Δευτέρα, 20/03/2023

Τοποθέτηση πόλων (Συστήματα με 1 είσοδο)

$\dot{x} = Ax + Bu$

επιθυμητές ιδιοτιμές

Αν (A,B) ελέγξιμο, $\exists K : \lambda_i(A+BK) = \lambda_{d,i}$

$u = Kx + r$, $\dot{x} = (A+BK)x + Br$

Μέθοδοι Τοποθέτησης ιδιοτιμών

1) Με απευθείας αντικατάσταση

$$\det(s\mathbb{I} - (A+BK)) = \underbrace{\prod_{i=1}^{n}(s - \lambda_{d,i})}_{\text{επιθυμητό χαρακτ. πολυώνυμο}}$$

$K = [K_1 \ K_2 \ \dots \ K_n]$

2) Με χρήση της κανονικής ελέγξιμης μορφής

$\bar{x} = Tx$

$$\dot{\bar{x}} = \overbrace{\begin{bmatrix} 0_{(n-1)\times 1} & \mathbb{I}_{n-1} \\ -a_0 & \dots \ -a_{n-1} \end{bmatrix}}^{A_c} \bar{x} + \overbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}^{B_c} u$$

$\chi_A(s) = \det(s\mathbb{I} - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0$

$u = \bar{K}\bar{x} = \underbrace{KT}_{\bar{K}}\bar{x}$, $BK = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [K_1 \ K_2 \dots K_n] = \begin{bmatrix} 0 \\ K_1 \ K_2 \dots K_n \end{bmatrix}$

$$A_c + B_c\bar{K} = \begin{bmatrix} 0 & \mathbb{I} \\ -a_0 + \bar{K}_1 & -a_{n-1} + \bar{K}_n \end{bmatrix}, \quad \chi_{A_c+B_c\bar{K}}(s) = s^n + (a_{n-1} - \bar{K}_n)s^{n-1} + \dots + (a_0 - \bar{K}_1)$$

$\chi_d(s) = s^n + a_{d,n-1}s^{n-1} + \dots + a_{d,0}$

$a_i - \bar{K}_{i+1} = a_{d,i}$

$$\Rightarrow \boxed{\bar{K}_{i+1} = a_i - a_{d,i}, \ i = 0, 1, \dots, n-1}$$

$\underset{n\times n}{\mathcal{C}} = [B \ AB \ \dots \ A^{n-1}B]$

$$\mathcal{C}^{-1} = \begin{bmatrix} \times \\ q^T \end{bmatrix}, \quad T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$

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$$\mathcal{C}^{-1}\mathcal{C} = \mathbb{I}_n \Leftrightarrow \begin{bmatrix} X \\ q^T \end{bmatrix} [B \ \ AB \ \cdots \ A^{n-1}B] = \mathbb{I}_n \Leftrightarrow$$

$$\begin{bmatrix} & X & \\ q^TB & q^TAB \ldots & q^TA^{n-1}B \end{bmatrix} = \begin{bmatrix} & & X & & \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

$$q^TA^{i-1}B = \begin{cases} 0, & i = 1, \ldots, n-1 \\ 1, & i = n \end{cases}$$

$$\dot{x} = Ax + Bu \xRightarrow{\bar{x} = Tx} \dot{\bar{x}} = \underbrace{TAT^{-1}}_{\bar{A}} \bar{x} + \underbrace{TB}_{\bar{B}} u$$

$$\bar{B} = TB = \begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix} B = \begin{bmatrix} q^TB \\ q^TAB \\ \vdots \\ q^TA^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$TA = \begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix} A = \begin{bmatrix} q^TA \\ \vdots \\ q^TA^{n-1} \\ q^TA^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} = \begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix}$$

$$A^n + a_{n-1}A^{n-1} + \ldots + a_0\mathbb{I}_n = 0 \Rightarrow A^n = -a_0\mathbb{I} - a_1A - \ldots - a_{n-1}A^{n-1} \Rightarrow$$

$$\Rightarrow q^TA^n = -a_0q^T - a_1q^TA - \ldots - a_{n-1}q^TA^{n-1}$$

$$K = \bar{K}T$$

$$A=\begin{bmatrix}1&0&1\\2&1&0\\0&0&3\end{bmatrix},\quad B=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\chi_A(s)=\det(sI-A)=(s-1)^2(s-3)=(s^2-2s+1)(s-3)=s^3-5s^2+7s-3$$

$$\bar{A}=\begin{bmatrix}0&1&0\\0&0&1\\3&-7&5\end{bmatrix},\quad \bar{B}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Είναι οι ιδιοτιμές του A ελέγξιμες;

$$\mathcal{C}=[B\ \ AB\ \ A^2B]=\begin{bmatrix}1&2&5\\1&3&7\\1&3&9\end{bmatrix}$$

$$\det\mathcal{C}=\begin{vmatrix}1&2&5\\0&1&2\\0&1&4\end{vmatrix}=\begin{vmatrix}1&2&5\\0&1&2\\0&0&2\end{vmatrix}=2\neq 0$$

$$\mathcal{C}^{-1}=\frac{1}{2}\begin{bmatrix} & & \\ & & \\ 0&-1&1\end{bmatrix},\quad q=[0\ \ -1/2\ \ 1/2]$$

$$T=\begin{bmatrix}q^T\\q^TA\\q^TA^2\end{bmatrix}=\begin{bmatrix}0&-1/2&1/2\\-1&-1/2&3/2\\-2&-1/2&7/2\end{bmatrix}$$

Επιθ. Ιδιοτιμές: $\{-1,-2,-3\}$

$$\chi_d(s)=(s+1)(s+2)(s+3)=s^3+6s^2+11s+6$$

$$\bar{K}=[-3\ \ 7\ \ -5]-[6\ \ 11\ \ 6]=[-9\ \ -4\ \ -11]$$

$$K=\bar{K}T=[-9\ \ -4\ \ -11]\begin{bmatrix}0&-1/2&1/2\\-1&-1/2&3/2\\-2&-1/2&7/2\end{bmatrix}=[26\ \ 12\ \ -49]$$

Αν θέλω ιδιοτιμές: $-5,-6,-7$ $\quad (s+5)(s+6)(s+7)=s^3+18s^2+107s+210$

$$a_{d,i}=[-210\ \ -107\ \ -18]\Rightarrow K=[146\ \ 168\ \ 337]$$

3) Τύπος του Ackermann

$$\boxed{K = -e_n^T \mathcal{C}^{-1} \chi_d(A)}$$

$$e_n^T = [0\ 0 \dots 0\ 1]$$

$$\left.\begin{aligned} &\chi_d(A) = A^n + a_{d,n-1}A^{n-1} + \dots + a_{d,0} I \\ &A^n + a_{n-1}A^{n-1} + \dots + a_0 I = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \chi_d(A) = (a_{d,n-1} - a_{n-1})A^{n-1} + \dots + (a_{d,0} - a_0) I$$

$$\bar{K} = [a_0 - a_{d,0} \ \dots \ a_{n-1} - a_{d,n-1}]$$

$$\bar{A} + \bar{B}\bar{K} = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ -a_{d,0} & -a_{d,n-1} \end{bmatrix}$$

$$\bar{\mathcal{C}} = [\bar{B}\ \ \bar{A}\bar{B}\ \dots\ \bar{A}^{n-1}\bar{B}], \quad \bar{A} = \left[\begin{array}{c|c} 0 & I_{n-1} \\ -a_0 & \dots -a_{n-1} \end{array}\right], \quad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & . & * \\ 0 & 0 & 1 & \vdots & \vdots \\ 0 & 1 & * & \vdots & * \\ 1 & -a_{n-1} & * & . & * \end{bmatrix}$$

$$e_1^T \bar{\mathcal{C}} = e_n^T \Rightarrow \boxed{e_1^T = e_n^T \bar{\mathcal{C}}^{-1}}$$

$$e_i^T \bar{A} = e_{i+1}^T$$

$$-e_n^T \bar{\mathcal{C}}^{-1} \chi_d(\bar{A}) = -e_1^T \chi_d(\bar{A}) = -e_1^T\left[(a_{d,0} - a_0) I + (a_{d,1} - a_1)\bar{A} + \dots + (a_{d,n-1} - a_{n-1})\bar{A}^{n-1}\right]$$

$$= -(a_{d,0} - a_0)e_1^T - (a_{d,1} - a_1)e_2^T - (a_{d,n-1} - a_{n-1})e_n^T = \bar{K}$$

$$K = \bar{K}T = -e_n^T \bar{\mathcal{C}}^{-1} \chi_d(\bar{A}) T$$

$$\bar{\mathcal{C}} = [\bar{B}\ \ \bar{A}\bar{B}\ \dots\ \bar{A}^{n-1}B] = [TB\ \ TAB\ \dots\ TA^{n-1}B] = T\mathcal{C}$$

$$\bar{\mathcal{C}}^{-1} = \mathcal{C}^{-1}T^{-1}$$

$$K = -e_n^T \mathcal{C}^{-1} T^{-1} \underbrace{\chi_d(TAT^{-1})}_{T\chi_d(A)T^{-1}} T \Rightarrow K = -e_n^T \mathcal{C}^{-1} \chi_d(A)$$

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4)

$$\det(sI-(A+BK)) = \chi_d(s)$$

$$\det((sI-A)-BK) = \chi_d(s)$$

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$$\det[(sI-A)(I-(sI-A)^{-1}BK)] = \chi_d(s) \Leftrightarrow \chi_A(s)\det(I-(sI-A)^{-1}BK) = \chi_d(s)$$

$$\det(I-xy^T) = 1-y^Tx$$

$(I-xy^T)v_i = v_i$ $\quad \exists\, v_1, v_2, \dots, v_{n-1}$ γρ. ανεξάρτητα

t $y^T v_i = 0$

$$(I-xy^T)x = (1-y^Tx)x$$

$$\chi_A(s)\left(1-K(sI-A)^{-1}B\right) = \chi_d(s)$$

$$\chi_A(s) - \chi_A(s)\frac{K\cdot \operatorname{adj}(sI-A)B}{\chi_A(s)} B = \chi_d(s)$$

$$K\cdot \operatorname{adj}(sI-A)B = \chi_A(s)-\chi_d(s)$$

π.χ. $\chi_A(s) = s^3-5s^2+7s-3$, $\quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

$\chi_d(s) = s^3+6s^2+11s+6$

$$sI-A = \begin{bmatrix} s-1 & 0 & -1 \\ -2 & s-1 & 0 \\ 0 & 0 & s-3 \end{bmatrix} \Rightarrow \operatorname{adj}(sI-A) = \begin{bmatrix} s^2-4s+3 & 0 & s-1 \\ 2s-6 & s^2-4s+3 & 2 \\ 0 & 0 & s^2-2s+1 \end{bmatrix}$$

$$[K_1 \quad K_2 \quad K_3]\begin{bmatrix} s^2-3s+2 \\ s^2-2s-1 \\ s^2-2s+1 \end{bmatrix} = -11s^2-4s-9$$

$$\Downarrow$$

$$\begin{cases} K_1+K_2+K_3 = -11 \\ -3K_1-2K_2-2K_3 = -4 \quad \rightarrow \dots \\ 2K_1-K_2+K_3 = -9 \end{cases}$$

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