Πέμπτη, 03/11/2022

(ΓΧΑ) Συστήματα Διακριτού Χρόνου

$$x(k+1) = Ax(k) + Bu(k) \qquad (*)$$
$$y(k) = Cx(k) + Du(k)$$

π.χ. $y(k+1) = a_1 y(k) + a_2 y(k-1) + bu(k)$

$$x(k) = \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} \Rightarrow x(k+1) = \begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(k)$$

$$y(k) = [0 \quad 1]\, x(k)$$

$x(0)$

$x(1) = Ax(0) + Bu(0)$

$x(2) = Ax(1) + Bu(1) = A^2 x(0) + Bu(1) + ABu(0)$

$\vdots$

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j)$$

Αν $\exists\, c > 0 : \|u(k)\| \le c \quad \forall k$ κ' $|\lambda_i(A)| < 1$, τότε

$$\left\| \sum_{j=0}^{k-1} A^{k-1-j} B u(j) \right\| \le \left\| \sum_{j=0}^{k-1} T J^{k-1-j} T^{-1} B u(j) \right\| \le$$
$$\le \sum_{j=0}^{k-1} \|T\| \, \underbrace{\|J^{k-1-j}\|}_{\le \lambda^{k-1-j}} \|T^{-1}\| \, \|B\| c$$

$$\left(\|A\|_2 = \sup_{\|x\|=1} \|Ax\|_2 = \sqrt{\lambda_{max}(AA^T)} \right)$$

$$\le c' \sum_{j=0}^{k-1} \lambda^{k-1-j} = 1 + \lambda + \lambda^2 + \ldots + \lambda^{k-1} = \frac{1-\lambda^k}{1-\lambda}$$

$A = TJT^{-1}$

Συνάρτηση μεταφοράς: $\dfrac{Y(z)}{U(z)}$

$$(*) \Rightarrow \begin{cases} zX(z) = AX(z) + BU(z) \\ Y(z) = CX(z) + DU(z) \end{cases}$$

$$\Rightarrow Y(z) = \underbrace{\left[C(zI - A)^{-1} B + D \right]}_{G(z)} U(z) + C(zI - A)^{-1} x(0)$$

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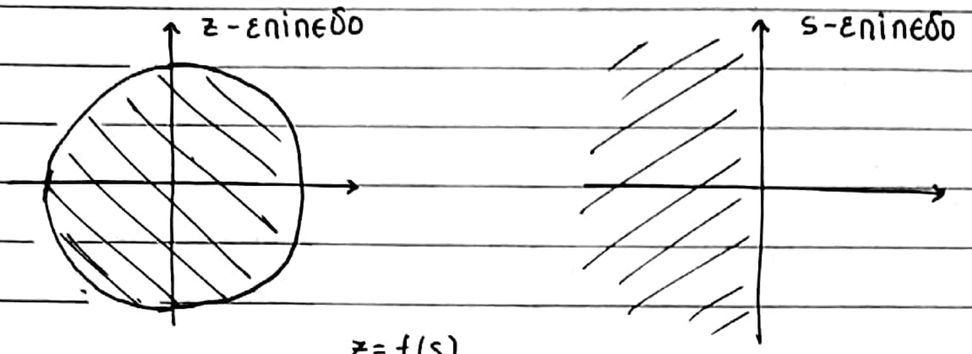
$$G(z) = \frac{C\,\mathrm{adj}(z\mathbb{I}-A)B}{\det(z\mathbb{I}-A)} + D$$



$$z = f(s)$$

(Δεν μας κάνει ο $z=e^{Ts}$, $\psi(z) = \psi(f(s))$)

Μετασχηματισμός Möbius ή Διγραμμικός Μετασχηματισμός

$$z = \frac{s+1}{s-1} \rightsquigarrow s = \frac{z+1}{z-1}$$

$$|z|<1 \Leftrightarrow \left|\frac{s+1}{s-1}\right| < 1 \Leftrightarrow |s+1|^2 < |s-1|^2 \Leftrightarrow \overline{(s+1)}(s+1) < (s-1)\overline{(s-1)}$$

$$\Leftrightarrow (\bar{s}+1)(s+1) < (s-1)(\bar{s}-1) \Leftrightarrow |s|^2 + s + \bar{s} + 1 < |s|^2 - (s+\bar{s}) + 1$$

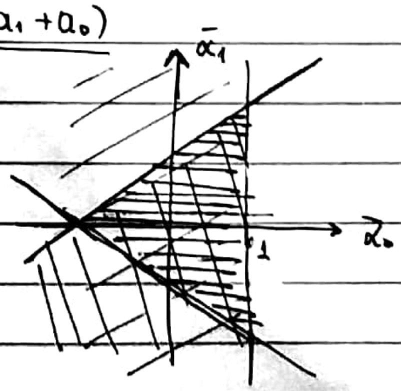
$$\Leftrightarrow s+\bar{s} < 0 \Leftrightarrow Re(s) < 0$$

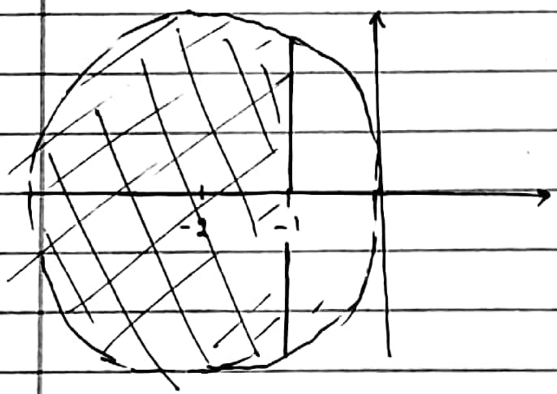
π.χ. $\Psi(z) = a_2 z^2 + a_1 z + a_0$ (Δευτεροβάθμια σύστημα) $= a_2\left(z^2 + \overbrace{a_1/a_2}^{\bar{a}_1} z + \overbrace{a_0/a_2}^{\bar{a}_0}\right)$

$$\Rightarrow \Psi\left(\frac{s+1}{s-1}\right) = a_2\left(\frac{s+1}{s-1}\right)^2 + a_1\left(\frac{s+1}{s-1}\right) + a_0 = \frac{a_2(s+1)^2 + a_1(s+1)(s-1) + a_0(s-1)^2}{(s-1)^2}$$

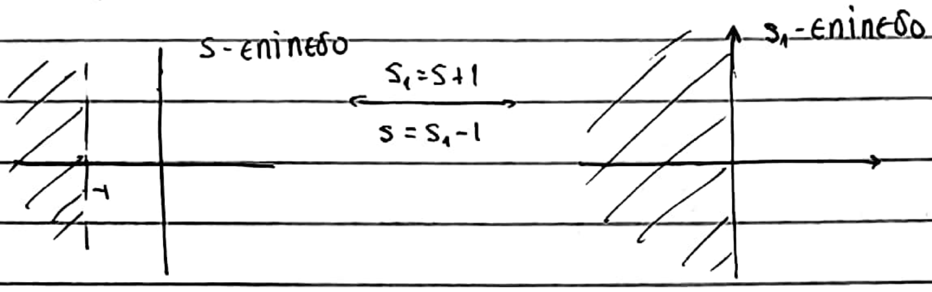
$$= \frac{(a_2+a_1+a_0)s^2 + (2a_2-2a_0)s + (a_2-a_1+a_0)}{(s-1)^2}$$

Αν $a_2 = 1$: $\begin{cases} 1+\bar{a}_0+\bar{a}_1 > 0 \\ 2-2\bar{a}_0 > 0 \\ 1+\bar{a}_0+\bar{a}_1 > 0 \end{cases}$





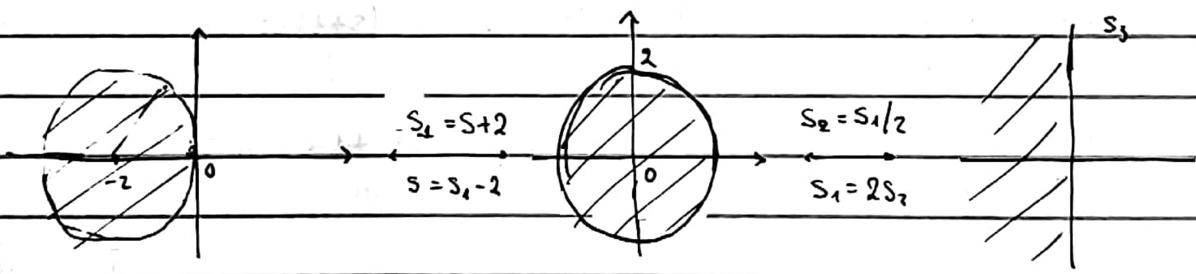
$$\psi(s) = s^2 + a_1 s + a_0$$



$$\psi(s_1 - 1) = (s_1 - 1)^2 + a_1(s_1 - 1) + a_0 = s_1^2 + (a_1 - 2)s_1 + (-a_1 + a_0 + 1)$$

$a_1 > +2$

$a_1 < 1 + a_0$



$$s_3 = \frac{s_2+1}{s_2-1} \Leftrightarrow s_2 = \frac{s_3+1}{s_3-1}$$

$$\Rightarrow s = 2\left(\frac{s_3+1}{s_3-1} - 1\right)$$

Ελεγξιμότητα

$\dot{x} = Ax + Bu \quad (\Sigma)$

Ελέγξιμο: Αν $\forall x_0, x_f, t_f > 0$: $\exists u : [0, t_f) \rightarrow \mathbb{R}^m$ που εξασφαλίζει στη $x(t_f ; 0, x_0) = x_f'$.

x_0: αρχική κατάσταση, x_f: τελική κατάσταση, t_f: τελική τιμή

(Σ) ελέγξιμο ή (A, B) ελέγξιμο

$$x(t_f) = e^{At} x_0 + \int_0^{t_f} e^{A(t-s)} B u(s) ds$$

$\exists u$ που λύνει την ολοκληρωτική εξίσωση ?

$\exists$ ανν $W_c(t) = \int_0^t e^{As} B B^T e^{A^T s} ds > 0$ θετικά ορισμένος $\forall t > 0$

$W_c^T(t) = W_c(t)$

$\underset{n \times n}{Q} = Q^T > 0$

$\lambda_{min}(Q) \|y\|^2 \leq y^T Q y \leq \lambda_{max}(Q) \|y\|^2$, $Q = \sum \lambda_i u_i u_i^T$

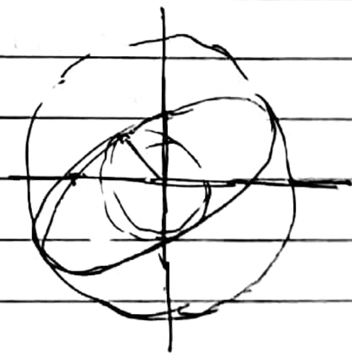
$y^T Q y = \sum \lambda_i (y^T u_i)^2$

$y = u_{max} \|y\|$

$$\lambda_{min}(Q) \leq \frac{y^T Q y}{\|y\|^2} \leq \lambda_{max}(Q)$$

$Q \in \mathbb{R}^{2 \times 2}$, $Q = Q^T > 0$ $\quad y^T Q y = c$

↳ έλλειψη



$$y^T W_c(t) y = \int_0^t \|B^T e^{A^T s} B\|^2 ds \geq 0$$

$$u(s) = B^T e^{A^T(t_f-s)} W_c^{-1}(t_f)\left[-e^{At_f}x_0 + x_f\right], \; s\in[0,t_f)$$

$$\int_0^{t_f} \underbrace{e^{A(t_f-s)} B B^T e^{A^T(t_f-s)} ds}\, W_c^{-1}(t_f)\left(-e^{At_f}x_0 + x_f\right)$$

$$/\!/ \; w = t_f - s$$

$$\int_{t_f}^{0} e^{Aw} BB^T e^{A^T w}(-dw)$$

$$\shortparallel$$

$$W_c(t_f)$$

(Απόδειξη για αναγκαία συνθήκη)

$x_0 = 0$

$$x(t_f) = \int_0^{t_f} e^{A(t_f-s)} Bu(s)ds \quad \ldots$$

αν $W_c(t^*)$ έχει ιδιοτιμή μηδέν, τότε $\exists x_f$ στο οποίο δεν μπορώ να οδηγηθώ

$$W_c(t^*)\,v = 0 \Rightarrow v^T W_c(t^*)\,v = 0 \Rightarrow$$

$$\Rightarrow \int_0^{t^*} \| B^T e^{A^T} v \| = 0 \quad \forall t\in[0,t^*] \Rightarrow v^T e^{A t} B = 0$$

$$v = \int_0^{t^*} e^{A(t^*-s)} Bu(s)ds \Rightarrow \underbrace{v^T v}_{\neq 0} = \int_0^{t^f} \underbrace{v^T e^{A(t^*-s)}}_{=0} Bu(s)ds, \text{ άτοπο.}$$

$$W_c(t) > 0 \Leftrightarrow \mathcal{C} = [B \;\; AB \;\; \ldots \;\; A^{n-1}B] \quad \text{rank}(\mathcal{C}) = n$$

$n\times(nm)$ ↳ Πίνακας ελεγξιμότητας

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