



UNIVERSITY OF  
**ILLINOIS**  
URBANA-CHAMPAIGN

## CS 555 Mini-App Presentation

### The $S_N$ Method for Solving the Linear Boltzmann Equation

05/06/2025

# The Linear Boltzmann Equation

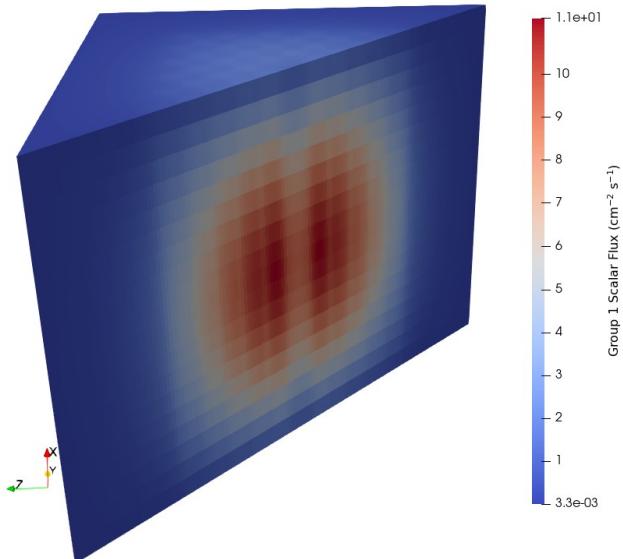
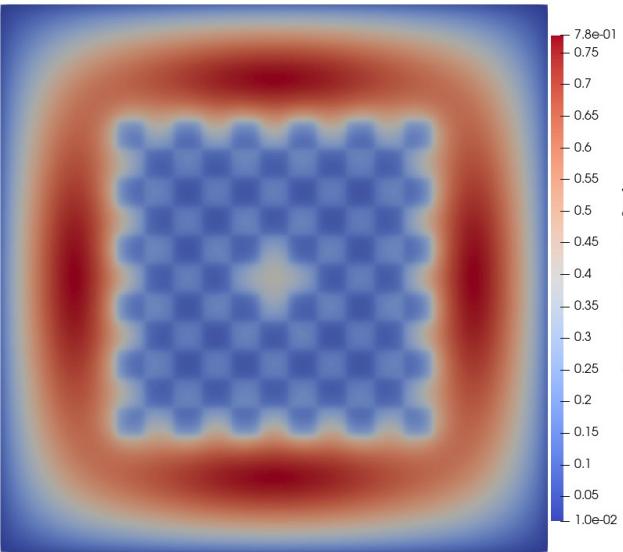


$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \sigma_t(x) \psi(x, \mu) = \frac{\sigma_s(x)}{2} \int_{-1}^1 \psi(x, \mu) d\mu + \frac{q(x)}{2} \quad x \in [-L, L] \quad \mu \in [-1, 1]$$



$$\begin{aligned} \psi(x, \mu) &= 0, & \mu < 0 \text{ and } x = L \\ \psi(x, \mu) &= 0, & \mu > 0 \text{ and } x = -L \end{aligned}$$

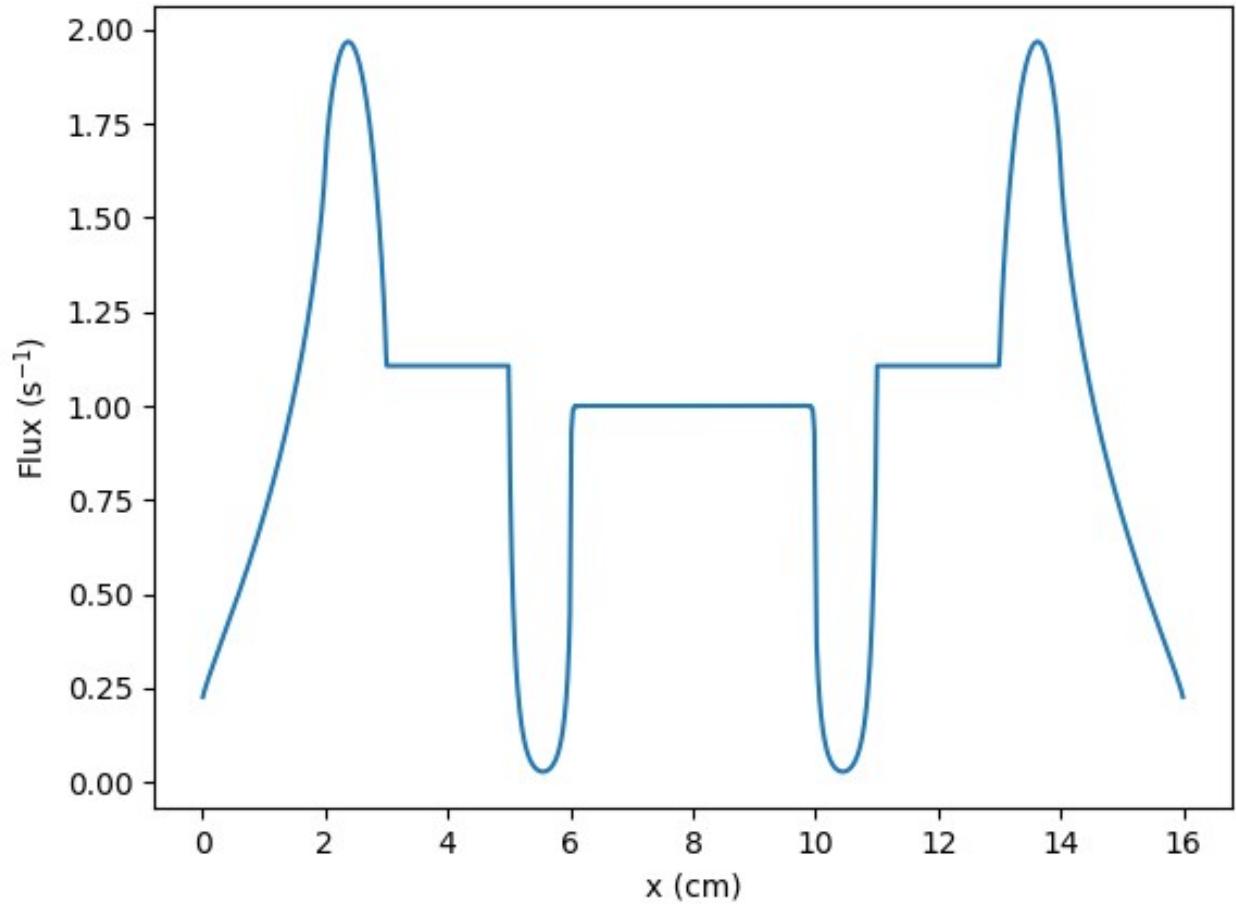
$$\phi(x) = \int_{-1}^1 \psi(x, \mu) d\mu$$



# The Test Problem: 1D Reed Benchmark



Region	Location	$\sigma_t$	$\sigma_s$	$q$
1	$-2 < x < 2$	50.0	0.0	50.0
2	$2 \leq x < 3, -3 < x \leq -2$	5.0	0.0	0.0
3	$3 \leq x < 5, -5 < x \leq -3$	0.0	0.0	0.0
4	$5 \leq x < 6, -6 < x \leq -5$	1.0	0.9	1.0
5	$6 \leq x < 8, -8 < x \leq -6$	1.0	0.9	0.0



# The S<sub>N</sub> Method with Richardson Iteration



- The S<sub>N</sub> method:

$$\mu \rightarrow \mu_n \quad \psi(x, \mu_n) = \psi_n(x)$$

$$\mu_n \frac{\partial \psi_n(x)}{\partial x} + \sigma_t \psi_n(x) = \frac{\sigma_s}{2} \phi(x) + \frac{q}{2} \quad \phi(x) = \int_{-1}^1 \psi(x, \mu) d\mu \approx \sum_{n=1}^N w_n \psi_n(x)$$

- Richardson iteration:

$$1. \mu_n \frac{\partial \psi_n^1(x)}{\partial x} + \sigma_t(x) \psi_n^1(x) = \frac{q(x)}{2}$$

$$2. \phi^1(x) = \sum_{n=1}^N w_n \psi_n(x)$$
  
$$3. \mu_n \frac{\partial \psi_n^2(x)}{\partial x} + \sigma_t(x) \psi_n^2(x) = \frac{\sigma_s(x)}{2} \phi^1(x)$$

$$\therefore \phi(x) = \sum_{\ell=0}^{\infty} \phi^\ell(x)$$

# Spatial Discretization and Solution Procedure

- Discretized with diamond differencing and upwinding:

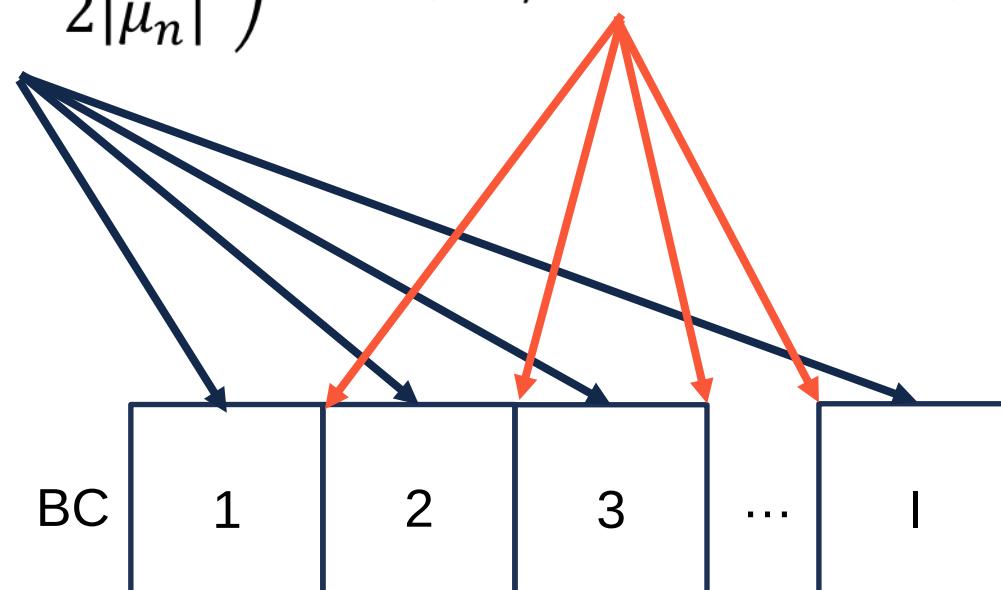
$$\mu_n > 0 \quad \psi_{n,i}^\ell = \left(1 + \frac{\sigma_{t,i} h_x}{2|\mu_n|}\right)^{-1} \left(\psi_{n,i-1/2}^\ell + \frac{q_i^{\ell-1} h_x}{2|\mu_n|}\right) \quad \psi_{n,i+1/2}^\ell = 2\psi_{n,i}^\ell - \psi_{n,i-1/2}^\ell$$

- Resulting system of equations is block lower triangular per direction!

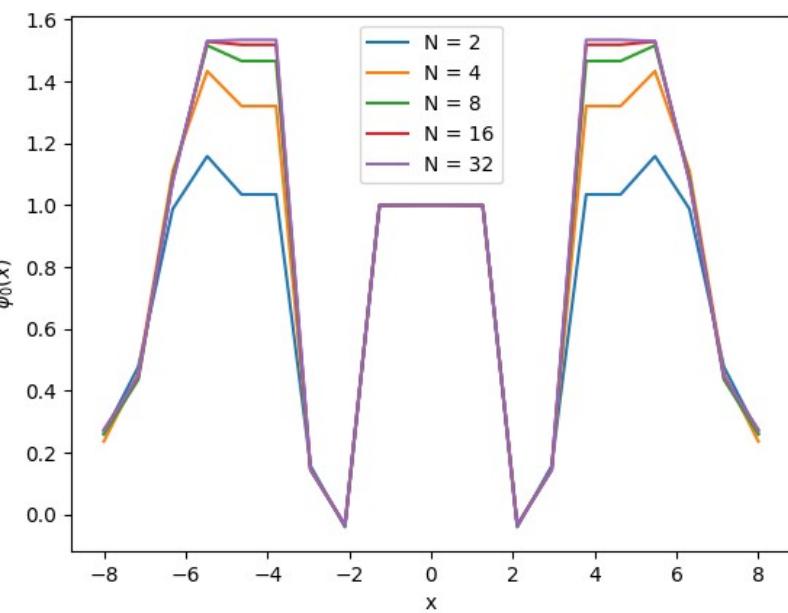
$$\begin{bmatrix} x \\ x & x \\ x & x & x \\ x & x & x & x \end{bmatrix} \begin{bmatrix} \textcircled{1} \\ \textcircled{2} \end{bmatrix} \begin{bmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \psi_{3,1} \\ \psi_{4,1} \\ \psi_{4,2} \end{bmatrix} = \begin{bmatrix} q_{1,1} \\ q_{2,1} \\ q_{3,1} \\ q_{4,1} \\ q_{4,2} \end{bmatrix}$$

$$\begin{bmatrix} x & & & \\ & x & & \\ & & x & \\ & & & x \end{bmatrix} \begin{bmatrix} \textcircled{3} \\ \textcircled{4} \end{bmatrix} \begin{bmatrix} \psi_{3,2} \\ \psi_{2,2} \\ \psi_{1,2} \end{bmatrix} = \begin{bmatrix} q_{3,2} \\ q_{2,2} \\ q_{1,2} \end{bmatrix}$$

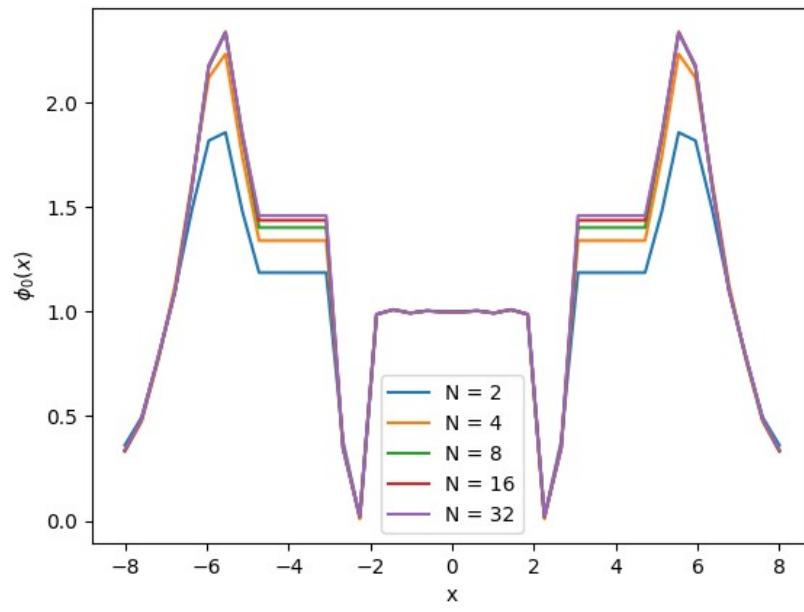
Block structure of the 1D  $S_N$  equations.  
(Lewis and Miller, Chapter 3)



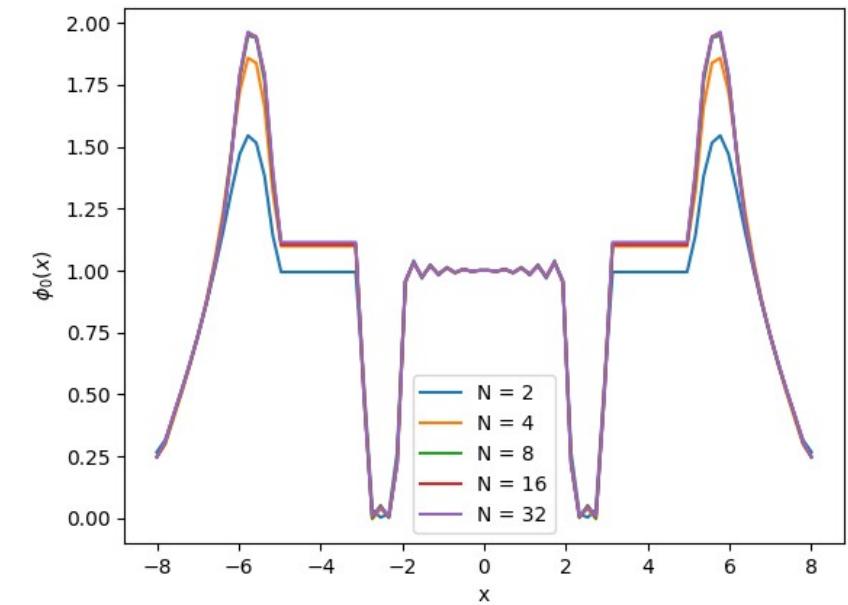
# Results: Low to Moderate Space-Angle Refinement



20 grid points

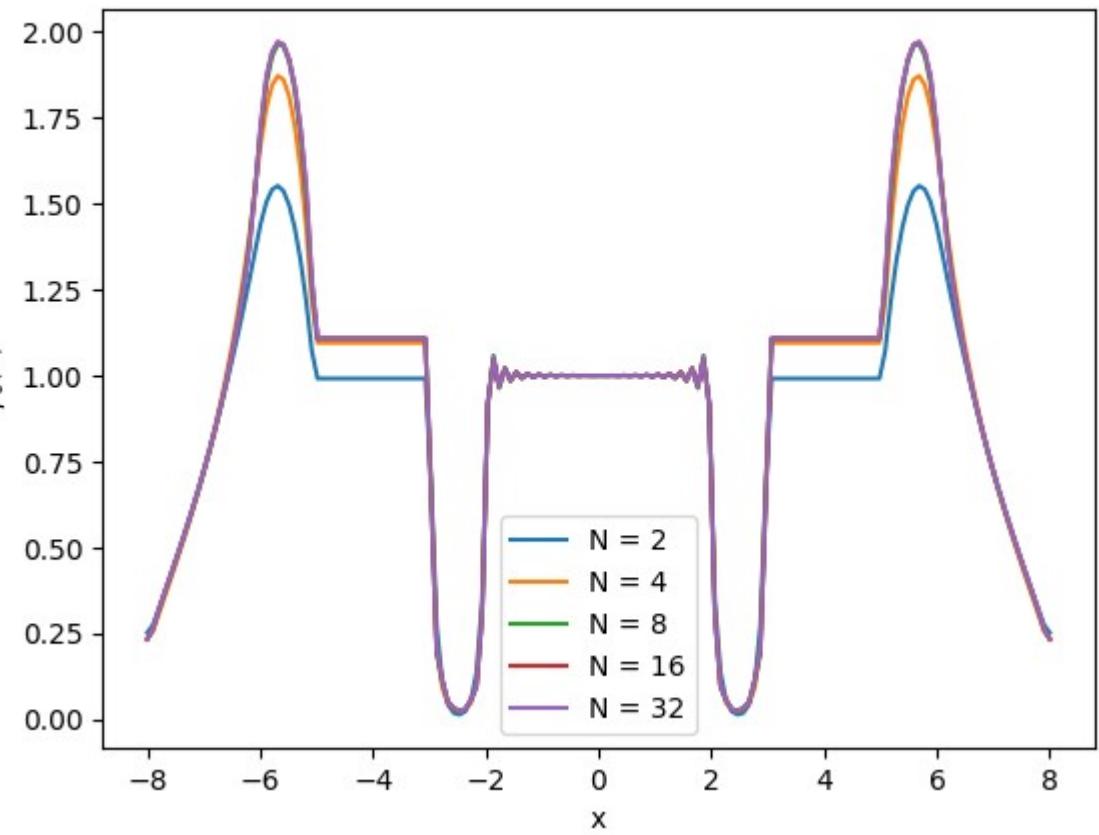


40 grid points

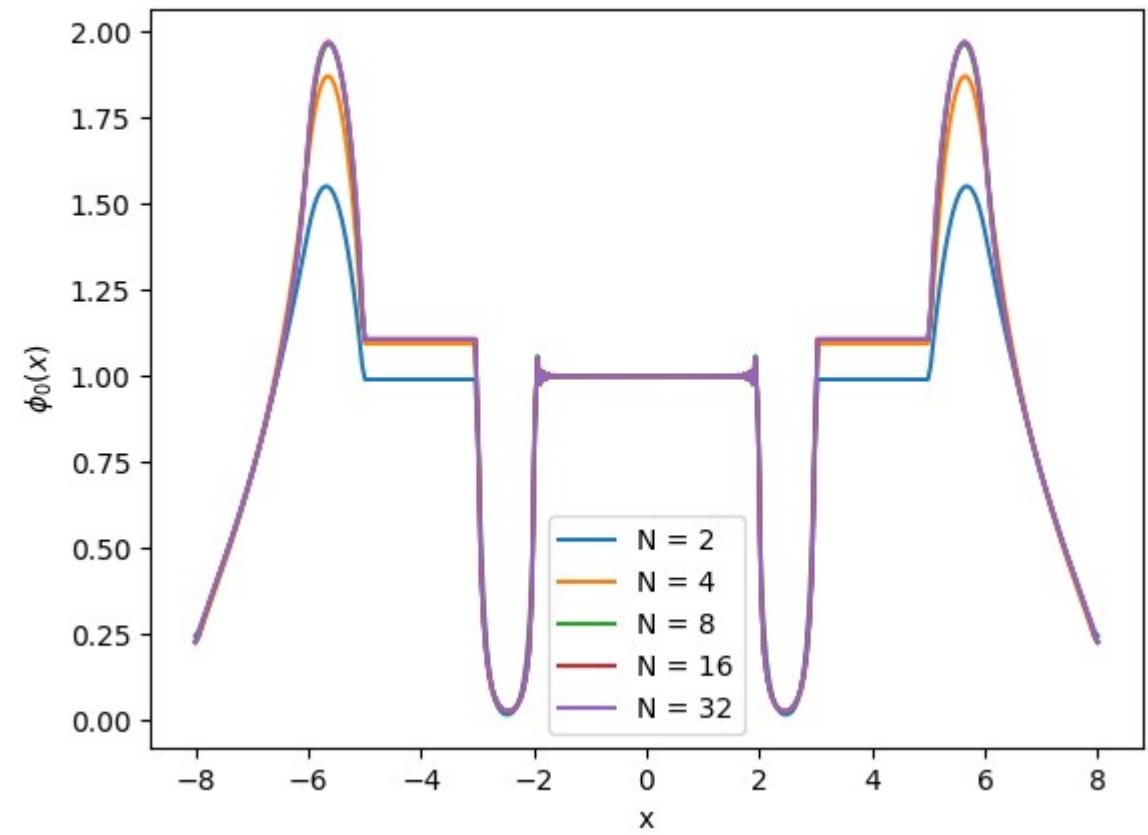


80 grid points

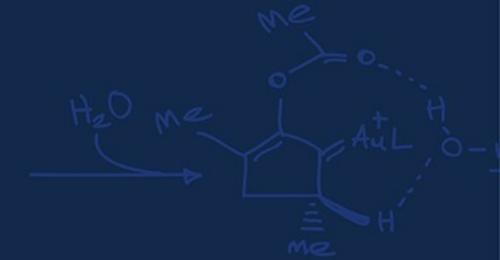
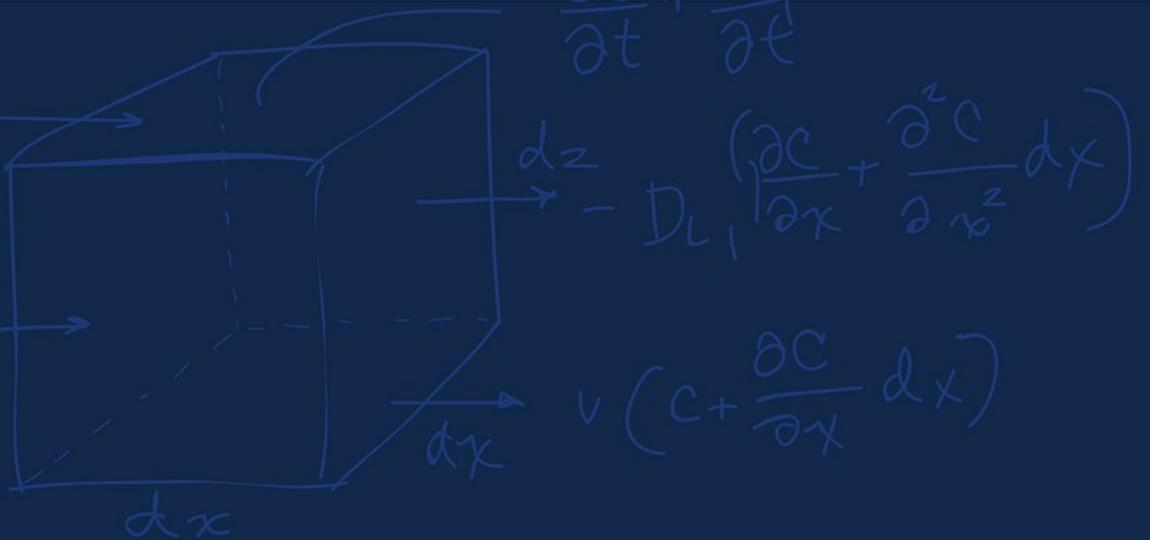
# Results: Resolved Space-Angle Refinement



160 grid points



320 grid points



Thank You!  
Any Questions?

