COM S 311 EXAM 2

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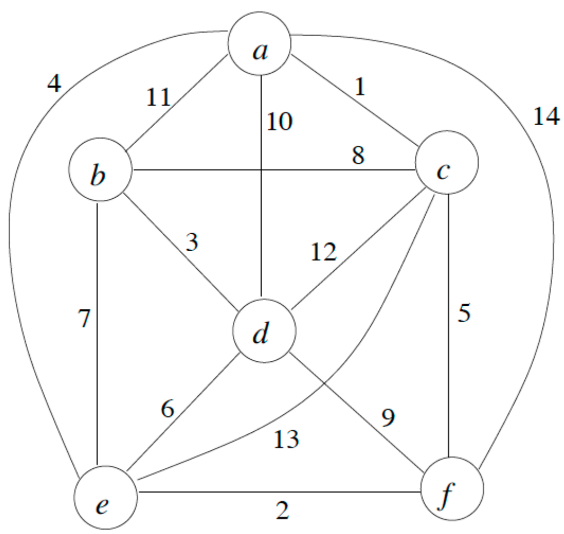
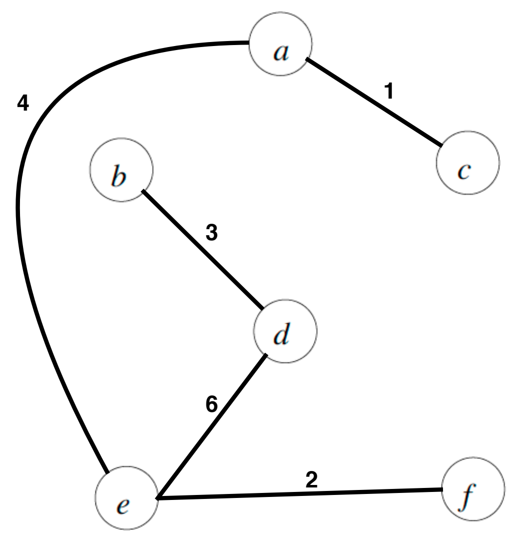
This assignment represents my own work in accordance with University regulations.

- Nathan Tucker

Problem 1: (Minimum Spanning Trees) (20 Points)

(a) Construct a minimum spanning tree for the graph below on the left. Draw the tree by adding edges to the graph on the right.





(b) Fill out the table below with the edges of the above minimum spanning tree according to their orders of selection by Kruskal's and Prim's algorithms, respectively.

	1	2	3	4	5
Kruskal's	(**a** , **c**)	(**e** , **f**)	(**b** , **d**)	(**a** , **e**)	(**e** , **d**)
Prim's	(**a** , **c**)	(**a** , **e**)	(**e** , **f**)	(**e** , **d**)	(**d** , **b**)

(c) Suppose all edges in a graph G have different edge weights. Show that G has a unique minimum spanning tree.

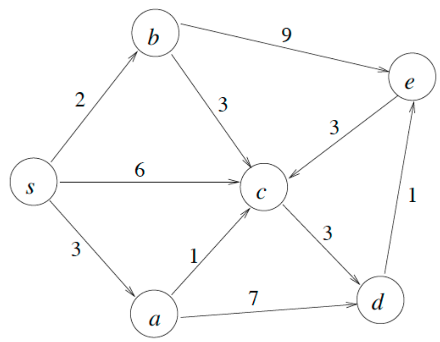
Let us assume we have two MSTs, T_1 and T_2. Then, consider the edge of minimum weight among all the edges that are contained in exactly on of T_1 *or* T_2. This edge will only appear in a single tree, say, T_1, so let's call it e_1. Then, $T_2 \cup \{e_1\}$ must contain a cycle, and one of the edges of this cycle, e_2, is not in T_1.

Because e_2 is an edge different from e_1 and is contained in only one of T_1 *or* T_2, it must be that $w(e_1) < w(e_2)$. $T = T_2 \cup \{e_1\} \backslash \{e_2\}$ is a spanning tree. The total weight of T is now smaller than the total weight of T_2, but this is a contradiction to our earlier assumption that T_2 is a minimum spanning tree.

$\therefore$ G has a unique minimum spanning tree if all of the edges in graph G have different costs.

Problem 2: (Shortest Paths) (20 Points)

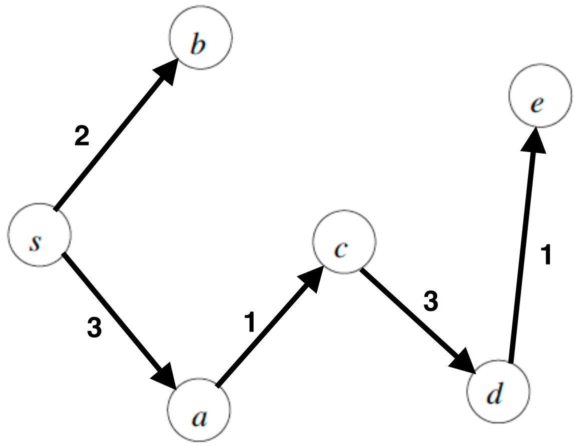
(a) Consider the graph G below.



Show the execution of Dijkstra's algorithm on G by filling out the table below. Assume the source is vertex s. You must show the changes in the d-array at the end of each iteration. Iteration 0 refers to the situation just before the first iteration of the **while** loop. Also, fill in the vertex that is selected (i.e., extracted from the priority queue) at each iteration.

iter.	d[ ]						selected node
	s	a	b	c	d	e	
0	**0**	**∞**	**∞**	**∞**	**∞**	**∞**	s
1	**0**	**3**	**2**	**6**	**∞**	**∞**	**s**
2	**0**	**3**	**2**	**5**	**∞**	**∞**	**b**
3	**0**	**3**	**2**	**4**	**∞**	**11**	**a**
4	**0**	**3**	**2**	**4**	**10**	**11**	**c**
5	**0**	**3**	**2**	**4**	**7**	**11**	**d**
6	**0**	**3**	**2**	**4**	**7**	**8**	**e**

(b) Draw a shortest path tree for the graph in (a) with source vertex s.



(c) Suppose that all edge weights in a given graph $G := (V, E)$ are NOT negative, and that the shortest path distances in G from a source $s \in V$ to each vertex $v \in V$ are unique. Let V_k denote the vertices in V with the k-closest shortest path distances from s.

Example: Let $V = \{s, u, v\}$ and $0, 2, 3$ be the shortest path distances of the vertices s, u, v from s in G respectively. Then, s is the 1-closest vertex, u is the 2-closest vertex, and v is the 3-closest vertex. Therefore we have $V_1 = \{s\}$, $V_2 = \{s, u\}$, $V_3 = V$.

Prove by contradiction that a shortest path from the source vertex $s \in V$ to a k-closest vertex $x \in V$ consists only of vertices in V_k.

Let us assume that we have vertexes V, we are trying to get to vertex $x \in V$, and our starting vertex is $s \in V$. We also have a shortest path of weight w from $s \in V$ to $x \in V$. Becase it is a shortest path, we know that w is the smallest weight path possible from the two vertexes.

Now let us assume that there is another vertex, V_{k+1} that is part of this shortest path from s to x. This would mean that the weight of the path would then be $w + e$ where e is the edge weight to the additional vertex V_{k+1}. This would also mean that the path from s to x would be the original weight w plus the distance from x to V_{k+1}, which is greater than our original shortest path, which is a contradiction of the earlier assumption that the shortest path would include V_{k+1}. This also stands with the knowledge that each shortest path from a source vertex to any other vertex is also made up of shortest paths.

$\therefore$ A shortest path from the source vertex $s \in V$ to a k-closest vertex $x \in V$ consists only of vertices in V_k.

Problem 3: (P is Closed under Reverse Complement) (20 Points)

Consider the operation of reverse complement on a language over the alphabet $\{0,1\}$. The reverse complement of a string $x \in \{0,1\}^*$ of length n is a string $y \in \{0,1\}^*$ of the same length such that $y_k = 1 - x_{n-k+1}$ for $k = 1, 2, ..., n$, where y_k is the bit at index k of string y. Let $\overline{x}$ denote the reverse complement of string x. For example, if $x = 011101$, then $\overline{x} = 010001$. Note that $\overline{x}$ is obtained from x by reversing its bit sequence and complementing each bit (changing 1 to 0 and 0 to 1). Let L be a language over the alphabet $\{0,1\}$. The reverse complement of L is defined as

$$\overline{L} = \{\overline{x} : x \in L\}.$$

Show that if L is in P, then $\overline{L}$ is also in P. Note that the reverse complement of L is not related to the set complement of L. Your algorithm takes as input a binary string in $\{0,1\}^*$.

ReverseComplement

Input: A binary string x in $\{0,1\}^*$

Let y be a new string of the same size of x, size n

for $k = 1$ *to* n **do**

$y_k = x_{k-n+1}$

return y

For this algorithm, we take in an input string x and, through the given formula of $y_k = x_{k-n+1}$, convert is into a "Reverse Complement" y of the original input string x. It is iterating through the original input size of n exactly once, from 1 to k where k is set equal to n. Because of this, it is able to produce output in $O(n^k)$ time where $k = 1$, so this problem is $O(n^1)$.

$\therefore \overline{L}$ is in P.

Problem 4: (String Transformation is in NP) (20 Points)

A string $x \in \{0,1\}^*$ is transformed into another string y by a sequence of operations of two types. Let x be a string of n bits, indexed from 1 to n, where $x_{i,j}$ denotes a substring of x, consisting of consecutive bits from index i to index j in the same order, with $1 \le i \le j \le n$. Let xyz denote the concatenation of strings x, y, and z. An inversion operation $r(i,j)$ turns string x into $x_{1,i-1}\overline{x_{i,j}}x_{j+1,n}$, with $1 \le i \le j \le n$, where $\overline{x_{i,j}}$ is the reverse complement of $x_{i,j}$ (see problem 3). A deletion operation $d(i,j)$ turns string x into $x_{1,i-1}x_{j+1,n}$. For example, an inversion operation $r(3,6)$ transforms string 11011100 into string 11000100 ($x_{3,6} = 0111$, $\overline{x_{3,6}} = 0001$), which is further transformed into string 100100 by a deletion operation $d(2,3)$. Show that the problem of deciding whether a string can be transformed into another string by a sequence of at most k operations of substring deletions and inversions is in NP. Specifically, the problem is defined as the formal language

ST $= \{< x, y, k >$: there exists a sequence of at most k operations
of deletions and inversions to transform binary strings x into y $\}$.

Show that ST is in NP. Note that your algorithm takes two types of input: one type is an ordinary input including two strings x and y along with an integer k, and the other is a certificate.

In order for ST to be in NP, there needs to be an efficient polynomial time certifier that can effectivey run a specific instance with the correct certificate. We are given a string x of n bits, an "end result" y, an integer k, and a certificate c. The language is defined as

L = $\{x \in \{0,1\}^*, y \in \{0,1\}^*, k :$ there exists a certificate, $c \in \{0,1\}^*$, with $|c| = O(|x|^c)$ such that StringTransformation(x,y,k,c) = 1$\}$ In essence, c is the certificate that will be the most optimal choice to operate on x with either deletion or reverse complement polynomial algorithms to get one step closer to y. This can be done at most k times to decide if x can be transformed into y.

```
StringTransformation
    Input: A string x of n bits, an "end result" string y, integer k, and certificate c
    Not possible in the number of total runs we've done
    if number of runs > k then
        return false
    if x == y then
        return true
    if x is smaller than y then
        return false
    We know that we need to perform delete operations to get it closer
    if size of x > size of c then
        for i = 0 to size of x do
            for j = 0 to size of x do
                Assuming non-destructive r and d methods
                if result of d(i, j) == c then
                    return true
    This is in the instance where we are the same size, and now we need to
     perform reverse complement operations on input
    for i = 0 to size of x do
        for j = 0 to size of x do
            Assuming non-destructive r and d methods
            if result of r(i, j) == c then
                return true
```

The overall runtime of this algorithm can be justified as being 2 runs on the input size of n meaning a polynomial runtime of n^2 as well as mutiplying polynomial time operations of $r(i, j)$ and $d(i, j)$, meaning a total runtime of $n^2 * n^k$ or n^{2k} which is still within a polynomial runtime. Additionally, we know that the certificate input string will yield the next best decision of either a $d(i, j)$ or an $r(i, j)$. Because we know we have an algorithm with an efficient polynomial certifier, we can say that ST is in NP.

Problem 5: (Restricted CNF-SAT is in P) (20 Points)

A boolean formula in conjunctive normal form, or CNF, is expressed as an AND of clauses, each of which is the OR of one or more literals. A restricted CNF formula meets the following requirements. For every clause with two or more literals in the formula, if the clause contain literal x, then no clause can contain $\neg x$, and if it contain $\neg x$, then no clause can contain x. If any clause with only one literal contains x or $\neg x$, then this literal cannot occur in any clause with two or more literals. For example, the boolean formulas,

$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_4 \vee x_5) \wedge x_6 \wedge \neg x_6$, and
$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_4 \vee x_5) \wedge x_6 \wedge \neg x_7$,

meet the requirements, and the following formulas do not,

$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_4 \vee x_5) \wedge x_2$, and
$(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_5)$.

Show that the problem of deciding whether a restricted boolean formula in CNF is satisfiable is in P. Specifically, the problem is defined as the formal language

RES-CNF $= \{< \theta >: \theta$ is a satisfiable restricted boolean formula in CNF$\}$.

Show that RES-CNF is in P. Note that input to your algorithm is a restricted boolean formula in CNF.

```
RES-CNF
    Input: A restricted boolean formula f in CNF
    Let SingleLiterals be a list to hold literals by themselves, meaning a single literal in a clause
    for i = 0 to input string f length do
        If i is a literal in the form x_n and is single, as explained above
        if i is complete clause and SingleLiteral(i) then
            Add i to SingleLiterals
    for j = 0 to length of SingleLiterals do
        for k = 0 to length of SingleLiterals do
            If we have a single literal in a clause with a ¬ counterpart, i.e. x_6 and ¬x_6
            if Input[i] == ¬Input[j] then
                return false
    return true
```

This algorithm will determine if an expression is satisfiable in the RES-CNF form given. We know that an expression will always be able to evalutate to TRUE with varied inputs for the values of x_1 to x_n except in the case where there are two standalone literals that are the negation of the other, such as in the case of $x_6 \land \neg x_6$ standalone clauses. This algorithm will go through the input CNF and find the clauses which are a standalone literal, which can be found in $O(n)$ time, then it iterates through the list of these standalone clauses in $O(n^2)$ time, leading to an overall runtime of $n^2 + n$ or a big O of $O(n^2)$ which is $O(n^k)$ where k is 2. Because of this, we can say that deciding whether a RES-CNF is satisfiable is in P.