

1. "Sum of sinusoids of equal frequencies is still a sinusoid of the same frequency."

(a) 任何 sinusoid 都能以 rotating complex phasor (旋轉複數相幅) 表示, 以 cosine signal 為例:

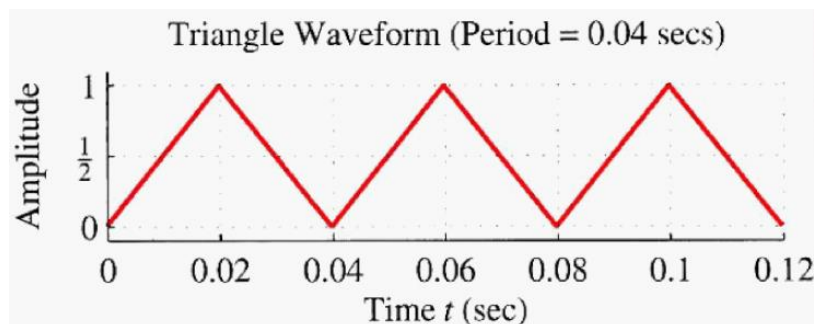
$$\begin{aligned} A \cos(\omega_0 t + \phi) &= \frac{A}{2}(e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}) \\ &= \frac{1}{2} A e^{j\phi} e^{j\omega_0 t} + \frac{1}{2} A e^{-j\phi} e^{-j\omega_0 t} \end{aligned}$$

(b) 我們可以看到, 將 sinusoid 以 complex exponential 來表示的話, 其相位差 ϕ 會被 complex amplitude ($Ae^{j\phi}$) 所吸收。

(c) 其實蠻直覺的, 兩個或多個具有相同頻率, 不同振幅和相位差的 sinusoids 的和, 可表示為對等的單個 sinusoid, 相加前各不相同的振幅和相位差在相加之後都被整合在 complex amplitude ($Ae^{j\phi}$) 裡了, 所以相加得到的 sinusoid 仍是原本的頻率 (ω_0)。

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_0 t + \phi_k) = A \cos(\omega_0 t + \phi)$$

2. Find the DC component of the following periodic signal with the period 0.04.



$$x(t) = \begin{cases} \frac{2t}{T_0}, & 0 \leq t \leq \frac{T_0}{2} \\ \frac{2(T_0 - t)}{T_0}, & \frac{T_0}{2} \leq t \leq T_0 \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_0^{T_0} x(t) dt \\ &= \frac{1}{T_0} (\text{Area of Triangle}) \\ &= \frac{1}{T_0} \left(\frac{1}{2} * T_0 * 1 \right) \end{aligned}$$

$$= 0.5$$

3. Derive that the following is a Continuous Fourier Transform pair ($a > 0$), where $u(t)$ is the unit step function (Heaviside step function)

<i>Time-Domain</i>		$\xleftrightarrow{\mathcal{F}}$		<i>Frequency-Domain</i>
$e^{-at}u(t)$				$\frac{1}{a + j\omega}$

$$x(t) = e^{-at}u(t)$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt \quad \left(\int e^{at} dt = \frac{1}{a} e^{at} \right)$$

$$= \left. \frac{e^{-at} e^{-j\omega t}}{-(a + j\omega)} \right|_0^{\infty}$$

$$= \frac{0 * 0 - 1 * 1}{-(a + j\omega)}$$

$$= \frac{1}{a + j\omega}$$



4. Prove that the Continuous Fourier transform of a Gaussian function is still a Gaussian function.

Gaussian Equation : $e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sigma\sqrt{2\pi}e^{-\frac{\sigma^2\omega^2}{2}}$

已知積分公式：

$$(1) \int_{-\infty}^{\infty} e^{-ct^2} dt = \sqrt{\frac{\pi}{c}}$$

$$(2) \int_{-\infty}^{\infty} e^{-c(t+b)^2} dt = \sqrt{\frac{\pi}{c}}, \text{ b 不影響積分結果}$$

$$(3) \int_{-\infty}^{\infty} ae^{-(\frac{1}{2c^2})(t+b)^2} dt = ac\sqrt{2\pi}$$

$$f(t) = e^{-\frac{t^2}{2\sigma^2}}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t^2 + 2j\sigma^2\omega t)} dt$$

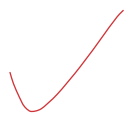
$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [t^2 + 2j\sigma^2\omega t + (j\sigma^2\omega)^2 - (j\sigma^2\omega)^2]} dt$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (t+j\sigma^2\omega)^2 + \frac{(j\sigma^2\omega)^2}{2\sigma^2}} dt \quad \text{把指數當中與 t 無關的項從積分抽出來}$$

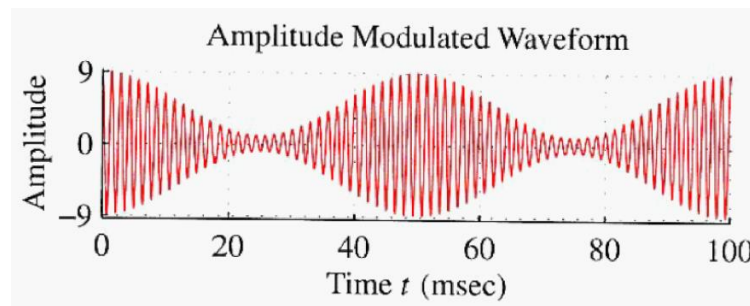
$$= \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} (t+j\sigma^2\omega)^2} dt \right] e^{\frac{(j\sigma^2\omega)^2}{2\sigma^2}} \quad \text{套用公式 (2) 得出下一個等式}$$

$$= \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} * e^{-\frac{\sigma^2\omega^2}{2}}$$

$$= \sigma\sqrt{2\pi} * e^{-\frac{\sigma^2\omega^2}{2}} \quad \text{若在前兩步直接套用公式 (3) 亦得出同樣的答案}$$



5. The amplitude-modulation (AM) signal is a product of the form, $x(t) = v(t) \cos(2\pi f_c t)$. Consider the case where $v(t) = 5 + 4\cos(40\pi t)$, and the carrier frequency $f_c = 700\text{Hz}$. The time-domain of the signal $x(t)$ is shown as



Question: Find and draw the spectrum of $x(t)$ in terms of Continuous Fourier Transform.

$$\begin{aligned} x(t) &= [5 + 4 \cos(40\pi t)] \cos(1400\pi t) \\ &= 5 \cos(1400\pi t) + 4 \cos(40\pi t) \cos(1400\pi t) \end{aligned}$$

已知公式: $2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = \cos x + \cos y$

則, $4 \cos\left(\frac{1440 - 1360}{2}\pi t\right) \cos\left(\frac{1440 + 1360}{2}\pi t\right) = 2 (\cos 1440\pi t + \cos 1360\pi t)$

$$x(t) = 5 \cos(1400\pi t) + 2 \cos(1440\pi t) + 2 \cos(1360\pi t)$$

Let $x_1(t) = \cos(1400\pi t)$, $x_2(t) = \cos(1440\pi t)$ and $x_3(t) = \cos(1360\pi t)$,

Fourier transform pair: $\cos(\omega_0 t) \longleftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

$$\begin{aligned} X(j\omega) &= 5 X_1(j\omega) + 2 X_2(j\omega) + 2 X_3(j\omega) \\ &= 5\pi\delta(\omega - 1400\pi) + 5\pi\delta(\omega + 1400\pi) \\ &\quad + 2\pi\delta(\omega - 1440\pi) + 2\pi\delta(\omega + 1440\pi) \\ &\quad + 2\pi\delta(\omega - 1360\pi) + 2\pi\delta(\omega + 1360\pi) \end{aligned}$$

Spectrum plot:

