Πέμπτη, 02/03/2023

Διάσπαση σε ελέγξιμο/μη ελέγξιμο υποσύστημα

$\dot{x} = Ax + Bu \quad , x \in \mathbb{R}^n, u \in \mathbb{R}^m$

$\underset{n \times nm}{\mathcal{C}} = [B \;\; AB \;\; \dots \;\; A^{n-1}B]$

Έστω ότι $\operatorname{rank}(\mathcal{C}) = r < n$.

Μετασχηματισμός: $\bar{x} = Tx$, $T_{n\times n}$ αντιστρέψιμος

Έστω $q_1, q_2, \dots, q_r \in \mathbb{R}^n$ γραμμικά ανεξάρτητες στήλες του $\mathcal{C}$

$Q = [q_1 \;\; q_2 \;\; \dots \;\; q_r \;\; \underbrace{q_{r+1} \;\; \dots \;\; q_n}]$

$T = Q^{-1}$ επιλέγονται ώστε ο T να είναι αντιστρέψιμος.

$\dot{\bar{x}} = T\dot{x} = T(Ax + Bu)$, $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$

$= TAx + TBu$

$\dot{\bar{x}} = \underset{\bar{A}}{\underbrace{TAT^{-1}}}\bar{x} + \underset{\bar{B}}{\underbrace{TB}}u$

$$B = [b_1 \;\; b_2 \;\; \dots \;\; b_m] = \left[\sum_{i=1}^{r} \beta_{i1} q_i \;\; \dots \;\; \sum_{i=1}^{r} \beta_{im} q_i\right] = [q_1 \; q_2 \dots q_r \mid q_{r+1} \dots q_r] \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \beta_{12} & \ddots & \vdots \\ \vdots & & \vdots \\ \beta_{r1} & \dots & \beta_{rm} \\ 0 & & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

$$B = Q \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$$\bar{B} = TB = TQ \begin{bmatrix} B_c \\ 0 \end{bmatrix} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$AQ = A[q_1 \;\; q_2 \;\; \dots \;\; q_r \;\; q_{r+1} \;\; \dots \;\; q_n] = [Aq_1 \;\; Aq_2 \dots Aq_r \;\; Aq_{r+1} \dots Aq_n]$

$\mathcal{C} = [b_1 \;\; b_2 \dots b_m \;\; Ab_1 \dots Ab_m \dots A^{n-1} b_m]$

$q_i = A^j b_k, \quad 0 \le j \le n-1, \; 1 \le k \le m, \; i = 1, \dots, r$

$$\rightarrow Aq_i = A^{j+1} b_k = \begin{cases} \text{στήλη του } \mathcal{C}, \text{ αν } j < n-1 \\ A^n b_k = -\sum_{\ell=0}^{n-1} a_\ell A^\ell b_k, \text{ αν } j = n-1 \end{cases} \longrightarrow \text{γρ. συνδυασμός των } q_1, \dots, q_r$$

Θ. Cayley-Hamilton: $A^n + a_{n-1}A^{n-1} + \dots + a_0 I = 0 \Leftarrow \det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0$

$$AQ = \left[\sum_{j=1}^{r} \gamma_{1j} q_j \;\; \dots \;\; \sum_{j=1}^{r} \gamma_{rj} q_j \;\Big|\; \sum_{j=1}^{n} \gamma_{r+1,j} q_j \;\; \dots \;\; \sum_{j=1}^{n} \gamma_{n,j} q_j\right]$$

$$= [q_1 \;\; q_2 \dots q_r \;\; q_{r+1} \dots q_n] \left[\begin{array}{ccc|ccc} \gamma_{11} & \dots & \gamma_{r1} & \gamma_{r+1,1} & \dots & \gamma_{n1} \\ \gamma_{12} & \dots & \gamma_{r2} & \gamma_{r+1,2} & \dots & \gamma_{n2} \\ \gamma_{1r} & \dots & \gamma_{rr} & \vdots & & \vdots \\ \hline 0 & & 0 & & & \\ \vdots & & \vdots & & \ddots & \\ 0 & & 0 & \gamma_{r+1,m} & & \gamma_{nm} \end{array}\right]$$

$$AT^{-1} = T^{-1}\left[\begin{array}{c|c} \bar{A}_c & \bar{A}_{12} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right] \Rightarrow \bar{A} = \left[\begin{array}{c|c} \bar{A}_c & \bar{A}_{12} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right]$$

$$\dot{\bar{x}} = \begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$\bar{x}_c$ → ελέγξιμο

$\bar{x}_{\bar{c}}$ → μη ελέγξιμο

$$\dot{\bar{x}}_{\bar{c}} = \bar{A}_{\bar{c}}\,\bar{x}_{\bar{c}} \Rightarrow \bar{x}_{\bar{c}}(t) = e^{\bar{A}_{\bar{c}} t}\,\bar{x}_{\bar{c}}(0)$$

$$\bar{\mathcal{C}} = [\bar{B}\ \ \bar{A}\bar{B}\ \dots\ \bar{A}^{n-1}B] = \left[\begin{array}{cccc} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}_c^{n-1}\bar{B}_c \\ \hline 0 & 0 & \dots & 0 \end{array}\right]$$

$$\operatorname{rank}(\bar{\mathcal{C}}) = \operatorname{rank}(\underbrace{[\bar{B}_c\ \ \bar{A}_c\bar{B}_c\ \dots\ \bar{A}_c^{n-1}\bar{B}_c]}_{r \times nr}) = \operatorname{rank}([\bar{B}_c\ \ \bar{A}_c\bar{B}_c\ \dots\ \bar{A}_c^{r-1}\bar{B}_c])$$

$$\bar{\mathcal{C}} = [\underbrace{TB}_{\bar{B}}\ \ \underbrace{TAT^{-1}TB}_{\bar{A}\bar{B}}\ \dots\ \underbrace{(TAT^{-1})\dots(TAT^{-1})}_{\bar{A}^{n-1}}\underbrace{TB}_{\bar{B}}]$$

$$= [TB\ \ TAB\ \dots\ TA^{n-1}B] = T\mathcal{C} \Rightarrow \operatorname{rank}(\mathcal{C}) = \operatorname{rank}(\bar{\mathcal{C}}) = r$$

$\Rightarrow (\bar{A}_c, \bar{B}_c)$ ελέγξιμο

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \bar{x} + Du$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI_{n-r} - \bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D$$

$$\hookrightarrow \begin{bmatrix} (sI_r - \bar{A}_{\bar{c}})^{-1} & (sI_r - \bar{A}_c)^{-1} A_{12} (sI_{n-r} - \bar{A}_{\bar{c}})^{-1} \\ 0 & (sI_{n-r} - \bar{A}_{\bar{c}})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{C}_c (sI_r - A_c)^{-1} & * \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D = \bar{C}_c (sI_r - \bar{A}_c)^{-1} B + D$$

↳ (βλέπουμε μόνο το ελέγξιμο κομμάτι στην έξοδο)

Παράδειγμα

μη ελέγξιμη ιδιοτιμή

$$\begin{cases} \dot{x}_1 = -ax_1 + x_2 + u \\ \dot{x}_2 = -x_2 + (a-1)u \\ y = x_1 \end{cases}, \quad A = \begin{bmatrix} -a & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ a-1 \end{bmatrix}$$

$$\text{rank}([A + aI_2 \;\; B]) = \text{rank}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & a-1 & a-1 \end{bmatrix}\right) = 1$$

$$\text{rank}([A + I_2 \;\; B]) = \text{rank}\left(\begin{bmatrix} 1-a & 1 & 1 \\ 0 & 0 & a-1 \end{bmatrix}\right) = 2 \quad \forall a \neq 1$$

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ a-1 & 1-a \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix} \Rightarrow T = Q^{-1} = \begin{bmatrix} 1 & 0 \\ 1-a & 1 \end{bmatrix}$$

$$\bar{x} = Tx = \begin{bmatrix} x_1 \\ (1-a)x_1 + x_2 \end{bmatrix}$$

$$\dot{\bar{x}}_2 = (1-a)\dot{x}_1 + \dot{x}_2 = (1-a)(-ax_1 + x_2 + u) - x_2 + (a-1)u = -a(1-a)x_1 - ax_2 = -a\bar{x}_2$$

(αν $a>0$, το σύστημα σταθεροποιήσιμο)

$$\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1-a & 1 \end{bmatrix} = [\underbrace{1}_{\bar{C}_c} \;\; 0], \quad \bar{B} = TB = \begin{bmatrix} \overbrace{1}^{\bar{B}_c} \\ 0 \end{bmatrix}$$

$$G(s) = 1 \cdot (s+1)^{-1} \cdot 1 = \frac{1}{s+1}$$

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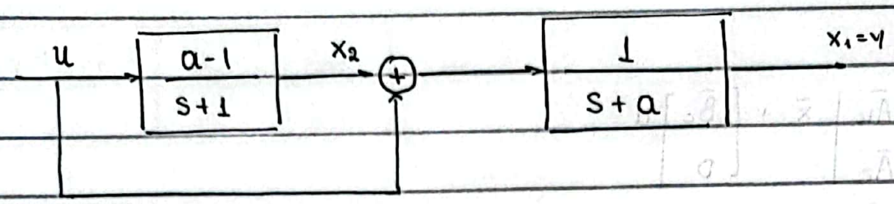

$$\frac{Y(s)}{U(s)} = \frac{1}{s+a}\left[1+\frac{a-1}{s+1}\right] = \frac{1}{s+a}\cdot\frac{s+a}{s+1} = \frac{1}{s+1}$$

$$G(s) = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1}s^{n} + \dots + a_0} \Rightarrow \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_0 y = b_m \frac{d^m u(t)}{dt} + \dots + b_0 u(t)$$

minimal realization : $x \in \mathbb{R}^n$

$$\Xi(s) = \frac{U(s)}{s^n + a_{n-1}s^{n-1} + \dots + a_0} \qquad u \longrightarrow \boxed{\frac{1}{s^n + \dots + a_0}} \longrightarrow \xi$$

$$\left.\begin{aligned} x_1 &= \xi \\ x_2 &= \dot{\xi} \\ &\vdots \\ x_n &= \xi^{(n-1)} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{\xi} = x_2 \\ \dot{x}_2 &= \ddot{\xi} = x_3 \\ &\vdots \\ \dot{x}_{n-1} &= \xi^{(n-1)} = x_n \\ \dot{x}_n &= \xi^{(n)} = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1}x_n + u \end{aligned}$$

$$\frac{d^n \xi}{dt^n} + a_{n-1}\frac{d^{n-1}\xi}{dt^{n-1}} + \dots + a_0 \xi = u$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$m<n : \frac{Y(s)}{U(s)} = \frac{Y(s)}{\Xi(s)} \cdot \frac{\Xi(s)}{U(s)} = (b_m s^m + \dots + b_0)\cdot \frac{\Xi(s)}{U(s)}$$

$$y(t) = [b_0 \ b_1 \ \dots \ b_m \ 0 \dots 0] \begin{bmatrix} \xi \\ \dot{\xi} \\ \vdots \\ \xi^{(m)} \\ \vdots \\ \xi^{(n-1)} \end{bmatrix}$$

$$m=n: \frac{b_n s^n + \dots + b_0}{s^n + \dots + a_0'} = b_n + \frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + \dots + (b_0 - b_n a_0)}{s^n + \dots + a_0}$$

$$y(t) = [b_0 - b_n a_0 \ \dots \ b_{n-1} - b_n a_{n-1}]\, x(t) + b_n u(t)$$

n.x. • $G(s) = \dfrac{2s-1}{s^3+3s^2+2s+1}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} x$$

• $G(s) = \dfrac{2s^3+2s+1}{s^3+3s^2+2s+1} = 2 + \dfrac{-6s^2-2s-3}{s^3+3s^2+2s+1}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -3 & -2 & -6 \end{bmatrix} x + 2u$$

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