CS 555 Mini-App Presentation

The S_N Method for Solving the Linear Boltzmann Equation

05/06/2025

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \sigma_t(x)\psi(x,\mu) = \frac{\sigma_s(x)}{2} \int_{-1}^{1} \psi(x,\mu)d\mu + \frac{q(x)}{2} \quad \begin{array}{l} x \in [-L, L] \\ \mu \in [-1,1] \end{array}$$
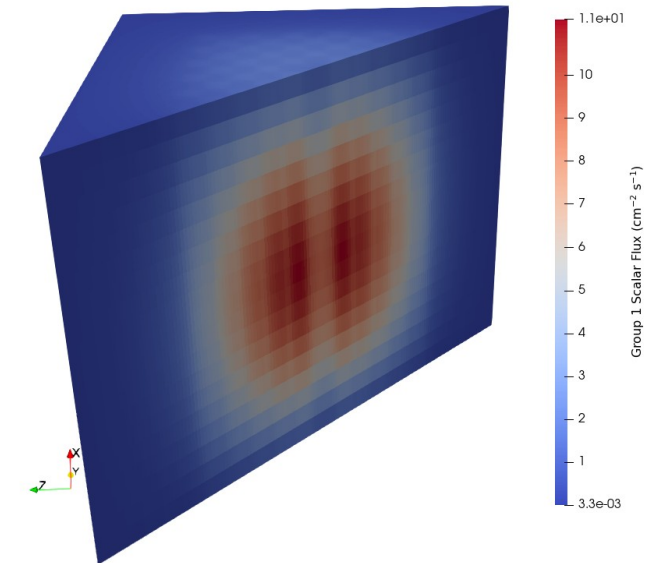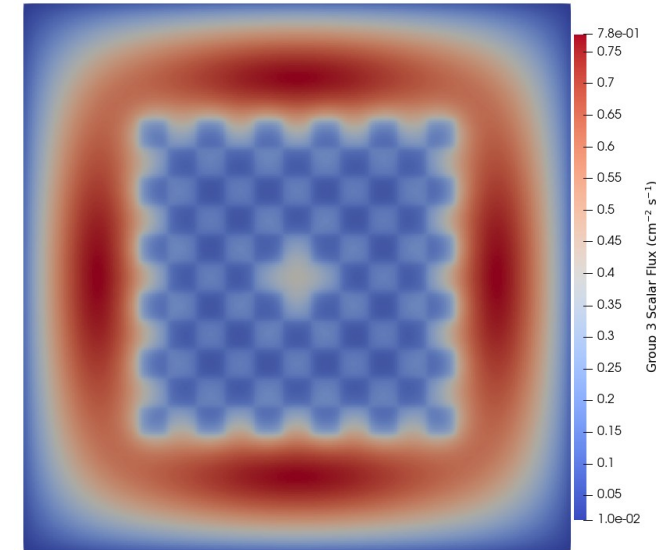
Self-advection

Total removal

Scattering source

External source

$$\psi(x,\mu) = 0, \qquad \mu < 0 \ and \ x = L$$
$$\psi(x,\mu) = 0, \qquad \mu > 0 \ and \ x = -L$$

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)d\mu$$

Region	Location	σ_t	σ_s	q
1	$-2 < x < 2$	50.0	0.0	50.0
2	$2 \leq x < 3, -3 < x \leq -2$	5.0	0.0	0.0
3	$3 \leq x < 5, -5 < x \leq -3$	0.0	0.0	0.0
4	$5 \leq x < 6, -6 < x \leq -5$	1.0	0.9	1.0
5	$6 \leq x < 8, -8 < x \leq -6$	1.0	0.9	0.0
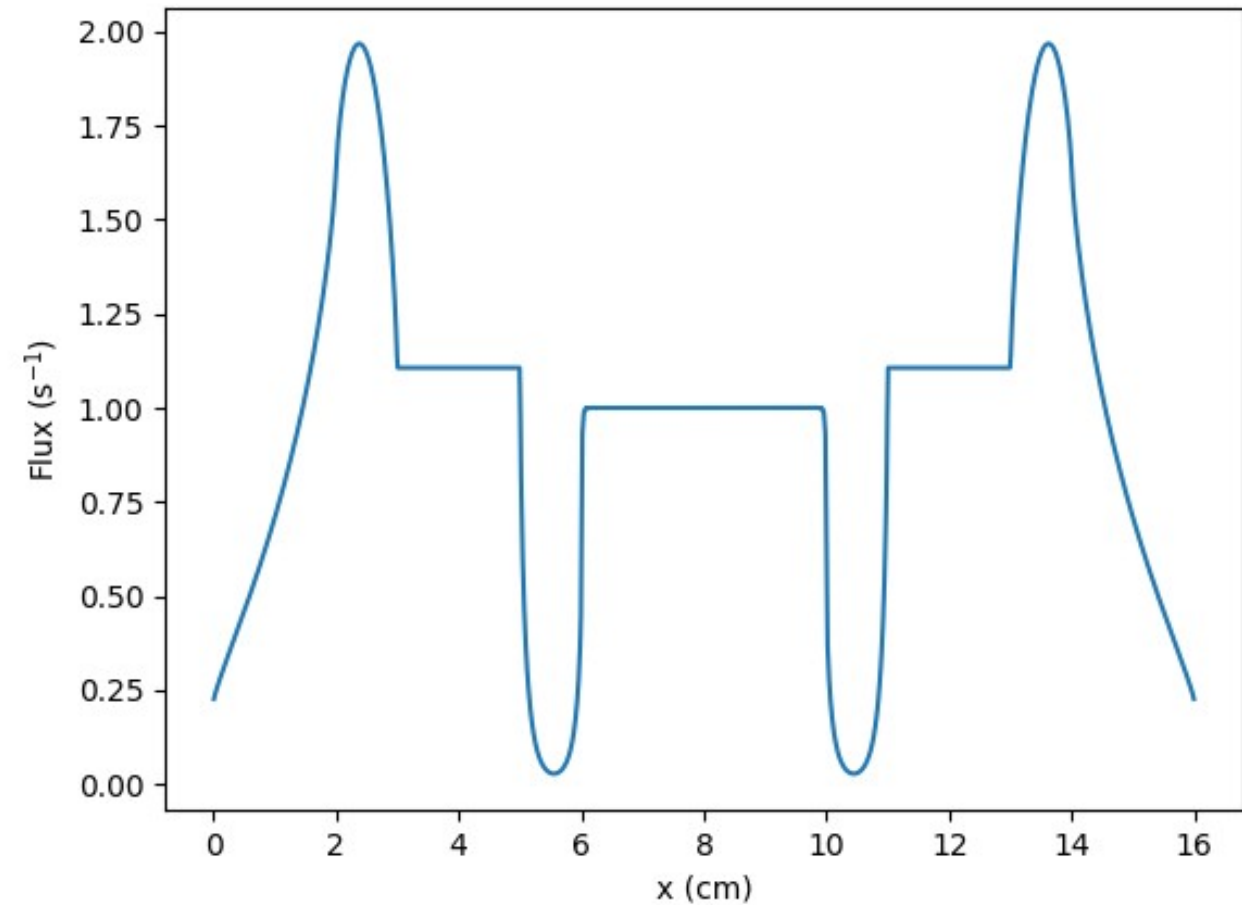
- The S$_N$ method:

$$\mu \rightarrow \mu_n \qquad \psi(x, \mu_n) = \psi_n(x)$$

$$\mu_n \frac{\partial \psi_n(x)}{\partial x} + \sigma_t \psi_n(x) = \frac{\sigma_s}{2} \phi(x) + \frac{q}{2} \quad \phi(x) = \int_{-1}^{1} \psi(x, \mu) d\mu \approx \sum_{n=1}^{N} w_n \psi_n(x)$$
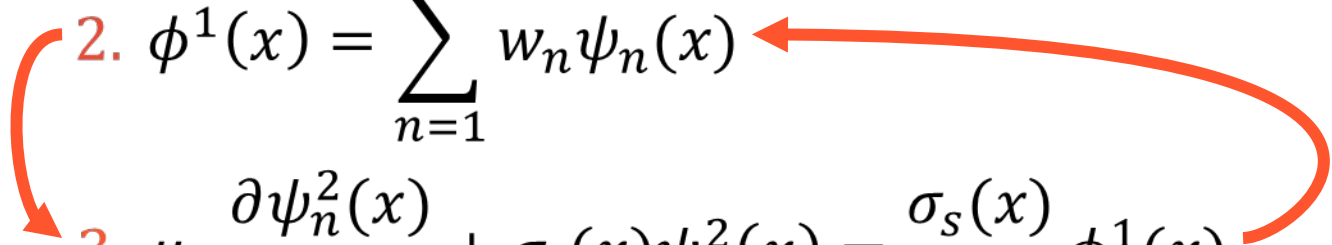
- Richardson iteration:

1. $\mu_n \dfrac{\partial \psi_n^1(x)}{\partial x} + \sigma_t(x) \psi_n^1(x) = \dfrac{q(x)}{2}$

2. $\phi^1(x) = \displaystyle\sum_{n=1}^{N} w_n \psi_n(x)$

3. $\mu_n \dfrac{\partial \psi_n^2(x)}{\partial x} + \sigma_t(x) \psi_n^2(x) = \dfrac{\sigma_s(x)}{2} \phi^1(x)$

$$\therefore \phi(x) = \sum_{\ell=0}^{\infty} \phi^\ell(x)$$

- Discretized with diamond differencing and upwinding:

$$\mu_n > 0 \quad \psi_{n,i}^{\ell} = \left(1 + \frac{\sigma_{t,i} h_x}{2|\mu_n|}\right)^{-1} \left(\psi_{n,i-1/2}^{\ell} + \frac{q_i^{\ell-1} h_x}{2|\mu_n|}\right) \qquad \psi_{n,i+1/2}^{\ell} = 2\psi_{n,i}^{\ell} - \psi_{n,i-1/2}^{\ell}$$
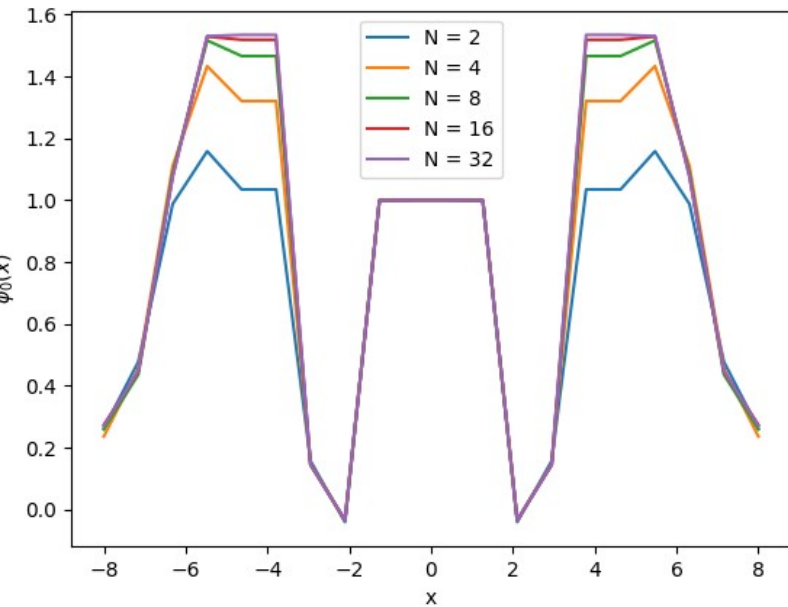
- Resulting system of equations is block lower triangular per direction!

$$\begin{bmatrix} x & & & & & & & & \\ x & x & & & & & & & \\ x & x & x & & & \bigcirc & & & \\ x & x & x & x & & & & & \\ & & & & x & & & & \\ & \bigcirc & & & x & x & & & \\ & & & & x & x & x & & \\ & & & & x & x & x & x \end{bmatrix} \begin{bmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \psi_{3,1} \\ \psi_{4,1} \\ \psi_{4,2} \\ \psi_{3,2} \\ \psi_{2,2} \\ \psi_{1,2} \end{bmatrix} = \begin{bmatrix} q_{1,1} \\ q_{2,1} \\ q_{3,1} \\ q_{4,1} \\ q_{4,2} \\ q_{3,2} \\ q_{2,2} \\ q_{1,2} \end{bmatrix}$$
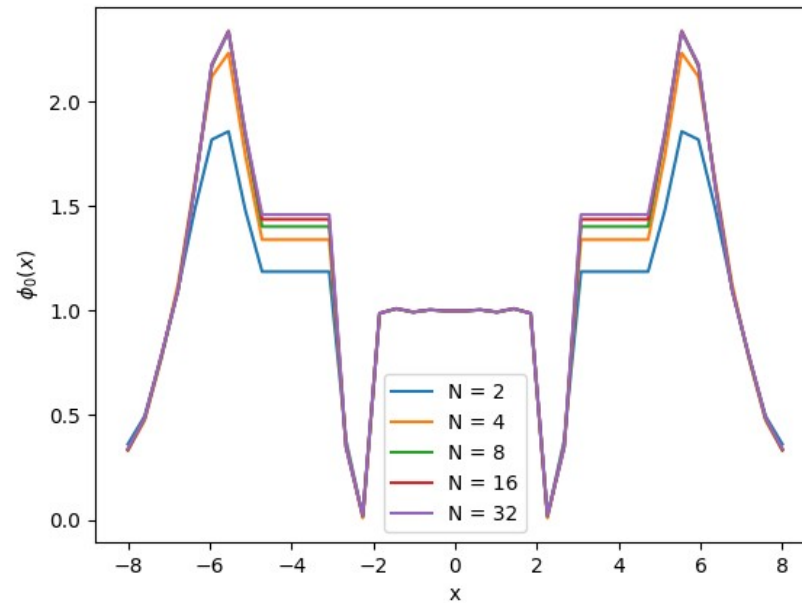
Block structure of the 1D S_N equations.
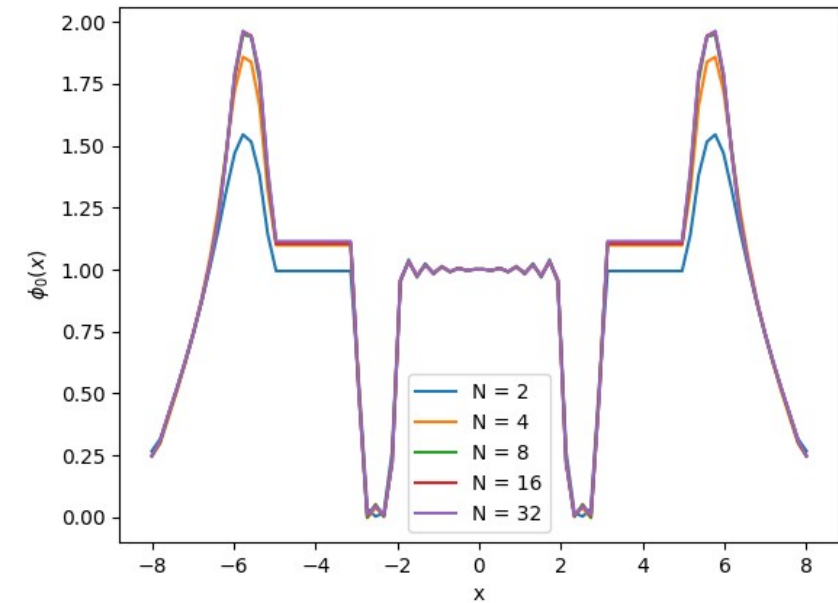(Lewis and Miller, Chapter 3)

BC | 1 | 2 | 3 | … | I
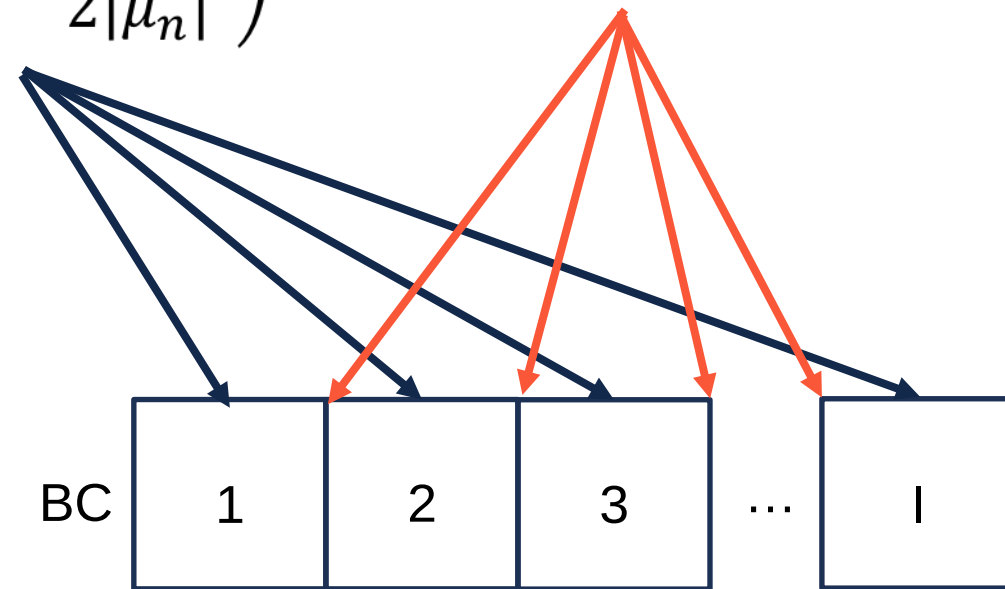
20 grid points

40 grid points

80 grid points
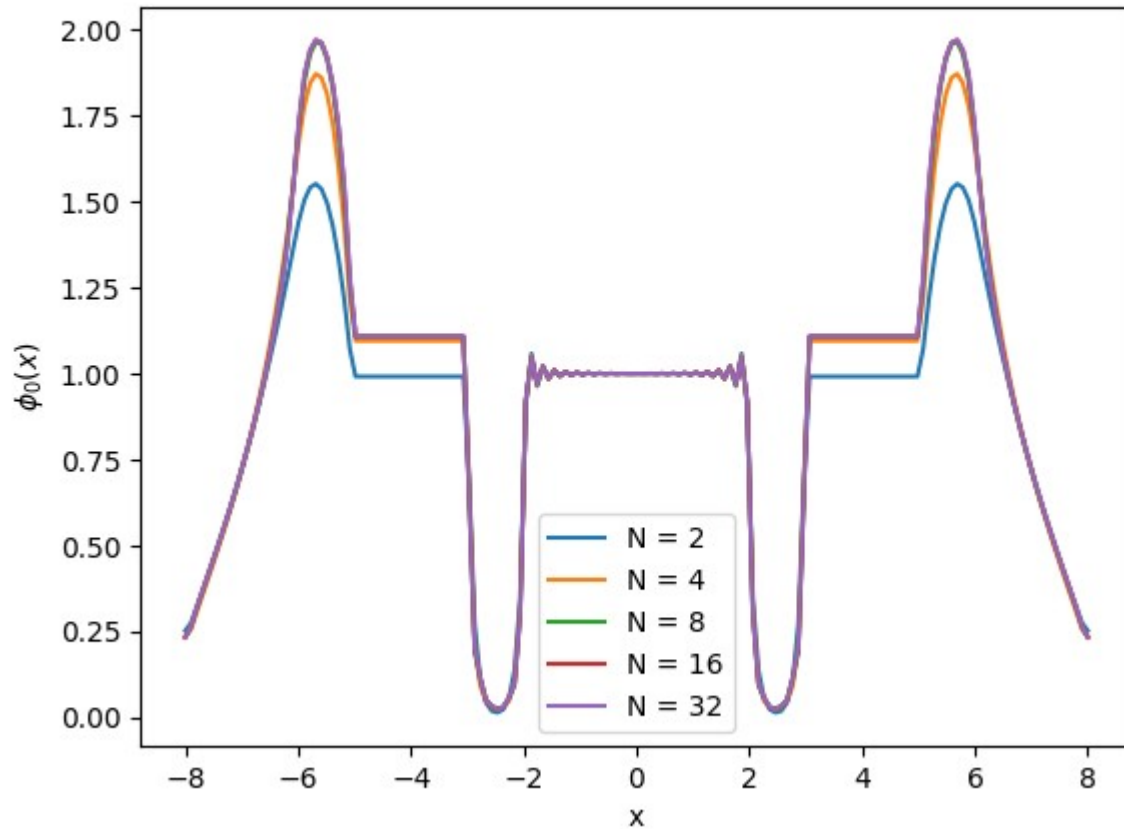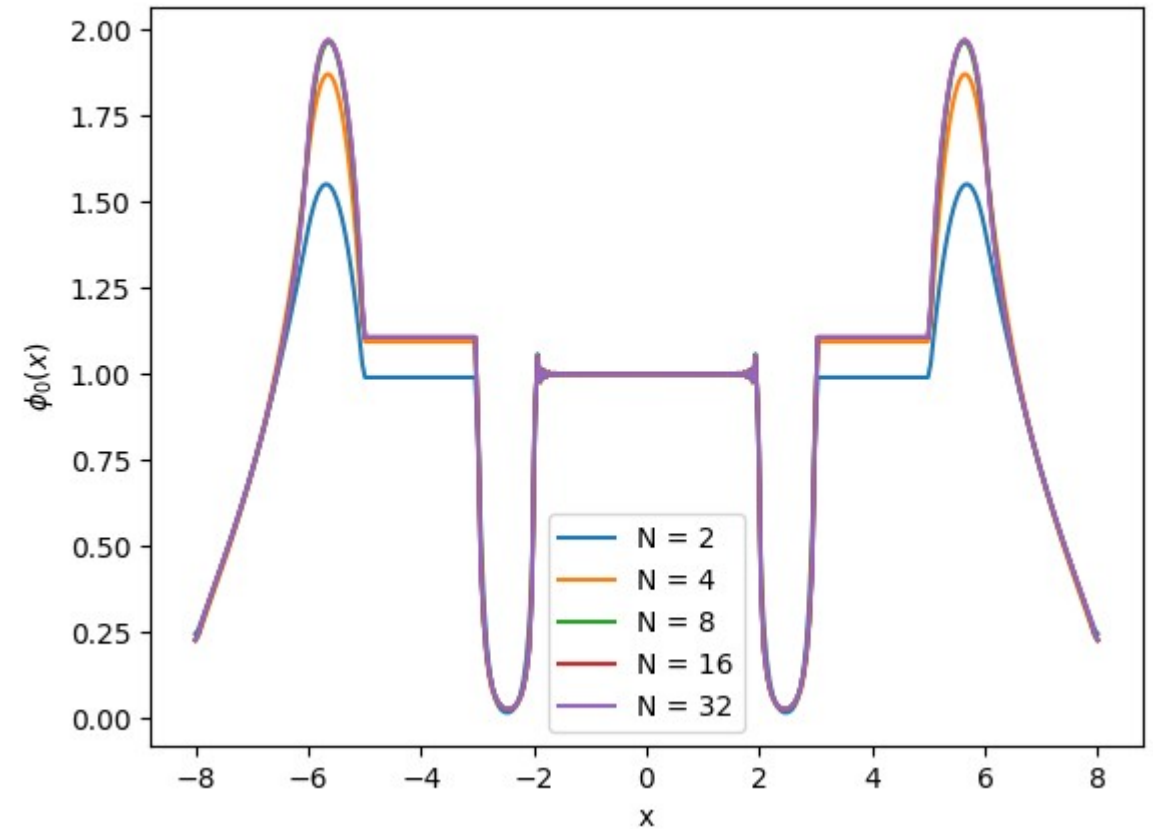
160 grid points

320 grid points

Thank You!
Any Questions?