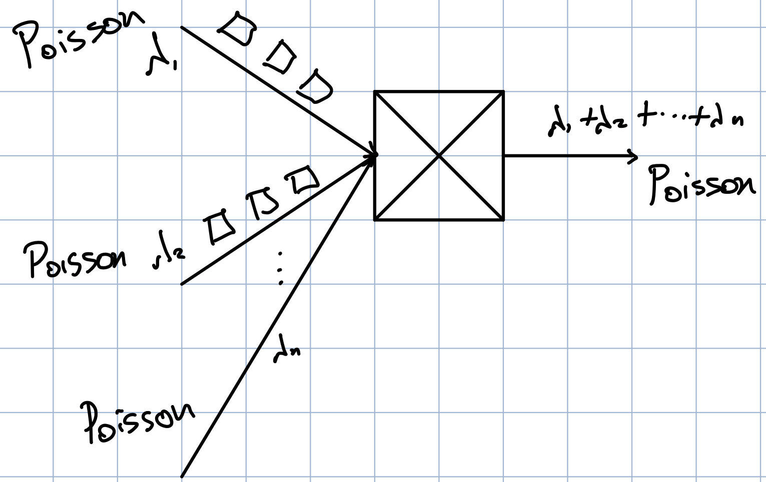
Poisson

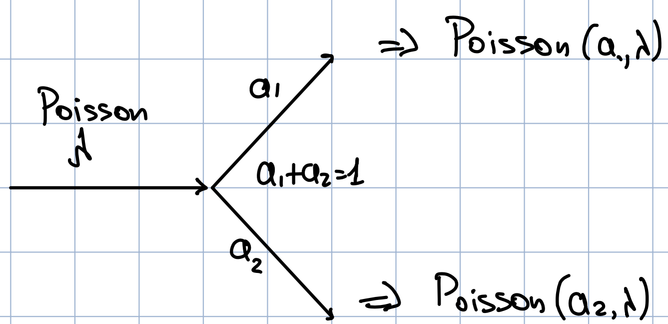
$$P[N(t)=n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n=0,1,\dots$$

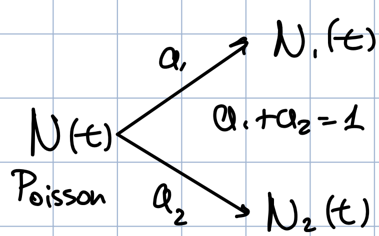


a) $X_1, X_2, \dots X_n$ εκθ. κατ. και στατιστικά ανεξάρτητες

$Y=\min\{X_1, X_2, \dots, X_n\}$ ⟶ εκθετική κατανομ.

b) Διάσπαση (τυχαία)





$$P[N_1(t)=n_1, N_2(t)=n_2] =$$

$$= P[N_1(t)=n_1, N_2(t)=n_2 / N(t)=n_1+n_2] * P[N(t)=n_1+n_2]$$

$$= \frac{\cancel{(n_1+n_2)!}}{n_1!\, n_2!}\cdot a_1^{n_1} a_2^{n_2}\, \frac{(\lambda t)^{n_1+n_2}}{\cancel{(n_1+n_2)!}}\, e^{-\lambda t}$$

$$P_k = \binom{n}{k} p^k (1-p)^{n-k}$$

p = πιθ επιτυχίας $\rightarrow a_1$

$1-p$ = πιθ αποτυχίας $\rightarrow a_2$ $(1-a_1)$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$k \rightarrow n_1$

$n-k \rightarrow n_2$

$$e^{-(a_1+a_2)\lambda t}$$

$$\text{άρα}, = \frac{(a_1\lambda t)^{n_1}}{n_1!} \frac{(a_2\lambda t)^{n_2}}{n_2!} e^{-\lambda t}$$

$$= \frac{(a_1\lambda t)^{n_1}}{n_1!} e^{-(a_1\lambda)t} \cdot \frac{(a_2\lambda t)^{n_2}}{n_2!} e^{-(a_2\lambda)t}$$

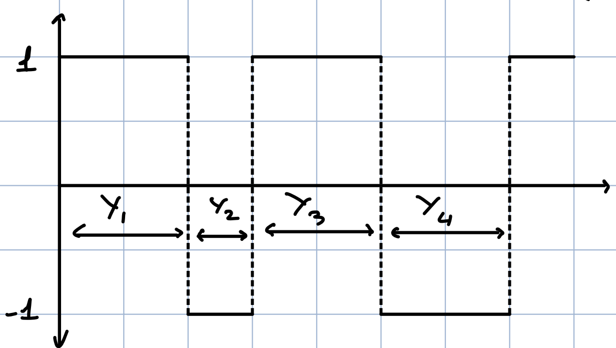
$$\Downarrow a_1\lambda = \lambda_1 \qquad \Downarrow a_2\lambda = \lambda_2$$

$$= \frac{(\lambda_1 t)^{n_1} e^{-\lambda_1 t}}{n_1!} \cdot \frac{(\lambda_2 t)^{n_2} e^{-\lambda_2 t}}{n_2!}$$

Άρα καταλήγουμε σε δύο επιμέρους Poisson.

Άσκηση: Έστω στοχαστική διαδικασία $X(t) = \pm 1$

$X(0) = \pm 1$ με πιθ. $= \frac{1}{2}$ $X(t)$ αλλάζει πολικότητες με εμφάνιση ενός γεγονότος Poisson(a)



A B

$$P[x(t)=1] = P[x(t)=1 \mid x(0)=1]\, P[\cancel{x(0)=1}]^{1/2} + P[x(t)=1 \mid x(0)=-1]\, P[\cancel{x(0)=-1}]^{1/2}$$

$$P[x(t)=-1] = P[x(t)=-1 \mid x(0)=-1]\, P[x(0)=-1] + P[x(t)=-1 \mid x(0)=1]\, P[x(0)=1]$$

(A) $P[x(t)=1 \mid x(0)=1] = P[\overset{\text{poisson}}{N(t)} = \text{even integer}] = \sum_{j=0}^{\infty} \frac{(at)^{2j}}{(2j)!} e^{-at} = \frac{1}{2}[1+e^{-2at}]$

(B) $P[x(t)=1 \mid x(0)=-1] = P[\overset{\text{poisson}}{N(t)} = \text{odd integer}] = \sum_{j=0}^{\infty} \frac{(at)^{2j+1}}{(2j+1)!} e^{-at} = \frac{1}{2}[1-e^{-2at}]$

Άρα $P[x(t)=1] = \frac{1}{2} \cdot \frac{1}{2}[1+e^{-2at}] + \frac{1}{2} \cdot \frac{1}{2}[1-e^{-2at}] = \frac{1}{2}$

και το $P[x(t)=-1] = 1 - P[x(t)=1]$

$M_x(t) = 1 \cdot P[x(t)=1] + (-1) P[x(t)=-1] = 0$

$E[x^2(t)] = 1^2 P[x(t)=1] + (-1)^2 P[x(t)=-1]$

μηδενική μέση τιμή $= 1$

$C_x(t_1, t_2) = E[X(t_1) X(t_2)] = 1\, P[x(t_1) = x(t_2)] + (-1) P[x(t_1) \neq x(t_2)]$

Άσκηση: Έστω $x(t) = \cos(2\pi f_c t + \Phi)$, όπου Φ τμ με $\Gamma(\lambda) = E[e^{j\lambda\Phi}]$

$\Gamma(1) = \Gamma(2) = 0$

$E[x(t)] = E[\cos(2\pi f_c t + \Phi)] = E[\cos(2\pi f_c t)] E[\cos\Phi] - E[\sin(2\pi f_c t)] E[\sin\Phi]$

$\Gamma(1) = 0 \Rightarrow E[e^{j\Phi}] = 0 \Rightarrow E[\cos\Phi + j\sin\Phi] = 0$

$E[\cos\Phi] = E[\sin\Phi] = 0$

$$R_{xx}(\tau) = E[x(t+\tau)\,x(t)] = E[\cos(2\pi f_c t + 2\pi f_c \tau + \varphi)\cos(2\pi f_c t + \varphi)]$$

$$= \frac{1}{2} E[\cos(2\pi f_c t + 2\pi f_c \tau + \varphi + 2\pi f_c t + \varphi) + \cos(\cancel{2\pi f_c t} + 2\pi f_c \tau \cancel{+\varphi} \cancel{-2\pi f_c t} \cancel{-\varphi})]$$

$$= \frac{1}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\varphi)] + \underbrace{\frac{1}{2} E[\cos 2\pi f_c \tau]}_{= \frac{1}{2}\cos(2\pi f_c \tau)}$$

$$= \frac{1}{2} \cdot \cos(2\pi f_c \tau) + \frac{1}{2} \cdot \frac{1}{2} E[e^{j(4\pi f_c t + 2\pi f_c \tau + 2\varphi)} + e^{-j(4\pi f_c t + 2\pi f_c \tau + 2\varphi)}]$$

$$= \frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{4} E[e^{j4\pi f_c t}\, e^{j2\pi f_c \tau}\, e^{j2\varphi}] + \frac{1}{4} E[e^{-j4\pi f_c t} \cdot e^{-j2\pi f_c \tau} \cdot e^{-j2\varphi}]$$

$$= \frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{4} E[e^{j4\pi f_c t}]\,E[e^{j2\pi f_c \tau}]\,\underbrace{E[\cancel{e^{j2\varphi}}]}_{0} + \frac{1}{4} E[e^{-j4\pi f_c t}]\,E[e^{-j2\pi f_c \tau}]\,\underbrace{E[\cancel{e^{-j2\varphi}}]}_{0}$$

(εξήγηση) μαζί

$$\Gamma(-2) = E[e^{-j2\varphi}] = E[\cos 2\varphi - j\sin 2\varphi] = E[\cos\varphi] - j\,E[\sin 2\varphi] = 0$$

άρα και $E[e^{-j2\varphi}] = 0$

$$E[x(t)] = \frac{1}{2}\cos(2\pi f_c \tau)$$

(μέση τιμή)