Πέμπτη, 22/12/2022

(2019)

$$\Sigma 1: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^2 - x_2 \end{cases}, \quad \Sigma 2: \ddot{x} + \dot{x} + x^3 = 0$$

α) Σ1: Σημεία ισορροπίας

$$\left.\begin{array}{l} x_{2e} = 0 \\ -x_{1e} - x_{1e}^2 - x_{2e} = 0 \end{array}\right\} \Rightarrow \begin{cases} x_{2e} = 0 \\ x_{1e}(1+x_{1e}) = 0 \end{cases} \Rightarrow \text{σ.ι. } (0,0), (-1,0)$$

$$\Sigma 2: \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \left.\begin{array}{l} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1^3 - y_2 \end{array}\right\} \overset{\text{Σ.Ι.}}{\Rightarrow} \left.\begin{array}{l} y_{2e} = 0 \\ -y_{1e}^3 - y_{2e} = 0 \end{array}\right\} \Rightarrow \text{μοναδ. σ.ι. } (0,0)$$

β) Σ1: $\dot{x} = f(x)$, $f(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 - x_1^2 \end{bmatrix}$

$$A_{(0,0)} = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}$$

$$\frac{\partial f}{\partial x_1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-2x_1 & -1 \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad Re(\lambda_i(A_{(0,0)})) < 0, \ i=1,2$$

$\Downarrow$ 1ο Θεώρημα Lyapunov

(0,0) ασυμπτωτικά ευσταθές

$$A_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\det(\lambda I - A_{(-1,0)}) = \lambda^2 + \lambda - 1$$

$\exists i$ με $Re(\lambda_i(A_{(-1,0)})) > 0 \Rightarrow (-1,0)$ ασταθές

$\Sigma 2: \quad f(y) = \begin{bmatrix} y_2 \\ -y_1^3 - y_2 \end{bmatrix}, \quad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

$V = x_1^4 + x_1^2 + 2x_1x_2 + 2x_2^2 = x_1^4 + (x_1+x_2)^2 + x_2^2$

↳ θετικά ορισμένη

$V(x_1,x_2) > 0 \quad \forall (x_1,x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\dot{V} = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 = (4y_1^3 + 2y_1 + 2y_2)y_2 + (2y_1 + 4y_2)(-y_1^3 - y_2)$

$= (\cancel{4y_1^3} + \cancel{2y_1} + 2y_2)y_2 - 2y_1^4 - \cancel{2y_1y_2} - \cancel{4y_1^3y_2} - 4y_2^2$

$= -2y_1^4 - 2y_2^2 < 0 \quad \forall (y_1,y_2) \in \mathbb{R}^2 \setminus \{0,0\}$

$(0,0)$ ολικά ασυμπτωτικά ευσταθές

γ) $\Delta\dot{x} = A\Delta x + bu$, $\quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad c = [1 \quad 0]$, $\quad \Delta x = x - x_e = \begin{bmatrix} x_1+1 \\ x_2 \end{bmatrix}$

$y = c\Delta x$

$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^2 - x_2 + u \end{cases}$ $\quad A_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

$y = x_1 - (-1) = x_1 + 1$

$\Delta\dot{x} = \begin{bmatrix} x_2 \\ -x_1\Delta x_1 - x_2 - u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 0 \\ -(\Delta x_1)^2 \end{bmatrix}$

$\det(\lambda I - A_{(0,0)}) = \lambda^2 + \lambda + 1$

$\mathcal{C} = [b \mid Ab] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, $\det(\mathcal{C}) = -1 \neq 0$ ελέγξιμο

$u = -2\Delta x_1$

$$x(k+1) = f(x(k))$$

$\downarrow$ 6.1.

$$x_e = f(x_e)$$

$$\Delta x(k) = x(k) - x_e$$

$$\Delta x(k+1) = f(x(k)) - x_e = f(x_k) - f(x_e) = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} \Delta x(k) + O(\|\Delta x(k)\|^2)$$

$$\Delta x(k+1) = A\,\Delta x(k)$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e}$$

<u>1º Θεώρημα Lyapunov</u> (ΣΔΧ)

$x(k+1) = f(x(k))$. Έστω $A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e}$

Αν x_e 6.1., δηλ. $x_e = f(x_e)$,

- Αν $|\lambda_i(A)| < 1 \quad \forall i$, τότε το x_e είναι <u>τοπικά ασυμπτωτικά ευσταθές</u>
- Αν $\exists i$ τ.ω. $|\lambda_i(A)| > 1$, τότε το x_e είναι <u>ασταθές</u>.

<u>2º Θεώρημα Lyapunov</u> (ΣΔΧ)

$x(k+1) = f(x(k))$ κ' x_e 1.σ.

Αν $\exists\ V: D \to \mathbb{R}_{\geq 0}$

$D \subseteq \mathbb{R}^n$ συνεκτικό, που περιέχει το x_e

$V(f(x(k))) - V(x(k)) < 0 \quad \forall x(k) \in D \setminus \{x_e\}$

τότε το x_e είναι ασυμπτωτικά ευσταθές.

$x(k+1) = \alpha x(k) + x^4(k)$, $\alpha \in \mathbb{R}$

α) σ.ι.

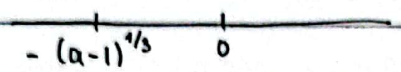
β) $V(x) = x^4$ για ποια $x(0), \alpha$, $x(t) \to 0$ καθώς $k \to \infty$

α) $x_e = \alpha x_e + x_e^4 \Rightarrow (1-\alpha)x_e = x_e^4$

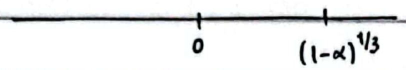
αν $\alpha = 1$, τότε $x_e = 0$ μοναδικό σ.ι.

αν $\alpha \neq 1$, $\begin{cases} 1-\alpha = x_e^3 \\ x_e = 0 \end{cases} \Rightarrow \begin{cases} x_e = (1-\alpha)^{1/3} \\ 0 \end{cases}$

$\alpha \neq 1$
$\alpha > 1$



$\alpha \neq 1$
$\alpha < 1$



$$A = \left.\frac{\partial f}{\partial x}\right|_0 = \alpha$$

• $\alpha \in (-1, 1)$ • $\alpha \in (-\infty, -1)$

$\alpha > 1$ ή $\alpha < -1$: ασταθές

ευστάθεια: $\alpha \in (-1, 1)$

$\alpha = 1$: $x(k+1) = x(k) + x^4(x)$ απορρίπτεται

$\alpha = -1$: $x(k+1) = -x(k) + x^4(k) = -(1 - x^3(k))x(k)$ απορρίπτεται

$x(0) = -\delta$

$x(1) = -(1+\delta^3)\delta$

$x(2) = -(1+\delta^3(1+\delta^3)^3)(1+\delta^3)\delta$

$\vdots$

$\Delta x(t) = x(t+1) - x(t) = -2x(t) + x^4(t)$

β) $V(x(t+1)) - V(x(t)) = (\alpha x(t) + x^4(t))^4 - x^4(t) =$

$= [(\alpha + x^3(t))^4 - 1]\, x^4(k) < 0$

όταν $-1 < \alpha + x^3(t) < 1 \Rightarrow -1-\alpha < x^3(t) < 1-\alpha \Rightarrow$

$\Rightarrow -(1+\alpha)^{1/3} < x(t) < (1+\alpha)^{1/3}$