ME455 Active Learning for Robotics

Author: Jingkun (Allen) Liu

Problem 1

Find: $a_z(t), b_v(t), M, m_1, m_2$

From the description, the $a_x(t)$, $b_u(t)$ and p_1 are:

$$\begin{aligned} a_x(t) &= D_1 l(x(t), u(t)) \\ b_u(t) &= D_2 l(x(t), u(t)) \\ p_1 &= Dm(x(T)) \end{aligned}$$

From equation (7), we can derive that

$$\begin{aligned} &\int_0^T \left[D_1 l(x(t), u(t)) \cdot z(t) + D_2 l(x(t), u(t)) \cdot v(t)\right] dt + Dm(x(T)) \cdot z(T) \\ +&\int_0^T \left[z(t)^\top Q_z z(t) + v(t)^\top R_v v(t)\right] dt \\ =&\int_0^T \left\{\left[a_x(t) + z(t)^\top Q_z\right] \cdot z(t) + \left[b_u(t) + v(t)^\top R_v\right] \cdot v(t)\right\} dt + p_1 \cdot z(T) \\ =&\int_0^T \left[D_1 l'(z(t), v(t)) \cdot z(t) + D_2 l'(z(t), v(t)) \cdot v(t)\right] dt + p_1 \cdot z(T) \\ =&\int_0^T \left[a_z(t) \cdot z(t) + b_v(t) \cdot v(t)\right] dt + p_1 \cdot z(T) \end{aligned}$$

Hence, the expression for $a_z(t)$ and $b_v(t)$ are

$$\begin{aligned} a_z(t) &= a_x(t) + z(t)^\top Q_z \\ &= D_1 l(x(t), u(t)) + z(t)^\top Q_z \\ b_v(t) &= b_u(t) + v(t)^\top R_v \\ &= D_2 l(x(t), u(t)) + v(t)^\top R_v \end{aligned}$$

Then, since we know that

$$\begin{aligned} 0 &= p(t)^\top B(t) + b_v(t)^\top \\ b_v(t) &= b_u(t) + v(t)^\top R_v \end{aligned}$$

We can find the expression for $v(t)$ by

$$\begin{aligned} 0 &= p(t)^\top B(t) + \left[b_u(t) + v(t)^\top R_v\right]^\top \\ &= p(t)^\top B(t) + b_u(t)^\top + R_v^\top v(t) \\ R_v^\top v(t) &= -p(t)^\top B(t) - b_u(t)^\top \\ v(t) &= -R_v^{-\top} p(t)^\top B(t) - R_v^{-\top} b_u(t)^\top \end{aligned}$$

Then since the ODE is expressed as:

$$\begin{aligned} \dot{p}(t) &= -A(t)^\top p(t) - a_z(t) \\ \dot{z}(t) &= A(t) z(t) + B(t) v(t) \end{aligned}$$

The ODE can be expressed in matrix form as:

$$\begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{p}(t) \end{bmatrix} &= \begin{bmatrix} A(t) & 0 \\ 0 & -A(t)^\top \end{bmatrix} \begin{bmatrix} z(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} -a_z(t) \\ B(t) v(t) \end{bmatrix} \\ &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \end{aligned}$$

Finally the M matrix and vector m_1, m_2 can be expressed as:

$$
\begin{aligned}
M &= \begin{bmatrix} A(t) & 0 \\ 0 & -A(t)^\top \end{bmatrix} \\
m_1 &= -a_z(t) \\
m_2 &= B(t)v(t) \\
&= B(t)\left[-R_v^{-\top} p(t)^\top B(t) - R_v^{-\top} b_u(t)^\top\right] \\
&= -B(t)R_v^{-\top}\left[p(t)^\top B(t) + b_u(t)^\top\right]
\end{aligned}
$$

After we solve the BVP to find $z(t)$ and $p(t)$ we can find $v(t)$ by

$$
\begin{aligned}
\dot{z}(t) &= \frac{d}{dt} z(t) \\
\dot{z}(t) &= A(t)z(t) + B(t)v(t) \\
v(t) &= B^{-1}(t)\left[\dot{z}(t) - A(t)z(t)\right]
\end{aligned}
$$