Multiphase Systems and Technologies

Lecture notes

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Chapter 1

Introduction

A **homogeneous system** is a system in which chemical and physical properties are independent of the position (uniform).

a **heterogeneous system** is a system in which chemical and physical properties depend on the position. A particular set of heterogeneous systems are multiphase systems.

A **multiphase system** can be thought as a heterogeneous system coming from the union of homogeneous systems. Each homogeneous portion is called **phase**.
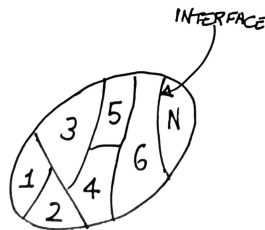

Figure 1.1: A multiphase system.

Phases are separated by boundaries, called **interfaces**. Interface are usually:

- deformable (moving);

- diathermal (diabatic);

- permeable to chemical species.

From the point of view of energy, an interface usually allows the exchange of work, heat and chemical energy.

1.1   Thermodynamic equilibrium

1.1.1   How many phases can coexist at equilibrium?

Let's consider a system with M phases and r chemical components (chemical species). Then let's write equilibrium equations:

- **Mechanical equilibrium**: for planar interfaces[1] pressure p is uniform

$$p^{(1)} = p^{(2)} = \cdots = p^{(M)}$$

- **Thermal equilibrium**: temperature T is uniform

$$T^{(1)} = T^{(2)} = \cdots = T^{(M)}$$

- **Chemical equilibrium**: we have $r \times M$ equations, chemical potential is the same throughout the heterogeneous system

$$\mu_1^{(1)} = \mu_1^{(2)} = \cdots = \mu_1^{(M)}$$
$$\mu_2^{(1)} = \mu_2^{(2)} = \cdots = \mu_2^{(M)}$$
$$\vdots$$
$$\mu_r^{(1)} = \mu_r^{(2)} = \cdots = \mu_r^{(M)}$$

This system of equations states the thermodynamic equilibrium and contains $M \times (r+2)$ variables and $(M-1) \times (r+2)$ equations.

Each one of the M phases is an homogeneous system by itself, hence each phase has to satisfy the Gibbs-Duhem Equation to fullfill the first and second principles of thermodynamics:

$$S dT - V dp + \sum_{i=1}^{r} N_i d\mu_i = 0$$

The real number of equations now becomes $(M-1) \times (r+2) + M$.

Given that the well-posedness of a system arises when # Variables $\geq$ # Equations,

$$\text{\# Equations} \leq \text{\# Variables}$$
$$(M-1)(r+2) + M \leq M(r+2)$$
$$M(r+2) - r - 2 + M \leq M(r+2)$$
$$M \leq r + 2$$

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[1]The number of equations is independent of the shape of the interface, but for the sake of simplicity we will consider flat boundaries. We will extend these considerations when dealing with bubbles or drops.

Examples

In the case $r = 1$ (single component system),

$$r = 1 \implies M \leq 3 \quad \begin{cases} M = 3 \to & \text{three phase system (triple state)} \\ M = 2 \to & \text{two phase system} \\ M = 1 \to & \text{single phase system} \end{cases}$$

In the case $r = 2$ (two component system), the number of possible phases is $M \leq 4$. An example could be moist air, given that we approximate dry air as if it was made by a single component. In a lab environment could be possible to observe a mixture of condensed air, condensed water and moist air.

1.1.2 How many quantities are required to describe equilibrium?

Let's now derive the number of indipendent variables (also known as *degrees of freedom*):

$$\begin{aligned} f &= \# \text{ Variables} - \# \text{ Equations} \\ &= M(r + 2) - (M - 1)(r + 2) - M \\ &= M(r + 2) - M(r + 2) + r + 2 - M \\ &= r - M + 2 \end{aligned}$$

Examples

For a single component system ($r = 1$) the degrees of freedom are $f = 3 - M$. Three cases:

- $M = 3 \to f = 0$ meaning that we cannot set any quantity among temperature, pressure and chemical potential;

- $M = 2 \to f = 1$ meaning we are only allowed to set one variable, typically the temperature or the pressure (*saturation condition*);

- $M = 1 \to f = 2$ meaning we are free to set temperature and pressure independently.

1.2   Basic quantities

We start by considering two-phase gas-liquid[2] or vapour-liquid flows[3]. To study fluid dynamics of a single phase system we can write EDPs that are able to give the distribution of all the quantities related to the flow field at any point inside the pipeline.

In the case of more than one phase flowing in a pipe, no analytical solutions are available, and moreover the numerical methods effective for single phase flows fail to properly simulate all the relevant quantity.

For design purposes, we need a simplified approach and hence we will adopt a *one-dimentional axial flow model* that lumps the information pertaining the distribution of the velocity of the phases using cross section averages. We are only interested in determining the behaviour of the relevant quantities along the pipe axis.

1.2.1   Total Volume Flow Rate

Volume flow rate is the rate at which volume flows in the pipe. Its unit is $\left[\frac{\mathrm{m}^3}{\mathrm{s}}\right]$ in SI units.

$$Q = Q_g + Q_l \tag{1.1}$$

where subscript g stands for *gas* whereas l stands for *liquid*.

1.2.2   Average Volume Quality

Based on the definition of total volume flow rate we can define a quantity related to phase fractions. If we divide each member of Eq. 1.1 by Q,

$$1 = \frac{Q_g}{Q} + \frac{Q_l}{Q}$$

We now define $\bar{x}_v$ (cross-section average) as follows:

$$\bar{x}_v = \frac{Q_g}{Q}$$

Average Volume Quality is adimensional.

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[2]Two components, $M \leq 4$.
[3]One component, $M \leq 3$.

1.2.3 Total Mass Flow Rate

If we call the phase densities ρ_g and ρ_l (in SI units $[\frac{\text{kg}}{\text{m}^3}]$) we can also define mass flow rates, Γ_g for the gas and Γ_l for the liquid:

$$\Gamma_g = \rho_g Q_g \qquad\qquad\qquad \Gamma_l = \rho_l Q_l$$

Since mass is additive, we can write the total mass flow rate (in SI units $[\frac{\text{kg}}{\text{s}}]$):

$$\Gamma = \Gamma_g + \Gamma_l \tag{1.2}$$

Recall that instead of phase densities we can use their reciprocal, *specific volumes*:

$$v_g = \frac{1}{\rho_g} \qquad\qquad\qquad v_l = \frac{1}{\rho_l}$$

1.2.4 Average Mass Quality

Dividing each member of Eq. 1.2 by Γ:

$$1 = \frac{\Gamma_g}{\Gamma} + \frac{\Gamma_l}{\Gamma}$$

We define $\bar{x}$, the average mass quality:

$$\bar{x} = \frac{\Gamma_g}{\Gamma}$$

Mass quality is adimensional.

1.2.5 Relationship between $\bar{x}$ and $\bar{x}_v$

Owing to the difference in densities (or specific volumes), the numerical values of mass quality $\bar{x}$ and volume quality $\bar{x}_v$ may differ greatly.

Starting from volume quality,

$$\bar{x}_v = \frac{Q_g}{Q} = \frac{Q_g}{Q_g + Q_l} = \frac{1}{1 + \frac{Q_l}{Q_g}} =$$

$$= \frac{1}{1 + \frac{\Gamma_l}{\rho_l}\frac{\rho_g}{\Gamma_g}} =$$

$$= \frac{1}{1 + \frac{\rho_g}{\rho_l}\frac{(1-\bar{x})\Gamma}{\bar{x}\Gamma}}$$

In a more elegant form:

$$\frac{1 - \bar{x}_v}{\bar{x}_v} = \frac{\rho_g}{\rho_l} \frac{1 - \bar{x}}{\bar{x}}$$

Generally $\rho_g \ll \rho_l$, hence $\bar{x}_v \gg \bar{x}$. For a pure liquid-vapour mixture $\rho_g - \rho_l$ lowers with T (or p) and at a critical point $\rho_g = \rho_l$.
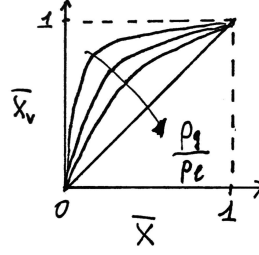

Figure 1.2: Volume Quality vs Mass Quality.

1.2.6   Bulk Density

The ratio between the mass flow rate and the volume flow rate has the dimensions of a density and is called bulk density. It is the density that would pertain to an equivalent single-phase fluid flowing with the same mass and volume flow rates of the two-phase mixture.

Referring to the cross-section average, the bulk density $\bar{\rho}_b$ is such that:

$$\bar{\rho}_b = \frac{\Gamma}{Q}$$

Let's specialize the numerator in the previous equation:

$$\bar{\rho}_b = \frac{\Gamma_g + \Gamma_l}{Q} = \frac{\rho_g Q_g + \rho_l Q_l}{Q} = \rho_g \bar{x}_v + \rho_l(1 - \bar{x}_v)$$

On the other hand, we can specialize the denominator:

$$\bar{\rho}_b = \frac{\Gamma}{Q_g + Q_l} = \frac{\Gamma}{v_g \Gamma_g + v_l \Gamma_l} = \frac{1}{v_g \bar{x} + v_l(1 - \bar{x})} = \frac{1}{\bar{v}_b}$$

where $\bar{v}_b$ is the *bulk specific volume*.

Bulk density and bulk specific volume are only apparent quantities of the flowing mixture and cannot be considered if we're interested in the real mass or volume content in the pipeline. This is because the two phases do not travel with the same velocities (phase slippage).

1.2.7 Elementary cross-section

Let's introduce the elementary cross-section element of the pipe (Fig. 1.3). Its area $d\Omega$ is the sum of the area occupied by the gas and the one occupied by the liquid:
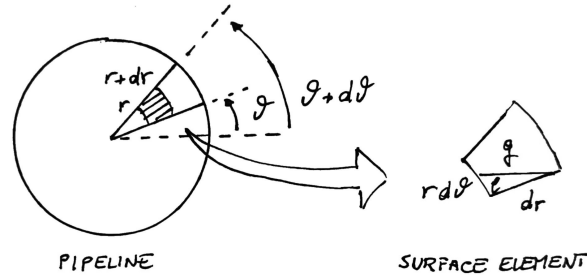
$$d\Omega = d\Omega_g + d\Omega_l \tag{1.3}$$

Figure 1.3: Elementary cross section inside a pipe.

1.2.8 Local Void Fraction

If we divide each member of Eq. 1.3 by Ω,

$$1 = \frac{d\Omega_g}{d\Omega} + \frac{d\Omega_l}{d\Omega}$$

We now define the local void fraction α as follows[4]:

$$\alpha = \frac{d\Omega_g}{d\Omega} \tag{1.4}$$

1.2.9 Local Actual Velocity

Given that dQ_g and dQ_l are the elementary volume flow rates, we can define the local actual velocity (in SI units $\left[\frac{m}{s}\right]$) of each phase by dividing the volume flow rate by the cross section area occupied by the phase:

$$u_g = \frac{dQ_g}{d\Omega_g} \qquad\qquad u_l = \frac{dQ_l}{d\Omega_l} \tag{1.5}$$

where u_g is the local actual gas velocity, u_l is the local actual liquid velocity.

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[4]Letter ϵ is entering widespread use in place of α.

1.2.10   Cross-section Average Void Fraction

We extract $d\Omega_g$ from Eq. 1.4:

$$\alpha = \frac{d\Omega_g}{d\Omega} \rightarrow d\Omega_g = \alpha d\Omega$$

Let's now integrate the previous result over the cross section:

$$\Omega_g = \int_\Omega \alpha d\Omega$$
$$= \bar{\alpha}\Omega$$

Accordingly, we define the cross-section average void fraction $\bar{\alpha}$ as follows:

$$\bar{\alpha} = \frac{\Omega_g}{\Omega} \tag{1.6}$$

Similarly, we define the *liquid hold-up* $1 - \bar{\alpha}$ as follows:

$$1 - \bar{\alpha} = \frac{\Omega_l}{\Omega}$$

1.2.11   Cross-section Average Actual Velocity

We extract dQ_g from Eq. 1.5:

$$u_g = \frac{dQ_g}{d\Omega_g} \rightarrow dQ_g = u_g d\Omega_g$$

Applying the theorem of the integral average,

$$Q_g = \int_\Omega u_g d\Omega_g$$
$$= \bar{u}_g \Omega_g$$

Accordingly, we define the gas average actual velocity $\bar{u}_g$ as follows:

$$\bar{u}_g = \frac{Q_g}{\Omega_g} \tag{1.7}$$

Similarly, we define the liquid average actual velocity $\bar{u}_l$ as follows:

$$\bar{u}_l = \frac{Q_l}{\Omega_l} \tag{1.8}$$

1.2.12 Superficial Velocity

If we have no information about the phase distribution (Ω_g and Ω_l), we cannot derive the actual velocities. Taking into account the definition of void fraction (Eq. 1.6), we can replace Ω_g with $\bar{\alpha}\Omega$ into Eq. 1.7:

$$\bar{u}_g = \frac{Q_g}{\bar{\alpha}\Omega}$$

The previous relationship puts in evidence an interesting quantity:

$$J_g = \frac{Q_g}{\Omega}$$

where Q_g can be set or measured and Ω is the total cross-section of the pipe. J_g is known as the gas superficial velocity.

Similarly, we can do the same for Eq. 1.7:

$$\bar{u}_l = \frac{Q_l}{(1 - \bar{\alpha})\Omega}$$

Then we introduce a new quantity, the liquid superficial velocity J_l:

$$J_l = \frac{Q_l}{\Omega}$$

Superficial velocity is the velocity that the phase would assume if it flowed alone in the pipe, disregarding the other one. It is an apparent quantity with no physical meaning.

Notice that the gas and liquid superficial and actual velocities are respectively related by the void fraction and the liquid hold-up:

$$\frac{J_g}{\bar{u}_g} = \frac{\cancel{Q_g}}{\Omega}\frac{\Omega_g}{\cancel{Q_g}} = \frac{\Omega_g}{\Omega} = \bar{\alpha}$$

1.2.13 Mixture Velocity

The mixture velocity[5] ($[\frac{m}{s}]$ in SI units) is defined as follows:

$$J = \frac{Q}{\Omega} = \frac{Q_g + Q_l}{\Omega}$$

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[5]Instead of mixture velocity, you could find literature that calls this quantity volume (or volumetric) flux. Recall that the term flux always refers to a flow rate per unit surface area. Volume flux is volume flow rate per unit cross-section area, hence it's a velocity ($[\frac{m^3}{s}\frac{1}{m^2}] = [\frac{m}{s}]$).

'Two-phase flows have been deeply studied for more than a century: you can find all sorts of confusing alternative names. If you try to make an agreement, you always find a collegue saying: "No! At our institution it is customary and traditional to use this name for this quantity!"' — Luigi Pietro Maria Colombo

Splitting Q into its two components Q_g and Q_l, we find that the mixture velocity is the sum of two apparent velocities:

$$J = \frac{Q_g + Q_l}{\Omega} = J_g + J_l$$

Consequently, volume fraction is expressed as the following ratio:

$$\bar{x}_v = \frac{J_g}{J}$$

1.2.14 Mass Flux

It is customary to also define mass fluxes[6] and, since we have actual and apparent velocities, we can obtain four different mass fluxes by multiplying each velocity by the corresponding phase density. SI unit of velocity is $\left[\frac{\mathrm{kg}}{\mathrm{m^2\,s}}\right]$.

Actual Mass Flux

$$G_g = \rho_g \bar{u}_g \qquad\qquad\qquad G_l = \rho_l \bar{u}_l$$

Apparent Mass Flux

$$G_g^* = \rho_g J_g \qquad\qquad\qquad G_l^* = \rho_l J_l$$

1.2.15 Photographic Mixture Density

Photographic mixture density is the density of the fluid that at a certain time occupies an elementary portion of a duct. It is then the ratio of the mass contained in the element and its volume.

Since there are often intermittend flow patterns, for instance plug flow, the photographic density is highly variable in time. It is then more useful to refer to a time averaged quantity.

Let's consider the whole cross-section, which is particularly suitable for one-dimensional models. The elementary pipe element depicted in Fig. 1.4 has a small volume $dV = \Omega dz$ and mass $dM = dM_g + dM_l$.

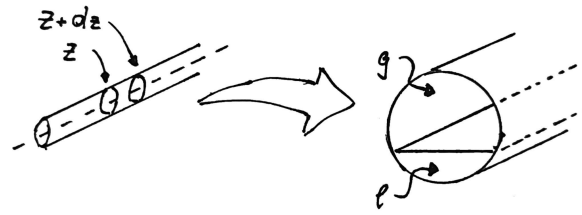[6]Instead of mass flux, this quantity may also be called mass velocity.

Figure 1.4: Elementary pipe element.

The cross-section averaged photographic density is defined as:

$$\bar{\rho}^* = \frac{dM}{dV} =$$
$$= \frac{\rho_g V_g + \rho_l V_l}{\Omega dz} = \frac{\rho_g \Omega_g dz + \rho_l \Omega_l dz}{\Omega dz} =$$
$$= \rho_g \bar{\alpha} + \rho_l (1 - \bar{\alpha})$$

i.e. a weighted average of the densities of the two phases, based on the void fraction.

1.2.16 Relationship between $\bar{\alpha}$ and $\bar{x}$ or $\bar{x}_v$