1. “Sum of sinusoids of equal frequencies is still a sinusoid of the same frequency.”

（a）任何 sinusoid 都能以 rotating complex phasor （旋轉複數相幅）表示， 以 cosine signal 為例:

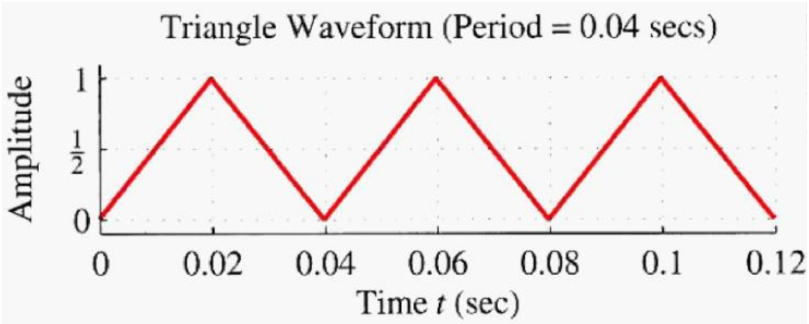
$$
\begin{aligned}
Acos(\omega_0 t + \phi) &= \frac{A}{2}(e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}) \\
&= \frac{1}{2}Ae^{j\phi}e^{j\omega_0 t} + \frac{1}{2}Ae^{-j\phi}e^{-j\omega_0 t}
\end{aligned}
$$

（b）我們可以看到，將 sinusoid 以 complex exponential 來表示的話，其相位差 ϕ 會被 complex amplitude ($Ae^{j\phi}$) 所吸收。

（c）其實蠻直覺的，兩個或多個具有相同頻率，不同振幅和相位差的 sinusoids 的和，可表示為對等的單個 sinusoid，相加前各不相同的振幅和相位差在相加之後都被整合在 complex amplitude ($Ae^{j\phi}$) 裡了，所以相加得到的 sinusoid 仍是原本的頻率 (ω_0)。

$$
x(t) = \sum_{k=1}^{N} A_k \ cos(\omega_0 t + \phi_k) = A \ cos(\omega_0 t + \phi)
$$

2. Find the DC component of the following periodic signal with the period 0.04.



$$
x(t) = \begin{cases} \frac{2t}{T_0} \ , 0 \le t \le \frac{T_0}{2} \\ \frac{2(T_0 - t)}{T_0} \ , \frac{T_0}{2} \le t \le T_0 \end{cases}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{T_0}\int_0^{T_0} x(t) \ \mathrm{d}t \\
&= \frac{1}{T_0}(Area \ of \ Triangle) \\
&= \frac{1}{T_0}(\frac{1}{2} * T_0 * 1)
\end{aligned}
$$

$= 0.5$

3. Derive that the following is a Continuous Fourier Transform pair (a > 0), where u(t) is the unit step function (Heaviside step function)

Time-Domain *Frequency-Domain*

$$e^{-at}u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{a+j\omega}$$

$x(t) = e^{-at}u(t)$

$$X(j\omega) = \int_0^\infty e^{-at}\ e^{-j\omega t}\ \mathrm{d}t$$

$$= \int_0^\infty e^{-(a+j\omega)t}\ \mathrm{d}t \qquad \left(\int e^{at}\ \mathrm{d}t \ = \ \frac{1}{a}\ e^{at}\right)$$

$$= \left.\frac{e^{-at}\ e^{-j\omega t}}{-(a+j\omega)}\right|_0^\infty$$

$$= \frac{0*0-1*1}{-(a+j\omega)}$$

$$= \frac{1}{a+j\omega}$$

4. Prove that the Continuous Fourier transform of a Gaussian function is still a Gaussian function.

$$Gaussian\ Equation\ :\ e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sigma\sqrt{2\pi}e^{-\frac{\sigma^2\omega^2}{2}}$$

已知積分公式:

（1） $\int_{-\infty}^{\infty} e^{-ct^2}\mathrm{d}t = \sqrt{\frac{\pi}{c}}$

（2） $\int_{-\infty}^{\infty} e^{-c(t+b)^2}\mathrm{d}t = \sqrt{\frac{\pi}{c}}$ ，b 不影響積分結果

（3） $\int_{-\infty}^{\infty} ae^{-(\frac{1}{2c^2})(t+b)^2}\mathrm{d}t = ac\sqrt{2\pi}$

$$f(t) = e^{-\frac{t^2}{2\sigma^2}}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\ e^{-j\omega t}\ \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\ e^{-j\omega t}\ \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t^2+2j\sigma^2\omega t)}\ \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[t^2+2j\sigma^2\omega t+(j\sigma^2\omega)^2-(j\sigma^2\omega)^2\right]}\ \mathrm{d}t$$

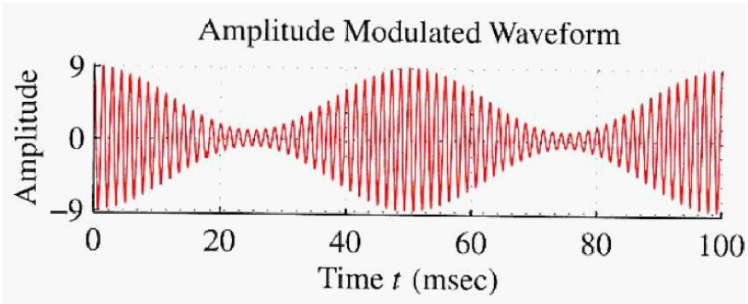
$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\ (t+j\sigma^2\omega)^2+\frac{(j\sigma^2\omega)^2}{2\sigma^2}}\ \mathrm{d}t$$ 把指數當中與 t 無關的項從積分抽出來

$$= \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\ (t+j\sigma^2\omega)^2}\ \mathrm{d}t\right]\ e^{\frac{(j\sigma^2\omega)^2}{2\sigma^2}}$$ 套用公式（2）得出下一個等式

$$= \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} * e^{-\frac{\sigma^2\omega^2}{2}}$$

$$= \sigma\sqrt{2\pi} * e^{-\frac{\sigma^2\omega^2}{2}}$$ 若在前兩步直接套用公式（3）亦得出同樣的答案

5. The amplitude-modulation (AM) signal is a product of the form, $x(t) = v(t)\ cos(2\pi f_c t)$. Consider the case where $v(t) = 5 + 4cos(40\pi t)$, and the carrier frequency $f_c = 700Hz$. The time-domain of the signal x(t) is shown as



Question: Find and draw the spectrum of x(t) in terms of Continuous Fourier Transform.

$$x(t) = \left[5 + 4\ cos(40\pi t)\right]\ cos(1400\pi t)$$

$$= 5\ cos(1400\pi t)\ +\ 4\ cos(40\pi t)\ cos(1400\pi t)$$

已知公式： $$2\ cos(\frac{x+y}{2})\ cos(\frac{x-y}{2}) = cos\ x + cos\ y$$

則， $$4\ cos(\frac{1440-1360}{2}\pi t)\ cos(\frac{1440+1360}{2}\pi t) = 2\ (cos\ 1440\pi t + cos\ 1360\pi t)$$

$$x(t) = 5\ cos(1400\pi t)\ +\ 2\ cos(1440\pi t)\ +\ 2\ cos(1360\pi t)$$

Let $x_1(t) = cos(1400\pi t)$ **,** $x_2(t) = cos(1440\pi t)$ **and** $x_3(t) = cos(1360\pi t)$**,**

Fourier transform pair : $cos(\omega_0 t) \longleftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

$$X(j\omega) = 5\ X_1(j\omega)\ +2\ X_2(j\omega)\ +2\ X_3(j\omega)$$

$$= 5\pi\delta(\omega - 1400\pi)\ +\ 5\pi\delta(\omega + 1400\pi)$$

$$+\ 2\pi\delta(\omega - 1440\pi) + 2\pi\delta(\omega + 1440\pi)$$

$$+\ 2\pi\delta(\omega - 1360\pi) + 2\pi\delta(\omega + 1360\pi)$$

Spectrum plot :