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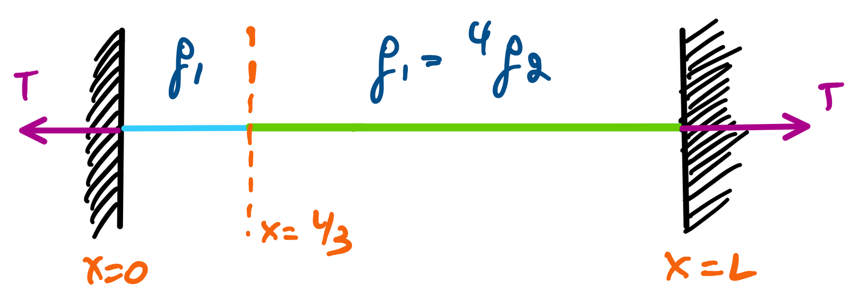
Παράδειγμα

Υπολογισμός των ΚΤΤ (στάσιμα κύματα) σε συνδυασμό 2 χορδών με διαφορετικές γραμμικές πυκνότητες.



ΚΤΤ: $y_{1,2}(x,t) = f_{1,2}(x)\cos(\omega t + \varphi)$ (1)

$c_{1,2} = \sqrt{\dfrac{T}{\rho_{1,2}}}$

$$\frac{1}{c_{1,2}}\frac{\partial^2 y_{1,2}}{\partial t^2} = \frac{\partial^2 y_{1,2}}{\partial x^2} \quad (2)$$

$$f_1'' + \left(\frac{\omega}{c_1}\right)^2 f_1 = 0 \longrightarrow f_1(x) = A\sin(k_1 x + \theta_1)$$

$$f_2'' + \left(\frac{\omega}{c_2}\right)^2 f_2 = 0 \longrightarrow f_2(x) = B\sin(k_2 x + \theta_2)$$

Συνοριακές συνθήκες

α) $f_1(x=0) = 0$ (ακλόνητο άκρο) $\Rightarrow \sin\theta_1 = 0 \Rightarrow \boxed{\theta_1 = 0}$

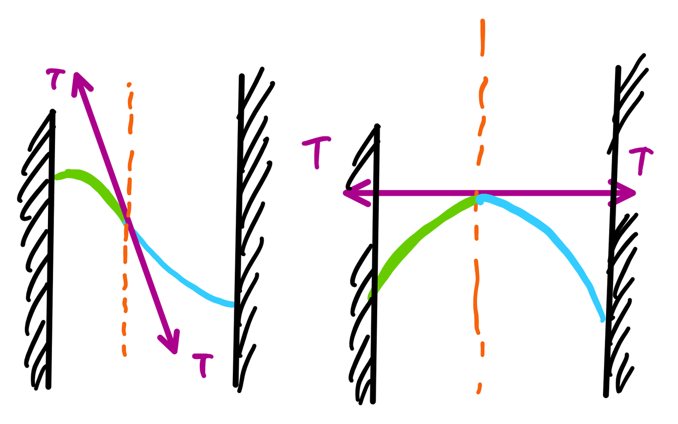
β) $f_2(x=L) = 0$ (ακλόνητο άκρο) $\Rightarrow \sin(k_2 L + \theta_2) \Rightarrow \boxed{\theta_2 = -k_2 L}$

γ) $f_1(x=L/3) = f_2(x=L/3) \Rightarrow A\sin(k_1 L/3) = B\sin(-k_2\, 2L/3)$ (3)

$\dfrac{k_2}{k_1} = \dfrac{c_1}{c_2} \Rightarrow \boxed{k_2 = \dfrac{k_1}{2}}$ (4)

$(3) \xrightarrow{(4)} A\sin(k_1 L/3) = -B\sin(k_1 L/3) \Rightarrow \begin{cases} A = -B & (1\alpha) \\ \sin(k_1 L/3) = 0 \Rightarrow \boxed{k_{1,n} = n\dfrac{3\pi}{L}} & (1\beta) \end{cases}$

δ)



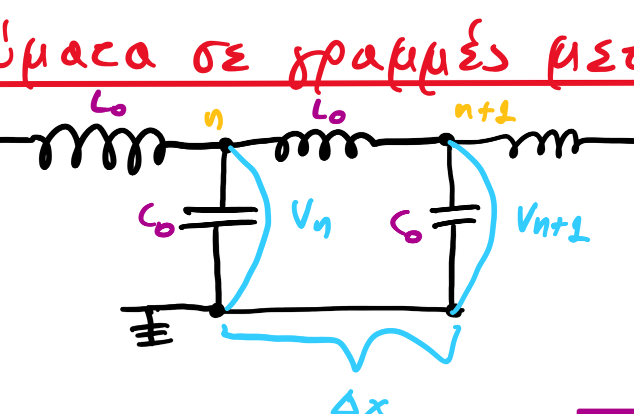
$m=0 \Rightarrow f_{o\lambda,y} = 0 \Rightarrow$

$$\left(\frac{\partial f_1}{\partial x}\right)_{x=L/3} = \left(\frac{\partial f_2}{\partial x}\right)_{x=L/3} \Rightarrow \begin{matrix} k_1 A\cos(k_1 L/3) = k_2 B\cos\left(-k_2 \frac{2L}{3}\right) \\ k_1 A\cos(k_1 L/3) = \frac{k_1}{2} B\cos(-k_1 L/3) \end{matrix}$$

(1α), (2) $\Rightarrow k_{1,n} = (2n-1)\dfrac{3\pi}{2L}$

(1β), (2) $\Rightarrow A = \pm B/2$

Κύματα σε γραμμές μεταφοράς



$I_{n-1} = I_n + \dfrac{d}{dt}Q_n$

$V_n - L\dfrac{d}{dt}I = V_{n+1}$

$\begin{Bmatrix} n \to x \\ n+1 \to x+dx \end{Bmatrix} \Rightarrow \boxed{V(x) - (L_0\,dx)\dfrac{\partial I}{\partial t} = V(x) + \dfrac{\partial V}{\partial x}dx}$ (1)

$I(x-dx) = I(x) + \dfrac{\partial}{\partial t}\underbrace{V\cdot C_0\cdot dx}_{c} \xrightarrow{\text{Taylor}}$

$I(x) - \dfrac{\partial I}{\partial x}dx = I(x) + C_0\dfrac{\partial V}{\partial t}dx \Rightarrow$

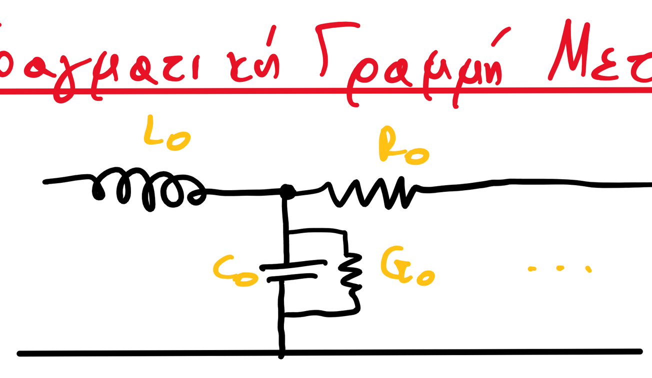
$\boxed{-\dfrac{\partial I}{\partial x}dx = C_0\dfrac{\partial V}{\partial t}dx}$ (2)

$\dfrac{\partial}{\partial t}$(1) $\Rightarrow -L_0\dfrac{\partial^2 I}{\partial t^2} = \dfrac{\partial}{\partial t}\dfrac{\partial V}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial V}{\partial t} \Rightarrow$

$-L_0\dfrac{\partial^2 I}{\partial t^2} = \dfrac{\partial}{\partial x}\left(-\dfrac{1}{C_0}\dfrac{\partial I}{\partial x}\right) \Rightarrow L_0 C_0\dfrac{\partial^2 I}{\partial t^2} = \dfrac{\partial^2 I}{\partial x^2}$, $c = \dfrac{1}{\sqrt{L_0 C_0}}$

$\dfrac{\partial}{\partial t}$(2) $\Rightarrow L_0 C_0\dfrac{\partial^2 V}{\partial t^2} = \dfrac{\partial^2 V}{\partial x^2}$, $c = \dfrac{1}{\sqrt{L_0 C_0}}$

Πραγματική Γραμμή Μεταφοράς



$\left.\begin{matrix} V(x) - (L_0\,dx)\dfrac{\partial I}{\partial t} - (R_0\,dx)I = V(x+dx) \\ I(x) = I(x+dx) + \dfrac{\partial}{\partial t}(C_0 dx V) + (G_0\,dx V) \end{matrix}\right\} \Rightarrow$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = L_0 C_0\frac{\partial^2 V}{\partial t^2} + (R_0 C_0 + L_0 G_0)\frac{\partial V}{\partial t} + R_0 G_0 V$$

Φαινόμενα διασποράς

Διασπορά: ταχύτητα φάσης ενός κύματος είναι διαφορετική για κάθε συχνότητα.

Ταχύτητα φάσης: $V_{ph} = \dfrac{\omega}{k} = \dfrac{\lambda}{T_{\pi\varepsilon\rho}} = \sqrt{\dfrac{T_{\sigma\nu}}{\rho}}$

Για ιδανική χορδή: $V_{ph} = \dfrac{\omega}{k} = c = \sqrt{\dfrac{T}{\rho}} \neq f(\omega)$

π.χ. $\dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + a\dfrac{\partial^4 y}{\partial x^4}$ (1)

Διάδοση αρμονικού κύματος

$\boxed{y = Ae^{i(kx-\omega t)}}$

(2) → (1) $\Rightarrow -\dfrac{\omega^2}{c^2}e^{i(kx-\omega t)} = -k^2 Ae^{i(kx-\omega t)} + ak^4 Ae^{i(kx-\omega t)} \Rightarrow$

$\Rightarrow \omega^2 = c^2k^2 - c^2ak^4 \Rightarrow \boxed{\omega = ck\sqrt{1-ak^2}} = \omega(k)$ σχέση διασποράς

$k = k(\omega)$: σχέση διασποράς

$V_{ph} = \dfrac{\omega}{k} = \dfrac{\omega}{k(\omega)} = V_{ph}(\omega)$

$k = \dfrac{2\pi}{\lambda} = k(\omega) \Rightarrow \boxed{\lambda = \dfrac{2\pi}{k(\omega)}}$

Παράδειγμα διασποράς:

Ιδανική χορδή (ρ, T) μέσα σε ένα "ελαστικό περιβάλλον" με σταθερά σκληρότητας ανά μονάδα μήκους $\dfrac{ds}{dx} = \sigma$

$dm = \rho\,dx \Rightarrow$

$dm\dfrac{\partial^2 y}{\partial t^2} = T\dfrac{\partial^2 y}{\partial x^2}dx - (\sigma dx)y \Rightarrow$

$\left.\begin{matrix} \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} - \dfrac{\sigma}{T}y \\ y = Ae^{i(kx-\omega t)} \end{matrix}\right\} \Rightarrow -\dfrac{\omega^2}{c^2}\cancel{y} = -k^2\cancel{y} - \dfrac{\sigma}{T}\cancel{y} \Rightarrow$

$\boxed{\omega^2 = c^2k^2 + c^2\dfrac{\sigma}{T}}$

$k^2 = \dfrac{\omega^2}{c^2} - \dfrac{\sigma}{T} \Rightarrow \dfrac{2\pi}{\lambda} = \sqrt{\dfrac{\omega^2}{c^2} - \dfrac{\sigma}{T}}$

$\boxed{\lambda = \dfrac{2\pi}{\sqrt{\dfrac{\omega^2}{c^2} - \dfrac{\sigma}{T}}}}$ $V_{ph} = \dfrac{\omega}{k} = \boxed{\dfrac{\omega}{\sqrt{\dfrac{\omega^2}{c^2} - \dfrac{\sigma}{T}}} = V_{ph}(\omega)}$

V_{ph}: πραγματική $\Rightarrow \dfrac{\omega^2}{(c')^2} > \dfrac{\sigma}{T} > \omega > \sigma\sqrt{\dfrac{\sigma}{T}} = \omega_c$

$\omega > \omega_c$: διάδοση κύματος

$k = \sqrt{\dfrac{\omega^2}{c^2} - \dfrac{\sigma}{T}}$: πραγματικό $\omega > \omega_c$

$\omega < \omega_c \Rightarrow k = ia \Rightarrow y = Ae^{i(ia - \omega t)} = (Ae^{-ax})e^{-i\omega t}$