1. Math foundation of fuzzy preferences

Based on results from [1, 2, 3, 4].

Definition 1.1 (A-FPS). Let (T_ϕ, S_ϕ, N_ϕ) be a Łukasiewicz triple. A *additive fuzzy preference structure* on a set of alternatives X is a triplet (P, I, J) of fuzzy relations in X such that:

- P is irreflexive, I is reflexive, J is irreflexive
- P is T_ϕ-asymmetric, I and J are symmetric
- $P \cap_{T_\phi} I = \emptyset$
- $P \cap_{T_\phi} J = \emptyset$
- $I \cap_{T_\phi} J = \emptyset$
- $co_n(P \cup_{S_\phi} I) = P^{-1} \cup_{S_\phi} J$

Definition 1.2 ((s, ϕ)-FPS). For $s \in [0, \infty]$ and ϕ automorphism of unit interval, triplet of fuzzy relations (P, I, J) defined by:

$$P(a,b) = \phi^{-1}(T^{1/s}(\phi(R(a,b), 1 - \phi(R(b,a))))$$

$$I(a,b) = \phi^{-1}(T^{s}(\phi(R(a,b), \phi(R(b,a))))$$

$$J(a,b) = \phi^{-1}(T^{s}(1 - \phi(R(a,b)), 1 - \phi(R(b,a))))$$

is A-FPS and it holds the large preference axiom. By using definition of Frank family t-norm above could be written explicit as:

$$P(a,b) = \phi^{-1}\Bigg(\log_{1/s} \left(1 + \frac{(1/s^{\phi(R(a,b))} - 1)(1/s^{1-\phi(R(b,a))} - 1)}{1/s - 1} \right) \Bigg)$$

$$I(a,b) = \phi^{-1}\Bigg(\log_{s} \left(1 + \frac{(s^{\phi(R(a,b))} - 1)(s^{\phi(R(b,a))} - 1)}{s - 1} \right) \Bigg)$$

$$J(a,b) = \phi^{-1}\Bigg(\log_{s} \left(1 + \frac{(s^{1-\phi(R(a,b))} - 1)(s^{1-\phi(R(b,a))} - 1)}{s - 1} \right) \Bigg)$$

Limit cases of Frank families for $0, 1, \infty$:

(0, ϕ)-FPS

$$P(a,b) = \phi^{-1}(max\{\phi(R(a,b)) - \phi(R(b,a)), 0\})$$

$$I(a,b) = \phi^{-1}(min\{\phi(R(a,b)), \phi(R(b,a))\}) = min\{R(a,b), R(b,a)\}$$

$$J(a,b) = \phi^{-1}(min\{1 - \phi(R(a,b)), 1 - \phi(R(b,a))\}) = min\{\phi^{-1}(1 - \phi(R(a,b))), \phi^{-1}(1 - \phi(R(b,a)))\}$$

(1, ϕ)-FPS

$$P(a,b) = \phi^{-1}(\phi(R(a,b))(1-\phi(R(b,a))))$$

$$I(a,b) = \phi^{-1}(\phi(R(a,b))\phi(R(b,a)))$$

$$J(a,b) = \phi^{-1}((1-\phi(R(a,b)))(1-\phi(R(b,a)))$$

(∞, ϕ)-FPS

$$P(a,b) = \phi^{-1}(min\{\phi(R(a,b)), 1-\phi(R(b,a)\}$$

$$I(a,b) = \phi^{-1}(max\{\phi(R(a,b)) + \phi(R(b,a) - 1, 0\}$$

$$J(a,b) = \phi^{-1}(max\{1-\phi(R(a,b)) - \phi(R(b,a), 0\}$$

2. FPS in decision making

Problem of fuzzy multi-criteria decision making according to [6] could be expressed as a problem of choosing alternative with maximum degree of non-dominance. It could be represented as an ordered par:

$$(A, R)$$

where $A = [A_1, A_2, ..., A_n]$ is a set of available alternatives (a crisp set) and $R = [R_1, R_2, ..., R_m]$ is a vector of fuzzy large preference relations for m different criteria. In other words we have a set of alternatives and "comparing tool" which allow us to assign any two options a truth value of sentence **a is not worse than b according to criterion k** .

Next step a decision maker must take is to find the set of non-dominated alternatives which will indicate degrees of non-dominated options under all criteria and choose the alternative with the highest degree.

To do this he will need strict preference and indifference relation obtained from large preferences R_k of criterion k. As we already know such relations could be easily constructed in form of (s, ϕ)-FPS. In practice we usually use the simplest $(0, id)$-FPS which provides us with following strict preference P and indifference relation I:

$$P(a, b) = max\{R_k(a, b) - R_k(b, a), 0\}$$

$$I(a, b) = minR_k(a, b), R_k(b, a)\}$$

One may question why not define incomparability relation? In practical decision making we prefer when each pair of alternatives could be compared what is not always true. In discussed case this assumption is masked in definition of (A, R) and this makes each R_k a completed relation hence indifference relation is an empty set.

As we already know that $P_k(a, b)$ could be interpreted as **alternative a is better than b** or **alternative a dominates than b** according to criterion k. The later interpretation leads directly to the way in which we can obtain a relation which describes a degree to which **b is not dominated by a** as $N_\phi(P_k(a, b)) = 1 - P_k(a, b)$ (because in our case $\phi = id$). In the result the following intersection

$$ND_k = \bigcap_{a_i \in A} (1 - P_k(a_i, a))$$

is a fuzzy set of alternatives by which a is not-dominated according to criterion k with the membership function $ND_k(a)$ given by:

$$ND_k(a) = \min_{a_i \in A}(1 - P_k(a, a_i)) = 1 - \max_{a_i \in A}(P(a, a_i))$$

Definition 2.1 (Non-dominance relation). A fuzzy relation which corresponds with set ND_k is called *non-dominance relation* and is denoted $ND_k(a)$.

Definition 2.2 (Optimal decision). Let A be a set of alternatives and $ND_k(a)$ is a non-dominance relation build from relations P_k, R_k. Then following set is called *optimal decision* according to criterion k:

$$a_k^{ND} = \{a_i \in A \mid ND_k(a_i) = \max_{a_i \in A}(ND_k(a_i))\}$$

In other words the optimal decision is a set of alternatives with highest degree of membership to the fuzzy set of non-dominated decision ND_k.

When our decision problem (A, R) is based only on one criterion ($k = 1$) then there is only one optimal decision set which is also the final decision. When $k > 1$ the two methods could be found in [6].

Method I (*Aggregation - Scoring*)
First method is based on the aggregated large preference:

$$R^G = \bigcap_{k=1}^{m} R_k$$

Membership function of this relation is given by:

$$R^G(a,b) = \min_{k \in \{1,\dots,m\}} R_k(a,b) \quad \text{for any } a, b \in A$$

With such relation our k-dimensional problem is brought to $k = 1$ and could be solved by finding the optimal decision as described previously. Then obtained set A^{ND} is a P-optimal set [6].

Method II (*Scoring - Aggregation*)
Second method firstly needs a decision maker to obtain non-dominance relation for each $k = 1, ..., m$ criterion:

$$ND_k(a) = 1 - \max_{a_i \in A}(P_k(a, a_i))$$

Then we seek for intersection of fuzzy sets ND_k:

$$ND(a) = \bigcap_{k=1}^{m} ND_k(a) = \min_{k \in \{1,\dots,m\}} ND_k(a) \quad \text{for any } a \in A$$

and then for the optimal decision a^{ND} as described previously in case of $k = 1$.

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