Determining the Velocity of a World Point in a Moving Camera Frame

Thien-Minh Nguyen, PhD

June 8, 2023

1 Preliminaries

1.1 Notations

In this note the following practice is adopted:

The left superscript is used to indicate the coordinate frame that some quantities are referenced with respect to (w.r.t.).

On the other hand, the left subscript is used to indicate the object to which the quantity is a characteristics of.

For example, we shall denote the angular velocity of a body frame $\mathtt{B}$ referenced in the world frame $\mathtt{W}$ as ${}^{\mathtt{W}}_{\mathtt{B}}\omega$.

The dot product between two vectors is denoted as $\langle\cdot,\cdot\rangle$, while $\times$ denotes the cross product.

The skew-symmetric matrix of a vector $\mathbf{v}$ is denoted as $\lfloor\mathbf{v}\rfloor_{\times}$. $()^{\vee}$ denotes the mapping from a skew-symmetric matrix to Euclidean vector. For example $\lfloor\mathbf{v}\rfloor_{\times}^{\vee} = \mathbf{v}$.

For an angle ϕ, we shall use $\mathrm{c}_{\psi}, \mathrm{s}_{\psi}, \mathrm{t}_{\psi}$ as shorthands for the cos, sin and tan of the angle ϕ.

1.2 Rigid body motion

Following the convention of flying robot [1], we use the term yaw, pitch, roll to refer to the Euler angles around the axes of the *moving coordinate frame* in ZYX order (intrinsic rotations). The yaw, pitch, roll angles are denoted as ψ, θ, ϕ. In other words:

$$ {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi). \tag{1} $$

In the studies of rigid body kinematics, the following equation plays an important role:

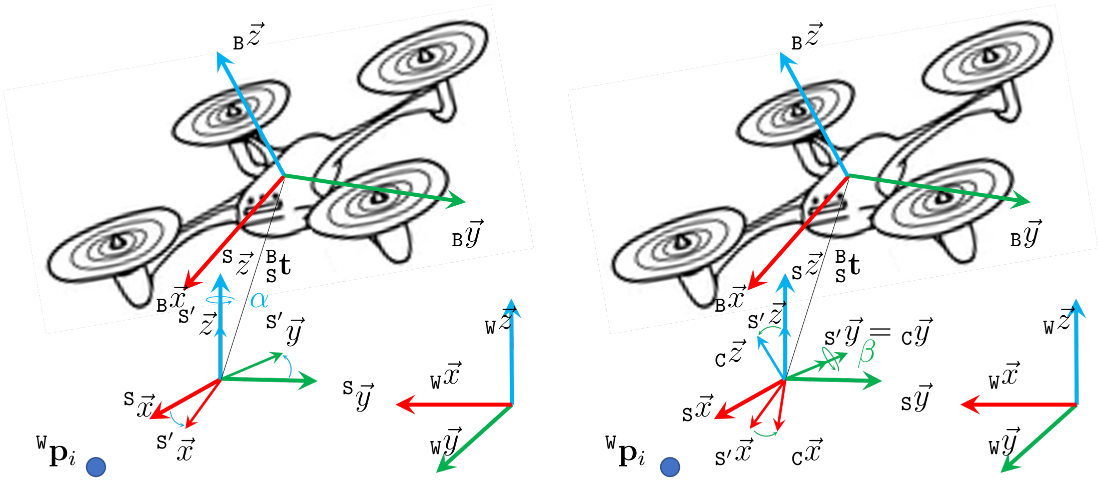

Figure 1: The World frame W, UAV body frame B, Stabilizer frame S, and the Camera frame C

$$\frac{d}{dt}\left({}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}\right) = {}^{\mathtt{W}}_{\mathtt{B}}\dot{\mathbf{R}} = {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}\left\lfloor {}^{\mathtt{B}}_{\mathtt{B}}\omega \right\rfloor_{\times} = \left\lfloor {}^{\mathtt{W}}_{\mathtt{B}}\omega \right\rfloor_{\times} {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}, \tag{2}$$

where ${}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}$ is the rotation matrix that converts coordinates in the body frame B to the world frame W, and $_{\mathtt{B}}\omega$ is the angular velocity of the body frame B. In practice, the quantity ${}^{\mathtt{B}}_{\mathtt{B}}\omega$ is what often obtained from the IMU.

2 Problem formulation

Fig. 1 illustrates the objects involved in our analysis.

- The world frame W.
- The body frame B, coinciding with the body frame center.
- The stabilizer frame S, which is offset from the body frame by a translation ${}^{\mathtt{B}}_{\mathtt{S}}\mathbf{t}$.
- The camera frame C, which does not have any translation from the S frame. Hence ${}^{\mathtt{B}}_{\mathtt{C}}\mathbf{t} = {}^{\mathtt{B}}_{\mathtt{S}}\mathbf{t}$.
- The camera yaw and pitch angles α and β, which are defined relative to the S frame.
- The interest point ${}^{\mathtt{W}}\mathbf{p}_i$.

The stabilizer frame is automatically controlled such that

- ${}_{\mathtt{S}}\vec{x} \times (\langle {}_{\mathtt{B}}\vec{x}, {}_{\mathtt{W}}\vec{x}\rangle \, {}_{\mathtt{W}}\vec{x} + \langle {}_{\mathtt{B}}\vec{x}, {}_{\mathtt{W}}\vec{y}\rangle \, {}_{\mathtt{W}}\vec{y}) = 0$, i.e. ${}_{\mathtt{S}}\vec{x}$ is always parallel with the projection of ${}_{\mathtt{B}}\vec{x}$ on the span$\{{}_{\mathtt{W}}\vec{x}, {}_{\mathtt{W}}\vec{y}\}$ plane. Simply speaking, the $\mathtt{S}$ frame has the same yaw angle with the $\mathtt{B}$ frame.
- $\langle {}_{\mathtt{S}}\vec{z}, {}_{\mathtt{W}}\vec{z}\rangle = 1$, i.e. the ${}_{\mathtt{S}}\vec{z}$ axis is always parallel with the world frame ${}_{\mathtt{W}}\vec{z}$ axis. We can also say that it does not exhibit any pitch or roll motion.

Problem statement Given the body frame's pose, velocity, angular velocity, the camera pose, the camera control inputs $\dot{\alpha}$, $\dot{\beta}$, and the coordinate of the interest point. Find the velocity of the interest point in the camera frame, i.e. ${}^{\mathtt{C}}\dot{\mathbf{p}}_i$.

3 Finding the angular velocity of the stabilizer frame

The stabilizer frame $\mathtt{S}$ is assumed to have a perfect feedback controller that negates all the roll and pitch of the body frame to keep its ${}^{\mathtt{S}}\vec{x}$, ${}^{\mathtt{S}}\vec{y}$ axes parallel with the xy plane of the world frame. Its orientation can be defined through this chain rotation:

$$
\begin{aligned}
{}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R} = {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}\mathbf{R}_x(-\phi)\mathbf{R}_y(-\theta) &= \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)\mathbf{R}_x(-\phi)\mathbf{R}_y(-\theta) && (3)\\
&= \mathbf{R}_z(\psi) = \begin{bmatrix} \mathrm{c}_\psi & -\mathrm{s}_\psi & 0 \\ \mathrm{s}_\psi & \mathrm{c}_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} && (4)
\end{aligned}
$$

Taking derivatives on both sides yields:

$$
\left\lfloor {}^{\mathtt{W}}_{\mathtt{S}}\omega \right\rfloor_\times \begin{bmatrix} \mathrm{c}_\psi & -\mathrm{s}_\psi & 0 \\ \mathrm{s}_\psi & \mathrm{c}_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\psi} \begin{bmatrix} -\mathrm{s}_\psi & -\mathrm{c}_\psi & 0 \\ \mathrm{c}_\psi & -\mathrm{s}_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}
$$

The LHS and RHS of (5) are 3×3 matrices. By comparing them entry by entry we can find that:

$$
{}^{\mathtt{W}}_{\mathtt{S}}\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}. \tag{6}
$$

We do NOT have $\dot{\psi}$. However it can be calculated from the angular velocity of he body frame ${}^{\mathtt{B}}_{\mathtt{B}}\omega$. From the Ross' Lecture, and equations (7) and (8) in [1], we have:

$$
\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -\mathrm{s}_\theta & 0 & 1 \\ \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi & 0 \\ \mathrm{c}_\phi \mathrm{c}_\theta & -\mathrm{s}_\phi & 0 \end{bmatrix}^{-1} {}^{\mathtt{B}}_{\mathtt{B}}\omega = \begin{bmatrix} 0 & \mathrm{s}_\phi/\mathrm{c}_\theta & \mathrm{c}_\phi/\mathrm{c}_\theta \\ 0 & \mathrm{c}_\phi & -\mathrm{s}_\psi \\ 1 & \mathrm{s}_\phi \mathrm{t}_\theta & \mathrm{c}_\phi \mathrm{t}_\theta \end{bmatrix} {}^{\mathtt{B}}_{\mathtt{B}}\omega \tag{7}
$$

From (6) and (7), the angular velocity of the stabilizer w.r.t. to the world frame is therefore:

$$
{}^{\mathtt{W}}_{\mathtt{S}}\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{s}_\phi/\mathrm{c}_\theta & \mathrm{c}_\phi/\mathrm{c}_\theta \end{bmatrix} {}^{\mathtt{B}}_{\mathtt{B}}\omega = \left[0,\ 0,\ \frac{\mathrm{s}_\phi}{\mathrm{c}_\theta}\,{}^{\mathtt{B}}_{\mathtt{B}}\omega_y + \frac{\mathrm{c}_\phi}{\mathrm{c}_\theta}\,{}^{\mathtt{B}}_{\mathtt{B}}\omega_z\right]^\top \tag{8}
$$

4 Finding the angular velocity of the camera

The camera is mounted on the stabilizer, from which extra DOFs in yaw α and pitch β are exerted. Hence, the orientation of the camera w.r.t. the world frame is obtained by the following rotation chain:

$$
\begin{aligned}
{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} = {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R}\mathbf{R}_z(\alpha)\mathbf{R}_y(\beta) &= {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R} \begin{bmatrix} \mathrm{c}_\alpha & -\mathrm{s}_\alpha & 0 \\ \mathrm{s}_\alpha & \mathrm{c}_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}_\beta & 0 & \mathrm{s}_\beta \\ 0 & 1 & 0 \\ -\mathrm{s}_\beta & 0 & \mathrm{c}_\beta \end{bmatrix} && (9) \\
&= {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R} \begin{bmatrix} \mathrm{c}_\alpha \mathrm{c}_\beta & -\mathrm{s}_\alpha & \mathrm{c}_\alpha \mathrm{s}_\beta \\ \mathrm{s}_\alpha \mathrm{c}_\beta & \mathrm{c}_\alpha & \mathrm{s}_\alpha \mathrm{s}_\beta \\ -\mathrm{s}_\beta & 0 & \mathrm{c}_\beta \end{bmatrix} \triangleq {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R}{}^{\mathtt{S}}_{\mathtt{C}}\mathbf{R}. && (10)
\end{aligned}
$$

Hence the angular velocity of the camera w.r.t. the world frame is:

$$
\begin{aligned}
&\left\lfloor {}^{\mathtt{W}}_{\mathtt{C}}\omega \right\rfloor_\times {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} = \frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} = \left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R}\right)\mathbf{R}_z(\alpha)\mathbf{R}_y(\beta) + {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R}\left(\frac{d}{dt}\mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\right) && (11) \\
&= \left\lfloor {}^{\mathtt{W}}_{\mathtt{S}}\omega \right\rfloor_\times {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R}\mathbf{R}_z(\alpha)\mathbf{R}_y(\beta) + {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R} \begin{bmatrix} -\dot\alpha \mathrm{s}_\alpha \mathrm{c}_\beta - \dot\beta \mathrm{c}_\alpha \mathrm{s}_\beta & -\dot\alpha \mathrm{c}_\alpha & -\dot\alpha \mathrm{s}_\alpha \mathrm{s}_\beta + \dot\beta \mathrm{c}_\alpha \mathrm{c}_\beta \\ \dot\alpha \mathrm{c}_\alpha \mathrm{c}_\beta - \dot\beta \mathrm{s}_\alpha \mathrm{s}_\beta & -\dot\alpha \mathrm{s}_\alpha & \dot\alpha \mathrm{c}_\alpha \mathrm{s}_\beta + \dot\beta \mathrm{s}_\alpha \mathrm{c}_\beta \\ -\dot\beta \mathrm{c}_\beta & 0 & -\dot\beta \mathrm{s}_\beta \end{bmatrix} && (12)
\end{aligned}
$$

Therefore the angular velocity of the camera is

$$
{}^{\mathtt{W}}_{\mathtt{C}}\omega = \left[\left\lfloor {}^{\mathtt{W}}_{\mathtt{S}}\omega \right\rfloor_\times + {}^{\mathtt{W}}_{\mathtt{S}}\mathbf{R} \begin{bmatrix} -\dot\alpha \mathrm{s}_\alpha \mathrm{c}_\beta - \dot\beta \mathrm{c}_\alpha \mathrm{s}_\beta & -\dot\alpha \mathrm{c}_\alpha & -\dot\alpha \mathrm{s}_\alpha \mathrm{s}_\beta + \dot\beta \mathrm{c}_\alpha \mathrm{c}_\beta \\ \dot\alpha \mathrm{c}_\alpha \mathrm{c}_\beta - \dot\beta \mathrm{s}_\alpha \mathrm{s}_\beta & -\dot\alpha \mathrm{s}_\alpha & \dot\alpha \mathrm{c}_\alpha \mathrm{s}_\beta + \dot\beta \mathrm{s}_\alpha \mathrm{c}_\beta \\ -\dot\beta \mathrm{c}_\beta & 0 & -\dot\beta \mathrm{s}_\beta \end{bmatrix} {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1} \right]^\vee
$$

Remark 1. *Because $\left\lfloor {}^{W}_{C}\omega \right\rfloor_\times$ is skew symmetric, the matrix in the bracket on the right hand side should also be skew symmetric. This can be used to check if the derivation is correct.*

5 Finding the linear velocity of the camera in the world frame

The velocity of the camera frame in the world frame will be useful in later calculations. This can be computed from the available information as follows:

$$
{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{v} = \frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{p} = \frac{d}{dt}({}^{\mathtt{W}}_{\mathtt{B}}\mathbf{p} + {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}{}^{\mathtt{B}}_{\mathtt{C}}\mathbf{t}) = {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{v} + {}^{\mathtt{W}}_{\mathtt{B}}\mathbf{R}\left\lfloor {}^{\mathtt{W}}_{\mathtt{B}}\omega \right\rfloor_\times {}^{\mathtt{B}}_{\mathtt{C}}\mathbf{t} \tag{13}
$$

6 Finding the linear velocity of the interest point in camera frame

We have prepared all ingredients to compute the interest point's velocity in camera frame. Let us start with the simple equation:

$$ {}^{\mathtt{C}}\mathbf{v}_i = \frac{d}{dt}{}^{\mathtt{C}}\mathbf{p}_i = \frac{d}{dt}\left[{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\left({}^{\mathtt{W}}\mathbf{p}_i - {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{p}\right)\right] \tag{14} $$

Hence:

$$ {}^{\mathtt{C}}\mathbf{v}_i = \left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\right){}^{\mathtt{W}}\mathbf{p}_i - \left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\right){}^{\mathtt{W}}_{\mathtt{C}}\mathbf{p} - {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{p}\right) \tag{15} $$

Note that:

$$ \left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}\right) = 0 \therefore \left(\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\right){}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} = -{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} = -{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\left\lfloor{}^{\mathtt{W}}_{\mathtt{C}}\omega\right\rfloor_{\times}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R} \tag{16} $$

Thus:

$$ \frac{d}{dt}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1} = -{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\left\lfloor{}^{\mathtt{W}}_{\mathtt{C}}\omega\right\rfloor_{\times} \tag{17} $$

Finally:

$$ {}^{\mathtt{C}}\mathbf{v}_i = {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}\left\lfloor{}^{\mathtt{W}}_{\mathtt{C}}\omega\right\rfloor_{\times}\left(-{}^{\mathtt{W}}\mathbf{p}_i + {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{p}\right) - {}^{\mathtt{W}}_{\mathtt{C}}\mathbf{R}^{-1}{}^{\mathtt{W}}_{\mathtt{C}}\mathbf{v} \tag{18} $$

References

[1] R. Beard, "Quadrotor dynamics and control rev 0.1," 2008.