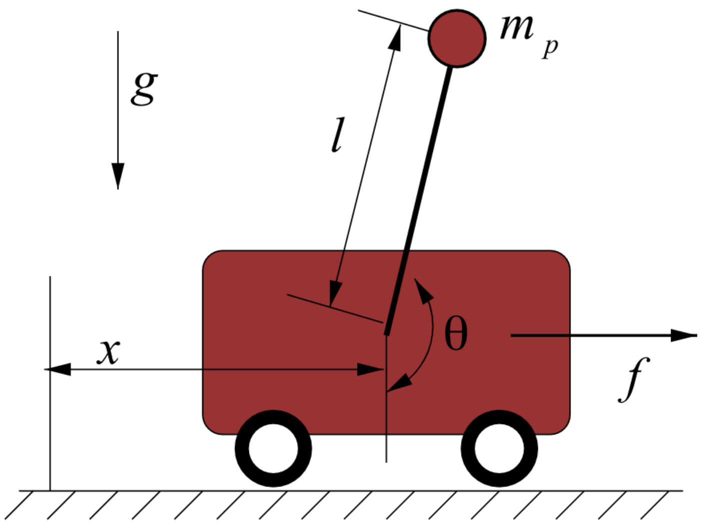
Cart-Pole



Consider the cart-pole system, consisting of a simple pendulum that can rotate about θ, affixed to a cart that can only move left and right along the x-axis. The goal is to balance the pendulum upright purely by moving the cart left and right.

Let:

$$\mathbf{q} = \begin{bmatrix} x \\ \theta \end{bmatrix}, \dot{\mathbf{q}} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} \text{ and } \ddot{\mathbf{q}} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}$$

be the generalised coordinates of the cart-pole, and corresponding velocities and accelerations, respectively.

We can use Lagrangian mechanics to determine the evolution of the system over time (with or without additional input torques). This requires us to find the kinetic- and potential-energy of the system (T and U, respectively), so that we can determine the torques from the Euler-Lagrange equation:

$$\boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}, \quad \text{where } \mathcal{L} = T - U$$

Using $\mathbf{x}_c$ and $\mathbf{x}_p$ to denote the locations of the mass concentrated at the centre of the cart, and at the end of the pole, respectively, then the kinematics of this system are as follows:

$$\mathbf{x}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \mathbf{x}_p = \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} x + l\sin(\theta) \\ -l\cos(\theta) \end{bmatrix}$$

$$\dot{\mathbf{x}}_c = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \dot{\mathbf{x}}_p = \begin{bmatrix} \dot{x} + l\dot{\theta}\cos(\theta) \\ l\dot{\theta}\sin(\theta) \end{bmatrix}$$

From this we can write the kinetic and potential energy:
Kinetic energy:

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m_c\dot{\mathbf{x}}_\mathbf{c}^\mathbf{T}\dot{\mathbf{x}}_\mathbf{c} + \frac{1}{2}m_p\dot{\mathbf{x}}_\mathbf{p}^\mathbf{T}\dot{\mathbf{x}}_\mathbf{p} \\
\frac{1}{2}m_c\dot{\mathbf{x}}_c^T\dot{\mathbf{x}}_c &= \frac{1}{2}m_c\begin{bmatrix}\dot{x} & 0\end{bmatrix}\begin{bmatrix}\dot{x} \\ 0\end{bmatrix} \\
&= \frac{1}{2}m_c\dot{x}^2 \\
\frac{1}{2}m_p\dot{\mathbf{x}}_p^T\dot{\mathbf{x}}_p &= \frac{1}{2}m_p\begin{bmatrix}\dot{x} + l\dot{\theta}\cos(\theta) & l\dot{\theta}\sin(\theta)\end{bmatrix}\begin{bmatrix}\dot{x} + l\dot{\theta}\cos(\theta) \\ l\dot{\theta}\sin(\theta)\end{bmatrix} \\
&= \frac{1}{2}m_p\left(\dot{x}^2 + l^2\dot{\theta}^2\cos^2(\theta) + 2\dot{x}l\dot{\theta}\cos(\theta) + l^2\dot{\theta}^2\sin^2(\theta)\right) \\
&= \frac{1}{2}m_p\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos(\theta) + l^2\dot{\theta}^2\right) && \because \sin^2(x) + \cos^2(x) \equiv 1
\end{aligned}$$

$$\therefore T = \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p\dot{x}l\dot{\theta}\cos(\theta) + \frac{1}{2}m_pl^2\dot{\theta}^2 \tag{1}$$

Potential energy (let g be a scalar, e.g. $g = 9.81$):

$$\begin{aligned}
U &= mgh \\
&= m_cgy_c + m_pgy_p \\
&= -m_pgl\cos(\theta)
\end{aligned} \tag{2}$$

From the Euler-Lagrange equation:

$$\begin{aligned}
\boldsymbol{\tau} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} - \frac{\partial\mathcal{L}}{\partial\mathbf{q}} \\
\mathcal{L} &= T - U \\
\implies \boldsymbol{\tau} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\mathbf{q}}} - \frac{\partial U}{\partial\dot{\mathbf{q}}}\right) - \left(\frac{\partial T}{\partial\mathbf{q}} - \frac{\partial U}{\partial\mathbf{q}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\mathbf{q}}} - \frac{\partial T}{\partial\mathbf{q}} + \frac{\partial U}{\partial\mathbf{q}} && \because \frac{\partial U}{\partial\dot{\mathbf{q}}} = 0
\end{aligned} \tag{3}$$

Calculating $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\mathbf{q}}}$, $\frac{\partial T}{\partial \mathbf{q}}$ and $\frac{\partial U}{\partial \mathbf{q}}$ for the Euler-Lagrange equation:
For reference:

$$
\begin{aligned}
T &= \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p \dot{x} l \dot{\theta}\cos(\theta) + \frac{1}{2} m_p l^2 \dot{\theta}^2 \\
U &= -m_p g l \cos(\theta)
\end{aligned}
$$

Partial derivatives for x:

$$\frac{\partial T}{\partial x} = 0 \tag{4}$$

$$\frac{\partial T}{\partial \dot{x}} = (m_c + m_p)\dot{x} + m_p l \dot{\theta}\cos(\theta) \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}} = (m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta) \tag{6}$$

$$\frac{\partial U}{\partial x} = 0 \tag{7}$$

$$\frac{\partial U}{\partial \dot{x}} = 0 \tag{8}$$

Inserting partial derivatives into Euler-Lagrange equation:

$$
\begin{aligned}
F_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} \\
&= \Big((m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta)\Big) - \Big(0\Big) + \Big(0\Big)
\end{aligned}
$$

$$\therefore F_1 = (m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta) \tag{9}$$

Notes on above:
$\frac{\partial T}{\partial x}$: there are no x terms in T.
$\frac{\partial T}{\partial \dot{\theta}}$: $\dot{x}^2 \to 2\dot{x}$, $\dot{x} \to 1$
$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_1}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_p l \dot{\theta}\cos(\theta)\right) = m_p l\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)\right)$$

$\because$

$$\begin{aligned}
\dot{\theta}\cos(\theta) &= f(t)g(t) \implies \text{product rule} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}\cos(\theta)\right) &= f(t)g'(t) + g(t)f'(t) \\
f'(t) &= \ddot{\theta} \\
g'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}\cos(\theta)\right) &= \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) + \cos(\theta)\ddot{\theta}
\end{aligned}$$

$$\begin{aligned}
\cos(\theta) &= f(g(t)) \implies \text{chain rule} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) &= f'(g(t)) \cdot g'(t) \\
f'(g(t)) &= -\sin(\theta(t)) \\
g'(t) &= \dot{\theta}(t) = \dot{\theta} \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) &= -\sin(\theta)\dot{\theta} \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}\cos(\theta)\right) &= -\dot{\theta}\sin(\theta)\dot{\theta} + \cos(\theta)\ddot{\theta} \\
&= \ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)
\end{aligned}$$

$\frac{\partial U}{\partial x}$: there are no x terms in U.
$\frac{\partial U}{\partial \dot{x}}$: there are no $\dot{x}$ terms in U.

For reference:

$$
\begin{aligned}
T &= \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p\dot{x}l\dot{\theta}\cos(\theta) + \frac{1}{2}m_p l^2\dot{\theta}^2 \\
U &= -m_p g l\cos(\theta)
\end{aligned}
$$

Partial derivatives for x:

$$
\begin{aligned}
\frac{\partial T}{\partial \theta} &= -m_p\dot{x}l\dot{\theta}\sin(\theta) && (10) \\
\frac{\partial T}{\partial \dot{\theta}} &= m_p\dot{x}l\cos(\theta) + m_p l^2\dot{\theta} && (11) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}} &= m_p\ddot{x}l\cos(\theta) - m_p\dot{x}l\dot{\theta}\sin(\theta) + m_p l^2\ddot{\theta} && (12) \\
\frac{\partial U}{\partial \theta} &= m_p g l\sin(\theta) && (13) \\
\frac{\partial U}{\partial \dot{\theta}} &= 0 && (14)
\end{aligned}
$$

Inserting partial derivatives into Euler-Lagrange equation:

$$
\begin{aligned}
\tau_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} \\
&= \Big(m_p\ddot{x}l\cos(\theta) - m_p\dot{x}l\dot{\theta}\sin(\theta) + m_p l^2\ddot{\theta}\Big) - \Big(-m_p\dot{x}l\dot{\theta}\sin(\theta)\Big) + \\
&\quad \Big(m_p g l\sin(\theta)\Big)
\end{aligned}
$$

$$
\therefore \tau_2 = 0 = m_p\ddot{x}l\cos(\theta) + m_p l^2\ddot{\theta} + m_p g l\sin(\theta) \qquad (15)
$$

Notes on above:
$\frac{\partial T}{\partial \theta}$: $\cos(\theta) \to -\sin(\theta)$
$\frac{\partial T}{\partial \dot{\theta}}$: $\dot{\theta} \to 1$, $\dot{\theta}^2 \to 2\dot{\theta}$
$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_p \dot{x} l \cos(\theta)\right) = m_p l \left(\ddot{x}\cos(\theta) - \dot{x}\dot{\theta}\sin(\theta)\right)$$

$\because$

$$\begin{aligned}
\dot{x}\cos(\theta) &= f(t)g(t) \implies \text{product rule} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}\cos(\theta)\right) &= f(t)g'(t) + g(t)f'(t) \\
f'(t) &= \ddot{x} \\
g'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}\cos(\theta)\right) &= \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) + \cos(\theta)\ddot{x}
\end{aligned}$$

$$\begin{aligned}
\cos(\theta) &= f(g(t)) \implies \text{chain rule} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) &= f'(g(t)) \cdot g'(t) \\
f'(g(t)) &= -\sin(\theta(t)) \\
g'(t) &= \dot{\theta}(t) = \dot{\theta} \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\theta)\right) &= -\sin(\theta)\dot{\theta} \\
\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}\cos(\theta)\right) &= -\dot{x}\sin(\theta)\dot{\theta} + \cos(\theta)\ddot{x} \\
&= \ddot{x}\cos(\theta) - \dot{x}\dot{\theta}\sin(\theta)
\end{aligned}$$

$\frac{\partial U}{\partial \theta}$: $\cos(x) \to -\sin(x)$
$\frac{\partial U}{\partial \dot{\theta}}$: there are no $\dot{\theta}$ terms in U.

Final equations for force:

$$\begin{aligned} F_1 &= (m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta) \\ 0 &= m_p \ddot{x} l \cos(\theta) + m_p l^2 \ddot{\theta} + m_p g l \sin(\theta) \end{aligned}$$

By grouping terms in $\ddot{\mathbf{q}}$ and $\dot{\mathbf{q}}$ we can obtain the manipulator equation form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}_g(\mathbf{q}) + \mathbf{B}\mathbf{u}$$

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_c + m_p & m_p l \cos(\theta) \\ m_p l \cos(\theta) & m_p l^2 \end{bmatrix}, \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_p l \dot{\theta} \sin(\theta) \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\tau}_g(\mathbf{q}) = \begin{bmatrix} 0 \\ -m_p g l \sin(\theta) \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\boldsymbol{\tau}_g(\mathbf{q})$ only applies to the pole, as our cart cannot move in the y-axis. As we can only control the cart's motion left and right, $\mathbf{B}$ ensures that our control input only enters into the first row of the force vector, F_1.

We could solve for the accelerations with:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q}) \left(\boldsymbol{\tau}_g(\mathbf{q}) + \mathbf{B}\mathbf{u} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \right)$$

But in this case, we can also solve directly for the accelerations:

$$\begin{aligned} F_1 &= (m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta) \\ \ddot{x} &= \frac{1}{(m_c + m_p)} \left(F_1 - m_p l \ddot{\theta} \cos(\theta) + m_p l \dot{\theta}^2 \sin(\theta) \right) && (16) \\ 0 &= m_p \ddot{x} l \cos(\theta) + m_p l^2 \ddot{\theta} + m_p g l \sin(\theta) \\ \ddot{\theta} &= \frac{-1}{m_p l^2} \left(m_p \ddot{x} l \cos(\theta) + m_p g l \sin(\theta) \right) && (17) \end{aligned}$$

Our equations are going to start getting long, so let $\mathrm{s} = \sin(\theta)$ and $\mathrm{c} = \cos(\theta)$; likewise $\mathrm{s}^2 = \sin^2(\theta)$ and $\mathrm{c}^2 = \cos^2(\theta)$, etc.

Inserting equation 17 into 16:

$$\begin{aligned} \ddot{x} &= \frac{1}{(m_c + m_p)} \left(F_1 - m_p l \frac{-1}{m_p l^2} \left(m_p \ddot{x} l \,\mathrm{c} + m_p g l \,\mathrm{s} \right) \mathrm{c} + m_p l \dot{\theta}^2 \,\mathrm{s} \right) \\ &= \frac{1}{(m_c + m_p)} \left(F_1 + m_p \ddot{x} \,\mathrm{c}^2 + m_p g \,\mathrm{s}\,\mathrm{c} + m_p l \dot{\theta}^2 \,\mathrm{s} \right) \end{aligned}$$

$$\implies \ddot{x} \left(1 - \frac{m_p \,\mathrm{c}^2}{(m_c + m_p)} \right) = \frac{1}{(m_c + m_p)} \left(F_1 + m_p g \,\mathrm{s}\,\mathrm{c} + m_p l \dot{\theta}^2 \,\mathrm{s} \right)$$

$$\begin{aligned} \implies \ddot{x} &= \frac{1}{m_c + m_p \,\mathrm{s}^2} \left(F_1 + m_p g \,\mathrm{s}\,\mathrm{c} + m_p l \dot{\theta}^2 \,\mathrm{s} \right) \\ &= \frac{1}{m_c + m_p \,\mathrm{s}^2} \left(F_1 + m_p \,\mathrm{s}(g \,\mathrm{c} + l \dot{\theta}^2) \right) && (18) \end{aligned}$$

$$\begin{aligned} \left(1 - \frac{m_p \,\mathrm{c}^2}{(m_c + m_p)} \right) &= \left(\frac{(m_c + m_p) - m_p \,\mathrm{c}^2}{(m_c + m_p)} \right) \\ &= \left(\frac{m_c + m_p(1 - \mathrm{c}^2)}{(m_c + m_p)} \right) \\ &= \left(\frac{m_c + m_p \,\mathrm{s}^2}{(m_c + m_p)} \right) \end{aligned}$$

$$\begin{aligned} \because \quad & \sin^2(x) + \cos^2(x) = 1 \\ \implies & (1 - \cos^2(x)) = \sin^2(x) \\ \implies & \frac{\frac{1}{(m_c + m_p)}}{\left(\frac{m_c + m_p \,\mathrm{s}^2}{(m_c + m_p)} \right)} = \frac{1}{m_c + m_p \,\mathrm{s}^2} \end{aligned}$$

Inserting equation 18 into 17:

$$\begin{aligned}
\ddot{\theta} &= \frac{-1}{m_p l^2}\left(m_p \frac{1}{m_c + m_p \,\mathrm{s}^2}\left(F_1 + m_p \,\mathrm{s}(g\,\mathrm{c} + l\dot{\theta}^2)\right) l\,\mathrm{c} + m_p g l\,\mathrm{s}\right) \\
&= \frac{-\,\mathrm{c}}{l(m_c + m_p \,\mathrm{s}^2)}\left(F_1 + m_p \,\mathrm{s}(g\,\mathrm{c} + l\dot{\theta}^2)\right) - \frac{g}{l}\,\mathrm{s} \\
&= \frac{1}{l(m_c + m_p \,\mathrm{s}^2)}\left(-F_1\,\mathrm{c} - m_p g\,\mathrm{s}\,\mathrm{c}^2 - m_p l\dot{\theta}^2\,\mathrm{s}\,\mathrm{c}\right) - \frac{g}{l}\,\mathrm{s} \\
&= \frac{-\,\mathrm{c}}{l(m_c + m_p \,\mathrm{s}^2)}\left(-F_1\,\mathrm{c} - m_p g\,\mathrm{s}\,\mathrm{c}^2 - m_p l\dot{\theta}^2\,\mathrm{s}\,\mathrm{c} - l(m_c + m_p \,\mathrm{s}^2)\frac{g}{l}\,\mathrm{s}\right) \\
&= \frac{1}{l(m_c + m_p \,\mathrm{s}^2)}\left(-F_1\,\mathrm{c} - m_p l\dot{\theta}^2\,\mathrm{s}\,\mathrm{c} - (m_c + m_p) g\,\mathrm{s}\right) \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&- m_p g\,\mathrm{s}\,\mathrm{c}^2 - l(m_c + m_p\,\mathrm{s}^2)\frac{g}{l}\,\mathrm{s} \\
&= -\left(m_p\,\mathrm{c}^2 + (m_c + m_p\,\mathrm{s}^2)\right) g\,\mathrm{s} \\
&= -(m_c + m_p) g\,\mathrm{s} \\
&\because \quad \sin^2(x) + \cos^2(x) \equiv 1
\end{aligned}$$