# Solutions to Homework No. 1 in Quantum Field Theory 2

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## Exercise 1–Free Particle Propagator

### (a)

- Claim: $\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$
- Proof:
  - Define $I := \int_{\mathbb{R}} dx\, e^{-ax^2}$
  - Then $I^2 = \left(\int_{\mathbb{R}} dx\, e^{-ax^2}\right)^2 = \left(\int_{\mathbb{R}} dx\, e^{-ax^2}\right)\left(\int_{\mathbb{R}} dy\, e^{-ay^2}\right) = \int_{\mathbb{R}^2} dxdy e^{-a\left(x^2+y^2\right)}$
  - Move to polar coordinates:
    $I^2 = \int_{\mathbb{R}^2} rdrd\theta e^{-ar^2} = \left(\int_{[0,\,2\pi]} d\theta\right)\left(\int_{[0,\,\infty)} rdre^{-ar^2}\right) = 2\pi\left(\int_{[0,\,\infty)} rdre^{-ar^2}\right)$
  - Make the change of variable $ar^2 \mapsto \rho$:
    $I^2 = 2\pi \int_{[0,\,\infty)} \frac{d\rho}{2a} e^{-\rho} = \frac{\pi}{a}\int_{[0,\,\infty)} d\rho e^{-\rho} = \frac{\pi}{a}\left(-\left.e^{-\rho}\right|_{\rho\to\infty} - \left(-\left.e^{-\rho}\right|_{\rho=0}\right)\right) = \frac{\pi}{a}(0+1) = \frac{\pi}{a}$
  - Thus we find that $I = \pm\sqrt{\frac{\pi}{a}}$, but because $\exp(\mathbb{R}) \subseteq [0,\,\infty]$, $I \geq 0$ and so $I = \sqrt{\frac{\pi}{a}}$. $\blacksquare$

### (b)

Assume that $a \in \mathbb{C}$ is such that $\frac{a-a^*}{2i} > 0$, that is, $a$'s imaginary part is strictly positive.

- Claim: $\int_{\mathbb{R}} dx\, e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}} e^{-\frac{ik^2}{4a}}$
- Proof:
  - Claim: $\int_{\mathbb{R}} dx\, e^{iax^2} = \sqrt{\frac{\pi i}{a}}$
  - Proof:
    Despite the hint we follow a similar approach to part (a):

* Define $I := \int_{\mathbb{R}} dx\, e^{iax^2}$
* Then $I^2 = \left(\int_{\mathbb{R}} dx\, e^{iax^2}\right)^2 = \int_{\mathbb{R}} dx\, e^{iax^2} \int_{\mathbb{R}} dy\, e^{iay^2} \overset{\text{Fubini's Theorem}}{=} \int_{\mathbb{R}^2} dxdye^{ia\left(x^2+y^2\right)}$
* Moving again to polar coordinates we obtain:
$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} drre^{iar^2} = 2\pi \left[\frac{e^{iar^2}}{2ia}\right]\Big|_0^{\infty} = -\frac{\pi i}{a}\left[\lim_{r\to\infty} e^{iar^2} - 1\right]$
  - Claim: $\lim_{r\to\infty} e^{iar^2} = 0$
  - Proof:
  Since $a$ has a positive imaginary part, we may write it without loss of generality as $a = \alpha + i\beta^2$ where $\{\alpha,\, \beta\} \subset \mathbb{R}$.
  Thus we get $\left|\lim_{r\to\infty} e^{iar^2}\right| = \left|\lim_{r\to\infty} e^{i\left(\alpha+i\beta^2\right)r^2}\right| = \left|\lim_{r\to\infty} e^{i\alpha r^2} e^{-\beta^2 r^2}\right| \leq \lim_{r\to\infty} e^{-\beta^2 r^2} = 0$
* Thus $I^2 = \frac{\pi i}{a}$.
* We thus find that $I = \pm\sqrt{\frac{\pi i}{a}}$. We reject the minus solution because $\exp\left(\mathbb{C}\right) \subseteq \mathbb{H}$.

- Going back to the original integral, $\int_{\mathbb{R}} dx\, e^{iax^2+ikx}$, we complete the square to get:
$iax^2+ikx = i\left(ax^2+kx\right) = i\left[ax^2 + 2\sqrt{a}x\frac{1}{2\sqrt{a}}k + \left(\frac{1}{2\sqrt{a}}k\right)^2 - \left(\frac{1}{2\sqrt{a}}k\right)^2\right] = i\left[\left(\sqrt{a}x + \frac{1}{2\sqrt{a}}k\right)^2 - \left(\frac{1}{2\sqrt{a}}k\right)^2\right] = i\left(\sqrt{a}x + \frac{k}{2\sqrt{a}}\right)^2 - \frac{ik^2}{4a}$
- and so:
$\int_{\mathbb{R}} dx\, e^{iax^2+ikx} = \int_{\mathbb{R}} dx\, e^{i\left(\sqrt{a}x+\frac{k}{2\sqrt{a}}\right)^2 - \frac{ik^2}{4a}} = e^{-\frac{ik^2}{4a}} \int_{\mathbb{R}} dx\, e^{ia\left(x+\frac{k}{2a}\right)^2}$
- We make a change of variable $x + \frac{k}{2a} \mapsto \rho$ to get:
$\int_{\mathbb{R}} dx\, e^{iax^2+ikx} \overset{\text{using the previous result}}{=} e^{-\frac{ik^2}{4a}}\sqrt{\frac{\pi i}{a}}$. ■

## (c)

- Claim: The free particle propagator, which is given by:

$$K\left(x,\, x';\, t-t'\right) = \sqrt{\frac{m}{2\pi i\hbar\left(t-t'\right)}} \exp\left[\frac{im\left(x-x'\right)^2}{2\hbar\left(t-t'\right)}\right] \qquad (1)$$

satisfies the completeness relation

$$K\left(x_b,\, x_a;\, t_b-t_a\right) = \int_{\mathbb{R}} dx\, \left\langle x_b,\, t_b \,\middle|\, x,\, t\right\rangle \left\langle x,\, t \,\middle|\, x_a,\, t_a\right\rangle \qquad (2)$$

for all $t \in \left(t_a,\, t_b\right)$.

- Proof:

1. Let $t \in (t_a,\, t_b)$ be given.
2. $\int_{\mathbb{R}} dx\, \langle x_b,\, t_b \,|\, x,\, t\rangle \langle x,\, t \,|\, x_a,\, t_a\rangle = \int_{\mathbb{R}} dx\, K\left(x_b,\, x;\, t_b - t\right) K\left(x,\, x_a;\, t - t_a\right) =$
   $\int_{\mathbb{R}} dx \sqrt{\frac{m}{2\pi i\hbar(t_b-t)}} \exp\left[\frac{im(x_b-x)^2}{2\hbar(t_b-t)}\right] \sqrt{\frac{m}{2\pi i\hbar(t-t_a)}} \exp\left[\frac{im(x-x_a)^2}{2\hbar(t-t_a)}\right] =$
3. $= \frac{m}{2\pi i\hbar}\sqrt{\frac{1}{(t_b-t)(t-t_a)}} \int_{\mathbb{R}} dx\, \exp\left\{\frac{im}{2\hbar}\left[\frac{(x_b-x)^2}{t_b-t} + \frac{(x-x_a)^2}{t-t_a}\right]\right\} =$
   (a) Rearrange $\frac{(x_b-x)^2}{t_b-t} + \frac{(x-x_a)^2}{t-t_a}$:
   $\frac{(x_b-x)^2}{t_b-t} + \frac{(x-x_a)^2}{t-t_a} = \frac{t_b-t_a}{(t_b-t)(t-t_a)}x^2 + \frac{-2[x_b(t-t_a)+x_a(t_b-t)]}{(t_b-t)(t-t_a)}x + \frac{(t-t_a)x_b^2+(t_b-t)x_a^2}{(t_b-t)(t-t_a)}$
4. Plugging this in we get:
   $= \frac{m}{2\pi i\hbar}\sqrt{\frac{1}{(t_b-t)(t-t_a)}} \exp\left\{\frac{im}{2\hbar(t_b-t)(t-t_a)}\left[(t-t_a)\,x_b^2 + (t_b-t)\,x_a^2\right]\right\}$
   $\times \int_{\mathbb{R}} dx\, \exp\left\{i\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}x^2 + i\frac{-2m[x_b(t-t_a)+x_a(t_b-t)]}{2\hbar(t_b-t)(t-t_a)}x\right\} =$
5. Add a small positive imaginary part to $\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}$ to be able to use our earlier results (it is valid to take $\lim_{\varepsilon\to0}$ out of the integral using the Lebesgue dominated convergence theorem):
   $= \lim_{\varepsilon\to0} \frac{m}{2\pi i\hbar}\sqrt{\frac{1}{(t_b-t)(t-t_a)}} \exp\left\{\frac{im}{2\hbar(t_b-t)(t-t_a)}\left[(t-t_a)\,x_b^2 + (t_b-t)\,x_a^2\right]\right\}$
   $\times \int_{\mathbb{R}} dx\, \exp\left\{i\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)} + i\varepsilon^2\right]x^2 + i\frac{-2m[x_b(t-t_a)+x_a(t_b-t)]}{2\hbar(t_b-t)(t-t_a)}x\right\} =$
6. Use our earlier result for the integral:
   $= \lim_{\varepsilon\to0} \frac{m}{2\pi i\hbar}\sqrt{\frac{1}{(t_b-t)(t-t_a)}} \exp\left\{\frac{im}{2\hbar(t_b-t)(t-t_a)}\left[(t-t_a)\,x_b^2 + (t_b-t)\,x_a^2\right]\right\}$
   $\times \sqrt{\frac{i\pi}{\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}+i\varepsilon^2\right]}} exp\left\{\frac{-i\left[\frac{-2m[x_b(t-t_a)+x_a(t_b-t)]}{2\hbar(t_b-t)(t-t_a)}\right]^2}{4\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}+i\varepsilon^2\right]}\right\} =$
7. Take the $\varepsilon \to 0$ limit because all involved functions are continuous:
   $= \frac{m}{2\pi i\hbar}\sqrt{\frac{1}{(t_b-t)(t-t_a)}} \exp\left\{\frac{im}{2\hbar(t_b-t)(t-t_a)}\left[(t-t_a)\,x_b^2 + (t_b-t)\,x_a^2\right]\right\}$
   $\times \sqrt{\frac{i\pi}{\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}\right]}} \exp\left\{\frac{-i\left[\frac{-2m[x_b(t-t_a)+x_a(t_b-t)]}{2\hbar(t_b-t)(t-t_a)}\right]^2}{4\left[\frac{m(t_b-t_a)}{2\hbar(t_b-t)(t-t_a)}\right]}\right\} =$
8. After rearranging and collecting terms:
   $= \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}} \exp\left[\frac{im(x_b-x_b)^2}{2\hbar(t_b-t_a)}\right]$
   ■

# Exercise 2–Free Particle Wave Function

- The free particle wave function at time $t=0$ is $\psi(x) = \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}$.
- According to our prescription, its value at later times $t>0$ is given by:
  $\psi(x,\, t) = \int_{\mathbb{R}} dx' K(x,\, x';\, t)\,\psi(x') \overset{\text{using the above}}{=} \int_{\mathbb{R}} dx' \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im\left(x-x'\right)^2}{2\hbar t}} \frac{e^{\frac{i}{\hbar}px'}}{\sqrt{2\pi\hbar}} =$
- $= \frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}} \int_{\mathbb{R}} dx' e^{\frac{im\left(x-x'\right)^2}{2\hbar t} + \frac{i}{\hbar}px'}$

- Next make the change of variables $x' - x \mapsto \rho$:
  $= \frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}}e^{\frac{i}{\hbar}px}\int_{\mathbb{R}} d\rho e^{\frac{im\rho^2}{2\hbar t}+\frac{i}{\hbar}p\rho} =$
  $= \frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}}e^{\frac{i}{\hbar}px}\int_{\mathbb{R}} d\rho \lim_{\varepsilon\to 0} e^{i\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)\rho^2+\frac{i}{\hbar}p\rho} =$
  $\overset{\text{Convergence Theorem}}{=} \frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}}e^{\frac{i}{\hbar}px}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} d\rho e^{i\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)\rho^2+\frac{i}{\hbar}p\rho} =$

- Use question 1 (b) to ascertain that: $\int_{\mathbb{R}} d\rho e^{i\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)\rho^2+\frac{i}{\hbar}p\rho} = \sqrt{\frac{i\pi}{\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)}}e^{-\frac{i(p/\hbar)^2}{4\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)}}$

- Thus we have $\psi(x,\,t) = \frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}}e^{\frac{i}{\hbar}px}\lim_{\varepsilon\to 0}\sqrt{\frac{i\pi}{\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)}}e^{-\frac{i(p/\hbar)^2}{4\left(\frac{m}{2\hbar t}+i\varepsilon^2\right)}} =$ $\frac{1}{2\pi\hbar}\sqrt{\frac{m}{it}}e^{\frac{i}{\hbar}px}\sqrt{\frac{i\pi}{m/2\hbar t}}e^{-\frac{i(p/\hbar)^2}{4(m/2\hbar t)}} = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px-\frac{i}{\hbar}\frac{p^2}{2m}t}$

- This is exactly what you would find in Sakurai page 523 on equation $B.1.1$ (making the proper $3D$ to $1D$ transition), hence we are self consistent with old quantum mechanics.

## Exercise 3–A Bit More on Propagators

We know that

$$K(x,\,x';\,t-t') = \sum_\beta e^{-\frac{i}{\hbar}E_\beta(t-t')}\langle x\,|\,\beta\rangle\langle\beta\,|\,x'\rangle \tag{3}$$

### (a)

Let us compute $K$ for a free particle:

- For a free particle, the energy eigenvalues are $\frac{p^2}{2m}$ where $p\in\mathbb{R}$ and so we have:

$$K(x,\,x';\,t-t') = \int_{\mathbb{R}} dp\, e^{-\frac{i}{\hbar}\frac{p^2}{2m}(t-t')}\langle x\,|\,p\rangle\langle p\,|\,x'\rangle \tag{4}$$

- Plug in $\langle x\,|\,p\rangle = \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}$ and $\langle p\,|\,x\rangle = \frac{e^{-\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}$:
  $K(x,\,x';\,t-t') = \int_{\mathbb{R}} dp\, e^{-\frac{i}{\hbar}\frac{p^2}{2m}(t-t')}\frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}\frac{e^{-\frac{i}{\hbar}px'}}{\sqrt{2\pi\hbar}} =$
  $= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp\,\exp\left\{+i\frac{(t'-t)}{2m\hbar}p^2 + i\frac{(x-x')}{\hbar}p\right\} =$
  $= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp\,\lim_{\varepsilon\to 0}\exp\left\{+i\left[\frac{(t'-t)}{2m\hbar}+i\varepsilon^2\right]p^2 + i\frac{(x-x')}{\hbar}p\right\} =$
  $\overset{\cdots}{=} \frac{1}{2\pi\hbar}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dp\,\exp\left\{+i\left[\frac{(t'-t)}{2m\hbar}+i\varepsilon^2\right]p^2 + i\frac{(x-x')}{\hbar}p\right\}$

- Now use the result of part 1. (b) with $a := \left[\frac{(t'-t)}{2m\hbar}+i\varepsilon^2\right]$ and $k = \frac{(x-x')}{\hbar}$ to get:

$= \frac{1}{2\pi\hbar} \lim_{\varepsilon\to 0} \sqrt{\frac{\pi i}{\left[\frac{(t'-t)}{2m\hbar}+i\varepsilon^2\right]}} e^{\frac{-i\left[\frac{\left(x-x'\right)}{\hbar}\right]^2}{4\left[\frac{(t'-t)}{2m\hbar}+i\varepsilon^2\right]}} =$

$= \frac{1}{2\pi\hbar} \sqrt{\frac{\pi i}{\frac{(t'-t)}{2m\hbar}}} e^{\frac{-i\left[\frac{\left(x-x'\right)}{\hbar}\right]^2}{4\frac{(t'-t)}{2m\hbar}}} =$

$= \sqrt{\frac{m}{i(t-t')2\pi\hbar}} \exp\left\{\frac{im\left(x-x'\right)^2}{2\hbar(t-t')}\right\}$

which is precisely what we started with in 1. (c).

## (b)

Let us compute $K$ for a simple harmonic oscillator (Sakurai page 527):

- For a simple harmonic oscillator, the energy eigenvalues are $\hbar\omega\left(n+\frac{1}{2}\right)$ where $n \in \mathbb{N} \cup \{0\}$ and so we have:

$$K\left(x,\, x';\, t-t'\right) = \sum_{n\in\mathbb{N}\cup\{0\}} e^{-\frac{i}{\hbar}\hbar\omega\left(n+\frac{1}{2}\right)\left(t-t'\right)} \left\langle x \,|\, n\right\rangle \left\langle n \,|\, x'\right\rangle \qquad (5)$$

- Plug in $\langle x \,|\, n\rangle = \left[\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right]^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ where $H_n(-)$ are the Hermite polynomials:
  $K\left(x,\, x';\, t-t'\right) = \sum_{n\in\mathbb{N}\cup\{0\}} e^{-\frac{i}{\hbar}\hbar\omega\left(n+\frac{1}{2}\right)\left(t-t'\right)} \left[\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right]^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \times$
  $\left[\left[\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right]^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}\left(x'\right)^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right)\right]^*$
- Since the Hermite polynomials are real we obtain:
  $= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\left(x'\right)^2\right) - \frac{i}{\hbar}\hbar\omega\frac{1}{2}\left(t-t'\right)\right] \sum_{n\in\mathbb{N}\cup\{0\}} \frac{1}{n!} \left\{\frac{\exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]}{2}\right\}^n \times$
  $H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right)$
- $\exists$ a formula called "Mehler's Hermite Polynomial Formula" (from http://mathworld.wolfram.com/MehlersHermitePolynomialFormula.html) which stipulates:

$$\sum_{n\ \in\mathbb{N}\cup\{0\}} \frac{1}{n!}\left(\frac{w}{2}\right)^n H_n(x)\, H_n(y) = \sqrt{\frac{1}{1-w^2}} \exp\left[\frac{2xyw-\left(x^2+y^2\right)w^2}{1-w^2}\right] \tag{6}$$

- We employ this formula with $w := \exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]$ to get:
  $= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\left(x'\right)^2\right) - \frac{i}{\hbar}\hbar\omega\frac{1}{2}\left(t-t'\right)\right] \sqrt{\frac{1}{1-\left\{\exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]\right\}^2}} \times$
  $\exp\left[\frac{2\sqrt{\frac{m\omega}{\hbar}}x\sqrt{\frac{m\omega}{\hbar}}x'\left\{\exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]\right\} - \left(\left[\sqrt{\frac{m\omega}{\hbar}}x\right]^2+\left[\sqrt{\frac{m\omega}{\hbar}}x'\right]^2\right)\left\{\exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]\right\}^2}{1-\left\{\exp\left[-\frac{i}{\hbar}\hbar\omega\left(t-t'\right)\right]\right\}^2}\right]$

- Rearranging we obtain finally:
$= \sqrt{\frac{m\omega}{\pi\hbar}\frac{1}{1-\exp\left[-2\frac{i}{\hbar}\hbar\omega(t-t')\right]}}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+(x')^2\right)-\frac{i}{\hbar}\hbar\omega\frac{1}{2}(t-t')+\frac{2\frac{m\omega}{\hbar}xx'\exp\left[-\frac{i}{\hbar}\hbar\omega(t-t')\right]}{1-\exp\left[-2\frac{i}{\hbar}\hbar\omega(t-t')\right]}\right]$
which can probably be simplified further?

# Exercise 4–Explicit Propagator Calculation

Let a Lagrangian be given as $L(x,\,\dot{x}) := \frac{1}{2}f(x)\,\dot{x}^2 + g(x)\,\dot{x} - V(x)$

## (a)

Let us calculate the Hamiltonian:

- $H \equiv p\dot{x} - L = p\dot{x} - \frac{1}{2}f(x)\,\dot{x}^2 - g(x)\,\dot{x} + V(x)$
- $p \equiv \frac{\partial L}{\partial \dot{x}} = f(x)\,\dot{x} + g(x)$, thus $\dot{x} = \frac{p-g(x)}{f(x)}$. Plug this into the expression of $H$ to get $H$ in terms of $p$ and $x$ instead of $\dot{x}$ and $x$:
$H = p\frac{p-g(x)}{f(x)} - \frac{1}{2}f(x)\left[\frac{p-g(x)}{f(x)}\right]^2 - g(x)\,\frac{p-g(x)}{f(x)} + V(x) =$
$= \frac{1}{2}\frac{[p-g(x)]^2}{f(x)} + V(x)$

## (b)

Let us calculate the propagator for some time $\varepsilon$ where we will eventually take the limit $\lim_{\varepsilon\to 0}$ and so during the calculation we shall ignore all orders above the linear order:

- $K(x,\,x';\,\varepsilon) \equiv \langle x,\,\varepsilon \mid x',\,0\rangle \equiv \left\langle x \;\middle|\; e^{-\frac{i}{\hbar}\hat{H}\varepsilon} \;\middle|\; x'\right\rangle = \left\langle x \;\middle|\; \sum_{n\in\mathbb{N}\cup\{0\}}\frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}\varepsilon\right)^n \;\middle|\; x'\right\rangle =$
$\left\langle x \;\middle|\; \left(1-\frac{i}{\hbar}\hat{H}\varepsilon\right) \;\middle|\; x'\right\rangle + \mathcal{O}\left(\varepsilon^2\right)$
- Insert $\mathbb{1} = \int_{\mathbb{R}} dp\ |p\rangle\langle p|$:
$= \langle x|\left(1-\frac{i}{\hbar}\hat{H}\varepsilon\right)\int_{\mathbb{R}} dp\ |p\rangle\langle p|\,|x'\rangle = \int_{\mathbb{R}} dp\,\langle x|\left(1-\frac{i}{\hbar}\hat{H}\varepsilon\right)\ |p\rangle\langle p \mid x'\rangle$
- Now, $\langle x|\,\hat{H}\ |p\rangle$ is easy to evaluate: $\langle x|\,\hat{H}\ |p\rangle = \langle x|\left(\frac{1}{2}\frac{[\hat{p}-g(\hat{x})]^2}{f(\hat{x})}+V(\hat{x})\right)|p\rangle \overset{\text{operators become numbers}}{=}$
$\left(\frac{1}{2}\frac{[p-g(x)]^2}{f(x)}+V(x)\right)\langle x \mid p\rangle$.
Plugging this in we get:
$= \left[1-\frac{i}{\hbar}\left(\frac{1}{2}\frac{[p-g(x)]^2}{f(x)}+V(x)\right)\varepsilon\right]\int_{\mathbb{R}} dp\,\langle x \mid p\rangle\langle p \mid x'\rangle =$
$= \exp\left\{-\frac{i}{\hbar}\varepsilon\left[\frac{1}{2}\frac{[p-g(x)]^2}{f(x)}+V(x)\right]\right\}\int_{\mathbb{R}} dp\frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}\frac{e^{-\frac{i}{\hbar}px'}}{\sqrt{2\pi\hbar}} + \mathcal{O}\left(\varepsilon^2\right)$
- Rearrange to arrive at:
$= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp\exp\left\{-\frac{i}{\hbar}\varepsilon\left[\frac{1}{2}\frac{[p-g(x)]^2}{f(x)}+V(x)\right]+\frac{i}{\hbar}p(x-x')\right\}$

- Make the change of variable $p - g(x) \mapsto p'$:
$= \frac{1}{2\pi\hbar} \int_{\mathbb{R}} dp' \exp\left\{ -\frac{i}{\hbar}\varepsilon \left[ \frac{1}{2} \frac{(p')^2}{f(x)} + V(x) \right] + \frac{i}{\hbar} (p' + g(x))(x - x') \right\}$
$= \frac{1}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') \right\} \int_{\mathbb{R}} dp' \exp\left\{ i \frac{-\varepsilon}{2\hbar f(x)} (p')^2 + i \frac{(x-x')}{\hbar} p' \right\} =$
$= \frac{1}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') \right\} \int_{\mathbb{R}} dp' \lim_{\delta\to 0} \exp\left\{ i \left[ \frac{-\varepsilon}{2\hbar f(x)} + i\delta^2 \right] (p')^2 + i \frac{(x-x')}{\hbar} p' \right\} =$
$\overset{\cdots}{=} \frac{1}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') \right\} \lim_{\delta\to 0} \int_{\mathbb{R}} dp' \exp\left\{ i \left[ \frac{-\varepsilon}{2\hbar f(x)} + i\delta^2 \right] (p')^2 + i \frac{(x-x')}{\hbar} p' \right\} =$

- Carry out the integration over $p'$ using the formula of 1. (b) with $a := \left[ \frac{-\varepsilon}{2\hbar f(x)} + i\delta^2 \right]$ and $k := \frac{(x-x')}{\hbar}$ to get:
$= \frac{1}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') \right\} \lim_{\delta\to 0} \sqrt{\frac{i\pi}{\left[ \frac{-\varepsilon}{2\hbar f(x)} + i\delta^2 \right]}} \exp\left\{ \frac{-i\left( \frac{(x-x')}{\hbar} \right)^2}{4\left[ \frac{-\varepsilon}{2\hbar f(x)} + i\delta^2 \right]} \right\} =$
$= \frac{1}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') \right\} \sqrt{\frac{i\pi}{\left( \frac{-\varepsilon}{2\hbar f(x)} \right)}} \exp\left\{ \frac{-i\left( \frac{(x-x')}{\hbar} \right)^2}{4\left( \frac{-\varepsilon}{2\hbar f(x)} \right)} \right\} =$
$= \sqrt{\frac{f(x)}{2\pi i\hbar\varepsilon}} \exp\left\{ -\frac{i}{\hbar}\varepsilon V(x) + \frac{i}{\hbar} g(x)(x - x') + \frac{i(x-x')^2 f(x)}{2\hbar\varepsilon} \right\}$
$= \sqrt{\frac{f(x)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon \left[ \frac{1}{2} f(x) \frac{(x-x')^2}{\varepsilon^2} + g(x) \frac{(x-x')}{\varepsilon} - V(x) \right] \right\}$
$= \sqrt{\frac{f(x)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon\, L(x, \dot{x})\big|_{\dot{x} = \frac{x-x'}{\varepsilon}} \right\}$

### (c)

Let us determine the path integral for a finite time interval $t_f - t_i$ ($i$ standing for initial and $f$ for final):

- $K(x_f, x_i; t_f - t_i) \equiv$
$\lim_{n\to\infty} \left( \int_{\mathbb{R}^n} \prod_{j=1}^n dx_j \right) \left\{ K(x_f, x_n; t_f - t_n) \left[ \prod_{j=n}^2 K(x_j, x_{j-1}; t_j - t_{j-1}) \right] K(x_1, x_i; t_1 - t_i) \right\}$
where $\prod_{j=1}^n dx_j$ stands for the Lebesgue measure $d\mu$ on $\mathbb{R}^n$.

- We take the partition of $[t_i, t_f]$ by dividing each $t_j - t_{j-1}$ length so that they are all equal to some positive quantity $\varepsilon$:
$= \lim_{n\to\infty} \left( \int_{\mathbb{R}^n} \prod_{j=1}^n dx_j \right) \left\{ K(x_f, x_n; \varepsilon) \left[ \prod_{j=n}^2 K(x_j, x_{j-1}; \varepsilon) \right] K(x_1, x_i; \varepsilon) \right\}$

- Plug in what we found above for the propagator for small times $\varepsilon$:
$= \lim_{n\to\infty} \left( \int_{\mathbb{R}^n} \prod_{j=1}^n dx_j \right) \{ \sqrt{\frac{f(x_f)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon L\left( x_f, \frac{x_f - x_n}{\varepsilon} \right) \right\}$
$\left[ \prod_{j=n}^2 \sqrt{\frac{f(x_j)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon L\left( x_j, \frac{x_j - x_{j-1}}{\varepsilon} \right) \right\} \right] \sqrt{\frac{f(x_1)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon L\left( x_1, \frac{x_1 - x_i}{\varepsilon} \right) \right\} \}$

- In the limit $n \to \infty$, $\frac{x_j - x_{j-1}}{\varepsilon} \to \dot{x}$, and so:
$= \lim_{n\to\infty} \left( \int_{\mathbb{R}^n} \prod_{j=1}^n dx_j \right) \left\{ \sqrt{\frac{f(x_f)}{2\pi i\hbar\varepsilon}} \exp\left\{ \frac{i}{\hbar}\varepsilon L(x_f, \dot{x}_f) \right\} \right\}$

$\left[\prod_{j=n}^{2}\sqrt{\frac{f(x_j)}{2\pi i\hbar\varepsilon}}\exp\left\{\frac{i}{\hbar}\varepsilon L\left(x_j,\,\dot{x}_j\right)\right\}\right]\sqrt{\frac{f(x_1)}{2\pi i\hbar\varepsilon}}\exp\left\{\frac{i}{\hbar}\varepsilon L\left(x_1,\,\dot{x}_1\right)\right\}$

- $=\lim_{n\to\infty}\left(\sqrt{\frac{1}{2\pi i\hbar\varepsilon}}\right)^{n+1}\left(\int_{\mathbb{R}^n}\prod_{j=1}^{n}dx_j\right)$
  $\left\{\sqrt{f\left(x_f\right)}\exp\left\{\frac{i}{\hbar}\varepsilon L\left(x_f,\,\dot{x}_f\right)\right\}\left[\prod_{j=n}^{1}\sqrt{f\left(x_j\right)}\exp\left\{\frac{i}{\hbar}\varepsilon L\left(x_j,\,\dot{x}_j\right)\right\}\right]\right\}$

- $=\sqrt{f\left(x_f\right)}\,\lim_{n\to\infty}\exp\left\{\frac{i}{\hbar}\varepsilon L\left(x_f,\,\dot{x}_f\right)\right\}\left(\sqrt{\frac{1}{2\pi i\hbar\varepsilon}}\right)^{n+1}$
  $\int_{\mathbb{R}^n}\left(\prod_{j=1}^{n}\sqrt{f\left(x_j\right)}dx_j\right)\exp\left\{\sum_{j=n}^{1}\frac{i}{\hbar}\varepsilon L\left(x_j,\,\dot{x}_j\right)\right\}$

- In that limit we could probably write (though I am not sure how to make this step rigorous nor how to compute $\mathcal{N}$):
  $=\mathcal{N}\int\mathcal{D}x\,\sqrt{f\left(x\right)}\,\exp\left\{\frac{i}{\hbar}\int_{\mathbb{R}}dt\,L\left(x\left(t\right),\,\dot{x}\left(t\right)\right)\right\}$