Δευτέρα, 24/04/2023

LQR

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)\, dt$$

$\dot{x} = Ax + Bu$, $u = -R^{-1}B^T P x$, $PA + A^T P - PBR^{-1}B^T P + Q = 0$

$$H_m = \begin{bmatrix} A & \overbrace{-BR^{-1}B^T}^{\bar{R}} \\ -Q & -A^T \end{bmatrix}$$

A-invariant υπόχωρος: $v \in V \Rightarrow Av \in V$

βάση: κάποια ιδιοδιανύσματα του A

$V = [v_1 \dots v_k]$

$\underset{n\times n}{A}\,\underset{n\times k}{V} = \underset{n\times k}{V}\,\underset{k\times k}{\bar{A}}$

$AV = [Av_1 \dots Av_k]$

$$v_i = [v_1 \dots v_k]\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{bmatrix}$$

Εξ. Riccati: $PA + A^T P + P\bar{R}P + Q = 0$

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad J^2 = -I \Rightarrow J^{-1}HJ = -JHJ = -H^*$$

$\mathrm{tr} H = 0$

Οι λύσεις της Riccati δεν είναι μοναδικές. Εμείς αναζητούμε τη stabilizing λύση

Θεωρούμε ότι ο H δεν έχει ιδιοτιμές με $Re(\lambda_i) = 0$.

$X(A + RX) + (A + RX)^T X = -Q + XRX$

$X_{\underset{(+)}{-}}(H)$: H-invariant υπόχωρος με βάση τα ιδιοδιανύσματα των ευσταθών (ασταθών) ιδιοτιμών

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$H_{2n\times 2n}$

$$\mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_-(H)$$

X_1: $n\times n$, X_2: $n\times n$

stabilizing solution: $X = X_2 X_1^{-1}$

Thm: $H \in \mathrm{dom}(\mathrm{Ric})$, $X = \mathrm{Ric}(H)$

(i) X is real symmetric

(ii) X ικανοποιεί τη Riccati

(iii) $A + RX$ stable

$$H_m \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \Rightarrow \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H$$

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} Q & A^T \\ A & R \end{bmatrix}$$

X συμμετρικός $\Rightarrow X = X_2 X_1^{-1} = (X_2^{-1})^* X_1^* X_2 X_1^{-1}$

$$\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} \mathbb{I} \\ X_2 X_1^{-1} \end{bmatrix} X_1 H_- X_1^{-1}$$

$A + RX = X_1 H_- X_1^{-1}$

$$[X \quad \mathbb{I}]\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} \mathbb{I} \\ X \end{bmatrix} = 0 \Rightarrow [XA - Q \quad XR - A^T]\begin{bmatrix} \mathbb{I} \\ X \end{bmatrix} = 0$$

$$\Rightarrow XA - Q + XRX - A^T X = 0$$

X has no eigenvalues with $\mathrm{Re}(\lambda_i) = 0$ when (R, A) stabilizable

If $H_m = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q^{1/2}Q^{1/2} & -A^T \end{bmatrix}$, λύση της Riccati θετικά ημιορισμένος

(A,B) stabil., (C,A) detect. $\Rightarrow$ Riccati μον. >0 λύση

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1) $PA + A^T P - PBR^{-1}B^T P + Q = 0$

2) $H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$

3) $\det[\lambda I_{2n} - H] = 0 \Rightarrow \lambda_1, \lambda_2, \dots \lambda_{2n}$ ιδιοτιμές

4) ιδιοδιαν. για ευσταθείς ιδιοτ. της H

5) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [U_1 \;\; U_2 \;\; \dots \;\; U_n]$

6) $P = X_2 X_1^{-1}$

$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0 = p(s)$

$$\boxed{tr\,\mathcal{L}\{e^{tA}\} = \frac{p'(s)}{p(s)}}$$

$$(sI-A)^{-1} = \frac{\overbrace{adj(sI-A)}^{N(s)}}{p(s)} = \frac{s^{n-1}}{p(n)}N_1 + \dots + \frac{1}{p(s)}N_n$$

$(sI-A)(sI-A)^{-1} = I \Leftrightarrow (sI-A)N(s) = p(s)I$

$(sI-A)(s^{n-1}N_1 + \dots + N_n) = (s^n + a_{n-1}s^{n-1} + \dots + a_0)I$

$\hookrightarrow s^n N_1 + (N_2 - AN_1)s^{n-1} + (N_3 - AN_2)s^{n-2} + \dots + (N_n - AN_{n-1})s - AN_n$

$\rightarrow N_1 = I$

$N_2 - AN_1 = a_{n-1}I \Rightarrow N_2 = AN_1 + a_{n-1}I$

$\vdots$

$N_n - AN_{n-1} = a_n I \Rightarrow N_n = AN_{n-1} + a_1 I$

$0 = AN_n + a_n I$

$$\boxed{\begin{aligned} &N_1 = I \\ &N_2 = AN_1 + a_{n-1}I \\ &\vdots \\ &N_n = AN_{n-1} + a_1 I \\ &0 = AN_1 + a_0 I \end{aligned}}$$

$$\frac{d}{dt}(e^{At}) = Ae^{At}$$

$$s\mathcal{L}[e^{At}] - I = A\mathcal{L}[e^{At}] \Rightarrow s\frac{p'(s)}{p(s)} - n = tr[A\underbrace{(sI-A)^{-1}}_{\frac{AN(s)}{p(s)}}]$$

$\Rightarrow sp'(s) - np(s) = tr\,AN(s)$

$sp'(s) = ns^n + (n-1)a_{n-1}s^{n-1} + \dots + a_1 s$

$sp'(s) - np(s) = -a_{n-1}s^{n-1} - \dots - (n-1)a_1 s - na_0$

$= tr(AN_1)s^{n-1} + tr(AN_2)s^{n-2} + \dots + tr(AN_1)$

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$a_0 = -\frac{1}{n} tr(AN_n)$

$a_{n-1} = -tr(AN_1)$

$a_k = -\frac{1}{(n-k)} tr(AN_{n-k})$

Για $n=2$ $\quad (trH = 0$ πάντα!)

$$\det(\lambda I - H) = \lambda^4 - \frac{1}{2} tr(H^2)\lambda^2 + \det(H)$$

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Παράδειγμα

$\dot{x}_1 = -x_1 + x_2 + u$

$\dot{x}_2 = -x_1 - 3x_2 + u$

$$\min J = \frac{1}{2}\int_0^{\infty} \left[7x_1^2 + 4x_2^2 + 6x_1x_2 + u^2\right] dt$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 7 & 3 \\ 3 & 4 \end{bmatrix}, \quad R = 1$$

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ \hline -7 & -3 & 1 & 1 \\ -3 & -4 & -1 & 3 \end{array}\right], \quad \det(H) = 4$$

$$H^2 = \begin{bmatrix} 4 & & & \\ & 9 & & \\ & & 4 & \\ & & & 9 \end{bmatrix}$$

$$\det(sI - H) = s^4 - 13s^2 + 36, \quad s = \pm 3, \pm 2$$

Για τις ευσταθείς

$s = -3: \quad (H + 3I)v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$s = -2: \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ \hline 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Οι ιδιοτιμές του κλ. βρόχου με $u = -R^{-1}B^T P x$ είναι $-2, -3$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

$$A_c = TAT^{-1}, \quad B_c = TB$$

$$PT^{-1}TAT^{-1} + T^T(TAT^{-1})^T(T^{-1})^T P - PT^{-1}TBR^{-1}(TB)^T(T^{-1})^T P + Q = 0$$

$$\Rightarrow PT^{-1}A_c T + T^T A_c^T (T^T)^{-1} P - PT^{-1}B_c R^{-1} B_c^T T^{-T} P + Q = 0$$

$$T^{-T}(\dots)T = 0 \Rightarrow T^{-T}PT^{-1}A_c + A_c^T T^{-T}PT^{-1} - T^{-T}PT^{-1}B_c R^{-1} B_c^T T^{-T} P T^{-1} + T^{-T} Q T^{-1} = 0$$

Άρα

$$\Rightarrow P_c A_c + A_c^T P_c - P_c B_c R^{-1} B_c^T P_c + Q_c = 0$$

για $Q_c = T^{-T} Q T^{-1}$, $P = T^T P_c T$

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