Άλγεβρα ΙΙ — Τετάρτη, 22/02/2023

I. Group actions
(Burnside theorem)

The 3 Sylow theorems
(Th. Lagrange ⇝ $|G|=n$ & $H \leq G \Rightarrow |H| = d \mid n$)
$\Leftarrow$?

II. Rings (Δακτύλιοι)
Subrings, Ideals (Ιδεώδη)
⇝ $(R/I, +, \cdot)$ quotient ring

Reminder: G: $N \leq G$ s.t. $\forall g \in G: gN = Ng$
⇝ $N \trianglelefteq G$ normal subgroup
$\Updownarrow$
$(G/N, *)$ quotient groups
↳ $L_N = R_N$ — well-defined
$\Leftrightarrow \exists h: G \to G'$ homomorphism s.t. $\boxed{N = \ker h}$
$\cong G/N$

Integral Domains, Fields
↳ (Ακέραιες περιοχές : $\mathbb{Z}, F(x), \ldots$)

III. Polynomial Rings, Irreducibility$|_F$
(x^2+1 in $\mathbb{R}$)

IV. (Non) constructions with rule & compasses
Field extensions, Prime Fields ($\mathbb{Z}_p, \mathbb{Q}$)
↳ Galois Theorem ⇝ The quintic is not solvable with radicals (no closed formula)

Book by John Fraleigh

Let $(G, \cdot)$ group and let X be a set. An action (δράση) of G on X is a map
$$*: G \times X \to X$$
$$*(g, x) = g * x = x' \in X, \text{ s.t.}$$

1. $e * x = x \quad \forall x \in X$ ... (naturality) identity
2. $\underbrace{(g_1 g_2)}_{\in G} * x = g_1 * \underbrace{(g_2 * x)}_{\in X}$ (consistency) compatibility

We say that X is a G-set.

e.g.1) X set
$\sigma: X \to X$ (1-1 & onto) = bijection
a permutation of X

($S_X = \{$all permutations of $X\}, \circ$) the permutation group of X
(composition)

(e.g. if $X = \{1, \ldots, n\}$ ⇝ $S_X = S_n$)

Then X is a S_X-set via $*: S_X \times X \to X$
$$(\sigma, x) \mapsto \sigma(x) := \sigma * x$$
$$(\sigma \circ \tau) * x \overset{def}{=} (\sigma \circ \tau)(x) = \sigma(\tau(x)) \underset{def}{=} \sigma(\underbrace{\tau * x}_{\in X}) \underset{def}{=} \sigma * (\tau * x)$$

e.g.2) $\forall$ G is a G-set : $\boxed{g_1 * g_2 := g_1 \cdot g_2}$
action — group operation
$$(g_1 g_2) * g_3 := (g_1 g_2) \cdot g_3 \underset{assoc.}{=} g_1 (g_2 g_3) \overset{def}{=} g_1 (g_2 * g_3)$$

Aside: A (binary) operation $*$ in a set S is a function: $*: S \times S \to S$
$\Leftrightarrow \forall a, b \in S \ \exists! \, a * b \in S$

e.g.3 G, $H \leq G$. Then G is a H-set: $h * g := h \cdot g \in G$

Be Careful: H not in general a G-set: $G \times H \to H$
$$g * h = g \cdot h \notin H$$

e.g.4: G is a G-set via: $\boxed{g * x := gxg^{-1}}$ (or $g^{-1}xg$) (conjugation)

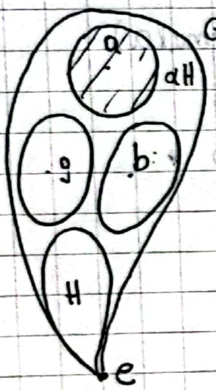
$$(g_1 g_2) * x \overset{\text{def}}{=} (g_1 g_2) x (g_1 g_2)^{-1} = g_1 g_2 x g_2^{-1} g_1^{-1}$$
$$\overset{\text{def}}{=} g_1 (g_2 * x) g_1^{-1} \overset{\text{def}}{=} g_1 * (g_2 * x)$$

($P, Q \in M_{n\times n}$ represent same linear endomorphism
$\Leftrightarrow$
$\exists B \in M_{n\times n}$ s.t. $Q = B^{-1} P B$)

e.g.5.: Let G group & $H \leq G$ with action "conjugation"
G is a H-set: $h * g := hgh^{-1} \in G$

The other way round works as: $\left.\begin{array}{l} H \text{ is a } G\text{-set} \\ g * h := ghg^{-1} \in H \end{array}\right\} \Leftrightarrow \boxed{H \trianglelefteq G}$

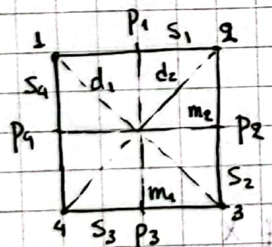
e.g.6. $H \leq G$ and let $\mathcal{L}_H := \{gH \mid g \in G\}$
$\hookrightarrow$(set)



then $\mathcal{L}_H$ is a G-set via: $* : G \times \mathcal{L}_H \to \mathcal{L}_H$
$g * aH := (ga) H$

$$(g_1 g_2) * aH \overset{\text{def}}{=} (g_1 g_2 a) H \overset{\text{def}}{=} [g_1 (g_2 a)] H$$
$$\overset{\text{def}}{=} g_1 * [g_2 a] H \overset{\text{def}}{=} g_1 * (g_2 * aH)$$

e.g.7.



$X = \{1, 2, 3, 4, p_1, p_2, p_3, p_4, C, m_1, m_2, s_1, s_2, s_3, s_4, d_1, d_2\}$

$D_4 = \{$ symmetries of the square $\}$
$\hookrightarrow$ dihedral group of order 8 $(= 4 \cdot 2)$

$D_n = \{$ symmetries of the regular n-gon $\}$

Then X is a D_4-set via: a symmetry $s \in D_4$ acts on an element $x \in X$ as: $s * x := s(x)$

For next time: read isotropy groups