_t)$



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Questions



Learning

1. How did we find these probabilities?

Inference

- 1. What's the probability that a random day is Sunny or Rainy?
- 2. If Baby is Happy today, what's the probability that it's Sunny or Rainy?
- 3. If for three days Baby is Happy, Sad, Happy, what was the weather?

Example

How did we find these probabilities?



Transition probabilities

WEATHER







































		counts	probabilities
<u></u>	→	8	0.8
*		2	0.2
	→	2	0.4
3000		3	0.6

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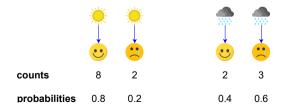
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How did we find these probabilities? (cont.)

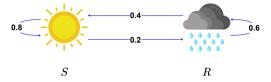


Emission probabilities





Sunny or Rainny



• We have the equations

$$S = 0.8S + 0.4R$$

 $R = 0.2S + 0.6R$

Solve the equations

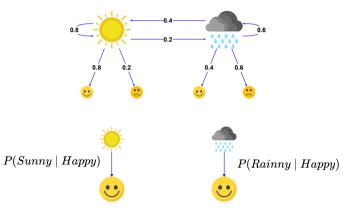
$$S = 2/3$$
$$R = 1/3$$

Most likely explanation

Example

Today, Baby is happy





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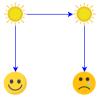
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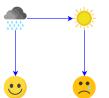
What was the weather?

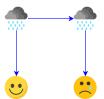


• If for two days Baby is Happy, Sad, what was the weather?









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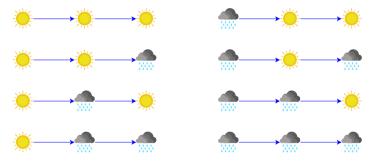
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What was the weather? (cont.)



• If for three days Baby is Happy, Sad, Happy, what was the weather?

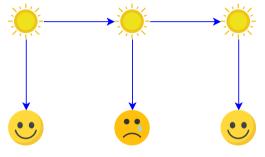


Most likely explanation

Example

What was the weather? (cont.)



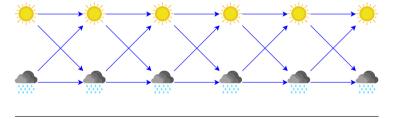


Example

Viterbi algorithm



 If for three days Baby is Happy, Happy, Sad, Sad, Sad and Happy, what was the most likely weather?



Observations













Hidden Markov models



Concept 4

Hidden Markov model is a temporal probabilistic model where X_t is a single, discrete variable (usually E_t is too) and domain of X_t is $\{1, \ldots, S\}$

• Transition matrix T_{ii} for each time step

$$T_{ij} = P(X_t = j \mid X_{t-1} = i)$$

• **Sensor matrix** O_t for each time step, diagonal elements

$$P(e_t \mid X_t = i)$$

• For example,
$$T = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$
, $O = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$ or $\begin{pmatrix} 0.1 & 0 \\ 0 & 0.8 \end{pmatrix}$

Forward-backward messages



• Forward and backward messages as column vectors:

$$f_{1:t+1} = \alpha O_{t+1} T^{\mathsf{T}} f_{1:t}$$

 $b_{k+1:t} = T O_{k+1} b_{k+2:t}$

• Forward-backward algorithm needs time $O(S^2t)$ and space O(St)

Country dance algorithm



Can avoid storing all forward messages in smoothing by running forward algorithm backwards:

$$\begin{array}{rcl} f_{1:t+1} & = & \alpha O_{t+1} T^{\mathsf{T}} f_{1:t} \\ O_{t+1}^{-1} f_{1:t+1} & = & \alpha T^{\mathsf{T}} f_{1:t} \\ \alpha'(T^{\mathsf{T}})^{-1} O_{t+1}^{-1} f_{1:t+1} & = & f_{1:t} \end{array}$$

Algorithm: forward pass computes f_t , backward pass does f_i , b_i

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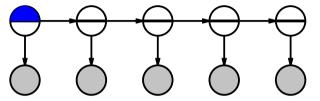
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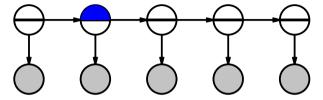
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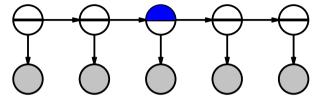
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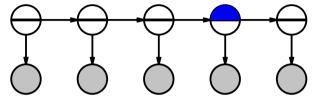
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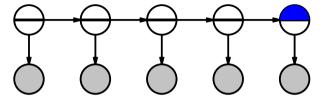
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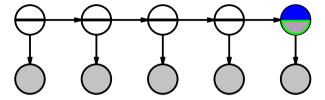
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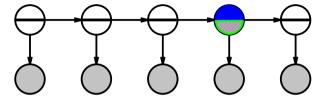
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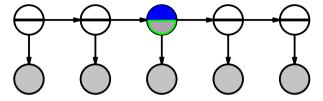
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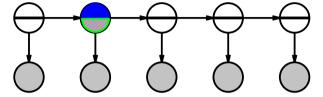
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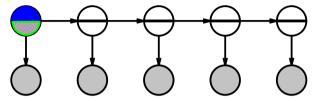
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Kalman filters



Concept 5

Kalman Filter is a temporal probabilistic model that represents systems described by a set of continuous variables with assumptions: Gaussian prior, linear Gaussian transition model and sensor model

Tracking a bird flying

$$\mathbf{X}_{t} = (position, velocity) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$$

$$\mathbf{X}_{t} = (\mathbf{x}_{t}, \mathbf{y}, \dot{z}, \dot{y}, \dot{z})$$

$$\mathbf{X}_{t} = (\mathbf{x}_{t}, \mathbf{y}, \dot{z}, \dot{y}, \dot{z})$$

Updating Gaussian distributions



Prediction step:

• If $P(X_t \mid e_{1:t})$ is Gaussian and the transition model $P(X_{t+1} \mid x_t)$ is linear Gaussian, then prediction

$$P(X_{t+1} \mid e_{1:t}) = \int_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t}) \, dx_t$$

is Gaussian.

• If $P(X_{t+1} \mid e_{1:t})$ is Gaussian and the sensor model $P(e_{t+1} \mid X_{t+1})$ is linear Gaussian, then the updated distribution

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

is Gaussian

• Hence $P(X_t \mid e_{1:t})$ is multivariate Gaussian $\mathcal{N}(\mu_t, \Sigma_t)$ for all t

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Example

Hidden Markov Mode

Kalman Filters

Bayesian Networks

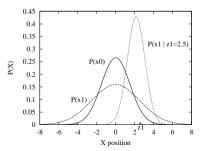
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Simple 1-D example



• Gaussian random walk on X-axis, transition s.d. σ_x , sensor s.d. σ_z

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$
$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2) \sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$



General Kalman update



Transition and sensor models:

$$P(\mathbf{x}_{t+1} \mid \mathbf{x}_t) = \mathcal{N}(F\mathbf{x}_t, \Sigma_x)(\mathbf{x}_{t+1}) P(\mathbf{z}_t \mid \mathbf{x}_t) = \mathcal{N}(H\mathbf{x}_t, \Sigma_z)(\mathbf{z}_t)$$
(9)

F is the matrix for the transition: Σ_{\times} the transition noise covariance H is the matrix for the sensors; Σ_{τ} the sensor noise covariance

Filter computes the following update:

$$\mu_{t+1} = F\mu_t + K_{t+1}(z_{t+1} - HF\mu_t)
\Sigma_{t+1} = (I - K_{t+1})(F\Sigma_t F^{\mathsf{T}} + \Sigma_x)$$
(10)

where $K_{t+1} = (F\Sigma_t F^{\mathsf{T}} + \Sigma_{\mathsf{v}})H^{\mathsf{T}}(H(F\Sigma_t F^{\mathsf{T}} + \Sigma_{\mathsf{v}})H^{\mathsf{T}} + \Sigma_{\mathsf{z}})^{-1}$ is the **Kalman** gain matrix

• Σ_t and K_t are independent of observation sequence, so compute offline

Temporal Pro

Markov proc

Inference Temporal

Models Filtering

Most likely explanation

Hidden Marko

Models

Hidden Markov Mode

Kalman Filters

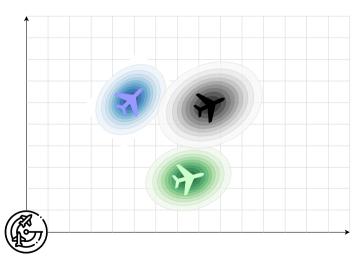
Bayesian Networks

Particle Filtering

Keeping Track of Many

2-D tracking example





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Kalman Filters

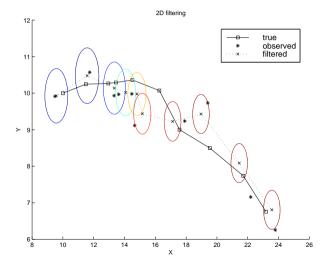
Dynamic

Network

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2-D tracking example: filtering





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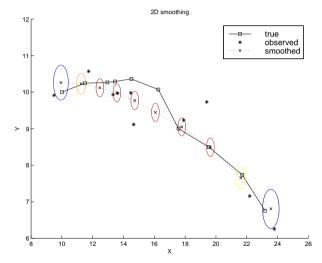
Networks

Particle Filter

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2-D tracking example: smoothing





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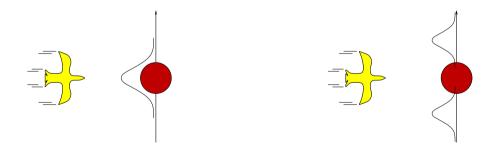
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Where it breaks



- Cannot be applied if the transition model is nonlinear
- ullet Extended Kalman Filter models transition as *locally linear* around $x_t = \mu_t$
 - Fails if systems is locally unsmooth



Dynamic Bayesian Networks

Particle Filtering

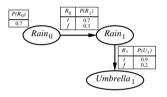
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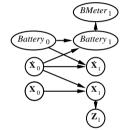
Bayesian Networks

Dynamic Bayesian networks



• X_t , E_t contain arbitrarily many variables in a replicated Bayes net





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Example

Kalman Filt

Dynamic Bayesian Networks

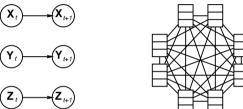
Particle Filterin

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DBNs vs. HMMs



• Every HMM is a single-variable DBN; every discrete DBN is an HMM



- Sparse dependencies ⇒ exponentially fewer parameters;
 e.g., 20 state variables, three parents each
- DBN has $20 \times 2^3 = 160$ parameters, HMM has $2^{20} \times 2^{20} \approx 10^{12}$

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Dynamic Bayesian Networks

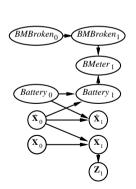
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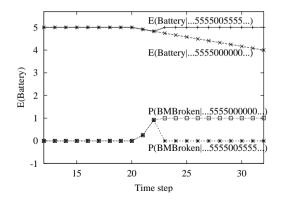
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DBNs vs Kalman filters



 Every Kalman filter model is a DBN, but few DBNs are KFs; real world requires non-Gaussian posteriors
 E.g., where are bin Laden and my keys? What's the battery charge?





Dynamic Bayesian Networks

Exact inference in DRNs



• Naive method: unroll the network and run any exact algorithm



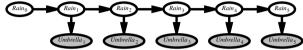
- Problem: inference cost for each update grows with t
- Rollup filtering: add slice t+1, "sum out" slice t using variable elimination
- Largest factor is $O(d^{n+1})$, update cost $O(d^{n+2})$ (cf. HMM update cost $O(d^{2n})$

Dynamic Bayesian Networks

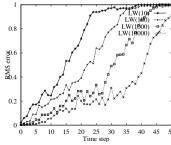
Likelihood weighting for DBNs



• Set of weighted samples approximates the belief state



- LW samples pay no attention to the evidence!
 - \Rightarrow fraction "agreeing" falls exponentially with t
 - \Rightarrow number of samples required grows exponentially with t



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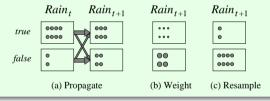
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Particle filtering



Concept 6

Particle filtering is a model to focus on the population of samples ("particles") that tracks the high-likelihood regions of the state-space



- Widely used for tracking nonlinear systems, esp. in vision
- \bullet Also used for simultaneous localization and mapping in mobile robots $10^5\text{-dimensional state space}$

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Models

Example Hidden Markov Mc

Kaiman Filter

Dynamic Bayesian Networks

Particle Filtering

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Filter task



Writing $N(\mathbf{x}_t \mid \mathbf{e}_{1:t})$ for the number of samples occupying state \mathbf{x}_t after observations $\mathbf{e}_{1:t}$ have been processed. Assume consistent at time t:

$$N(\mathbf{x}_t \mid \mathbf{e}_{1:t})/N = P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$
(11)

• **Propagate** forward populations of x_{t+1} :

$$N(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} P(\mathbf{x}_{t+1} \mid \mathbf{x}_t) N(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$
(12)

• Weight samples by their likelihood for e_{t+1} :

$$W(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t+1}) = P(\mathbf{e}_{t+1} \mid \mathbf{x}_{t+1}) N(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t})$$
(13)

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Models

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Kalman Filte

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Filter task (cont.)



• **Resample** to obtain populations proportional to *W*:

$$N(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t+1})/N = \alpha W(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t+1})$$

$$= \alpha P(\mathbf{e}_{t+1} \mid \mathbf{x}_{t+1}) N(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t})$$

$$= \alpha P(\mathbf{e}_{t+1} \mid \mathbf{x}_{t+1}) \sum_{\mathbf{x}_{t}} P(\mathbf{x}_{t+1} \mid \mathbf{x}_{t}) N(\mathbf{x}_{t} \mid \mathbf{e}_{1:t})$$

$$= \alpha P(\mathbf{e}_{t+1} \mid \mathbf{x}_{t+1}) \sum_{\mathbf{x}_{t}} P(\mathbf{x}_{t+1} \mid \mathbf{x}_{t}) P(\mathbf{x}_{t} \mid \mathbf{e}_{1:t})$$

$$= P(\mathbf{x}_{t+1} \mid \mathbf{e}_{1:t+1})$$
(14)

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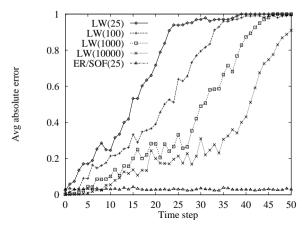
Particle Filtering

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Particle filtering performance



 Approximation error of particle filtering remains bounded over time, at least empirically—theoretical analysis is difficult



Keeping Track of Many Objects



Keeping Track of Many Objects

Summary



- Temporal models use state and sensor variables replicated over time
- Markov assumptions and stationarity assumption, so we need
 - transition model $P(X_t \mid X_{t-1})$
 - sensor model $P(E_t \mid X_t)$
- Tasks are filtering, prediction, smoothing, most likely sequence; all done recursively with constant cost per time step
- Hidden Markov models have a single discrete state variable; used for speech recognition
- Kalman filters allow n state variables, linear Gaussian, $O(n^3)$ update
- Dynamic Bayes nets subsume HMMs, Kalman filters; exact update intractable
- Particle filtering is a good approximate filtering algorithm for DBNs

References



Goodfellow, I., Bengio, Y., and Courville, A. (2016). Deep learning. MIT press.

Lê, B. and Tô, V. (2014).
Cở sở trí tuệ nhân tạo.
Nhà xuất bản Khoa học và Kỹ thuật.

Russell, S. and Norvig, P. (2021).

Artificial intelligence: a modern approach.
Pearson Education Limited.