Closest Pair of Points
(from "Algorithm Design" by J.Kleinberg and E.Tardos)

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

1-D version. $O(n \log n)$ easy if points are on a line.

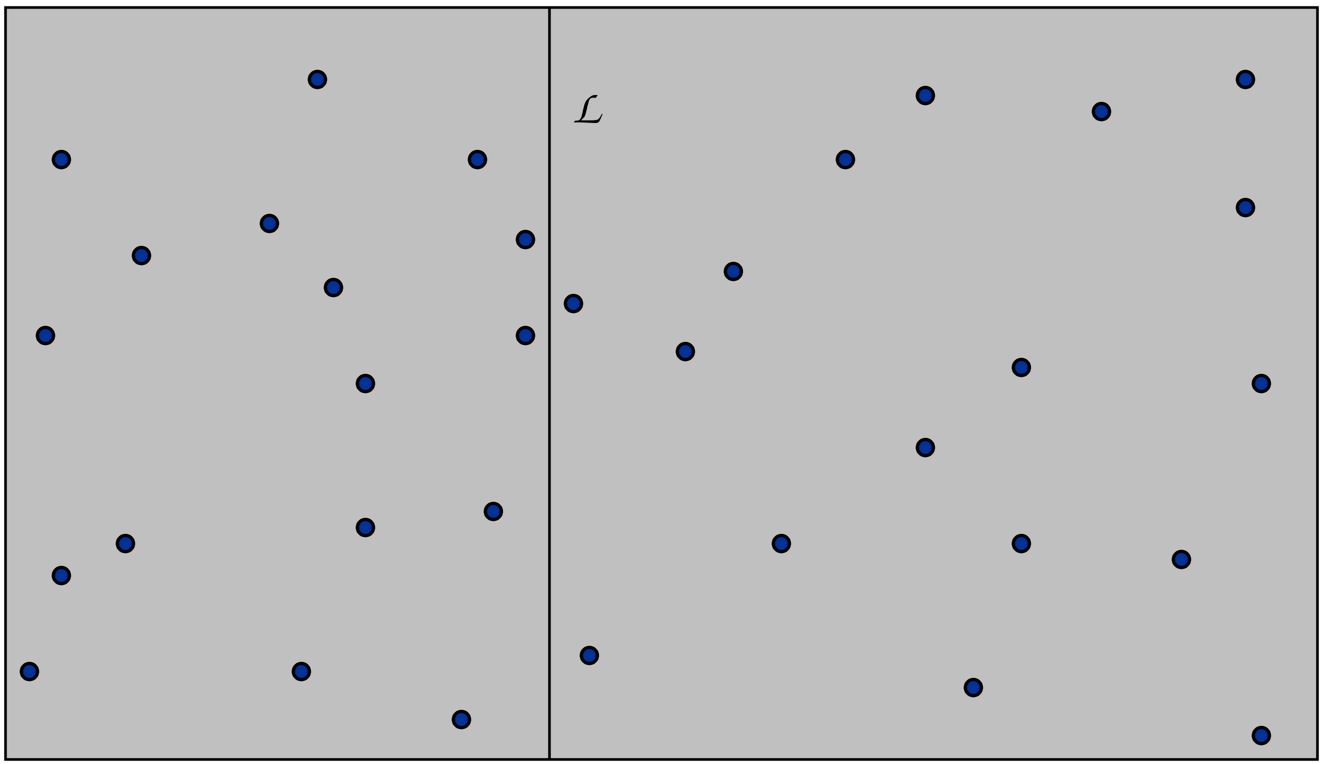
Assumption. No two points have same x coordinate.

to make presentation cleaner

Closest Pair of Points

Algorithm.

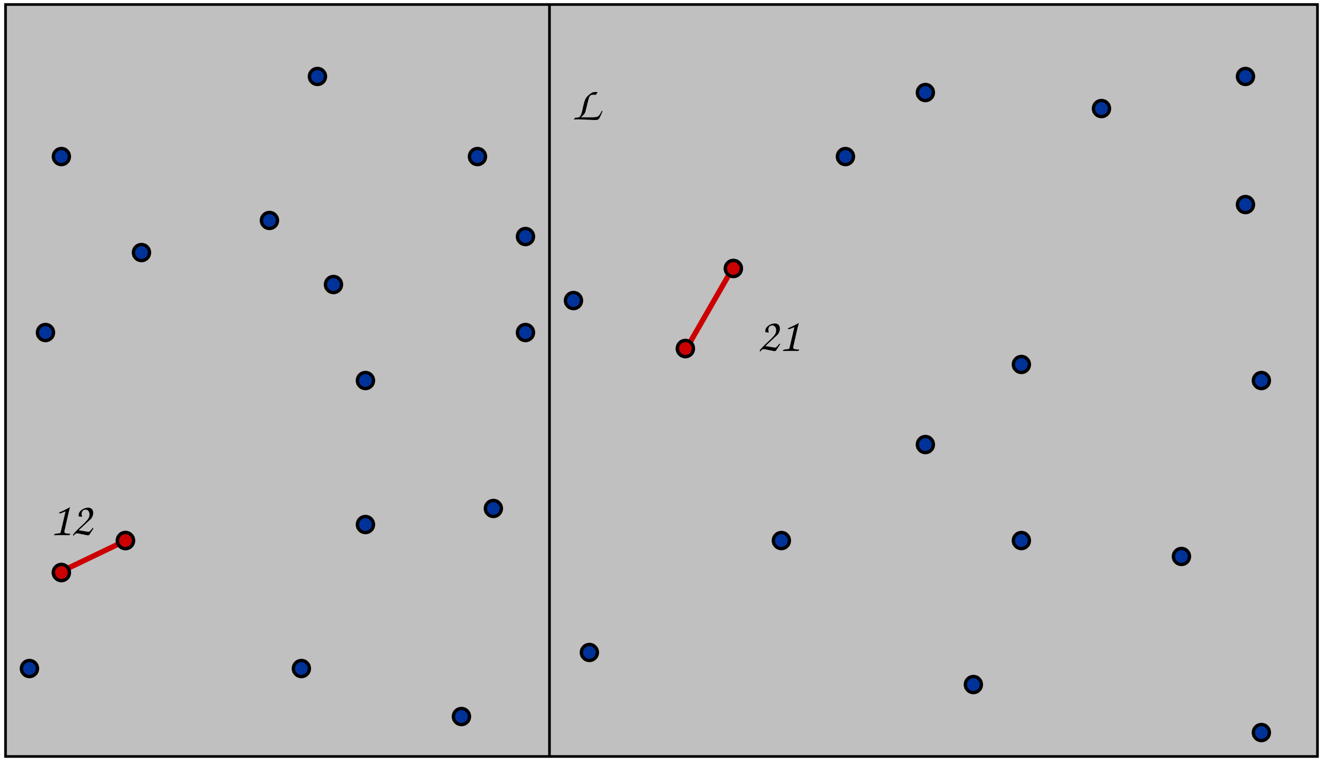
- Divide: draw vertical line L so that roughly ½n points on each side.



Closest Pair of Points

Algorithm.

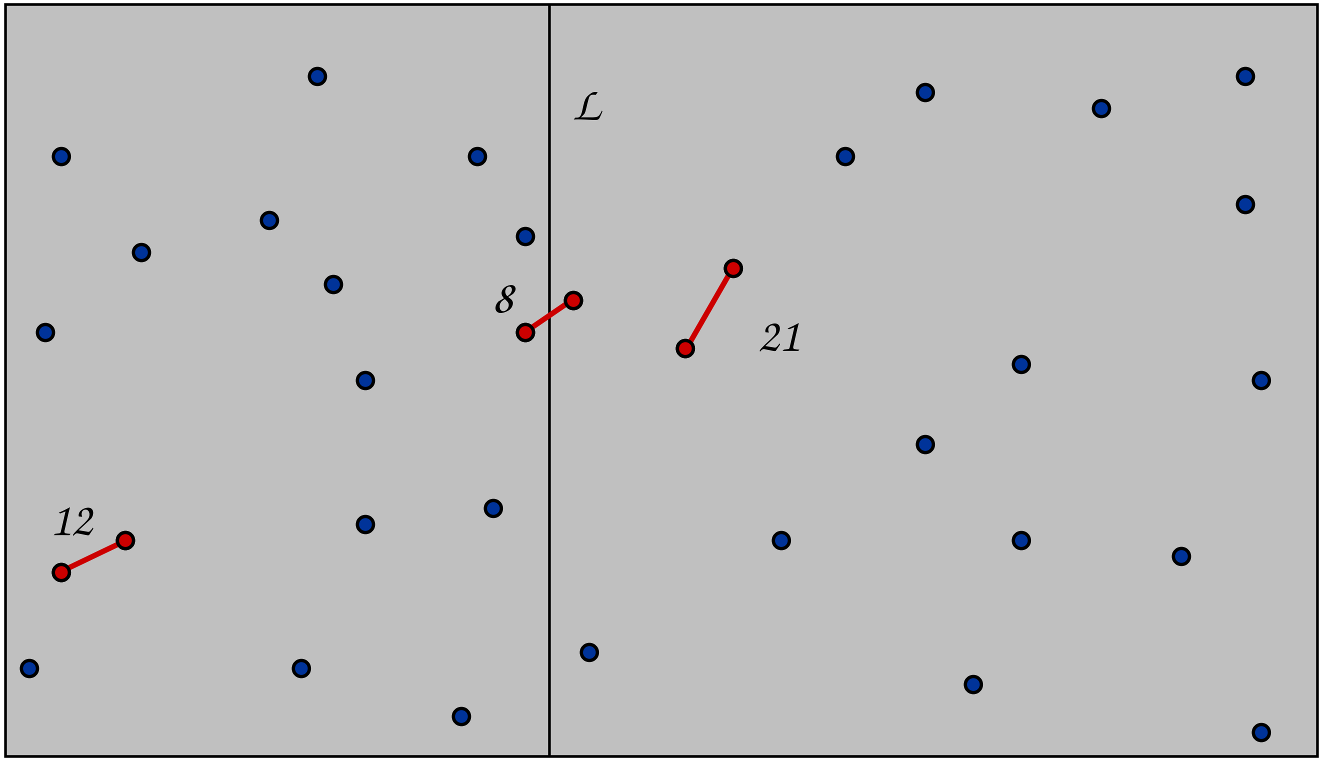
- Divide: draw vertical line L so that roughly ½n points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

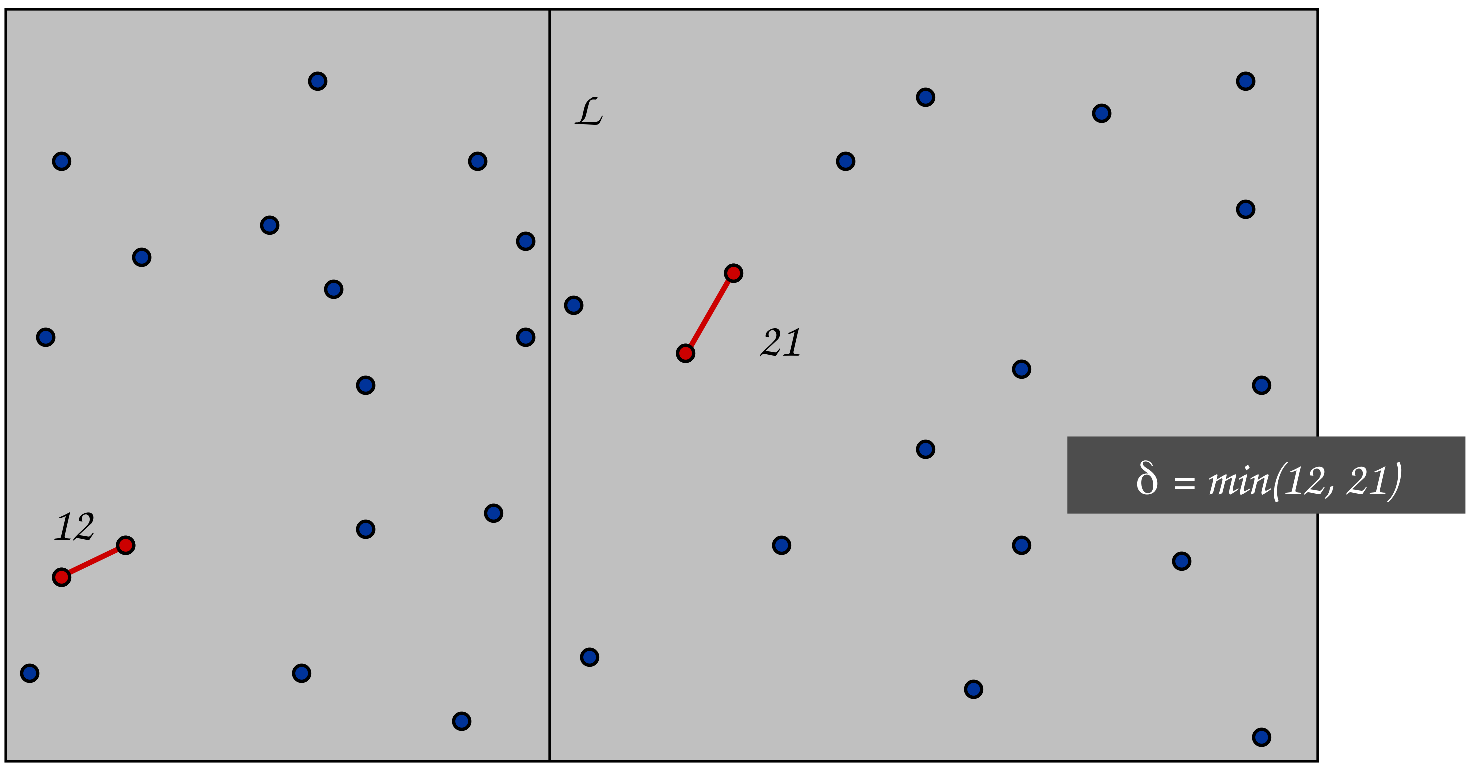
Algorithm.

- Divide: draw vertical line L so that roughly ½n points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

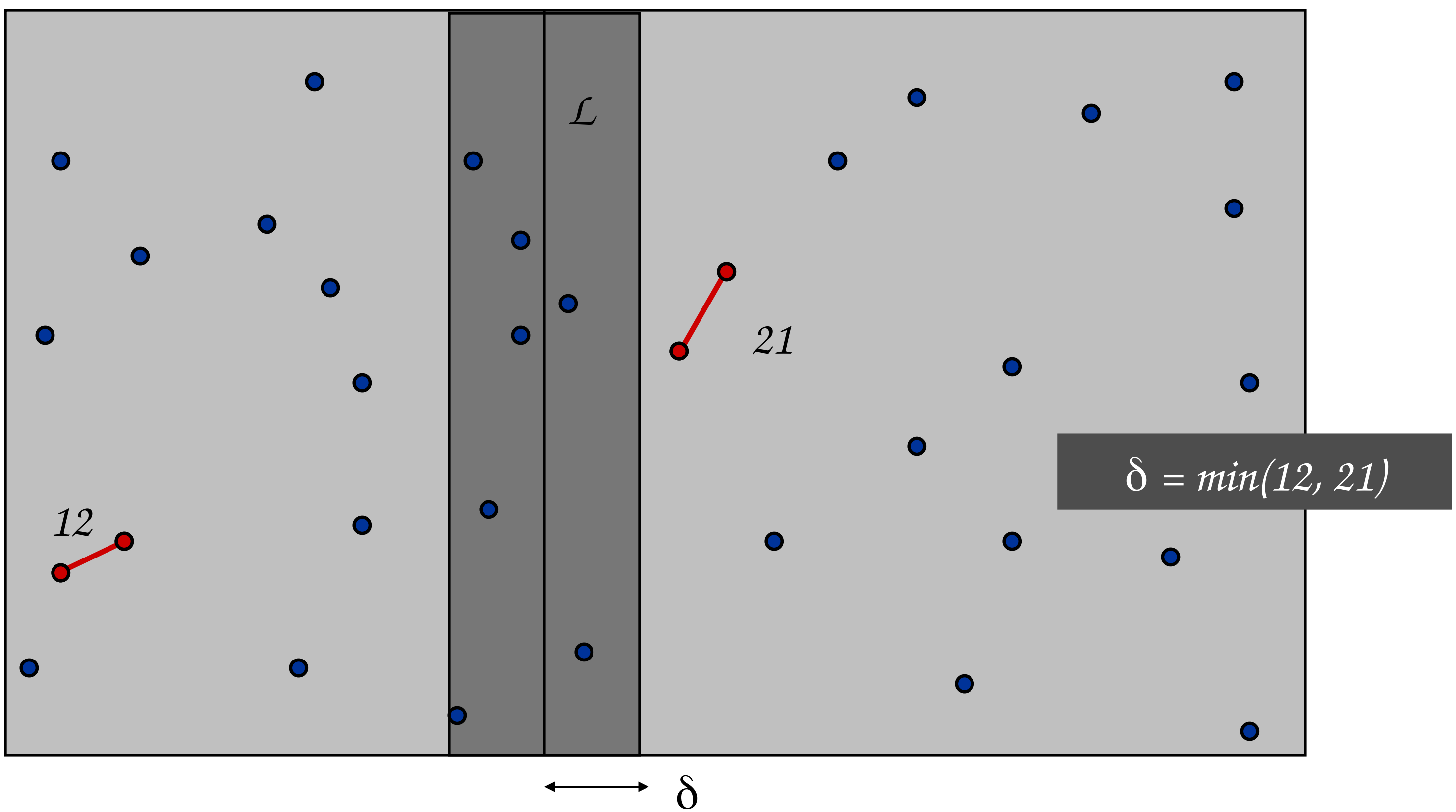
Find closest pair with one point in each side, *assuming that distance* < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

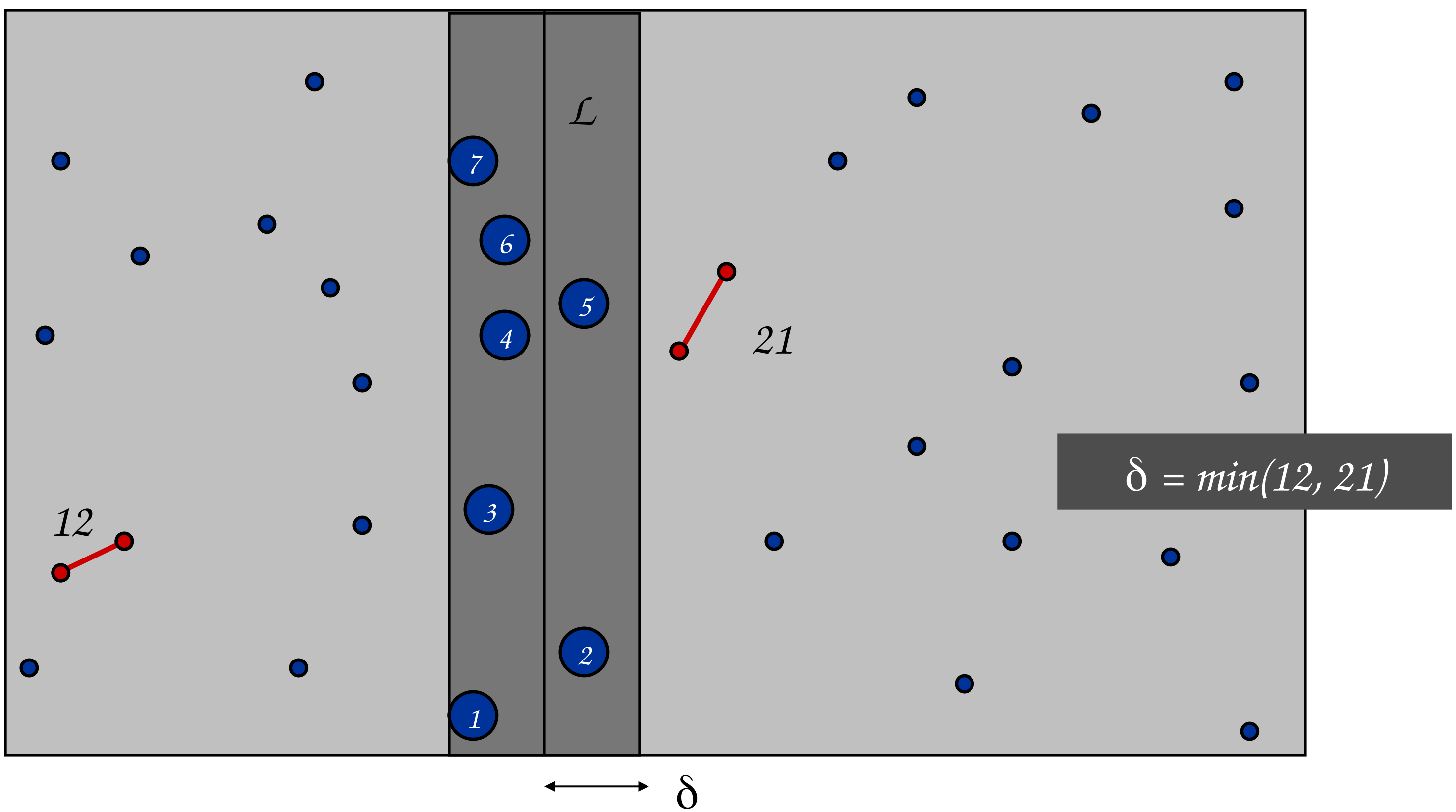
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

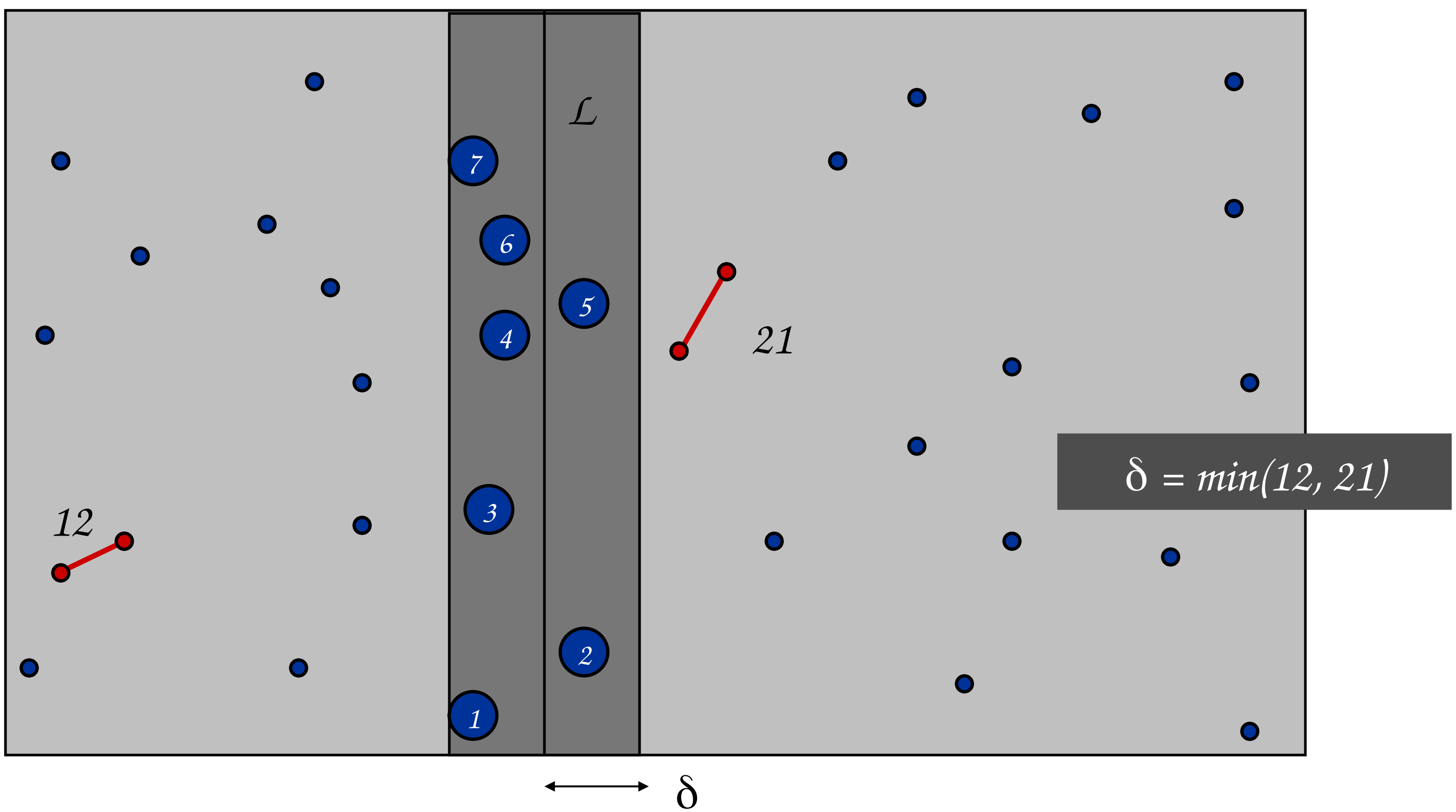
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- *Observation: only need to consider points within δ of line L.*
- *Sort points in 2δ-strip by their y coordinate.*
- *Only check distances of those within 11 positions in sorted list!*

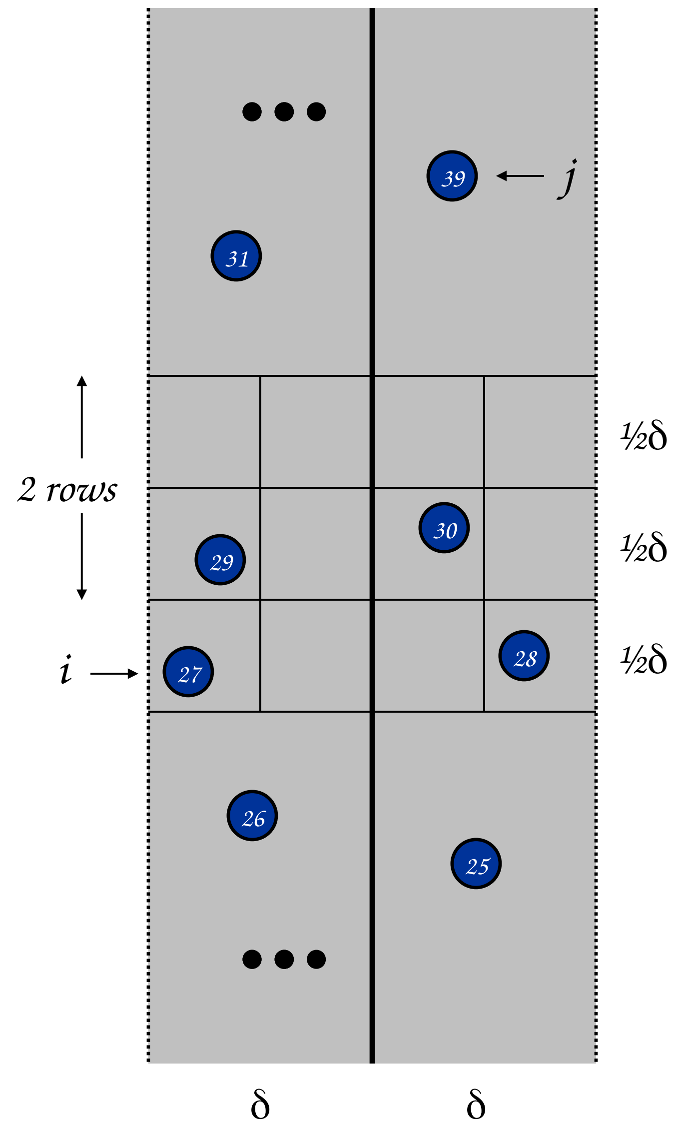


Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same ½δ-by-½δ box.
- Two points at least 2 rows apart have distance $\geq 2(½\delta)$. ■



Fact. Still true if we replace 12 with 7.

Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

O(n log n)

2T(n / 2)

O(n)

O(n log n)

O(n)

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \;\Rightarrow\; T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns all points sorted by y coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \;\Rightarrow\; T(n) = O(n \log n)$$