

# **Lecture 2: Sampling-based Approximations And Function Fitting**

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# Quick One-Slide Recap

- Optimal Control

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given an MDP  $(S, A, P, R, \gamma, H)$

find the optimal policy  $\pi^*$

- Exact Methods:



***Value Iteration***



***Policy Iteration***

**Limitations:**

- Update equations require access to dynamics model
- Iteration over / Storage for all states and actions: requires small, discrete state-action space

-> **sampling-based approximations**

-> **Q/V function fitting**

# Sampling-Based Approximation

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- Q Value Iteration
- Value Iteration?
- Policy Iteration
  - Policy Evaluation
  - Policy Improvement?

# Recap Q-Values

$Q^*(s, a)$  = expected utility starting in  $s$ , taking action  $a$ , and (thereafter) acting optimally

Bellman Equation:

$$Q^*(s, a) = \sum_{s'} P(s'|s, a)(R(s, a, s') + \gamma \max_{a'} Q^*(s', a'))$$

Q-Value Iteration:

$$Q_{k+1}(s, a) \leftarrow \sum_{s'} P(s'|s, a)(R(s, a, s') + \gamma \max_{a'} Q_k(s', a'))$$

# (Tabular) Q-Learning

- Q-value iteration:  $Q_{k+1}(s, a) \leftarrow \sum_{s'} P(s'|s, a) (R(s, a, s') + \gamma \max_{a'} Q_k(s', a'))$
- Rewrite as expectation:  $Q_{k+1} \leftarrow \mathbb{E}_{s' \sim P(s'|s, a)} \left[ R(s, a, s') + \gamma \max_{a'} Q_k(s', a') \right]$
- (Tabular) Q-Learning: replace expectation by samples
  - For an state-action pair  $(s, a)$ , receive:  $s' \sim P(s'|s, a)$
  - Consider your old estimate:  $Q_k(s, a)$
  - Consider your new sample estimate:  
$$\text{target}(s') = R(s, a, s') + \gamma \max_{a'} Q_k(s', a')$$
  - Incorporate the new estimate into a running average:  
$$Q_{k+1}(s, a) \leftarrow (1 - \alpha) Q_k(s, a) + \alpha [\text{target}(s')]$$

# (Tabular) Q-Learning

Algorithm:

Start with  $Q_0(s, a)$  for all  $s, a$ .

Get initial state  $s$

For  $k = 1, 2, \dots$  till convergence

    Sample action  $a$ , get next state  $s'$

    If  $s'$  is terminal:

$$\text{target} = R(s, a, s')$$

    Sample new initial state  $s'$

    else:

$$\text{target} = R(s, a, s') + \gamma \max_{a'} Q_k(s', a')$$

$$Q_{k+1}(s, a) \leftarrow (1 - \alpha)Q_k(s, a) + \alpha [\text{target}]$$

$$s \leftarrow s'$$

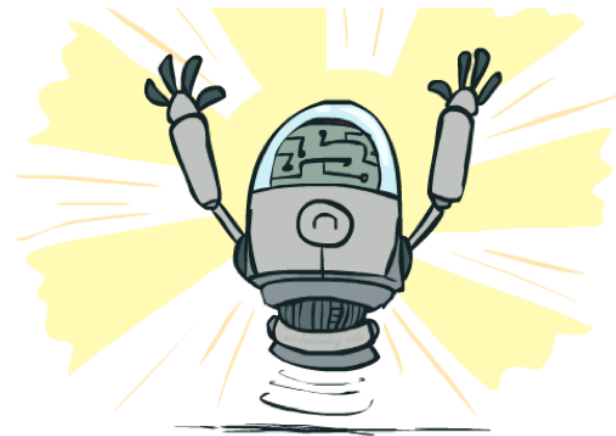
# How to sample actions?

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- Choose random actions?
- Choose action that maximizes  $Q_k(s, a)$  (i.e. greedily)?
- $\epsilon$ -Greedy: choose random action with prob.  $\epsilon$ , otherwise choose action greedily

# Q-Learning Properties

- Amazing result: Q-learning converges to optimal policy -- even if you're acting suboptimally!
- This is called **off-policy learning**
- Caveats:
  - You have to explore enough
  - You have to eventually make the learning rate small enough
  - ... but not decrease it too quickly



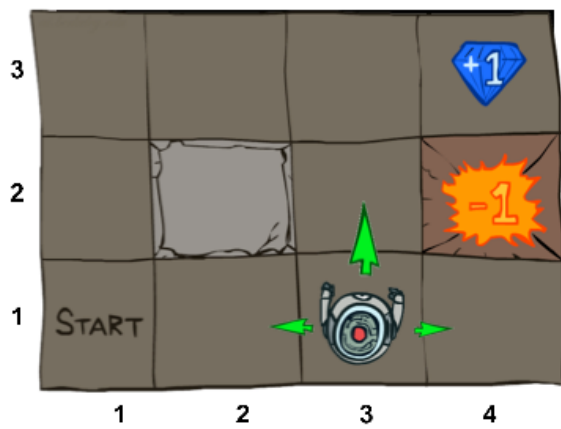


# Q-Learning Properties

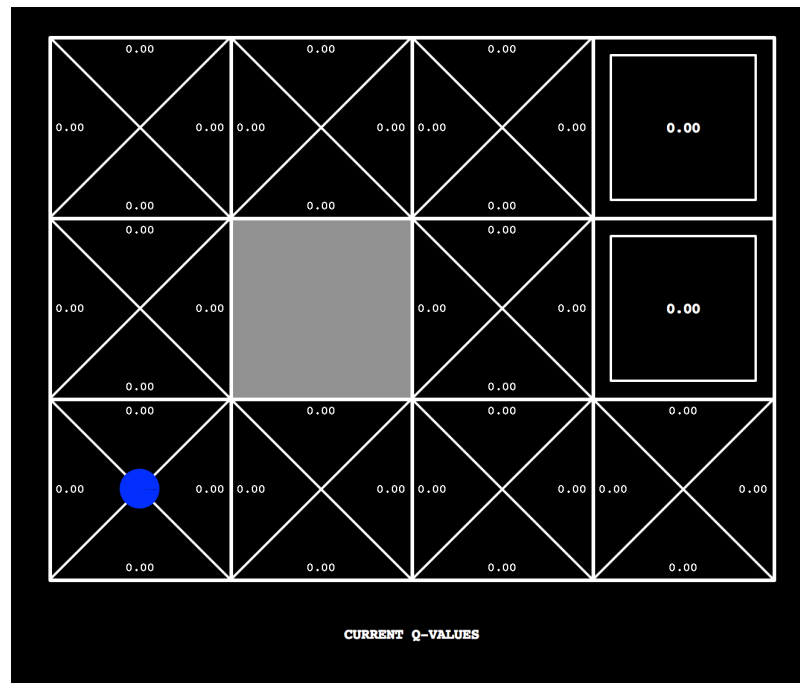
- Technical requirements.
  - All states and actions are visited infinitely often
    - Basically, in the limit, it doesn't matter how you select actions (!)
  - Learning rate schedule such that for all state and action pairs (s,a):

$$\sum_{t=0}^{\infty} \alpha_t(s, a) = \infty \qquad \sum_{t=0}^{\infty} \alpha_t^2(s, a) < \infty$$

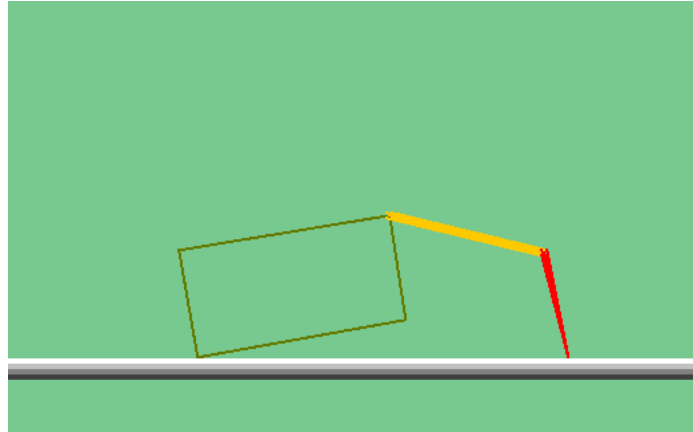
# Q-Learning Demo: Gridworld



- **States:** 11 cells
- **Actions:** {up, down, left, right}
- **Deterministic transition function**
- **Learning rate:** 0.5
- **Discount:** 1
- **Reward:** +1 for getting diamond, -1 for falling into trap



# Q-Learning Demo: Crawler



- **States:** discretized value of 2d state: (arm angle, hand angle)
- **Actions:** Cartesian product of {arm up, arm down} and {hand up, hand down}
- **Reward:** speed in the forward direction

# Sampling-Based Approximation

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✓ Q Value Iteration → (Tabular) Q-learning

- Value Iteration?
- Policy Iteration
  - Policy Evaluation
  - Policy Improvement?

# Value Iteration w/ Samples?

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- Value Iteration

$$V_{i+1}^*(s) \leftarrow \max_a \mathbb{E}_{s' \sim P(s'|s,a)} [R(s, a, s') + \gamma V_i^*(s')]$$

- unclear how to draw samples through max.....

# Sampling-Based Approximation

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✓ Q Value Iteration → (Tabular) Q-learning

■ ~~Value Iteration?~~

■ Policy Iteration

■ Policy Evaluation

■ Policy Improvement?

# Recap: Policy Iteration

## One iteration of policy iteration:

- Policy evaluation for current policy  $\pi_k$  :

- Iterate until convergence

$$V_{i+1}^{\pi_k}(s) \leftarrow \mathbb{E}_{s' \sim P(s'|s, \pi_k(s))} [R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')]$$

Can be approximated by samples

This is called Temporal Difference (TD) Learning

- Policy improvement: find the best action according to one-step look-ahead

$$\pi_{k+1}(s) \leftarrow \arg \max_a \mathbb{E}_{s' \sim P(s'|s, a)} [R(s, a, s') + \gamma V^{\pi_k}(s')]$$

Unclear what to do with the max (for now)

# Sampling-Based Approximation

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- ✓ ■ Q Value Iteration → (Tabular) Q-learning
- ~~Value Iteration?~~
- Policy Iteration
  - ✓ ■ Policy Evaluation → (Tabular) TD-learning
  - ~~Policy Improvement (for now)~~



# Quick One-Slide Recap

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***Policy Iteration***

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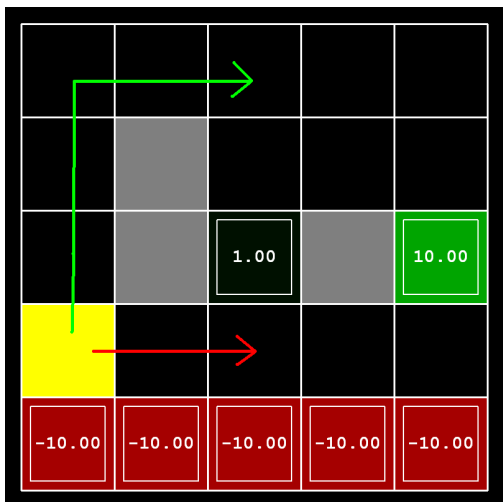


**sampling-based approximations**

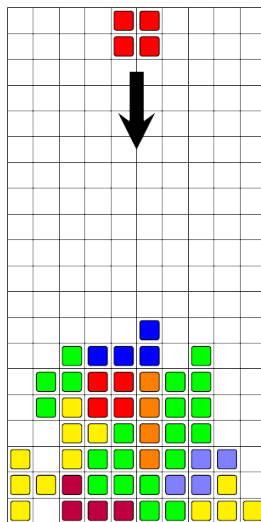
**-> Q/V function fitting**

# Can tabular methods scale?

- Discrete environments



Gridworld  
 $10^1$



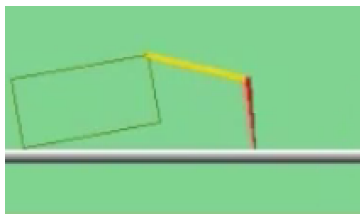
Tetris  
 $10^{60}$



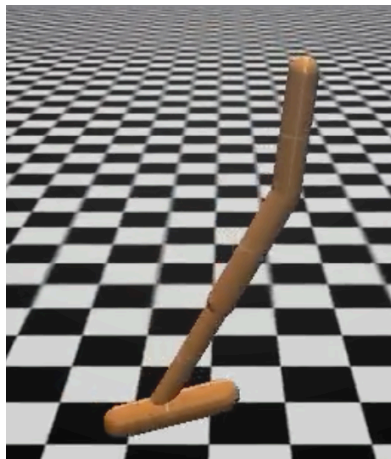
Atari  
 $10^{308}$  (ram)  $10^{16992}$  (pixels)

# Can tabular methods scale?

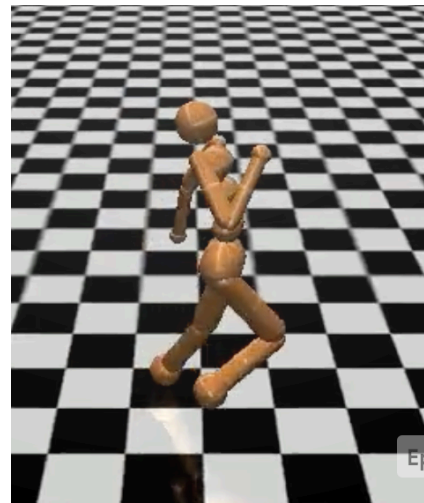
- Continuous environments (by crude discretization)



Crawler  
 $10^2$



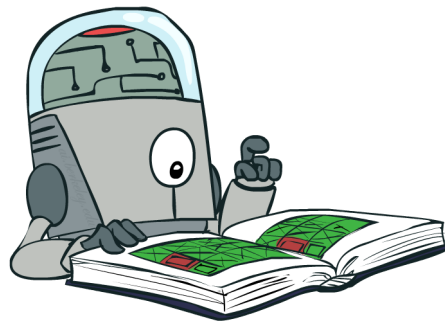
Hopper  
 $10^{10}$



Humanoid  
 $10^{100}$

# Generalizing Across States

- Basic Q-Learning keeps a table of all q-values
- In realistic situations, we cannot possibly learn about every single state!
  - Too many states to visit them all in training
  - Too many states to hold the q-tables in memory
- Instead, we want to generalize:
  - Learn about some small number of training states from experience
  - Generalize that experience to new, similar situations
  - This is a fundamental idea in machine learning, and we'll see it over and over again



# Approximate Q-Learning

- Instead of a table, we have a parametrized Q function:  $Q_{\theta}(s, a)$

- Can be a linear function in features:

$$Q_{\theta}(s, a) = \theta_0 f_0(s, a) + \theta_1 f_1(s, a) + \cdots + \theta_n f_n(s, a)$$

- Or a complicated neural net

- Learning rule:

- Remember:  $\text{target}(s') = R(s, a, s') + \gamma \max_{a'} Q_{\theta_k}(s', a')$

- Update:

$$\theta_{k+1} \leftarrow \theta_k - \alpha \nabla_{\theta} \left[ \frac{1}{2} (Q_{\theta}(s, a) - \text{target}(s'))^2 \right] \Big|_{\theta=\theta_k}$$

# Connection to Tabular Q-Learning

- Suppose  $\theta \in \mathbb{R}^{|S| \times |A|}$ ,  $Q_\theta(s, a) \equiv \theta_{sa}$

$$\begin{aligned} & \nabla_{\theta_{sa}} \left[ \frac{1}{2} (Q_\theta(s, a) - \text{target}(s'))^2 \right] \\ &= \nabla_{\theta_{sa}} \left[ \frac{1}{2} (\theta_{sa} - \text{target}(s'))^2 \right] \\ &= \theta_{sa} - \text{target}(s') \end{aligned}$$

- Plug into update:  $\theta_{sa} \leftarrow \theta_{sa} - \alpha(\theta_{sa} - \text{target}(s'))$   
 $= (1 - \alpha)\theta_{sa} + \alpha[\text{target}(s')]$

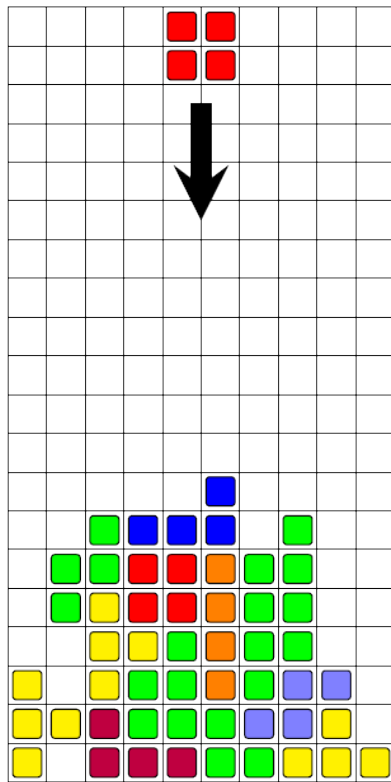
- Compare with Tabular Q-Learning update:

$$Q_{k+1}(s, a) \leftarrow (1 - \alpha)Q_k(s, a) + \alpha [\text{target}(s')]$$

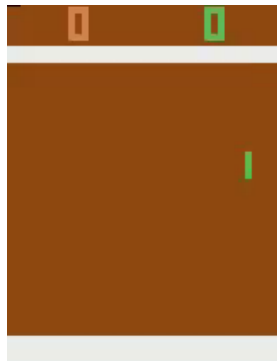
# Engineered Approximation Example: Tetris

- state: naïve board configuration + shape of the falling piece  $\sim 10^{60}$  states!
- action: rotation and translation applied to the falling piece
- 22 features aka basis functions  $\phi_i$ 
  - Ten basis functions,  $0, \dots, 9$ , mapping the state to the height  $h[k]$  of each column.
  - Nine basis functions,  $10, \dots, 18$ , each mapping the state to the absolute difference between heights of successive columns:  $|h[k+1] - h[k]|$ ,  $k = 1, \dots, 9$ .
  - One basis function, 19, that maps state to the maximum column height:  $\max_k h[k]$
  - One basis function, 20, that maps state to the number of 'holes' in the board.
  - One basis function, 21, that is equal to 1 in every state.

$$\hat{V}_\theta(s) = \sum_{i=0}^{21} \theta_i \phi_i(s) = \theta^\top \phi(s)$$



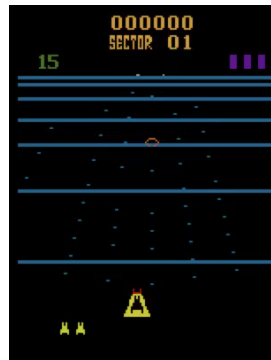
# Deep Reinforcement Learning



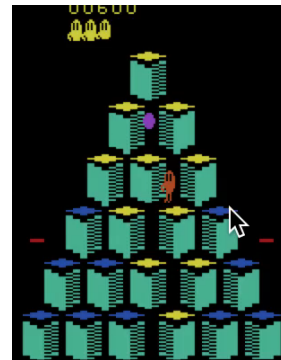
Pong



Enduro



Beamrider



Q\*bert

- From pixels to actions
- Same algorithm (with effective tricks)
- CNN function approximator, w/ 3M free parameters



# Lab 1

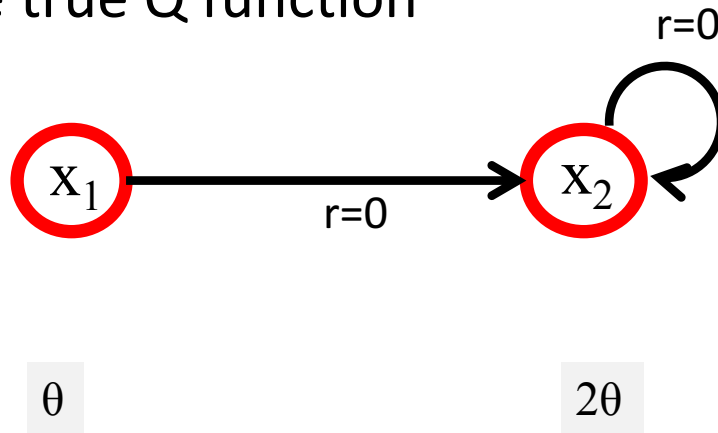
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- We have now covered enough materials for Lab 1.
- Will be released on Piazza by this afternoon.
- Covers value iteration, policy iteration, and tabular Q-learning.



# Convergence of Approximate Q-Learning

- The bad: it is not guaranteed to converge...
  - Even if the function approximation is expressive enough to represent the true Q function



Function approximator:  $[1 \ 2] * \theta$

# Simple Example\*\*

$$\bar{J}_\theta = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \theta$$

$$\begin{aligned}\bar{J}^{(1)}(x_1) &= 0 + \gamma \hat{J}_{\theta^{(0)}}(x_2) = 2\gamma\theta^{(0)} \\ \bar{J}^{(1)}(x_2) &= 0 + \gamma \hat{J}_{\theta^{(0)}}(x_2) = 2\gamma\theta^{(0)}\end{aligned}$$

Function approximation with least squares fit:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \theta^{(1)} \approx \begin{bmatrix} 2\gamma\theta^{(0)} \\ 2\gamma\theta^{(0)} \end{bmatrix}$$

Least squares fit results in:

$$\theta^{(1)} = \frac{6}{5}\gamma\theta^{(0)}$$

Repeated back-ups and function approximations result in:

$$\theta^{(i)} = \left(\frac{6}{5}\gamma\right)^i \theta^{(0)}$$

which diverges if  $\gamma > \frac{5}{6}$  even though the function approximation class can represent the true value function.]

# Composing Operators\*\*

- **Definition.** An operator  $G$  is a *non-expansion* with respect to a norm  $|| \cdot ||$  if

$$\|GJ_1 - GJ_2\| \leq \|J_1 - J_2\|$$

- **Fact.** If the operator  $F$  is a  $\gamma$ -contraction with respect to a norm  $|| \cdot ||$  and the operator  $G$  is a non-expansion with respect to the same norm, then the sequential application of the operators  $G$  and  $F$  is a  $\gamma$ -contraction, i.e.,

$$\|GFJ_1 - GFJ_2\| \leq \gamma \|J_1 - J_2\|$$

- **Corollary.** If the supervised learning step is a non-expansion, then iteration in value iteration with function approximation is a  $\gamma$ -contraction, and in this case we have a convergence guarantee.

# Averager Function Approximators Are Non-Expansions\*\*

DEFINITION: A real-valued function approximation scheme is an *averager* if every fitted value is the weighted average of zero or more target values and possibly some predetermined constants. The weights involved in calculating the fitted value  $\hat{Y}_i$  may depend on the sample vector  $X_0$ , but may not depend on the target values  $Y$ . More precisely, for a fixed  $X_0$ , if  $Y$  has  $n$  elements, there must exist  $n$  real numbers  $k_i$ ,  $n^2$  nonnegative real numbers  $\beta_{ij}$ , and  $n$  nonnegative real numbers  $\beta_i$ , so that for each  $i$  we have  $\beta_i + \sum_j \beta_{ij} = 1$  and  $\hat{Y}_i = \beta_i k_i + \sum_j \beta_{ij} Y_j$ .

- Examples:
  - nearest neighbor (aka state aggregation)
  - linear interpolation over triangles (tetrahedrons, ...)

# Averager Function Approximators Are Non-Expansions\*\*

*Proof:* Let  $J_1$  and  $J_2$  be two vectors in  $\Re^n$ . Consider a particular entry  $s$  of  $\Pi J_1$  and  $\Pi J_2$ :

$$\begin{aligned} |(\Pi J_1)(s) - (\Pi J_2)(s)| &= |\beta_{s0} + \sum_{s'} \beta_{ss'} J_1(s') - \beta_{s0} + \sum_{s'} \beta_{ss'} J_2(s')| \\ &= |\sum_{s'} \beta_{ss'} (J_1(s') - J_2(s'))| \\ &\leq \max_{s'} |J_1(s') - J_2(s')| \\ &= \|J_1 - J_2\|_\infty \end{aligned}$$

This holds true for all  $s$ , hence we have

$$\|\Pi J_1 - \Pi J_2\|_\infty \leq \|J_1 - J_2\|_\infty$$

# Linear Regression ☹️ \*\*

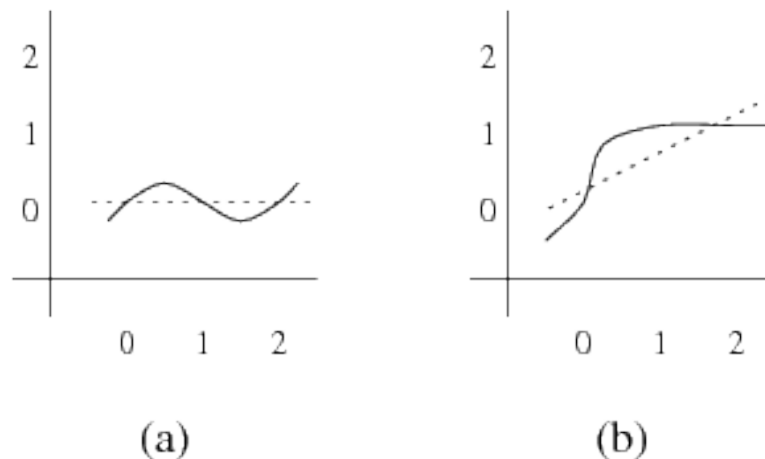


Figure 2: The mapping associated with linear regression when samples are taken at the points  $x = 0, 1, 2$ . In (a) we see a target value function (solid line) and its corresponding fitted value function (dotted line). In (b) we see another target function and another fitted function. The first target function has values  $y = 0, 0, 0$  at the sample points; the second has values  $y = 0, 1, 1$ . Regression exaggerates the difference between the two functions: the largest difference between the two target functions at a sample point is 1 (at  $x = 1$  and  $x = 2$ ), but the largest difference between the two fitted functions at a sample point is  $\frac{7}{6}$  (at  $x = 2$ ).



# Guarantees for Fixed Point\*\*

**Theorem.** Let  $J^*$  be the optimal value function for a finite MDP with discount factor  $\gamma$ . Let the projection operator  $\Pi$  be a non-expansion w.r.t. the infinity norm and let  $\tilde{J}$  be any fixed point of  $\Pi$ . Suppose  $\|\tilde{J} - J^*\|_\infty \leq \epsilon$ . Then  $\Pi T$  converges to a value function  $\bar{J}$  such that:

$$\|\bar{J} - J^*\| \leq 2\epsilon + \frac{2\gamma\epsilon}{1 - \gamma}$$

- I.e., if we pick a non-expansion function approximator which can approximate  $J^*$  well, then we obtain a good value function estimate.
- To apply to discretization: use continuity assumptions to show that  $J^*$  can be approximated well by chosen discretization scheme.