Solutions to Homework No. 2 in Quantum Field Theory II

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1. Free Particle Two-Point Function I
   Calculate the two-point function $\langle x=0,\, t_b \,|\, T\left(\hat{x}\left(t\right)\hat{x}\left(t'\right)\right) \,|\, x=0,\, t_a\rangle$ for a free particle explicitly.

   (a) Some integral identities:

   - Claim: $\int_{\mathbb{R}} dx\, x\, \exp\left(iax^2+ikx\right) = -\frac{k}{2a}\sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
     Proof:
     i. From the HW2Q1(b) we know that
        $\int_{\mathbb{R}} dx\, \exp\left(iax^2+ikx\right) = \sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
     ii. Differentiate both sides of the equation with respect to k:
        $\frac{d}{dk}\int_{\mathbb{R}} dx\,\ \exp\left(iax^2+ikx\right) = -i\frac{k}{2a}\sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
        Thus[1]
        $\int_{\mathbb{R}} dx\, \frac{d}{dk}\,\exp\left(iax^2+ikx\right) = -i\frac{k}{2a}\sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
        $\int_{\mathbb{R}} dx\, ix\, \exp\left(iax^2+ikx\right) = -i\frac{k}{2a}\sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
        $\int_{\mathbb{R}} dx\, x\, \exp\left(iax^2+ikx\right) = -\frac{k}{2a}\sqrt{\frac{i\pi}{a}}\exp\left(-\frac{ik^2}{4a}\right)$
        $\blacksquare$

   - Claim: $\int_{\mathbb{R}} dx\, x^2\, e^{iax^2} = -\frac{1}{2ia}\sqrt{\frac{i\pi}{a}}$
     Proof:
     i. Starting from
        $\int_{\mathbb{R}} dx\, \exp\left(iax^2\right) = \sqrt{\frac{i\pi}{a}}$
     ii. Differentiate both sides with respect to a:
        $\int_{\mathbb{R}} dx\, ix^2\, \exp\left(iax^2\right) = \sqrt{\frac{i\pi}{1}}\left(-\frac{1}{2}\right)a^{-3/2}$
     iii. Rearranging we get:
        $\int_{\mathbb{R}} dx\, x^2\, \exp\left(iax^2\right) = -\frac{1}{2ia}\sqrt{\frac{i\pi}{a}}$
        $\blacksquare$

[1] http://math.stackexchange.com/a/11716/61151

(b) Claim: $\langle x=0,\, t=t_b \mid \hat{x}(t_1)\,\hat{x}(t_2) \mid x=0,\, t=t_a\rangle = \sqrt{\frac{i\hbar}{2\pi m}}\,(t_b-t_a)^{-\frac{3}{2}}\,(t_b-t_1)\,(t_2-t_a)$

Proof:

i. Employ the definition $\hat{x}(t) \equiv \int_{\mathbb{R}} dx\, x\, |x,\, t\rangle\langle x,\, t|$:

$$\langle 0,\, t_b \mid \hat{x}(t_1)\,\hat{x}(t_2) \mid 0,\, t_a\rangle =$$
$$= \left\langle 0,\, t_b \,\middle|\, \left(\int_{\mathbb{R}} dx_1\, x_1\, |x_1,\, t_1\rangle\langle x_1,\, t_1|\right)\left(\int_{\mathbb{R}} dx_2\, x_2\, |x_2,\, t_2\rangle\langle x_2,\, t_2|\right) \,\middle|\, 0,\, t_a\right\rangle =$$
$$= \int_{\mathbb{R}^2} dx_1\, dx_2\, x_1\, x_2\, \langle 0,\, t_b \mid |x_1,\, t_1\rangle\langle x_1,\, t_1|\, |x_2,\, t\rangle\langle x_2,\, t| \mid 0,\, t_a\rangle =$$
$$= \int_{\mathbb{R}^2} dx_1\, dx_2\, x_1\, x_2\, K(0,\, x_1;\, t_b-t_1)\, K(x_1,\, x_2;\, t_1-t_2)\, K(x_2,\, 0;\, t_2-t_a) =$$

ii. Plug in the definition of the propagator of the free particle from HW1Q3 ($K(x,\, x';\, t-t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right]$):

$$= \int_{\mathbb{R}^2} dx_1\, dx_2\, x_1\, x_2 \sqrt{\frac{m}{i(t_b-t_1)2\pi\hbar}}\exp\left[\frac{im(0-x_1)^2}{2\hbar(t_b-t_1)}\right] \times$$
$$\times \sqrt{\frac{m}{i(t_1-t_2)2\pi\hbar}}\exp\left[\frac{im(x_1-x_2)^2}{2\hbar(t_1-t_2)}\right]\sqrt{\frac{m}{i(t_2-t_a)2\pi\hbar}}\exp\left[\frac{im(x_2-0)^2}{2\hbar(t_2-t_a)}\right] =$$
$$= \left(\frac{m}{i2\pi\hbar}\right)^{\frac{3}{2}}\left[(t_b-t_1)\,(t_1-t_2)\,(t_2-t_a)\right]^{-\frac{1}{2}} \times$$
$$\times \int_{\mathbb{R}^2} dx_1\, dx_2\, x_1\, x_2\, \exp\left[\frac{im\, x_1^2}{2\hbar(t_b-t_1)} + \frac{im(x_1-x_2)^2}{2\hbar(t_1-t_2)} + \frac{im\, x_2^2}{2\hbar(t_2-t_a)}\right] =$$
$$= \left(\frac{m}{i2\pi\hbar}\right)^{\frac{3}{2}}\left[(t_b-t_1)\,(t_1-t_2)\,(t_2-t_a)\right]^{-\frac{1}{2}} \times$$
$$\times \int_{\mathbb{R}} dx_1\, x_1\, \exp\left[\frac{im\, x_1^2}{2\hbar(t_b-t_1)}\right]\int_{\mathbb{R}} dx_2\, x_2\, \exp\left[\frac{im(x_1-x_2)^2}{2\hbar(t_1-t_2)} + \frac{im\, x_2^2}{2\hbar(t_2-t_a)}\right] =$$

A. Rearrange the term in the exponent of the latter integral to get it to a form of part (a):

$$\frac{(x_1-x_2)^2}{(t_1-t_2)} + \frac{x_2^2}{(t_2-t_a)} = \frac{(t_1-t_a)}{(t_1-t_2)(t_2-t_a)}x_2^2 - \frac{2x_1}{(t_1-t_2)}x_2 + \frac{1}{(t_1-t_2)}x_1^2$$

iii. Thus we get:

$$= \left(\frac{m}{i2\pi\hbar}\right)^{\frac{3}{2}}\left[(t_b-t_1)\,(t_1-t_2)\,(t_2-t_a)\right]^{-\frac{1}{2}} \times$$
$$\times \int_{\mathbb{R}} dx_1\, x_1\, \exp\left[\frac{im\, x_1^2}{2\hbar(t_b-t_1)} + \frac{im}{2\hbar}\frac{x_1^2}{(t_1-t_2)}\right]\int_{\mathbb{R}} dx_2\, x_2\, \exp\left[\frac{im}{2\hbar}\frac{(t_1-t_a)x_2^2}{(t_1-t_2)(t_2-t_a)} + \frac{im}{2\hbar}\frac{-2x_1x_2}{(t_1-t_2)}\right] =$$
$$= \left(\frac{m}{i2\pi\hbar}\right)^{\frac{3}{2}}\left[(t_b-t_1)\,(t_1-t_2)\,(t_2-t_a)\right]^{-\frac{1}{2}} \times$$
$$\times \int_{\mathbb{R}} dx_1\, x_1\, \exp\left[\frac{im\, x_1^2}{2\hbar}\left(\frac{1}{t_b-t_1} + \frac{1}{t_1-t_2}\right)\right]\int_{\mathbb{R}} dx_2\, x_2\, \exp\left[\frac{im}{2\hbar}\frac{(t_1-t_a)x_2^2}{(t_1-t_2)(t_2-t_a)} + \frac{im}{\hbar}\frac{x_1x_2}{t_2-t_1}\right] =$$

iv. Now employ the results of the first integral of part (a) with $a := \frac{m}{2\hbar}\frac{t_1-t_a}{(t_1-t_2)(t_2-t_a)}$ and $k := \frac{m}{\hbar}\frac{x_1}{t_2-t_1}$ to get:

$$= \left(\frac{m}{i2\pi\hbar}\right)^{\frac{3}{2}}\left[(t_b-t_1)\,(t_1-t_2)\,(t_2-t_a)\right]^{-\frac{1}{2}}\int_{\mathbb{R}} dx_1\, x_1\, \exp\left[\frac{im\, x_1^2}{2\hbar}\left(\frac{1}{t_b-t_1} + \frac{1}{t_1-t_2}\right)\right] \times$$
$$\times \left\{-\frac{\left(\frac{m}{\hbar}\frac{x_1}{t_2-t_1}\right)}{2\left[\frac{m}{2\hbar}\frac{t_1-t_a}{(t_1-t_2)(t_2-t_a)}\right]}\sqrt{\frac{i\pi}{\left[\frac{m}{2\hbar}\frac{t_1-t_a}{(t_1-t_2)(t_2-t_a)}\right]}}\exp\left(-\frac{i\left(\frac{m}{\hbar}\frac{x_1}{t_2-t_1}\right)^2}{4\left[\frac{m}{2\hbar}\frac{t_1-t_a}{(t_1-t_2)(t_2-t_a)}\right]}\right)\right\} =$$
$$= \left(\frac{m}{i2\pi\hbar}\right)\left[(t_1-t_a)\,(t_b-t_1)\right]^{-\frac{1}{2}}\frac{t_2-t_a}{t_1-t_a}\int_{\mathbb{R}} dx_1\, x_1^2\, \exp\left[\frac{im\, x_1^2(t_b-t_a)}{2\hbar(t_b-t_1)(t_1-t_a)}\right]$$

v. Now employ the result of the second integral of part (a) with $a = \frac{m(t_b-t_a)}{2\hbar(t_b-t_1)(t_1-t_a)}$:

$$= \left(\frac{m}{i2\pi\hbar}\right)\left[(t_1-t_a)\,(t_b-t_1)\right]^{-\frac{1}{2}}\frac{t_2-t_a}{t_1-t_a}\left\{-\frac{1}{2i\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t_1)(t_1-t_a)}\right]}\sqrt{\frac{i\pi}{\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t_1)(t_1-t_a)}\right]}}\right\} =$$

vi. After some algebraic manipulations we arrive at:

$= \sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} (t_b - t_1)(t_2 - t_a)$
$\blacksquare$

(c) Claim: $\langle x = 0,\, t_b \mid T(\hat{x}(t')\hat{x}(t)) \mid x = 0,\, t_a\rangle =$
$= \sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} [\theta(t - t')(t_b - t)(t' - t_a) + \theta(t' - t)(t_b - t')(t - t_a)]$
Proof:

i. The definition of the time ordered product is
$T(\hat{x}(t')\hat{x}(t)) \equiv \theta(t' - t)\hat{x}(t')\hat{x}(t) + \theta(t - t')\hat{x}(t)\hat{x}(t')$

ii. Thus using the results of part (b) we have:
$\langle x = 0,\, t_b \mid T(\hat{x}(t')\hat{x}(t)) \mid x = 0,\, t_a\rangle =$
$= \langle x = 0,\, t_b \mid [\theta(t' - t)\hat{x}(t')\hat{x}(t) + \theta(t - t')\hat{x}(t)\hat{x}(t')] \mid x = 0,\, t_a\rangle =$
$= \theta(t' - t)\langle x = 0,\, t_b \mid \hat{x}(t')\hat{x}(t) \mid x = 0,\, t_a\rangle + \theta(t - t')\langle x = 0,\, t_b \mid \hat{x}(t)\hat{x}(t') \mid x = 0,\, t_a\rangle =$
$= \theta(t' - t)\sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} (t_b - t')(t - t_a) + \theta(t - t')\sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} (t_b - t)(t' - t_a) =$
$= \sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} [\theta(t - t')(t_b - t)(t' - t_a) + \theta(t' - t)(t_b - t')(t - t_a)]$
$\blacksquare$

2. We are given the identity:

$$\boxed{\langle x = 0,\, t_b \mid T(\hat{x}(t')\hat{x}(t)) \mid x = 0,\, t_a\rangle = \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t')\,\delta J(t)} \langle 0,\, t_b \mid 0, t_a\rangle_J\right]\Bigg|_{J(t)=0,\, J(t')=0}}$$

where $\mathcal{Z}[J] := \langle 0,\, t_b \mid 0, t_a\rangle_J = \int_{\text{all paths with both end points at x=0}} \mathcal{D}x \exp\left\{\frac{i}{\hbar} S[x,\, J]\right\}$
where $S[x,\, J] = \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\dot{x}^2 + J(t)\,x\right]$

(a) Define $\eta(t) := x(t) - x_{cl}(t)$ where $x_{cl}(t)$ satisfies the equation of motion with boundary conditions $x_{cl}(t_a) = x_a = x_{cl}(t_b) = x_b = 0$ and $\eta(t)$ being "quantum fulctuations", which satisfy also $\eta(t_a) = \eta(t_b) = 0$.
Claim: $S[x,\, J] = S[x_{cl},\, J] + S[\eta,\, 0]$
Proof:

i. $S[x,\, J] \equiv \int_{t_a}^{t_b} dt \left[\frac{1}{2} m\dot{x}^2 + J(t)\,x\right] \overset{\text{plug in x}}{=} \int_{t_a}^{t_b} dt \left\{\frac{1}{2} m [\dot{x_{cl}}(t) + \dot{\eta}(t)]^2 + J(t)[x_{cl}(t) + \eta(t)]\right\} =$
$= \int_{t_a}^{t_b} dt \left\{\frac{1}{2} m \left[\dot{x_{cl}}(t)^2 + \dot{\eta}(t)^2 + 2\dot{x_{cl}}(t)\dot{\eta}(t)\right] + J(t)[x_{cl}(t) + \eta(t)]\right\} =$
$= S[x_{cl},\, J] + S[\eta,\, 0] + \int_{t_a}^{t_b} dt \{m\dot{x_{cl}}(t)\dot{\eta}(t) + J(t)\eta(t)\} =$
$= S[x_{cl},\, J] + S[\eta,\, 0] + [m\dot{x_{cl}}(t)\eta(t)]|_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \{-m\ddot{x_{cl}}(t)\eta(t) + J(t)\eta(t)\} =$

ii. Now use the fact that $\eta(t_a) = \eta(t_b) = 0$ to ascertain that $[m\dot{x_{cl}}(t)\eta(t)]|_{t_a}^{t_b} = 0$ and the fact that $m\ddot{x}_{cl}(t) = J(t)$ by definition of the classical path $x_{cl}(t)$, and so we are left with:
$= S[x_{cl},\, J] + S[\eta,\, 0]$
$\blacksquare$

(b) Claim: $\mathcal{Z}[J] = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \exp\left\{\frac{i}{\hbar} S[x_{cl},\, J]\right\}$
Proof:

i. $\mathcal{Z}[J] \equiv \int_{x_a=0}^{x_b=0} \mathcal{D}x \exp\left\{\frac{i}{\hbar} S[x,\, J]\right\} \overset{\text{(a)}}{=}$
$= \int_{x_a=0}^{x_b=0} \mathcal{D}x \exp\left\{\frac{i}{\hbar}\left\{S[x_{cl},\, J] + S[\eta,\, 0]\right\}\right\} =$

ii. But $x_{cl}(t)$ is a constant path (determined by the equations of motion) which doesn't vary during the integration process, so we can pull it out of the integral:
$= \exp\left\{\frac{i}{\hbar} S[x_{cl},\, J]\right\} \times \int_{\text{paths that have both end points at zero}} \mathcal{D}x \exp\left\{\frac{i}{\hbar} S[\eta,\, 0]\right\} =$

iii. Now we make a "change of variable" $x(t) \mapsto x(t) - x_{cl}(t) = \eta(t)$. Since "$\mathcal{D}x_{cl} = 0$", $\mathcal{D}x = \mathcal{D}\eta$. (this needs more mathematical rigor) So we get:
$= \exp\left\{\frac{i}{\hbar} S[x_{cl},\, J]\right\} \times \int_{\text{paths that have both end points at zero}} \mathcal{D}\eta \exp\left\{\frac{i}{\hbar} S[\eta,\, 0]\right\} =$
$= \exp\left\{\frac{i}{\hbar} S[x_{cl},\, J]\right\} \times \int_{\text{paths that have both end points at zero}} \mathcal{D}\eta \exp\left\{\frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \dot{\eta}(t)^2\right]\right\}$

iv. But $\int_{\text{paths that have both end points at zero}} \mathcal{D}\eta \exp\left\{\frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \dot{\eta}(t)^2\right]\right\}$ is just the propagator for the free particle, with both end points at $x = 0$, which have computed in HW1Q3: $\int \mathcal{D}\eta \exp\left\{\frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \dot{\eta}(t)^2\right]\right\} =$
$K_{\text{free particle}}(0,\, 0;\, t_b - t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \exp\left[\frac{im(0-0)^2}{2\hbar(t_b - t_a)}\right] = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}$
and so our result follows.
■

(c) Claim: $S[x_{cl},\, J] = \frac{1}{2m} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, J(t)\, G(t,\, t')\, J(t')$ where $G(t,\, t')$ is the Green's function of $\frac{d^2}{dt^2}$.
Proof:

i. We start from $S[x_{cl},\, J] \equiv \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \dot{x_{cl}}^2 + J x_{cl}\right] =$

ii. We integrate $\dot{x_{cl}}^2 \equiv \dot{x_{cl}} \dot{x_{cl}}$ by parts:
$$= \underbrace{\left.\frac{1}{2} m x_{cl} \dot{x_{cl}}\right|_{t_a}^{t_b}}_{\text{zero as } x_{cl} \text{ is zero on the boundaries}} - \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \ddot{x_{cl}} x_{cl} - J x_{cl}\right] =$$
$= -\int_{t_a}^{t_b} dt x_{cl} \left[\frac{1}{2} m \ddot{x_{cl}} - J\right] =$

iii. Use the classical equation of motion which says $m\ddot{x_{cl}} = J$:
$= -\int_{t_a}^{t_b} dt x_{cl} \left[\frac{1}{2} J - J\right] = \frac{1}{2} \int_{t_a}^{t_b} dt x_{cl} J$

iv. Use the relation $x_{cl}(t) = \frac{1}{m} \int_{t_a}^{t_b} dt' G(t,\, t')\, J(t')$, which is *the* defining property of the Green's function of $\frac{d^2}{dt^2}$:
$= \frac{1}{2} \int_{t_a}^{t_b} dt \frac{1}{m} \int_{t_a}^{t_b} dt' G(t,\, t')\, J(t')\, J(t) =$
$= \frac{1}{2m} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' G(t,\, t')\, J(t')\, J(t)$
■

(d) Claim: $G(t,\, t') = (t_b - t_a)^{-1} \left[\theta(t - t')(t' - t_a)(t - t_b) + \theta(t' - t)(t' - t_b)(t - t_a)\right]$ where the boundary conditions are such that $G(t_a,\, t') = G(t_b,\, t') = 0$.
Proof:

i. Note that we are assuming that $t_a < t_b$ and that $\{t,\, t'\} \subseteq [t_a,\, t_b]$. These assumptions stem from the physical setup of the calculation (a transition amplitutde between two states (at t_a and t_b), where something (at t and t') happens in the middle).

ii. The Green's function obeys: $\frac{d^2}{dt^2} G(t,\, t') = \delta(t - t')$.

iii. Solve this equation in the region (I) where $t < t'$, where we have $\frac{d^2}{dt^2} G_I(t,\, t') = 0$ and so $G_I(t,\, t') = A(t') + B(t') \cdot t$.

iv. Similarly, in the region (II) where $t > t'$ we get $G_{II}(t,\, t') = C(t') + D(t') \cdot t$.

v. Employ the boundary conditions to get that $G_I(t_a,\, t') = A(t') + B(t') \cdot t_a \overset{!}{=} 0$ and $G_{II}(t_b,\, t') = C(t') + D(t') \cdot t_b \overset{!}{=} 0$.
(Note that each boundary condition applies to *either* I or II due to our assumptions about the ordering of t_a, t_b, t and t').
Thus we have that
$A(t') = -B(t')\, t_a$ and $C(t') = -D(t')\, t_b$
and so we can write
$G_I(t,\, t') = B(t')(t - t_a)$ and $G_{II}(t,\, t') = D(t')(t - t_b)$.

vi. Assume that $G(t,\, t')$ is continuous at $t = t'$, thus, $G_I(t,\, t) = G_{II}(t,\, t)$ and so $B(t)(t - t_a) = D(t)(t - t_b)$.

vii. We can integrate the equation of the Green's function from $t' - \varepsilon$ to $t' + \varepsilon$ for some small $\varepsilon > 0$ to get: $\int_{t'-\varepsilon}^{t'+\varepsilon} \frac{d^2}{dt^2} G(t,\, t')\, dt = 1$. Using the fundamental theorem of calculus we get $\frac{d}{dt} G_{II}(t',\, t') - \frac{d}{dt} G_I(t',\, t') = 1$ from whence we get another condition on B and D: $D(t') - B(t') = 1$. All together we have:
$$\begin{cases} B(t)(t - t_a) - D(t)(t - t_b) = 0 \\ B(t) - D(t) = -1 \end{cases}$$

viii. Thus we find that $D(t) = \frac{t - t_a}{t_b - t_a}$ and so $B(t) = \frac{t - t_a}{t_b - t_a} - 1 = \frac{t - t_b}{t_b - t_a}$.

ix. So finally $G(t,\, t') = (t_b - t_a)^{-1} \left[\theta(t - t')(t' - t_a)(t - t_b) + \theta(t' - t)(t' - t_b)(t - t_a)\right]$.
$\blacksquare$

(e) <u>Claim:</u> $\langle x = 0,\, t_b \mid T(\hat{x}(t')\, \hat{x}(t)) \mid x = 0,\, t_a \rangle =$
$= \sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} \left[\theta(t - t')(t_b - t)(t' - t_a) + \theta(t' - t)(t_b - t')(t - t_a)\right]$
(exactly what we found in the previous question using the explicit calculation)
<u>Proof:</u>

i. $\langle x = 0,\, t_b \mid T(\hat{x}(t')\, \hat{x}(t)) \mid x = 0,\, t_a \rangle = \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t') \delta J(t)} \langle 0,\, t_b \mid 0,\, t_a \rangle_J\right]\Big|_{J(t)=0,\, J(t')=0} =$
$= \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t') \delta J(t)} \mathcal{Z}[J]\right]\Big|_{J(t)=0,\, J(t')=0} =$
$= \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t') \delta J(t)} \sqrt{\frac{m}{2\pi i\hbar (t_b - t_a)}} \exp\left\{\frac{i}{\hbar} S[x_{cl},\, J]\right\}\right]\Big|_{J(t)=0,\, J(t')=0} =$

$= \sqrt{\frac{m}{2\pi i\hbar (t_b - t_a)}} \left(\frac{\hbar}{i}\right)^2 \left[\frac{\delta^2}{\delta J(t') \delta J(t)} \exp\left\{\frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, J(t_1)\, G(t_1,\, t_2)\, J(t_2)\right\}\right]\Big|_{J(t)=0,\, J(t')=0} =$

A. At this stage it would be a good time to carry out the functional derivatives:
$\frac{\delta^2}{\delta J(t') \delta J(t)} \exp\left\{\frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, J(t_1)\, G(t_1,\, t_2)\, J(t_2)\right\} =$
$= \frac{\delta}{\delta J(t')} \frac{\delta}{\delta J(t)} \exp\left\{\frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, J(t_1)\, G(t_1,\, t_2)\, J(t_2)\right\} =$
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{\delta}{\delta J(t)} \frac{i}{\hbar} \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, J(t_1)\, G(t_1,\, t_2)\, J(t_2) =$
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, \frac{\delta}{\delta J(t)} J(t_1)\, G(t_1,\, t_2)\, J(t_2) =$
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, G(t_1,\, t_2) \left[\delta(t - t_1)\, J(t_2) + J(t_1)\, \delta(t - t_2)\right] =$
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{i}{\hbar} \frac{1}{2m} \left[\int_{t_a}^{t_b} dt_2\, G(t,\, t_2)\, J(t_2) + \int_{t_a}^{t_b} dt_1\, G(t_1,\, t)\, J(t_1)\right] =$
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{i}{\hbar} \frac{1}{2m} \int_{t_a}^{t_b} dt_1 J(t_1) \left[G(t,\, t_1) + G(t_1,\, t)\right] =$

B. However, note that $G(t,\, t_1) = G(t_1,\, t)$ for Green's functions, so that we have:
$= \frac{\delta}{\delta J(t')} \exp\{\dots\} \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) =$
$= \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) \frac{\delta}{\delta J(t')} \exp\{\dots\} + \exp\{\dots\} \frac{\delta}{\delta J(t')} \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) =$

C. The first term is:
$\frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) \frac{\delta}{\delta J(t')} \exp\{\dots\} =$
$= \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) \exp\{\dots\} \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_2 J(t_2)\, G(t',\, t_2) =$
When we will set $J(t) = J(t') = 0$, this term will vanish.

D. The second term is:
$\exp\{\dots\} \frac{\delta}{\delta J(t')} \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 J(t_1)\, G(t,\, t_1) =$
$= \exp\{\dots\} \frac{i}{\hbar} \frac{1}{m} \int_{t_a}^{t_b} dt_1 \delta(t' - t_1)\, G(t,\, t_1) =$
$= \exp\{\dots\} \frac{i}{\hbar} \frac{1}{m} G(t,\, t')$

ii. So we are left with (after setting $J = 0$):
$\sqrt{\frac{m}{2\pi i\hbar (t_b - t_a)}} \left(\frac{\hbar}{i}\right)^2 \frac{i}{\hbar} \frac{1}{m} G(t,\, t') =$
$= \sqrt{\frac{i\hbar}{2\pi m}} (t_b - t_a)^{-\frac{3}{2}} \left[\theta(t - t')(t_b - t)(t' - t_a) + \theta(t' - t)(t_b - t')(t - t_a)\right]$
■

3. Euclidean Time

(a) Claim: The Green's function equation for the Euclidean simple harmonic oscillator is given by $\left(-\frac{d^2}{d\tau^2} + \omega^2\right) G_E(\tau,\, \tau') = \delta(\tau - \tau')$.
Proof:

i. The Green's function equation for the simple harmonic oscillator is given by $\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,\, t') = \delta(t - t')$

ii. Now we make a change of variables: $\boxed{\tau := it}$:

A. Then $\frac{d}{dt} = \frac{dt}{d\tau} \frac{d}{d\tau} = -i \frac{d}{d\tau}$ and so $\frac{d^2}{dt^2} = -\frac{d^2}{d\tau^2}$.

B. $\delta(t - t') = \delta(-i\tau + i\tau') = \delta(-i(\tau - \tau')) = \frac{\delta(\tau - \tau')}{|-i|} = \delta(\tau - \tau')$.

C. Define $G_E(\tau,\,\tau') := G(-i\tau,\,-i\tau')$.

$\blacksquare$

(b) Claim: $G_E(\tau,\,\tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}$
Proof:

i. Take the Fourier transform of both sides of $\left(-\frac{d^2}{d\tau^2}+\omega^2\right)G_E(\tau,\,\tau') = \delta(\tau-\tau')$:

A. $\delta(\tau-\tau') = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ik(\tau-\tau')}dk$

B. $G_E(\tau,\,\tau') \overset{\text{follows from translational invariance}}{=} G_E(\tau-\tau') = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ik(\tau-\tau')}\tilde{G_E}(k)\,dk$

C. $-\frac{d^2}{d\tau^2}G_E(\tau,\,\tau') = -\frac{d^2}{d\tau^2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-ik(\tau-\tau')}\tilde{G_E}(k)\,dk = \frac{1}{2\pi}\int_{\mathbb{R}}\left(-\frac{d^2}{d\tau^2}\right)e^{-ik(\tau-\tau')}\tilde{G_E}(k)\,dk = \frac{1}{2\pi}\int_{\mathbb{R}} k^2 e^{-ik(\tau-\tau')}\tilde{G_E}(k)\,dk$

ii. Thus in Fourier space our equation is:
$\left(k^2+\omega^2\right)\tilde{G_E}(k) = 1$
and so we conclude that:
$\tilde{G_E}(k) = \frac{1}{k^2+\omega^2}$

iii. In order to compute $G_E(\tau,\,\tau')$ we need the inverse transform of $\tilde{G_E}(k)$:
$G_E(\tau,\,\tau') = \int_{\mathbb{R}} e^{ik(\tau-\tau')}\tilde{G}_E(k)\,dk = \int_{\mathbb{R}} e^{ik(\tau-\tau')}\frac{1}{k^2+\omega^2}dk$

A. The integrand has two poles at $k^2+\omega^2=0$ that is, at $k=\pm i\omega$.

B. Write $\int_{\mathbb{R}} e^{ik(\tau-\tau')}\frac{1}{k^2+\omega^2}dk = \int_{\mathbb{R}}\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)}dk$.

C. Using Cauhcy's theorem we know that $\int_{\mathbb{R}}\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)}dk + \int_{\text{semi-circle}}\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)}dk = \sum_{\text{residues}} 2\pi i Res\left(\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)},\,k_i\right)$ where the contour on the semi-circle is meant as a semicircle (either above or below $\mathbb{R}$, depending on the sign of $\tau-\tau'$–see below) with radius which tends to infinity.

D. Using Jordan's lemma we know that $\int_{\text{semi-circle}}\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)}dk = 0$ as the radius of the semi-circule tends to infinity.

E. If $\tau-\tau'>0$, we need to take the upper semicircle, in which case we get the residue at $k=i\omega$, which, since this is a simple pole, is given by $\lim_{k\to i\omega}(k-i\omega)\frac{e^{ik\left(\tau-\tau'\right)}}{(k-i\omega)(k+i\omega)} = \frac{e^{i(i\omega)\left(\tau-\tau'\right)}}{(2i\omega)}$ and so our integral becomes, in that case, $2\pi i\frac{e^{-\omega\left(\tau-\tau'\right)}}{2i\omega} = \frac{e^{-\omega\left(\tau-\tau'\right)}}{2\omega}$

F. Similarly, if $\tau-\tau'<0$ we have that the integral results in $\frac{e^{\omega\left(\tau-\tau'\right)}}{2\omega}$

iv. $\blacksquare$

(c) Our Green's function was $\frac{1}{k^2+\omega^2}$ which is related to the one of Minkowski space by $\frac{1}{-k^2+\omega^2}$. We used the Feynman prescription to evaluate the integral. We can only do Wick rotation if we don't encounter any poles while rotating: that is only in the Feynman prescription.