Introduction to Statistics: Problems

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1 Kolmogorov axioms

Show that the definitions of *frequentist* and *Bayesian* probabilities given in the lectures satisfy the three Kolmogorov axioms. I don't expect any rigorous proofs here!

2 No correlation does not mean independence

In the lectures, we said that two random variables which are independent will have a zero correlation coefficient.

1. Show that two continuous random variables X and Y, with $(X, Y) \sim f(X, Y)$ which are independent will have a 0 correlation coefficient.
2. Let X be a continuous random variable symmetrically distributed around 0 with a density function $f(X)$. Let $Y = X^2$. Show that despite the fact that Y and X are clearly dependent, their correlation coefficient is 0.

3 Cauchy distribution

In the lectures, we showed how the sum of two Gaussian distributed random variables is itself Gaussian distributed. Suppose now that $X \sim \phi(X; 0, 1)$ and $Y \sim \phi(Y; 0, 1)$ are independent random variables. Show that the distribution of $Z = \dfrac{X}{Y}$ is Cauchy, i.e that $p(Z) = \dfrac{1}{\pi(1 + Z^2)}$.

Hint: You should start by deriving the marginal distribution formula for the ratio of two independent random variables. Careful that the case $Y = 0$ will cause a problem, so split the marginal distribution into two cases, one for $Y > 0$ and one for $Y < 0$. The sum of these marginal distributions will be the total distribution.

4 Normal approximation to a Poisson

Show that the Poisson probability density distribution $P(k) = \dfrac{\lambda^k}{k!}e^{-\lambda}$ converges (in distribution) to a normal probability density distribution as $\lambda \to \infty$. What are the mean and variance of the normal distribution that it converges to?

Hint: Start by considering a "standardized" Poisson random variable $K = \dfrac{k - \lambda}{\sqrt{\lambda}}$.