Πέμπτη, 30/03/2023

Κανονική Παρατηρήσιμη μορφή (για συστήματα μίας εξόδου)

$$\dot{x} = \underbrace{\left[\begin{array}{c|c} & -a_0 \\ & -a_1 \\ I_{n-1} & \vdots \\ & -a_{n-1} \end{array}\right]}_{A_0} x + Bu$$

$$y = \underbrace{[0 \ 0 \ \dots \ 0 \ 1]}_{C_0} x + Du \ , \ y \in \mathbb{R}$$

$$A_0 + \underbrace{L_0}_{n\times 1} C_0$$

$$L_0 C_0 = L_{0_{n\times 1}} \cdot [0 \ 0 \ \dots \ 1] = [\underbrace{0_n \ 0_n \ \cdots \ 0_n}_{0_{n\times(n-1)}} \ L_0] \ , \quad L_0 = \begin{bmatrix} l_{01} \\ l_{02} \\ \vdots \\ l_{0n} \end{bmatrix}$$

$$A_0 + L_0 C_0 = \left[\begin{array}{c|c} & l_{01} - a_0 \\ & l_{02} - a_1 \\ I_{n-1} & \vdots \\ & l_{0n} - a_{n-1} \end{array}\right]$$

$$\det(sI_n - (A_0 + L_0 C_0)) = s^n + (a_{n-1} - l_{0,n}) s^{n-1} + \dots + a_0 - l_{0,1}$$

$$\chi_{d,o}(s) = s^n + a_{d,n-1} s^{n-1} + \dots + a_{d,0} = \prod_{i=1}^{n} (s - p^{o}_{d,i})$$

$p^{o}_{d,i}$: επιθ. πόλοι παρατηρητή

$$l_{0,i} = a_{i-1} - a_{d,i-1} \ , \ i = 1, 2, \dots, n$$

$$\bar{x} = T_0 x \ , \ A_0 = T_0 A T_0^{-1} \ , \ C_0 = C T_0^{-1} \Rightarrow A = T_0^{-1} A T_0 \ , \ C = C_0 T_0$$

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} + Du - y)$$

$$e = \hat{x} - x \Rightarrow \dot{e} = (A + LC)e \Rightarrow \dot{e} = (T_0^{-1} A_0 T_0 + L C_0 T_0) e = T_0^{-1} (A_0 + \overset{L_0}{T_0 L} C_0) T_0 e$$

$$L_0 = T_0 L \Rightarrow \boxed{L = T_0^{-1} L_0}$$

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \Rightarrow O^{-1} = \left[\begin{array}{c|c} \times & q \end{array}\right] \ , \quad T_0^{-1} = [q \ \ Aq \ \dots \ A^{n-1} q]$$

$$\rightarrow C T_0^{-1} = C[q \ Aq \ \dots \ A^{n-1} q] = [0 \ 0 \ \dots \ 0 \ 1]$$

$$O O^{-1} = I_n$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \left[\begin{array}{c|c} \times & q \end{array}\right] = I_n \rightarrow \left[\begin{array}{c|c} & Cq \\ \times & CAq \\ & \vdots \\ & CA^{n-1} q \end{array}\right] = \left[\begin{array}{c|c} & 0 \\ \times & 0 \\ & \vdots \\ & 1 \end{array}\right] \Rightarrow$$

$$\rightarrow CA^{i-1} q = \begin{cases} 0, & i = 1, \dots, n-1 \\ 1, & i = n \end{cases}$$

$$AT_0^{-1} = A[q \ Aq \ \dots \ A^{n-1}q] = [Aq \ A^2q \ \dots \ A^n q]$$

$$= [q \ Aq \ \dots \ A^{n-1}q] \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$A^n q = -\sum_{i=0}^{n-1} a_i A^i q = -a_0 q - a_1 Aq - \dots - a_{n-1}A^{n-1}q$$

$$A+LC = (A^T + C^T \underset{K}{\underset{\downarrow}{L^T}})^T$$

<u>Διάσπαση σε παρατηρήσιμο/μη παρατηρήσιμο υποσύστημα</u>

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$y \in \mathbb{R}^p, x \in \mathbb{R}^n, u \in \mathbb{R}^n$

$$\underset{np \times n}{O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}$$

Έστω ότι $\operatorname{rank} O = r < n$.

Έστω $O_1 \ O_2 \ \dots \ O_r$ γρ. ανεξάρτητες γραμμές του O

$$T = \begin{bmatrix} O_1 \\ O_2 \\ \vdots \\ O_r \\ \hline O_{r+1} \\ \vdots \\ O_n \end{bmatrix}$$

} επιλέγονται ώστε ο T να είναι αντιστρέψιμος

$$TA = \begin{bmatrix} O_1 \\ O_2 \\ \vdots \\ O_r \\ \vdots \\ O_n \end{bmatrix} A = \begin{bmatrix} O_1 A \\ O_2 A \\ \vdots \\ O_r A \\ \vdots \\ O_n A \end{bmatrix}, \quad O = \begin{bmatrix} C_1 \\ \vdots \\ C_p \\ C_1 A \\ \vdots \\ C_p A \\ \vdots \\ C_p A^{n-1} \end{bmatrix}, \quad O_i = C_j A^{k-1}, \ k \in \{1, 2, \dots n\}, \ j \in \{1, 2, \dots, p\}$$

$$O_i A = c_j A^k = \begin{cases} \text{γραμμή του } O \text{ αν } k<n \\ c_j A^n = -\sum_{\ell=0}^{n-1} c_j a_\ell A^\ell = -\sum a_\ell \underbrace{c_j A^\ell}_{\text{γραμμή του } O} \end{cases}$$

γραμμικός συνδυασμός των $O_1, \dots, O_r$

$$TA = \begin{array}{c} \quad r\times r \\ \left[\begin{array}{c|cccc} \bar{A}_o & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline \bar{A}_{21} & & \bar{A}_{\bar{o}} & & \end{array}\right] \\ (n-r)\times r \quad (n-r)\times(n-r) \end{array} \begin{bmatrix} o_1 \\ o_2 \\ \vdots \\ o_r \\ \hline o_{r+1} \\ \vdots \\ o_n \end{bmatrix}^{T} \Rightarrow TAT^{-1} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}$$

$$C = [\,\bar{C}_o \quad 0\,] \begin{bmatrix} o_1 \\ o_2 \\ \vdots \\ o_r \\ \hline o_{r+1} \\ \vdots \\ o_n \end{bmatrix}$$

παρατηρήσιμο

$$\dot{x} = \begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u$$

μη παρατηρήσιμο

$$y = [\,\bar{C}_o \quad 0\,] \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Du$$

$$\bar{A} + \bar{L}\bar{C} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{L}_o \\ \bar{L}_{\bar{o}} \end{bmatrix} [\,\bar{C}_o \quad 0\,] = \begin{bmatrix} \bar{A}_o + \bar{L}_o\bar{C}_o & 0 \\ \bar{A}_{21} + \bar{L}_{\bar{o}}\bar{C}_o & \bar{A}_{\bar{o}} \end{bmatrix}$$

<u>Άσκηση</u> $(\bar{C}_o, \bar{A}_o)$ παρατηρήσιμο (αν rank $C = r$)

$$G(s) = [\,\bar{C}_o \quad 0\,] \begin{bmatrix} sI_r - \bar{A}_o & 0 \\ -\bar{A}_{21} & sI_{n-r} - \bar{A}_{\bar{o}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} + D$$

$$= [\,\bar{C}_o \quad 0\,] \begin{bmatrix} (sI_r - \bar{A}_o)^{-1} & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} + D = \bar{C}_o (sI_r - \bar{A}_o)^{-1} \bar{B}_o + D$$

θα έχει τις ιδιοτιμές των $\bar{A}_{co}, \bar{A}_{c\bar{o}}, \bar{A}_{\bar{c}o}, \bar{A}_{\bar{c}\bar{o}}$

$$\bar{x} = \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix}, \quad \dot{\bar{x}} = \begin{bmatrix} \dot{\bar{x}}_{co} \\ x_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \left[\begin{array}{cc|cc} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ \hline 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{array}\right] \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

$$G(s) = \bar{C}_{co}(s\mathbb{I} - \bar{A}_{co})^{-1} \bar{B}_{co} + D$$

Διάσπαση Kalman

$$\begin{cases} \dot{x} = Ax + Bu + d \\ y = Cx \end{cases}$$

$y \to y_{ref}$

$$u = K_p x + K_I \underbrace{\int_0^t (y(s) - y_{ref})\, ds}_{z(t)}$$ PI ελεγκτής

$$x_{ag} = \begin{bmatrix} x \\ z \end{bmatrix} \Rightarrow \dot{x}_{ag} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax + d + B(K_p + K_I z) \\ y - y_{ref} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \dot{x}_{ag} = \underbrace{\begin{bmatrix} A + BK_p & BK_I \\ C & 0 \end{bmatrix}}_{A_{ag}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} d \\ -y_{ref} \end{bmatrix}}_{d_{ag}}$$

(A, B) ελέγξιμο (ή σταθεροποιήσιμο)

$$\dot{x}_{ag} = A_{ag} x_{ag} + d_{ag}$$

σημείο ισορροπίας: $0 = A_{ag} x_{ag}^* + d_{ag} \Rightarrow \boxed{x_{ag}^* = -A_{ag}^{-1} d_{ag}}$

$$x_{ag}^* = \begin{bmatrix} x^* \\ z^* \end{bmatrix} \quad \begin{matrix} (A + BK_p)x^* + BK_I z^* = -d^* \\ y^* = C x^* = y_{ref} \end{matrix}$$

$$A_{ag} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} BK_p & BK_I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K_p & K_I \end{bmatrix}$$

$\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix} \right)$ σταθεροποιήσιμο

$$\text{rank} \begin{bmatrix} A - \lambda \mathbb{I}_n & 0 & B \\ C & -\lambda \mathbb{I}_p & 0 \end{bmatrix} = n + p \quad \forall \lambda \in \mathbb{C}_+$$

Για $\lambda \neq 0$ $\text{rank}\begin{bmatrix} C & -\lambda \mathbb{I}_p & 0 \end{bmatrix} = p$, άρα αρκεί (A, B) σταθερ.

Για $\lambda = 0$: $\begin{bmatrix} A & 0 & B \\ C & 0 & 0 \end{bmatrix}$, αρκεί (A, B) σταθερ. + $\text{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p$

$\dot{x}_1 = x_2 + d_1$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dot{x}_2 = x_2 + u + d_2$

(A,B) ελέγξιμο

$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] = 3 \checkmark$$