

HCM_(Hard c-means)

optimization problem

minimize $\sum_{i=1}^C \sum_{k=1}^N u_{i,k} \|x_k - v_i\|_2^2$ subject to $\sum_{i=1}^C u_{i,k} = 1$ and $u_{i,k} \in \{0, 1\}$

$$u_{i,k} = \begin{cases} 1 & (i = \arg \min_{1 \leq j \leq C} \{\|x_k - v_j\|_2^2\}) \\ 0 & (\text{otherwise}) \end{cases},$$

$$v_i = \frac{\sum_{k=1}^N u_{i,k} x_k}{\sum_{k=1}^N u_{i,k}}.$$

BFCM_(Bezdek-type fuzzy c-means with variables controlling cluster sizes)

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m \|x_k - v_i\|_2^2$$

$$d_{i,k} = \|x_k - v_i\|_2^2 = \left(\sqrt{\sum_{\ell=1}^M (x_{k,\ell} - v_{i,\ell})^2} \right)^2,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad v_i = \frac{\sum_{k=1}^N (u_{i,k})^m x_k}{\sum_{k=1}^N (u_{i,k})^m},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m d_{j,k}}{(u_{i,k})^m d_{i,k}} \right)^{\frac{1.0}{m}}}.$$

KLFCM_(KL-divergence based Fuzzy c-means with variables controlling cluster sizes)

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \|x_k - v_i\|_2^2 + \lambda^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log\left(\frac{u_{i,k}}{\pi_i}\right)$$

$$d_{i,k} = \|x_k - v_i\|_2^2,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad v_i = \frac{\sum_{k=1}^N u_{i,k} x_k}{\sum_{k=1}^N u_{i,k}},$$

$$\pi_i = \frac{\sum_{k=1}^N u_{i,k}}{N}.$$

QFCM_(q-divergence based Fuzzy c-means with variables controlling cluster sizes)

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m \|x_k - v_i\|_2^2 + \frac{\lambda^{-1}}{m-1} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m$$

$$d_{i,k} = \|x_k - v_i\|_2^2,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{1.0 - \lambda(1.0-m)d_{j,k}}{1.0 - \lambda(1.0-m)d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad v_i = \frac{\sum_{k=1}^N (u_{i,k})^m x_k}{\sum_{k=1}^N (u_{i,k})^m},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m (1.0 - \lambda(1.0-m)d_{j,k})}{(u_{i,k})^m (1.0 - \lambda(1.0-m)d_{i,k})} \right)^{\frac{1.0}{m}}}.$$

ref : https://www.jstage.jst.go.jp/article/fss/30/0/30_452/_pdf

FCCM

optimization problem

$$\begin{aligned} \underset{U, W, \pi}{\text{minimize}} \quad & \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M -u_{i,k} w_{i,\ell} x_{k,\ell} \\ & + \lambda_1^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log\left(\frac{u_{i,k}}{\pi_i}\right) + \lambda_2^{-1} \sum_{i=1}^C \sum_{\ell=1}^M w_{i,\ell} \log(w_{i,\ell}) \end{aligned}$$

$$d_{i,k} = - \sum_{\ell=1}^M w_{i,\ell} x_{k,\ell}, \quad (1)$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad (2)$$

$$w_{i,\ell} = \frac{1.0}{\sum_{r=1}^M \exp(\lambda_2 \sum_{k=1}^N (u_{i,k} x_{k,r} - u_{i,k} x_{k,\ell}))}, \quad \pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}. \quad (3)$$

ref : <http://ieeexplore.ieee.org/document/944403/>

FCCMb

optimization problem

$$\underset{u, w, \pi}{\text{minimize}} - \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M (\pi_i)^{1-m_1} (u_{i,k})^{m_1} (w_{i,\ell})^{m_2} x_{k,\ell}$$

$$d_{i,k} = - \sum_{\ell=1}^M (w_{i,\ell})^{m_2} x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m_1}}}, \quad w_{i,\ell} = \frac{1.0}{\sum_{r=1}^M \left(\sum_{k=1}^N \frac{(u_{i,k})^{m_1} x_{k,r}}{(u_{i,k})^{m_1} x_{k,\ell}} \right)^{\frac{1.0}{1.0-m_2}}},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^{m_1} d_{j,k}}{(u_{i,k})^{m_1} d_{i,k}} \right)^{\frac{1.0}{m_1}}}.$$

ref : <https://www.fujipress.jp/jaciii/jc/jaciii001900060852/>

BFCCM

optimization problem

$$\underset{u, v, \pi}{\text{minimize}} - \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M (\pi_i)^{1-m} (u_{i,k})^m \log(w_{i,\ell}) x_{k,\ell}$$

$$d_{i,k} = - \sum_{\ell=1}^M \log(w_{i,\ell}) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m_1}}}, \quad w_{i,\ell} = \frac{\sum_{k=1}^N (u_{i,k})^m x_{k,\ell}}{\sum_{r=1}^M \sum_{k=1}^N (u_{i,k})^m x_{k,r}},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^{m_1} d_{j,k}}{(u_{i,k})^{m_1} d_{i,k}} \right)^{\frac{1.0}{m_1}}}.$$

KLFCCM

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} - \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} \log(w_{i,\ell}) x_{k,\ell} + \lambda^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log\left(\frac{u_{i,k}}{\pi_i}\right)$$

$$d_{i,k} = - \sum_{\ell=1}^M \log(w_{i,\ell}) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad w_{i,\ell} = \frac{\sum_{k=1}^N u_{i,k} x_{k,\ell}}{\sum_{r=1}^M \sum_{k=1}^N u_{i,k} x_{k,r}},$$

$$\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}.$$

ref : <http://ieeexplore.ieee.org/abstract/document/6891747/>

QFCCM

optimization problem

$$\begin{aligned} \underset{u,v,\pi}{\text{minimize}} \quad & - \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M (\pi_i)^{1-m} (u_{i,k})^m \log(w_{i,\ell}) x_{k,\ell} \\ & + \frac{\lambda^{-1}}{m-1} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m \end{aligned}$$

$$d_{i,k} = - \sum_{\ell=1}^M \log(w_{i,\ell}) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{1.0 - \lambda(1.0-m)d_{j,k}}{1.0 - \lambda(1.0-m)d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad w_{i,l} = \frac{\sum_{k=1}^N (u_{i,k})^m x_{k,\ell}}{\sum_{r=1}^M \sum_{k=1}^N (u_{i,k})^m x_{k,r}},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m (1.0 - \lambda(1.0-m)d_{j,k})}{(u_{i,k})^m (1.0 - \lambda(1.0-m)d_{i,k})} \right)^{\frac{1.0}{m}}}.$$

ref : <http://ieeexplore.ieee.org/document/7337853/>

BFCCMM

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} -\frac{1}{t} \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M (u_{i,k})^m \left((w_{i,\ell})^t - 1 \right) x_{k,\ell}$$

$$d_{i,k} = -\frac{1.0}{t} \sum_{\ell=1}^M \left((w_{i,\ell})^t - 1.0 \right) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m_1}}}, \quad w_{i,\ell} = \frac{1.0}{\sum_{r=1}^M \left(\sum_{k=1}^N \frac{(u_{i,k})^m x_{k,r}}{(u_{i,k})^m x_{k,\ell}} \right)^{\frac{1.0}{1.0-t}}},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^{m_1} d_{j,k}}{(u_{i,k})^{m_1} d_{i,k}} \right)^{\frac{1.0}{m_1}}}.$$

KLFCCMM

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} -\frac{1}{t} \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} ((w_{i,\ell})^t - 1) x_{k,\ell} + \lambda^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log \left(\frac{u_{i,k}}{\pi_i} \right)$$

$$d_{i,k} = -\frac{1.0}{t} \sum_{\ell=1}^M ((w_{i,\ell})^t - 1.0) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad w_{i,\ell} = \frac{1.0}{\sum_{r=1}^M \left(\sum_{k=1}^N \frac{u_{i,k} x_{k,r}}{u_{i,k} x_{k,\ell}} \right)^{\frac{1.0}{1.0-t}}},$$

$$\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}.$$

ref : <http://ieeexplore.ieee.org/document/6982814/>

QFCCMM

optimization problem

$$\begin{aligned} \underset{u,v,\pi}{\text{minimize}} \quad & -\frac{1}{t} \sum_{i=1}^C \sum_{k=1}^N \sum_{\ell=1}^M (\pi_i)^{1-m} (u_{i,k})^m ((w_{i,\ell})^t - 1) x_{k,\ell} \\ & + \frac{\lambda^{-1}}{m-1} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m \end{aligned}$$

$$d_{i,k} = -\frac{1.0}{t} \sum_{\ell=1}^M ((w_{i,\ell})^t - 1.0) x_{k,\ell},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{1.0 - \lambda(1.0-m)d_{j,k}}{1.0 - \lambda(1.0-m)d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad w_{i,\ell} = \frac{1.0}{\sum_{r=1}^M \left(\sum_{k=1}^N \frac{(u_{i,k})^m x_{k,r}}{(u_{i,k})^m x_{k,\ell}} \right)^{\frac{1.0}{1.0-t}}},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m (1.0 - \lambda(1.0-m)d_{j,k})}{(u_{i,k})^m (1.0 - \lambda(1.0-m)d_{i,k})} \right)^{\frac{1.0}{m}}}.$$

ref : <http://ieeexplore.ieee.org/document/7337853/> (式注意)

BFCS

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m (1 - x_k^\top v_i)$$

$$d_{i,k} = 1.0 - x_k^\top v_i,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad v_i = \frac{\sum_{k=1}^N (u_{i,k})^m x_k}{\| \sum_{k=1}^N (u_{i,k})^m x_k \|_2},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m d_{j,k}}{(u_{i,k})^m d_{i,k}} \right)^{\frac{1.0}{m}}}.$$

ref : <http://ieeexplore.ieee.org/document/6891670/>

KLFCFS

optimization problem

$$\underset{u, v, \pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} (1 - x_k^\top v_i) + \lambda^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log\left(\frac{u_{i,k}}{\pi_i}\right)$$

$$d_{i,k} = 1.0 - x_k^\top v_i,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad v_i = \frac{\sum_{k=1}^N u_{i,k} x_k}{\|\sum_{k=1}^N u_{i,k} x_k\|_2},$$

$$\pi_i = \frac{\sum_{k=1}^N u_{i,k}}{N}.$$

QFCS

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m (1 - x_k^\top v_i) + \frac{\lambda^{-1}}{m-1} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m$$

$$d_{i,k} = 1.0 - x_k^\top v_i,$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{1.0 - \lambda(1.0-m)d_{j,k}}{1.0 - \lambda(1.0-m)d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad v_i = \frac{\sum_{k=1}^N (u_{i,k})^m x_k}{\| \sum_{k=1}^N (u_{i,k})^m x_k \|_2},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m (1.0 - \lambda(1.0-m)d_{j,k})}{(u_{j,k})^m (1.0 - \lambda(1.0-m)d_{i,k})} \right)^{\frac{1.0}{m}}}.$$

BPCS

optimization problem

$$\underset{u,v}{\text{minimize}} \sum_{k=1}^N (u_{1,k})^m (1 - x_k^\top v_1) - \alpha \sum_{k=1}^N (1 - u_{1,k})^m$$

$$d_{1,k} = 1.0 - x_k^\top v_1,$$

$$u_{1,k} = \frac{1.0}{1.0 + \left(\frac{m d_{1,k}}{\alpha}\right)^{\frac{1.0}{m-1.0}}}, \quad v_1 = \frac{\sum_{k=1}^N (u_{1,k})^m x_k}{\|\sum_{k=1}^N (u_{1,k})^m x_k\|_2},$$

ref : <https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=531779>

EPCS

optimization problem

$$\underset{u,v}{\text{minimize}} \sum_{k=1}^N u_{1,k} (1 - x_k^\top v_1) + \lambda^{-1} \sum_{k=1}^N u_{1,k} \log(u_{1,k}) - \alpha \sum_{k=1}^N u_{1,k}$$

$$d_{1,k} = 1.0 - x_k^\top v_1,$$

$$u_{1,k} = \exp(\lambda(\alpha - d_{1,k}) - 1.0), \quad v_1 = \frac{\sum_{k=1}^N u_{1,k} x_k}{\|\sum_{k=1}^N u_{1,k} x_k\|_2},$$

refhttps://link.springer.com/chapter/10.1007/978-3-319-23240-9_10

QPCS

optimization problem

$$\underset{u,v}{\text{minimize}} \sum_{k=1}^N (u_{1,k})^m (1 - x_k^\top v_1) + \frac{\lambda^{-1}}{m-1} \sum_{k=1}^N ((u_{1,k})^m - u_{1,k}) - \alpha \sum_{k=1}^N u_{1,k}$$

$$d_{1,k} = 1.0 - x_k^\top v_1,$$

$$u_{1,k} = \left(\frac{1.0 - \lambda(1.0 - m)d_{1,k}}{1.0 + \alpha\lambda(1.0 - m^{-1})} \right)^{\frac{1.0}{1.0-m}}, \quad v_1 = \frac{\sum_{k=1}^N (u_{1,k})^m x_k}{\|\sum_{k=1}^N (u_{1,k})^m x_k\|_2},$$

KLFCCMP

optimization problem

minimize $J_{KLFCCMP}$
 u, α, π

$$J_{KLFCCMP} = - \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \left(\log \Gamma(s_i) - \log \Gamma(s_i + \sum_{\ell=1}^M x_{k,\ell}) + \sum_{\ell: (x_{k,\ell}) \geq 1} \log(\alpha_{i,\ell}) \right) \\ + \lambda^{-1} \sum_{i=1}^C \sum_{k=1}^N u_{i,k} \log\left(\frac{u_{i,k}}{\pi_i}\right), \quad s_i = \sum_{\ell=1}^M \alpha_{i,\ell}$$

$$d_{i,k} = -\log \Gamma(s_i) + \log \Gamma(s_i + \sum_{\ell=1}^M x_{k,\ell}) - \sum_{\ell: (x_{k,\ell}) \geq 1} \log(\alpha_{i,\ell})$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \exp(-\lambda(d_{j,k} - d_{i,k}))}, \quad \pi_i = \frac{\sum_{k=1}^N u_{i,k}}{N}$$

$$\alpha_{i,\ell} = \frac{\sum_{k=1}^N u_{i,k}}{\sum_{k=1}^N u_{i,k} \psi(s_i + x_{k,\ell}) - \sum_{k=1}^N u_{i,k} \psi(s_i)}.$$

ref : https://link.springer.com/chapter/10.1007/978-3-319-67422-3_9

OCS-BFCS

optimization problem

$$\underset{u,v,\pi}{\text{minimize}} \sum_{i=1}^C \sum_{k=1}^N (\pi_i)^{1-m} (u_{i,k})^m (1 - x_k^\top v_i)$$

$$d_{i,k} = 1.0 - x_k^\top v_i,$$

$$x_{k,\ell} = \frac{\left(\sum_{i=1}^C (\pi_i)^{1.0-m} (u_{i,k})^m v_{i,\ell} \right) \sqrt{1.0 - \sum_{r=1}^M (x_{k,\ell})^{2.0} (1.0 - y_{k,r})}}{\sqrt{\sum_{r=1}^M \left(\sum_{i=1}^C (\pi_i)^{1.0-m} (u_{i,k})^m v_{i,r} (1.0 - y_{k,r}) \right)^{2.0}}},$$

$$u_{i,k} = \frac{1.0}{\sum_{j=1}^C \frac{\pi_j}{\pi_i} \left(\frac{d_{j,k}}{d_{i,k}} \right)^{\frac{1.0}{1.0-m}}}, \quad v_i = \frac{\sum_{k=1}^N (u_{i,k})^m x_k}{\| \sum_{k=1}^N (u_{i,k})^m x_k \|_2},$$

$$\pi_i = \frac{1.0}{\sum_{j=1}^C \left(\sum_{k=1}^N \frac{(u_{j,k})^m d_{j,k}}{(u_{i,k})^m d_{i,k}} \right)^{\frac{1.0}{m}}}.$$

ref : <https://ieeexplore.ieee.org/document/956035/>

TFIDF

TF-IDF

- $tf_{k,\ell}$: k 番目の文書の ℓ 番目の単語の頻出回数
- N : 全文書数
- Df_ℓ : ℓ 番目の単語が出現する文書数

$$TF_{k,\ell} = \begin{cases} 1 + \log(tf_{k,\ell}) & tf_{k,\ell} > 0 \text{ のとき} \\ 0 & \text{それ以外} \text{ のとき} \end{cases}$$

$$IDF_\ell = \log\left(\frac{N}{Df_\ell}\right)$$

$$TF.IDF_{k,\ell} = TF_{k,\ell} \times IDF_\ell$$

LBM(Bernoulli) (Latent Block Model)

optimization problem

$$\begin{aligned}
 & \underset{u, w, \pi, \rho, \alpha}{\text{maximize}} \sum_{i=1}^{C^{row}} \sum_{j=1}^{C^{col}} \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell} \log(\text{prob}(x_{k,\ell}; \alpha_{i,j})) \\
 & \quad + \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log\left(\frac{\pi_i}{u_{i,k}}\right) + \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log\left(\frac{\rho_j}{w_{j,\ell}}\right) \\
 & \quad \alpha_{i,j} \in \mathbb{R}^{C^{row} \times C^{col}}, \text{ subject to } \sum_{j=1}^{C^{col}} \rho_j = 1 \text{ and } \sum_{j=1}^{C^{col}} w_{j,\ell} = 1
 \end{aligned}$$

$$\text{Bernoulli : } \text{prob}(x_{k,\ell}; \alpha_{i,j}) = (\alpha_{i,j})^{x_{k,\ell}} (1 - (\alpha_{i,j}))^{(1-x_{k,\ell})}$$

$$u_{i,k} = \frac{\pi_i \exp\left(\sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell} \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1 - \alpha_{i,j})\right)}{\sum_{i'=1}^{C^{row}} \pi_{i'} \exp\left(\sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell} \log\left(\frac{\alpha_{i',j}}{1-\alpha_{i',j}}\right) + \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1 - \alpha_{i',j})\right)},$$

$$w_{j,\ell} = \frac{\rho_j \exp\left(\sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} x_{k,\ell} \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1 - \alpha_{i,j})\right)}{\sum_{j'=1}^{C^{col}} \rho_{j'} \exp\left(\sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} x_{k,\ell} \log\left(\frac{\alpha_{i,j'}}{1-\alpha_{i,j'}}\right) + \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1 - \alpha_{i,j'})\right)},$$

$$\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}, \quad \rho = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}, \quad \alpha_{i,j} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{\sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$$

Algorithm 1 LBM(Bernoulli)

- 1: initialize $u, w, \pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}$, $\rho_j = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}$
 - 2: **repeat**
 - 3: $s_{i,\ell}^u \in \mathbb{R}^{C^{row} \times M} = \sum_{k=1}^N u_{i,k} x_{k,\ell}$
 - 4: **repeat**
 - 5: $w_{j,\ell} = \frac{\rho_j \exp\left(\sum_{i=1}^{C^{row}} s_{i,\ell}^u \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1-\alpha_{i,j})\right)}{\sum_{j'=1}^{C^{col}} \rho_{j'} \exp\left(\sum_{i=1}^{C^{row}} s_{i,\ell}^u \log\left(\frac{\alpha_{i,j'}}{1-\alpha_{i,j'}}\right) + \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1-\alpha_{i,j'})\right)}$
 - 6: $\rho_j = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}$, $\alpha_{i,j} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{\sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$
 - 7: **until** convergence $\{w, \rho, \alpha\}$
 - 8: $s_{j,k}^w \in \mathbb{R}^{C^{col} \times N} = \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell}$
 - 9: **repeat**
 - 10: $u_{i,k} = \frac{\pi_i \exp\left(\sum_{j=1}^{C^{col}} s_{j,k}^w \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1-\alpha_{i,j})\right)}{\sum_{i'=1}^{C^{row}} \pi_{i'} \exp\left(\sum_{j=1}^{C^{col}} s_{j,k}^w \log\left(\frac{\alpha_{i',j}}{1-\alpha_{i',j}}\right) + \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1-\alpha_{i',j})\right)}$
 - 11: $\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}$, $\alpha_{i,j} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{\sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$
 - 12: **until** convergence $\{u, \pi, \alpha\}$
 - 13: **until** convergence $\{s^u, s^w\}$
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FuzzyLBM(Binomial) (Latent Block Model)

optimization problem

$$\begin{aligned} \underset{u, w, \pi, \rho, \alpha}{\text{maximize}} \quad & \sum_{i=1}^{C^{row}} \sum_{j=1}^{C^{col}} \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell} \log(\text{prob}(x_{k,\ell}; \alpha_{ij})) \\ & + \lambda_1^{-1} \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log\left(\frac{\pi_i}{u_{i,k}}\right) + \lambda_2^{-1} \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log\left(\frac{\rho_j}{w_{j,\ell}}\right) \end{aligned}$$

$$\text{Binomial : } \text{prob}(x_{k,\ell}; \alpha_{ij}) = \binom{n}{x_{k,\ell}} (\alpha_{ij})^{x_{k,\ell}} (1 - (\alpha_{ij}))^{(n-x_{k,\ell})}$$

$$u_{i,k} = \frac{\pi_i \exp\left(\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell} \log\left(\frac{\alpha_{ij}}{1-\alpha_{ij}}\right) + n\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1 - \alpha_{ij})\right)}{\sum_{i'=1}^{C^{row}} \pi_{i'} \exp\left(\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell} \log\left(\frac{\alpha_{i'j}}{1-\alpha_{i'j}}\right) + n\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1 - \alpha_{i'j})\right)},$$

$$w_{j,\ell} = \frac{\rho_j \exp\left(\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} x_{k,\ell} \log\left(\frac{\alpha_{ij}}{1-\alpha_{ij}}\right) + n\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1 - \alpha_{ij})\right)}{\sum_{j'=1}^{C^{col}} \rho_{j'} \exp\left(\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} x_{k,\ell} \log\left(\frac{\alpha_{ij'}}{1-\alpha_{ij'}}\right) + n\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1 - \alpha_{ij'})\right)},$$

$$\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}, \quad \rho = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}, \quad \alpha_{ij} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{n \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$$

Algorithm 2 FuzzyLBM(Binomial)

- 1: initialize $u, w, \pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}$, $\rho_j = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}$
 - 2: **repeat**
 - 3: $s_{i,\ell}^u \in \mathbb{R}^{C^{row} \times M} = \sum_{k=1}^N u_{i,k} x_{k,\ell}$
 - 4: **repeat**
 - 5: $w_{j,\ell} = \frac{\rho_j \exp\left(\lambda_2 \sum_{i=1}^{C^{row}} s_{i,\ell}^u \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + n\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1-\alpha_{i,j})\right)}{\sum_{j'=1}^{C^{col}} \rho_{j'} \exp\left(\lambda_2 \sum_{i=1}^{C^{row}} s_{i,\ell}^u \log\left(\frac{\alpha_{i,j'}}{1-\alpha_{i,j'}}\right) + n\lambda_2 \sum_{i=1}^{C^{row}} \sum_{k=1}^N u_{i,k} \log(1-\alpha_{i,j'})\right)}$
 - 6: $\rho_j = \frac{1}{M} \sum_{\ell=1}^M w_{j,\ell}$, $\alpha_{i,j} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{n \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$
 - 7: **until** convergence $\{w, \rho, \alpha\}$
 - 8: $s_{j,k}^w \in \mathbb{R}^{C^{col} \times N} = \sum_{\ell=1}^M w_{j,\ell} x_{k,\ell}$
 - 9: **repeat**
 - 10: $u_{i,k} = \frac{\pi_i \exp\left(\lambda_1 \sum_{j=1}^{C^{col}} s_{j,k}^w \log\left(\frac{\alpha_{i,j}}{1-\alpha_{i,j}}\right) + n\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1-\alpha_{i,j})\right)}{\sum_{j'=1}^{C^{col}} \pi_{j'} \exp\left(\lambda_1 \sum_{j=1}^{C^{col}} s_{j,k}^w \log\left(\frac{\alpha_{i,j'}}{1-\alpha_{i,j'}}\right) + n\lambda_1 \sum_{j=1}^{C^{col}} \sum_{\ell=1}^M w_{j,\ell} \log(1-\alpha_{i,j'})\right)}$
 - 11: $\pi_i = \frac{1}{N} \sum_{k=1}^N u_{i,k}$, $\alpha_{i,j} = \frac{\sum_{i=1}^N \sum_{j=1}^M u_{i,k} w_{j,\ell} x_{k,\ell}}{n \sum_{k=1}^N \sum_{\ell=1}^M u_{i,k} w_{j,\ell}}$
 - 12: **until** convergence $\{u, \pi, \alpha\}$
 - 13: **until** convergence $\{s^u, s^w\}$
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