# Series 1.

Samstag, 22. Februar 2014 19:30

**Name:** Nuhro Ego
**Course:** Quantum Field Theory II (FS 2014)
**Lecturer:** Prof. Dr. Charalampos Anastasiou
**Exercise class:** Caterina Specchia

uzh | eth | zürich

**QFT II**
**Series 1.**

FS 2014
Prof. Anastasiou

**Exercise 1.** ***Free particle propagator***

(a) Calculate the following integral

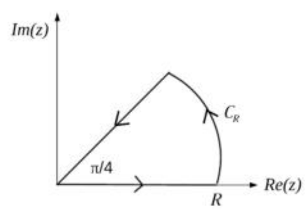
$$\int_{-\infty}^{\infty} dx\; e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

*Hint. Square the left hand side and use polar coordinates.*

(b) Prove the Fresnel integral relation:

$$\int_{-\infty}^{\infty} dx\; e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}} e^{-ik^2/4a}, \qquad (\text{Im } a > 0)$$

*Hint. First compute* $\int_{-\infty}^{\infty} dx\; e^{iax^2}$ *by considering the integral* $\oint_{C_R} dz\; e^{iaz^2}$ *in the complex plane around the contour given below.*




(c) Show explicitly that the free particle propagator

$$K(x,x';t-t') = \left(\frac{m}{2\pi i\hbar(t-t')}\right)^{1/2} e^{\frac{im(x-x')^2}{2\hbar(t-t')}}$$

satisfies the following completeness relation:

$$K(x_b,x_a;t_b-t_a) = \int_{-\infty}^{\infty} \mathrm{d}x\; \langle x_b,t_b|x,t\rangle\langle x,t|x_a,t_a\rangle, \qquad \forall t \in (t_a,t_b)$$

*Hint. Use the integral relation in part (b).*

**Exercise 2.** ***Free particle wave function***

Consider a free particle at $t=0$ (i.e. a wave function $\propto e^{ipx/\hbar}$). Calculate its time evolution using the propagator, i.e.

$$\psi(x,t) = \int dx'\; K(x,x';t)\; \psi(x',t'=0)$$

*Hint. Use the Fresnel integral relation in part (b) of exercise 1.*

**Exercise 3.** ***A bit more on propagators***

Using

$$K(x, x'; t - t') = \sum_\beta e^{-\frac{i}{\hbar} E_\beta (t-t')} \langle x|\beta\rangle\langle\beta|x'\rangle$$

Compute the propagator for

(a) a free particle

(b) the simple harmonic oscillator. Why is this computation more difficult than the previous one? Probably we need a different approach. Let's think about it!

**Exercise 4.** ***Explicit propagator calculation***

Given a Lagrangian of the form:

$$L = \frac{1}{2} f(x)\dot{x}^2 + g(x)\dot{x} - V(x)$$

(a) Calculate the Hamiltonian.

(b) Calculate the propagator for some small time interval $\delta t$.

(c) Determine the path integral expression for the propagator at large times, and show that the measure of the path integration is modified by a factor $\sqrt{f(x)}$ compared with the measure in the free particle case.

# Exercise 1. Free particle propagator

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(a) Z: $\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

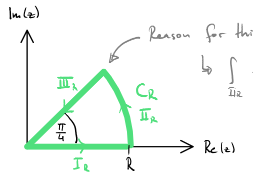
Pf.: $\left(\int_{\mathbb{R}} dx\, e^{-ax^2}\right)^2 = \iint_{\mathbb{R}^2} dx\,dy\; e^{-a(x^2+y^2)} = \int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, e^{-ar^2} \cdot \frac{(-2a)}{(-2a)} = \cancel{2}\pi \cdot \frac{1}{(-\cancel{2}a)} \left[e^{-ar^2}\right]_0^{\infty} = \frac{\pi}{-a}(-1) = \frac{\pi}{a}$

$$\Rightarrow \boxed{\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}}$$

(b) Z: $\int_{\mathbb{R}} dx\, e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}}\, e^{-i\frac{k^2}{4a}}$, $\cancel{\mathrm{Re}}\, a$ ⚠ $\cancel{\mathrm{Im}}\, a > 0$   Fresnel integral relation

Pf: (1) $\oint_{C_R} dz\, e^{iaz^2} = 0$, since $e^{iaz^2} \in \mathrm{Hol}(\mathbb{C})$

$\oint_{C_R} dz\, e^{iaz^2} = \left(\int_{I_R} + \int_{II_R} + \int_{III_R}\right) e^{iaz^2} dz = 0$



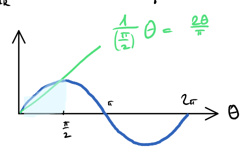

Reason for this path: $e^{ia\underline{\underline{z^2}}}$ & $\underline{a>0}$ ⚠
↳ $\int_{II_R} \to 0$ for $R\to\infty$

$$\Rightarrow \boxed{\int_{I_R} dz\, e^{iaz^2} = -\int_{II_R} dz\, e^{iaz^2} - \int_{III_R} dz\, e^{iaz^2}} \quad (*)$$

- $\int_{II_R} dz\, e^{iaz^2} = \int_0^{\frac{\pi}{4}} dt\, \dot{\gamma}_R(t)\, e^{ia\gamma(t)^2} = \int_0^{\frac{\pi}{4}} dt\, iRe^{it} e^{iaR^2 e^{2it}}$

  $\gamma_R(t) = Re^{it}$, $t\in[0,\frac{\pi}{4}]$; $\dot{\gamma}_R(t) = iRe^{it}$

$\left|\int_{II_R} dz\, e^{iaz^2}\right| = \left|\int_0^{\frac{\pi}{4}} dt\; iR\, e^{it} e^{iaR^2 e^{2it}}\right| \le R\int_0^{\frac{\pi}{4}} e^{-aR^2 \sin 2t}\, dt = (**)$




- $\sin\theta \ge \frac{2\theta}{\pi}$, $\theta\in[0,\frac{\pi}{2}]$ (konkav)

  $-\alpha\sin\theta \le -\alpha\frac{2\theta}{\pi}$, $\alpha>0$

- $e^x$ monotone rising

$\Rightarrow e^{-\alpha\sin\theta} \le e^{-\alpha\frac{2\theta}{\pi}}$, $\theta\in[0,\frac{\pi}{2}]$, $\alpha>0$ (1)

$(**) \overset{(1)}{\le} R\int_0^{\frac{\pi}{4}} e^{-aR^2\frac{2(2t)}{\pi}} dt = R\int_0^{\frac{\pi}{4}} e^{-\frac{4aR^2}{\pi}t} dt = \frac{R}{-\frac{4aR^2}{\pi}}\, e^{-\frac{4aR^2}{\pi}t}\Big|_0^{\frac{\pi}{4}} =$

$= +\frac{\pi}{4aR}\left(1 - e^{-\frac{\cancel{4}aR^2}{\cancel{\pi}}\frac{\cancel{\pi}}{\cancel{4}}}\right) = \frac{\pi}{4aR}\left(1-e^{-aR^2}\right) \xrightarrow[R\to\infty]{a>0\ ⚠} 0$

( or via Jordan's lemma: $\int_{II} e^{iaz^2} dz = \int_{II} \underbrace{e^{ia(z^2-z)}}_{g(z)} e^{iaz} dz = \int_{II} g(z)\, e^{iaz} dz \overset{\text{Jordan's lemma}}{\le} \frac{\pi}{a}\max_{t\in[0,\frac{\pi}{4}]} \left|g(Re^{it})\right|$

$\left|g(Re^{it})\right| = \left|e^{iaRe^{it}(Re^{it}-1)}\right| = e^{-aR^2\sin 2t}\, e^{aR\sin t} = e^{-aR^2(\sin 2t - \frac{1}{R}\sin t)} = \ldots$ )

- $\int_{III_R} dz\, e^{iaz^2} = \int_R^0 dt\, e^{i\frac{\pi}{4}} e^{iat^2 \overset{=i}{e^{i\frac{\pi}{2}}}} = e^{i\frac{\pi}{4}}\int_R^0 e^{-at^2} = -e^{i\frac{\pi}{4}}\int_0^R e^{-at^2} \xrightarrow{R\to\infty} -e^{i\frac{\pi}{4}}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a}}$

  $\gamma_R(t) = t\,e^{i\frac{\pi}{4}}$, $\dot{\gamma}_R(t) = e^{i\frac{\pi}{4}}$

$\Rightarrow \int_{\mathbb{R}} dx\, e^{iax^2} = 2\cdot\lim_{R\to\infty}\int_{I_R} e^{iaz^2} dz \overset{(*)}{=} 2\cdot\lim_{R\to\infty}\left(-\int_{II_R} e^{iaz^2} dz - \int_{III_R} e^{iaz^2} dz\right) =$

$= 2\cdot\left(0 + e^{i\frac{\pi}{4}}\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)$

$$\Rightarrow \boxed{\int_{\mathbb{R}} dx\, e^{iax^2} = e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{a}} = \sqrt{\frac{i\pi}{a}},\quad a>0}$$

(2) $ax^2 + kx = a\left(x^2 + 2\frac{k}{2a}x + \left(\frac{k}{2a}\right)^2 - \left(\frac{k}{2a}\right)^2\right) = a\underbrace{\left(x + \frac{k}{2a}\right)}_{\tilde{x}}{}^2 - \frac{k^2}{4a}$

$$\boxed{\int_{\mathbb{R}} dx\, e^{iax^2+ikx}} = \int_{\mathbb{R}} dx\, e^{ia(\underbrace{x+\frac{k}{2a}}_{\tilde{x}})^2} e^{-i\frac{k^2}{4a}} = e^{-i\frac{k^2}{4a}} \int_{\mathbb{R}} d\tilde{x}\, e^{ia\tilde{x}^2} = e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{a}}\, e^{-i\frac{k^2}{4a}} = \boxed{\sqrt{\frac{i\pi}{a}}\, e^{-i\frac{k^2}{4a}}} \quad a>0$$

(c) Free particle propagator (1D): $K(x, x'; t-t') = \left(\frac{m}{2\pi i\hbar (t-t')}\right)^{\frac{1}{2}} e^{i\frac{m(x-x')^2}{2\hbar(t-t')}}$

**Z:** $K(x_b, x_a; t_b - t_a) = \int_{\mathbb{R}} dx\, \langle x_b, t_b | x, t\rangle\langle x, t | x_a, t_a\rangle \quad \forall t \in (t_a, t_b)$

**Pf.:** Heisenberg picture: $|x,t\rangle = \hat{U}^{+}(t,0)|x\rangle \overset{\hat{H}(t)=\hat{H}}{=} e^{+\frac{i}{\hbar}\hat{H}t}|x\rangle$

$|x_a, t_a\rangle = \hat{U}^{+}(t_a, 0)|x_a\rangle \overset{\hat{H}(t)=\hat{H}}{=} e^{+\frac{i}{\hbar}\hat{H}t_a}|x_a\rangle$

General proof: $\boxed{\int_{\mathbb{R}} dx\, \langle x_b, t_b | x, t\rangle\langle x, t | x_a, t_a\rangle} = \underbrace{\int_{\mathbb{R}} dx\, \langle x_b, t_b | \hat{U}^{+}(t,0) |x\rangle\langle x|}_{=\mathbb{1}} \hat{U}(t,0) |x_a, t_a\rangle =$

$= \langle x_b, t_b | \underbrace{\hat{U}^{+}(t,0)\hat{U}(t,0)}_{\mathbb{1}} |x_a, t_a\rangle = \boxed{\langle x_b, t_b | x_a, t_a\rangle} = \langle x_b | U(t_b, 0)\, U^{+}(t_a, 0) | x_a\rangle =$

$= \langle x_b | U(t_b, 0)\, U(0, t_a) | x_a\rangle = \boxed{\langle x_b | U(t_b, t_a) | x_a\rangle \equiv K(x_b, t_b; x_a, t_a)}$ $\square$

Calculated: $\int_{\mathbb{R}} dx\, \langle x_b, t_b | x, t\rangle\langle x, t | x_a, t_a\rangle = \langle x_b | U(t_b, t_a) | x_a\rangle =$

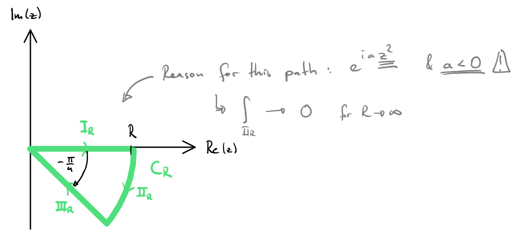
$= \langle x_b | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}(t_b - t_a)} | x_a\rangle = \int_{\mathbb{R}} dp\, e^{-\frac{i}{\hbar}\frac{p^2}{2m}(t_b-t_a)} \langle x_b | p\rangle\langle p | x_a\rangle =$

$= \int_{\mathbb{R}} dp\, e^{i\left(\frac{-(t_b-t_a)}{2m\hbar}\right)p^2} \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^2 e^{\frac{i}{\hbar}p(x_b - x_a)} =$

$= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp\, e^{i\underbrace{\left(\frac{-(t_b-t_a)}{2m\hbar}\right)}_{⚠ a<0 ⚠}p^2} e^{i\underbrace{\left(\frac{x_b - x_a}{\hbar}\right)}_{k\in\mathbb{R}}p} = (*)$

Claim: $\boxed{\int_{\mathbb{R}} dx\, e^{iax^2+ikx} = e^{-i\frac{\pi}{4}}\sqrt{\frac{\pi}{|a|}}\, e^{-i\frac{k^2}{4a}} = \sqrt{\frac{\pi}{i|a|}}\, e^{+i\frac{k^2}{4|a|}} \quad \text{for } \underline{a<0} ⚠}$

Proof: Same proof as in b), just take this path & modify formulas slightly:




$$\boxed{\text{Claim: } \int_{\mathbb{R}} dx\, e^{iax^2+ikx} = \sqrt{\frac{i\pi}{a}}\, e^{-i\frac{k^2}{4a}}, \text{ for } \underline{a\in\mathbb{R}\setminus\{0\}} ⚠}$$

Proof: follows from both results $(a>0)$ & $(a<0)$ and $i^2 = -1$ $\square$

$(*) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp\, e^{i\underbrace{\left(\frac{-(t_b-t_a)}{2m\hbar}\right)}_{a}p^2} e^{i\underbrace{\left(\frac{x_b-x_a}{\hbar}\right)}_{k\in\mathbb{R}}p} \overset{\text{generalised F-S-Rel.}}{=}$

$= \frac{1}{2\pi\hbar}\cdot\sqrt{\frac{i\pi}{\left(\frac{-(t_b-t_a)}{2m\hbar}\right)}}\, e^{-i\frac{\frac{(x_b-x_a)^2}{\hbar^2}}{4\left(\frac{-(t_b-t_a)}{2m\hbar}\right)}} =$

$= \frac{1}{2\pi\hbar}\sqrt{\frac{2m\hbar\pi}{i(t_b-t_a)}}\, e^{+i\frac{\cancel{2}m\cancel{\hbar}(x_b-x_a)^2}{\cancel{4}_2\hbar^{\cancel{2}}(t_b-t_a)}} =$

$= \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}\, e^{i\frac{m(x_b-x_a)^2}{2\hbar(t_b-t_a)}} \equiv K(x_b, x_a; t_b - t_a)$ $\square$

# Exercise 2. Free particle wave function

Sonntag, 23. Februar 2014 15:13

Free particle wave function: $\quad \psi(x,0) = \langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$

General free particle wave function: $\quad \psi(x,0) = \langle x | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \phi(p)\, e^{\frac{i}{\hbar}px}\, dp \quad$, where $\underset{\downarrow}{\hat{=}\text{ frequencies}}\ \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \psi(x,0)\, e^{-\frac{i}{\hbar}px}\, dx$ (by Plancherel's theorem)

Free particle propagator: $\quad K(x,x';t-t') = \left( \frac{m}{2\pi i \hbar (t-t')} \right)^{\frac{1}{2}} e^{i \frac{m(x-x')^2}{2\hbar(t-t')}}$

**To do:** Calculate $\psi(x,t) = \langle x | \hat{U}(t,0) | p \rangle = \int dx' \langle x | \hat{U}(t,0) | x' \rangle \langle x' | p \rangle = \int dx'\, K(x,x';t)\, \psi(x',0)$

**Solution:**

$$\boxed{\psi(x,t)} = \int_{\mathbb{R}} dx'\, K(x,x';t)\, \psi(x',0) =$$

$\tilde{x} = x' - x \quad \Rightarrow x' = \tilde{x} + x$

$d\tilde{x} = dx'$

$$= \int_{\mathbb{R}} dx' \left( \frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}} e^{i \frac{m}{2\hbar t}(x-x')^2} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px'} \overset{\downarrow}{=}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left( \frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar}px} \int_{\mathbb{R}} d\tilde{x}\, e^{i \underbrace{\left(\frac{m}{2\hbar t}\right)}_{a} \tilde{x}^2 + i \underbrace{\left(\frac{p}{\hbar}\right)}_{u} \tilde{x}} \quad \overset{\text{F-}\int\text{-Rel.}}{=}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px} \left( \frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}} \sqrt{\frac{i\pi}{\frac{m}{2\hbar t}}}\; e^{-i \frac{\frac{p^2}{\hbar^2}}{4\left(\frac{m}{2\hbar t}\right)}} =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px} \cancel{\left( \frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}}} \cancel{\left( \frac{2\pi i \hbar t}{m} \right)^{\frac{1}{2}}} e^{-i \frac{\cancel{2\hbar} t\, p^2}{\underset{2}{\cancel{4}} m \hbar^{\cancel{2}}}} =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} =$$

$$= \boxed{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)}} \quad \square$$

or much shorter: $\quad \underline{\psi(x,t)} = \langle x | \hat{U}(t,0) | p \rangle = \langle x | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t} | p \rangle = e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \overbrace{\langle x | p \rangle}^{\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}} = \underline{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)}} \quad \square$

# Exercise 3. A bit more on propagators

Sonntag, 23. Februar 2014 16:55

Propagator (general): $K(\vec{x},t;\vec{x}',t') = \langle \vec{x} | \hat{U}(t,t') | \vec{x}' \rangle$

$\rightarrow$ Propagator ($\hat{H}(t) = \hat{H}$ & 1D): $K(x,x';t-t') = \langle x | \hat{U}(t-t') | x' \rangle = \langle x | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | x' \rangle \overset{\hat{H}|\beta\rangle = E_\beta|\beta\rangle}{\overset{d}{=}} \sum_\beta e^{-\frac{i}{\hbar}E_\beta(t-t')} \langle x|\beta\rangle\langle\beta|x\rangle$

**To do:** Compute the Propagator for

(a) a free particle

(b) simple harmonic oscillator

**Solution:** (a) $\hat{H}(t) \equiv \hat{H} = \frac{\hat{p}^2}{2m} \Rightarrow [\hat{H},\hat{p}] = 0 \Rightarrow$ complete set of common EV: $\{|p\rangle\}$

$$\boxed{K(x,x';t-t')} = \langle x | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}(t-t')} | x' \rangle = \int dp \; e^{-\frac{i}{\hbar}\frac{p^2}{2m}(t-t')} \langle x|p\rangle\langle p|x'\rangle =$$

$$= \frac{1}{2\pi\hbar}\int dp \; e^{i\underbrace{\left(-\frac{(t-t')}{2m\hbar}\right)}_{a} p^2 + i\underbrace{\left(\frac{x-x'}{\hbar}\right)}_{b} p} \overset{\text{F-}\int\text{-Rel.}}{=}$$

$$= \frac{1}{2\pi\hbar}\left(\frac{i\pi}{\left(-\frac{(t-t')}{2m\hbar}\right)}\right)^{\frac{1}{2}} e^{-i\frac{\left(\frac{x-x'}{\hbar}\right)^2}{4\left(-\frac{(t-t')}{2m\hbar}\right)}} = \frac{1}{2\pi\hbar}\left(\frac{2\pi m\hbar}{i(t-t')}\right)^{\frac{1}{2}} e^{+i\frac{2m\cancel{\hbar}(x-x')^2}{2\cancel{4}\hbar^{\cancel{2}}(t-t')}} =$$

$$= \boxed{\left(\frac{m}{2\pi i\hbar(t-t')}\right)^{\frac{1}{2}} e^{+i\frac{m(x-x')^2}{2\hbar(t-t')}}}$$

(b) $\hat{H}(t) \equiv \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{x}^2 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2})$

$[\hat{H},\hat{N}] = 0 \Rightarrow$ complete set of common EV: $\{|n\rangle\}$

- $\hat{N}|n\rangle = n|n\rangle$, $n \in \mathbb{N}$
- $\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle = E_n|n\rangle$, $E_n = \hbar\omega(n+\frac{1}{2})$
- $\langle x|n\rangle = \alpha\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\frac{\xi^2}{2}}$, $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$, $\xi = \sqrt{\frac{m\omega}{\hbar}}x$
- $H_n(x) = (-1)^n e^{-x^2}\frac{d^n}{dx^n}(e^{-x^2})$

$$K(x,x';t-t') = \langle x | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | x' \rangle = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$$

$$1 = \int dp \; \langle x | e^{-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{x}^2\right)(t-t')} | p \rangle\langle p|x'\rangle =$$

Question: $\langle x | e^{f_1(\hat{p}) + f_2(\hat{x})} | p \rangle \overset{?}{=} e^{f_1(p)+f_2(x)}\langle x|p\rangle$ ?

$$= \int dp \; e^{-\frac{i}{\hbar}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega x^2\right)(t-t')} \langle x|p\rangle\langle p|x'\rangle =$$

$$= \int_{\mathbb{R}} dp \; e^{-\frac{i}{\hbar}\left(\frac{1}{2}m\omega x^2\right)(t-t')} e^{i\underbrace{\left(-\frac{(t-t')}{2m\hbar}\right)}_{a} p^2 + i\underbrace{\left(\frac{x-x'}{\hbar}\right)}_{b} p} \overset{\text{F-}\int\text{-Rel.}}{=}$$

$$= e^{-\frac{i}{\hbar}\left(\frac{1}{2}m\omega x^2\right)(t-t')} \left(\frac{m}{2\pi i\hbar(t-t')}\right)^{\frac{1}{2}} e^{+\frac{im(x-x')^2}{2\hbar(t-t')}} =$$

could also be $x'^2$ $\hookrightarrow$ error?

$$= \boxed{\left(\frac{m}{2\pi i\hbar(t-t')}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\frac{m}{2}\left(\frac{(x-x')^2}{(t-t')^2} - \omega x^2\right)(t-t')}}$$

(I think it's wrong, see )

$$2 = \sum_{n\in\mathbb{N}} e^{-\frac{i}{\hbar}E_n(t-t')} \langle x|n\rangle\langle n|x'\rangle = \sum_{n=0}^{\infty} e^{-i\omega(n+\frac{1}{2})(t-t')} |\langle x|n\rangle|^2 =$$

$$= \sum_{n=0}^{\infty} \alpha^2\frac{1}{2^n n!}H_n^2(\xi)e^{-\xi^2} e^{-i\omega(n+\frac{1}{2})(t-t')} =$$

$$= \alpha^2 e^{-\xi^2} e^{-i\omega\frac{t-t'}{2}} \sum_{n=0}^{\infty} \frac{1}{2^n n!}\left(\underbrace{(-1)^{2n}}_{1} e^{-\xi^2}\left(\frac{d^n}{d\xi^n}(e^{-\xi^2})\right)^2\right) e^{-i\omega n(t-t')} =$$

$$= \alpha^2 e^{-\xi^2} e^{-i\omega\frac{t-t'}{2}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left( \underbrace{(-1)^{2n}}_{1} e^{-\xi^2} \left( \frac{d^n}{d\xi^n} \left( e^{-\xi^2} \right) \right)^2 \right) e^{-i\omega n (t-t')} =$$

$$= \alpha^2 e^{-2\xi^2} e^{-i\omega\frac{t-t'}{2}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left[ \frac{d^n}{d\xi^n} \left( e^{-\xi^2} \right) \right]^2 e^{-i\omega n (t-t')} = \ldots \quad ?$$

③ Baker – Campbell – Hausdorff – Formula: $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{[\hat{A},\hat{B}]}{2}} = e^{\hat{B}} e^{\hat{A}} e^{\frac{[\hat{A},\hat{B}]}{2}}$ if $\begin{cases} [\hat{A}, [\hat{A},\hat{B}]] = 0 \\ [\hat{B}, [\hat{B},\hat{A}]] = 0 \end{cases}$

- $[\hat{x}^2, \hat{p}^2] = \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x} = \hat{x}\hat{p}[\hat{x},\hat{p}] + \hat{x}[\hat{x},\hat{p}]\hat{p} + \hat{p}[\hat{x},\hat{p}]\hat{x} + [\hat{x},\hat{p}]\hat{p}\hat{x}$
  $= i\hbar\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} + i\hbar\hat{p}\hat{x} + i\hbar\hat{p}\hat{x} = 2i\hbar\{\hat{x},\hat{p}\}$

- $[\hat{x}, \{\hat{x},\hat{p}\}] = [\hat{x}, \hat{x}\hat{p}] + [\hat{x}, \hat{p}\hat{x}] = \hat{x}[\hat{x},\hat{p}] + \underbrace{[\hat{x},\hat{x}]}_{=0}\hat{p} + \hat{p}\underbrace{[\hat{x},\hat{x}]}_{=0} + [\hat{x},\hat{p}]\hat{x} =$
  $= 2i\hbar\hat{x}$

- $[\hat{p}, \{\hat{x},\hat{p}\}] = [\hat{p}, \hat{x}\hat{p}] + [\hat{p}, \hat{p}\hat{x}] = \hat{x}\underbrace{[\hat{p},\hat{p}]}_{=0} + [\hat{p},\hat{x}]\hat{p} + \hat{p}[\hat{p},\hat{x}] + \underbrace{[\hat{p},\hat{p}]}_{=0}\hat{x} =$

  $= -2i\hbar\hat{p}$

- $[\hat{x}^2, [\hat{x}^2,\hat{p}^2]] = 2i\hbar\left( \hat{x}[\hat{x}, \{\hat{x},\hat{p}\}] + [\hat{x}, \{\hat{x},\hat{p}\}]\hat{x} \right) = 2i\hbar \cdot 2 \cdot (2i\hbar\hat{x})\hat{x} =$
  $= 2(2i\hbar\hat{x})^2 = -8\hbar^2\hat{x}^2 \neq 0$

- $[\hat{p}^2, [\hat{p}^2,\hat{x}^2]] = -2i\hbar\left( \hat{p}[\hat{p}, \{\hat{x},\hat{p}\}] + [\hat{p}, \{\hat{x},\hat{p}\}]\hat{p} \right) = -2i\hbar \cdot 2 \cdot (-2i\hbar\hat{p})\hat{p} =$
  $= 2(-2i\hbar\hat{p})^2 = -8\hbar^2\hat{p}^2 \neq 0$

$\Rightarrow$ B-C-H-formula not useful :(

# Exercise 4. Explicit propagator calculation

Sonntag, 23. Februar 2014 20:07

**Given:** $\cdot\ \mathcal{L}(x,\dot{x},t) = \frac{1}{2} f(x)\,\dot{x}^2 + g(x)\,\dot{x} - V(x)$

**(a) To do:** Calculate $\mathcal{H}(x,p,t)$

**Solution:**
$$\mathcal{H}(x,p,t) = \mathcal{L}^{*(\dot{x}\to p)}(x,p,t) = \sup_{\dot{x}\in\mathbb{R}} \left\{ \dot{x}p - \mathcal{L}(x,\dot{x},t) \right\} =$$
$$= \dot{x}(x,p,t)\,p - \mathcal{L}(x,\dot{x}(x,p,t),t) \qquad \text{for } \dot{x}(x,p,t) \text{ solution to } \frac{\partial\mathcal{L}(x,\dot{x},t)}{\partial\dot{x}} = p$$

**Here:** $\frac{\partial\mathcal{L}}{\partial\dot{x}}(x,\dot{x},t) = f(x)\,\dot{x} + g(x) = p \quad \Rightarrow \quad \boxed{\dot{x}(x,p,t) = \frac{p-g(x)}{f(x)}}$

$$\boxed{\mathcal{H}(x,p,t)} = \dot{x}(x,p,t)\,p - \mathcal{L}(x,\dot{x}(x,p,t),t) = f(x)\,\dot{x}^2 + \cancel{g(x)\dot{x}} - \frac{1}{2}f(x)\,\dot{x}^2 - \cancel{g(x)\dot{x}} + V(x) =$$
$$= \frac{1}{2}f(x)\,\dot{x}^2 + V(x) =$$
$$= \frac{1}{2}f(x)\left(\frac{p-g(x)}{f(x)}\right)^2 + V(x) =$$
$$= \boxed{\frac{(p-g(x))^2}{2f(x)} + V(x)}$$

**(b) To do:** Calculate $K(x,t+\delta t;x',t)$ for $\delta t \to 0$

**Solution:** $K(x,x';\delta t) = \langle x|\,\hat{U}(\delta t)\,|x'\rangle = \langle x|\,e^{-\frac{i}{\hbar}\hat{H}\delta t}\,|x'\rangle = (*)$

$\cdot\ \hat{H} = \mathcal{H}(\hat{x},\hat{p}) = \frac{(\hat{p}-g(\hat{x}))^2}{2f(\hat{x})} + V(\hat{x}) = \frac{\hat{p}^2 + g(\hat{x})^2 + \{\hat{p},g(\hat{x})\}}{2f(\hat{x})} + V(\hat{x})$

⚠ careful !