Πέμπτη, 06/04/2023

$\dot{x} = f(x,u,t)$

$J = \int_{t_0}^{t_f} L(x,u,t)\,dt + \varphi(x_f, t_f)$

s.t. $\psi(x_f, t_f) = 0$

$H(x,u,t) = L(x,u,t) + p^T f(x,u,t)$

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial \varphi} = 0 \end{cases}$$

Οριακές συνθήκες:

↓

Αναγκαίες συνθήκες

$$\delta t_f = H + \frac{\partial \varphi}{\partial t} + \sum \lambda_i \frac{\partial \psi_i}{\partial t}\Big|_{t=t_f} = 0$$

$$\delta x_f : p - \frac{\partial \phi}{\partial x} - \sum \lambda_i \frac{\partial \psi_i}{\partial x}\Big|_{t_f} = 0$$

Ικανή συνθήκη: $\dfrac{\partial^2 H}{\partial u^2} > 0$

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q x + u^T R u)\,dt + \underbrace{\frac{1}{2} x_f^T S x_f}_{\Phi(x_f, t_f)}, \quad t_f \text{ fixed}, \ x_f \text{ free}$$

$\dot{x} = Ax + Bu$

$H = \frac{1}{2}(x^T Q x + u^T R u) + p^T (Ax + Bu)$

$$\frac{\partial H}{\partial u} = Ru + B^T p \rightarrow \frac{\partial^2 H}{\partial u^2} = R > 0$$

$$\frac{\partial H}{\partial u} = 0 \rightarrow u = -R^{-1} B^T p(t)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^T p$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix}}_{2n\times 1} = \begin{bmatrix} Ax - BR^{-1}B^T p \\ -Qx - A^T p \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H_m} \begin{bmatrix} x \\ p \end{bmatrix} \rightarrow \text{λύση ανοικτού βρόχου}$$

$P(t_f) = Sx(t_f)$, $x(t_0) = x_0$ } $2n$ οριακές συνθήκες → Two Point Boundary Value Problems (TPBVP)

$2n$ Δ.Ε.

Λύση κλειστού βρόχου

Αναζητώ λύσεις της μορφής $p(t) = P(t)\,x(t)$

$$p(t_f) = P(t_f)\,x(t_f) = S\,x(t_f)$$

$$\dot{p} = \dot{P}x + P\dot{x} \Rightarrow -Qx - A^T P x = \dot{P}x + P(x - BR^{-1}B^T P x)$$

$$\left[\frac{dP}{dt} + PA + A^T P - PBR^{-1}B^T P + Q\right] x = 0$$

$$\begin{cases} \dfrac{dP}{dt} + PA + A^T P - PBR^{-1}B^T P + Q = 0 & \rightarrow \text{διαφορική Riccati} \\ P(t_f) = S \end{cases}$$

$$u = -\underbrace{R^{-1}B^T P(t)}_{K(t)}\,x$$

$K(t) \rightarrow$ βέλτιστο κέρδος (μπορώ να το προϋπολογίζω στο σύστημά μου)

Λύση απείρου χρόνου

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)\,dt \,, \quad (A,B) \text{ stabilizable}, \quad (Q^{1/2}, A) \text{ detectable}$$

τότε $x(\infty) \rightarrow 0$

$$\tilde{J} = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)\,dt + \underbrace{\frac{1}{2}\int_0^\infty \frac{d}{dt}(x^T P x)\,dt}_{-1/2\, x_0^T P x_0}$$

$$\tilde{J} = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u + \underbrace{\dot{x}^T}_{(Ax+Bu)^T} P x + x^T P \underbrace{\dot{x}}_{(Ax+Bu)})\,dt \Rightarrow$$

$$\Rightarrow \tilde{J} = \frac{1}{2}\int_0^\infty [\underbrace{u^T R u + u^T B^T P x + x^T P B u}_{} + x^T(Q + PA + A^T P)x]\,dt$$

$P = P^T$: $\quad (u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x) - x^T PBR^{-1}B^T P x$

$$\tilde{J} = \frac{1}{2}\int_0^\infty [(u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x) + x^T(PA + A^T P - PBR^{-1}B^T P + Q)x]\,dt$$

Επιλέγω P τ.ω. $PA + A^T P - PBR^{-1}B^T P + Q = 0 \rightarrow$ αλγεβρική εξίσωση Riccati

$\varphi^* = \arg\min \tilde{J} = -R^{-1}B^T P$

$\min \tilde{J} = \frac{1}{2} x_0^T P x_0 \geq 0$, $P > 0$ (MATLAB are(A,K,Q))

(Kleinman's algorithm)

Για το πρόβλημα απείρου χρόνου, αρκεί να λύσουμε τη:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \rightarrow$$

$$u^* = -\underbrace{R^{-1}B^T P}_{K} x$$

$$\Rightarrow P(A - \underbrace{BR^{-1}B^T P}_{BK}) + (A - BR^{-1}B^T P)^T P = \underbrace{-(Q + PBR^{-1}B^T P)}_{<0}$$

$$\dot{x} = (A+BK)x$$

$$P(A+BK) + (A+BK)^T P < 0$$

$$P = P^T > 0$$

$A+BK$ Hurwitz $x \to 0$

[Ο $-(Q+PBR^{-1}B^T P)$ είναι <0 επειδή $(Q^{1/2}, A)$ ανιχνεύσιμος]

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ & p_{22} & \dots & \\ & & \ddots & \vdots \\ & & & p_{nn} \end{bmatrix}$$

$PA + A^T P - PBR^{-1}B^T P + Q = [\nabla]$

$1+2+\dots+n = \frac{n(n+1)}{2}$ άγνωστοι

Παράδειγμα

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases}, \quad J = \frac{1}{2}\int_0^\infty (y^2 + \rho u^2)\,dt, \ \rho > 0, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y^2 = x_1^2 = x^T \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{Q} x, \quad Q^{1/2} = Q$$

$$\begin{bmatrix} A - 0\cdot I \\ Q^{1/2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$ παρατηρήσιμο $(Q^{1/2}, A)$

τάξη 2

3 άγνωστοι $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$, $P > 0 \rightarrow p_1 > 0$

$p_1 p_2 > p_{12}^2$

$$PA = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p_1 \\ 0 & p_{12} \end{bmatrix}$$

$$PB = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{12} \\ p_2 \end{bmatrix}$$

εξ. Riccati: $PA + A^T P - PBR^{-1}B^T P + Q = \begin{bmatrix} 0 & p_1 \\ p_1 & 2p_{12} \end{bmatrix} - \frac{1}{\rho}\begin{bmatrix} p_{12} \\ p_2 \end{bmatrix}\begin{bmatrix} p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} 1-\frac{1}{\varrho}p_{12}^2=0 \Rightarrow p_{12}=\pm\varrho^{1/2} \rightarrow \text{δεκτή μόνο η } \boxed{p_{12}=\varrho^{1/2}} \\ 2p_{12}-\frac{1}{\varrho}p_2^2=0 \Rightarrow p_{12}=\frac{1}{2\rho}p_2^2\geq 0 \Rightarrow p_2^2=2\varrho p_{12}=2\varrho^{3/2} \Rightarrow \boxed{p_2=\sqrt{2}\,\varrho^{3/4}} \\ p_1-\frac{1}{\varrho}p_{12}p_2=0 \Rightarrow p_1=\frac{1}{\varrho}\sqrt{\varrho}\cdot\sqrt{2}\,\varrho^{3/4} \Rightarrow \boxed{p_1=\sqrt{2}\,\varrho^{1/4}} \end{cases}$$

$$P=\begin{bmatrix} \sqrt{2}\,\varrho^{1/4} & \varrho^{1/2} \\ \varrho^{1/2} & \sqrt{2}\,\varrho^{3/4} \end{bmatrix}, \quad \det P=\varrho>0$$

Βέλτιστος νόμος ελέγχου: $u=-R^{-1}B^TPx=-\frac{1}{\varrho}[0 \quad 1]\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}x \Rightarrow$

$$\Rightarrow u=-\frac{1}{\varrho}\left(\varrho^{1/2}x_1+\sqrt{2}\,\varrho^{3/4}x_2\right) \Rightarrow u=-\frac{1}{\sqrt{\varrho}}x_1-\frac{\sqrt{2}}{\varrho^{1/4}}x_2$$

$$J^*=\frac{1}{2}x_0^TPx_0$$