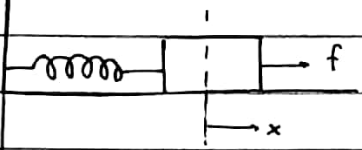
Πέμπτη, 13/10/2022



$$G(s) = \frac{1}{s^2 + ds + k} = \frac{N(s)}{D(s)}$$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$G(s) = C(s\mathbb{I} - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ k & s+d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 = \frac{1}{s^2 + ds + k}$$

$$\det(s\mathbb{I} - A) = D(s)$$

$$\bar{x} = Tx$$

T nxn αντιστρέψιμος

$$\dot{\bar{x}} = T\dot{x} = T(A\underset{T^{-1}\bar{x}}{\underbrace{x}} + Bu) = \underset{\bar{A}}{\underbrace{(TAT^{-1})}}\bar{x} + \underset{\bar{B}}{\underbrace{TB}}u$$

↳ μετασχηματισμός ομοιότητας

δεν αλλάζουν οι ιδιοτιμές κ' το χαρ. πολυώνυμο

$$y = Cx + Du = CT^{-1}\bar{x} + Du$$

$$\bar{G}(s) = \bar{C}(s\mathbb{I} - \bar{A})^{-1}\bar{B} + \bar{D} = CT^{-1}(s\mathbb{I} - TAT^{-1})^{-1}TB + D$$

$$= CT^{-1}\left[T(s\mathbb{I} - A)T^{-1}\right]^{-1}TB + D$$

$$= CT^{-1}T(s\mathbb{I} - A)^{-1}T^{-1}TB + D$$

$$= C(s\mathbb{I} - A)^{-1}B + D$$

Πόλοι της $G(s)$ $\equiv$ ιδιοτιμές του A

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + u \\ \dot{x}_2 = -x_2 \longrightarrow x_2(t) = e^{-t} x_2(0) \\ y = x_1 \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+1}{s^2-1} = \frac{1}{s-1}$$

$$Y(s) = \left[C(sI - A)^{-1} B + D \right] U(s) + C(sI - A)^{-1} x(0)$$

Σ.Μ. $\longrightarrow$ Περιγραφή στον χώρο κατάστασης

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_0}{a_n s^n + \dots + a_0}, \qquad x = ?$$

$$X_1(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_0} U(s) \Rightarrow \frac{d^n x_1}{dt^n} + a_{n-1}\frac{d^{n-1} x_1}{dt^{n-1}} + \dots + a_0 x_1 = u(t)$$

$$x = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ \vdots \\ x_1^{(n-1)} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_i = \frac{d^{i-1} x_1}{dt^{i-1}}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ u(t) - a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u$$

↳ Κανονική ελέγξιμη μορφή

$m \leq n \longrightarrow$ Περίπτωση 1: $m<n$

$$Y(s) = (b_m s^m + \ldots + b_0) X_1$$

$$y = [\, b_0 \ b_1 \ldots b_m \ 0 \ldots 0 \,] x + 0 \cdot u$$

↳ δεν εξαρτάται από την είσοδο

$\longrightarrow$ Περίπτωση 2: $m=n$

$$Y(s) = (b_{n-1} s^{n-1} + \ldots + b_0) X_1 + b_n (U - a_0 X_1 - a_1 X_2 - \ldots - a_{n-1} X_n)$$

$$y = [\, b_0 - a_0 b_n \,,\ b_1 - a_1 b_n \,,\ \ldots \,,\ b_{n-1} - a_{n-1} b_n \,] x + b_n u$$

Χρονική Απόκριση

$\dot{x} = Ax + Bu$, $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $x(t_0)$

$x = x(t)$

$x(t) = x_{ομ}(t) + x_{μερ}(t)$

λύση του $\dot{x}_{ομ} = A x_{ομ} \Rightarrow x_{ομ}(t) = \Phi(t - t_0)\, x_{ομ}(t_0)$

↳ πίνακας $n \times n$ μετάβασης

Θεωρούμε $t_0 = 0$.

$\Phi(t) = \Phi_0 + \Phi_1 t + \Phi_2 t^2 + \ldots$

Ανάπτυγμα Taylor

$$\frac{d}{dt}\left(\Phi(t) \cdot x(0)\right) = A\Phi(t)\, x(0) \quad \forall x(0) \in \mathbb{R}^n$$

$$\frac{d}{dt}\Phi(t) = A\Phi(t)$$

$$\frac{d\Phi}{dt} = A\Phi$$

$$\frac{d}{dt}\left(\Phi_0 + \Phi_1 t + \ldots\right) = A\left(\Phi_0 + \Phi_1 t + \ldots\right)$$

$$\Phi_1 + 2\Phi_2 t + 3\Phi_3 t^2 + \ldots + k\Phi_k t^{k-1} + \ldots$$

$\Phi_1 = A\Phi_0 = A$

$2\Phi_2 = A\Phi_1 = A^2$

$3\Phi_3 = A\Phi_2 = \frac{A^3}{2}$

$\vdots$

$k\Phi_k = A\Phi_{k-1} = A^k$

$\Phi_0 = I$, $\Phi_1 = A$, $\Phi_2 = \frac{A^2}{2!}$, $\Phi_3 = \frac{A^3}{3!}$

$$\Phi_k = \frac{A^k}{k!}$$

$$\Phi(t) = \mathbb{I} + At + \frac{A^2 t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots$$

$$= \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

$$\Phi(t) \triangleq e^{At} \neq [e^{a_{ij} t}]$$

expm(A)   exp(A)   στο MATLAB

$$x(t) = x_{ομ}(t) + x_{μερ}$$

$$= e^{At} x(0)$$

$$x_{μερ}(t) = \int_0^t e^{A(t-s)} Bu(s)\,ds$$

$$\frac{dx_{μερ}}{dt} = Bu(t) + \int_0^t \underbrace{\frac{d}{dt}\left(e^{A(t-s)}\right)}_{Ae^{A(t-s)}} Bu(s)\,ds \rightarrow Ax_{μερ}$$

$$e^{At} = \mathbb{I} + At + \frac{A^2 t^2}{2!}$$

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2 t + \frac{A^3}{2!} t^2 + \ldots = Ae^{At} = e^{At} A$$

$$\dot{x} = Ax + Bu$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} Bu(s)\,ds$$

$$y(t) = Ce^{At} x(0) + \int_0^t \overbrace{Ce^{A(t-s)} Bu(s)\,ds}^{e^{At}\, B * u(t)} + Du(t)$$

$$Y(s) = C(s\mathbb{I} - A)^{-1} x(0) + \left[C(s\mathbb{I} - A)^{-1} B + D\right] U(s)$$

$$\boxed{e^{At} = \mathcal{L}^{-1}\{(s\mathbb{I} - A)^{-1}\}}$$

Υπολογισμός e^{At}

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \mathbb{I} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N}, \quad N^2 = 0$$

$$A^2 = (\mathbb{I}+N)(\mathbb{I}+N) = \mathbb{I}+2N$$

$$A^3 = (\mathbb{I}+2N)(\mathbb{I}+N) = \mathbb{I}+3N$$

$$A^k = \mathbb{I}+kN$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0} \frac{(\mathbb{I}+kN)t^k}{k!} = e^t\mathbb{I} + te^tN$$

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(s\mathbb{I}-A)^{-1}\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}\right\} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$