Solutions to Homework No. 3 in Quantum Field Theory II

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1. The Lagrangian for a free scalar field is given by $\mathcal{L} = \frac{1}{2}\eta\left(\partial\left(\phi\right),\, \partial\left(\phi\right)\right) - \frac{1}{2}m^2\phi^2$.

   (a) <u>Claim:</u> The equation of motion for ϕ is $\boxed{trace\left(\partial\left(\partial\left(\phi\right)\right)\right) + m^2\phi = 0}$.
   <u>Proof:</u>

      i. We begin from the Euler-Lagrange equation, which is (now switching back to index-notation) $\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0$:

         A. $\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial}{\partial\phi}\left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2\right] = -m^2\phi$

         B. $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}\left(\partial_\nu\phi\right)\left(\partial^\nu\phi\right) - \frac{1}{2}m^2\phi^2\right] = \frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}\eta^{\alpha\beta}\left(\partial_\alpha\phi\right)\left(\partial_\beta\phi\right)\right] = \frac{1}{2}\eta^{\alpha\beta}\left[\left(\partial_\alpha\phi\right)\delta^\mu{}_\beta + \delta^\mu{}_\alpha\left(\partial_\beta\phi\right)\right] = \partial^\mu\phi$
         and hence:
         $\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = \partial_\mu\left[\partial^\mu\phi\right] \equiv trace\left(\partial\left(\partial\left(\phi\right)\right)\right)$.
         ■

   (b) <u>Claim:</u> $\langle 0 \mid T\left(\phi\left(x\right)\phi\left(y\right)\right) \mid 0\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^4}\frac{d^4q}{(2\pi)^4}\frac{ie^{-i\eta(q,\,(x-y))}}{\eta(q,\,q)-m^2+i\varepsilon}$
   <u>Proof:</u>

      i. The conjugate momentum-field to ϕ is $\pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi$

      ii. Compute the Hamiltonian:
      $H \equiv \int_{\mathbb{R}^3} d^3\vec{x}\left[\pi\left(t,\,\vec{x}\right)\partial_0\phi\left(t,\,\vec{x}\right) - \mathcal{L}\left(t,\,\vec{x}\right)\right] =$
      $= \int_{\mathbb{R}^3} d^3\vec{x}\left\{\frac{1}{2}\left[\pi\left(t,\,\vec{x}\right)\right]^2 + \frac{1}{2}\left[\vec{\nabla}\phi\left(t,\,\vec{x}\right)\right]^2 + \frac{1}{2}m^2\left[\phi\left(t,\,\vec{x}\right)\right]^2\right\}$

      iii. To quantize the theory, let ϕ and π now be *operators* (or rather operator valued distributions). Assume *equal time* commutation relations hold: $\left[\phi\left(t,\,\vec{x}\right),\,\pi\left(t,\,\vec{y}\right)\right] \stackrel{!}{=} i\delta^{(3)}\left(\vec{x}-\vec{y}\right)$

      iv. Writing the equation of motion for ϕ in momentum space:
      $\left(\partial_t^2 + \left|\vec{p}\right|^2 + m^2\right)\tilde{\phi}\left(t,\,\vec{p}\right) = 0$
      results in an equation of a simple harmonic oscillator with frequency $\omega_{\vec{p}} := \sqrt{\left|\vec{p}\right|^2 + m^2}$. Hence we conclude that for each value of $\vec{p}$ there is an indepedent oscillator. We may retrace the

solution of the simple harmonic oscillator for a single particle of frequency ω to our case, generalizing to an infinite number of harmonic oscillators indexed by $\vec{p}$.

v. Accordingly, $\forall\vec{p}\in\mathbb{R}^3$ and $\forall t\in\mathbb{R}$, define two new operator valued distributions:
   A. $a_{\vec{p}}(t) := \frac{1}{2}\left[\sqrt{2\omega_{\vec{p}}}\int_{\mathbb{R}^3} d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\phi(t\,,\vec{x}) + i\sqrt{\frac{2}{\omega_{\vec{p}}}}\int_{\mathbb{R}^3} d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\pi(t\,,\vec{x})\right]$
   B. $a^\dagger_{\vec{p}}(t) := \frac{1}{2}\left[\sqrt{2\omega_{\vec{p}}}\int_{\mathbb{R}^3} d^3\vec{x}e^{+i\vec{p}\cdot\vec{x}}\phi(t\,,\vec{x}) - i\sqrt{\frac{2}{\omega_{\vec{p}}}}\int_{\mathbb{R}^3} d^3\vec{x}e^{+i\vec{p}\cdot\vec{x}}\pi(t\,,\vec{x})\right]$

vi. Thus we assume $\exists$ we have a Hilbert space, $\mathcal{H}$, which is the "infinite direct sum" of the Hilbert space of each oscillator indexed by $\vec{p}$: $\mathcal{H}=\bigoplus_{\vec{p}\in\mathbb{R}^3}\mathcal{H}_{\vec{p}}$. This is called the Fock space. We assume that in it, $\exists$ a special state, $|0\rangle$, such that $a_{\vec{p}}|0\rangle = 0\forall\vec{p}\in\mathbb{R}^3$. Then, using the same procedure as for a system with a *single* harmonic oscillator, we can show that $a^\dagger_{\vec{p}}|0\rangle$ corresponds to a state of particle of "type" $\vec{p}$ etc.

vii. Write the field / momentum-field in terms of these two newly defined operators:
   $\phi(t,\,\vec{x}) = \int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)\,e^{-i\vec{p}\cdot\vec{x}}\right)$ and
   $\pi(t,\,\vec{x}) = \int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}(t)\,e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}(t)\,e^{-i\vec{p}\cdot\vec{x}}\right)$

viii. Plug in those expressions of ϕ and π in terms of a and $a^\dagger$ into H to obtain:
   $H=\int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}\left(a_{\vec{p}}(t)\,a^\dagger_{\vec{p}}(t) + a^\dagger_{\vec{p}}(t)\,a_{\vec{p}}(t)\right)$

ix. Compute the commutator $\left[a_{\vec{p}}(t)\,,\,a^\dagger_{\vec{q}}(t)\right]$ using the knowledge of $[\phi(t,\,\vec{x})\,,\,\pi(t,\,\vec{y})]$:
   $\left[a_{\vec{p}}(t)\,,\,a^\dagger_{\vec{q}}(t)\right] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})$

x. Use that to rewrite the Hamiltonian now as:

$$H=\int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\omega_{\vec{p}}\left[a^\dagger_{\vec{p}}(t)\,a_{\vec{p}}(t) + \underbrace{\frac{1}{2}(2\pi)^3\,\delta^{(3)}(0)}_{\text{constant infinite term which we shift away}}\right] =$$

   $\int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\omega_{\vec{p}}a^\dagger_{\vec{p}}(t)\,a_{\vec{p}}(t)$

xi. Using this result we may compute the following commutators:
   A. $[H,\,a_{\vec{p}}(t)] = -\omega_{\vec{p}}a_{\vec{p}}(t)$
   B. $\left[H,\,a^\dagger_{\vec{p}}(t)\right] = \omega_{\vec{p}}a^\dagger_{\vec{p}}(t)$

xii. Thus we may compute the ladder operators at different times t given their value at some reference time t_0:
   A. $a_{\vec{p}}(t)\equiv e^{iH(t-t_0)}a_{\vec{p}}(t_0)\,e^{-iH(t-t_0)} \overset{\text{Using the above relations recursively}}{=} e^{-i(t-t_0)\omega_{\vec{p}}}a_{\vec{p}}(t_0)$
   B. And similarly $a^\dagger_{\vec{p}}(t) = e^{+i(t-t_0)\omega_{\vec{p}}}a^\dagger_{\vec{p}}(t_0)$

xiii. From which we can get the value of the field and momentum-field operators at different times t given their value at some reference time t_0:

A. $\phi(x) = \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}}(t_0) e^{-i\eta(p,\,x)} + a^\dagger_{\vec{p}}(t_0) e^{+i\eta(p,\,x)}\right]$

B. $\pi(x) = \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a_{\vec{p}}(t_0) e^{-i\eta(p,\,x)} - a^\dagger_{\vec{p}}(t_0) e^{+i\eta(p,\,x)}\right]$

where we have used the convention that $p^0 := \omega_{\vec{p}}$.

xiv. Compute $\langle 0 \,|\, \phi(y)\phi(x) \,|\, 0\rangle$:
$\langle 0 \,|\, \phi(y)\phi(x) \,|\, 0\rangle = \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{-i\eta(p,\,y-x)}}{2\omega_{\vec{p}}}$

xv. Compute $\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{dE}{(2\pi)} \frac{ie^{-iEt}}{E^2-\omega_{\vec{p}}^2+i\varepsilon}$.

A. This integral has two poles on the *real* axis which are slightly shifted up and down by the $i\varepsilon$ term.

B. In order to compute this integral we use Cauchy's lemma to ascertain that an integral with a semicircular contour (with base on the real axis, with radius approaching infinity, up or down depending on the sign of t–so that the integral on the arc alone will tend to zero) is equal to the residue inside the contour.

C. Thus all together we find that $\int_{\mathbb{R}} \frac{dE}{(2\pi)} \frac{ie^{-iEt}}{E^2-\omega_{\vec{p}}^2+i\varepsilon} = \theta(t)\frac{e^{-i\omega_{\vec{p}}t}}{2\omega_{\vec{p}}} + \theta(-t)\frac{e^{+i\omega_{\vec{p}}t}}{2\omega_{\vec{p}}} = \frac{e^{-i\omega_{\vec{p}}|t|}}{2\omega_{\vec{p}}}$.

D. <u>Note:</u> This can also be achieved by Wick-rotating $E \mapsto -iE$ to get $-\int_{\mathbb{R}} \frac{dE}{(2\pi)} \frac{e^{-Et}}{-E^2-\omega_{\vec{p}}^2}$.

xvi. So that we have finally:

$\langle 0 \mid T(\phi(x)\phi(y)) \mid 0\rangle \equiv \theta(x^0-y^0)\langle 0 \mid \phi(x)\phi(y) \mid 0\rangle + \theta(y^0-x^0)\langle 0 \mid \phi(y)\phi(x) \mid 0\rangle =$

$\theta(x^0-y^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,x-y)}}{2\omega_{\vec{p}}} + \theta(y^0-x^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,y-x)}}{2\omega_{\vec{p}}} \underset{=}{\vec{p}\mapsto-\vec{p} \text{ in the 2nd integral}}$

$\theta(x^0-y^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,x-y)}}{2\omega_{\vec{p}}} + \theta(y^0-x^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta\left(\left(p^0,\,-\vec{p}\right),\,y-x\right)}}{2\omega_{\vec{p}}} =$

$\theta(x^0-y^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,x-y)}}{2\omega_{\vec{p}}} + \theta(y^0-x^0)\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta\left(p,\,\left(y^0-x^0,\,\vec{x}-\vec{y}\right)\right)}}{2\omega_{\vec{p}}} =$

$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left[\theta(x^0-y^0)\frac{e^{-ip^0\left(x^0-y^0\right)}}{2\omega_{\vec{p}}} + \theta(y^0-x^0)\frac{e^{-ip^0\left(y^0-x^0\right)}}{2\omega_{\vec{p}}}\right] =$

$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left[\theta(x^0-y^0)\frac{e^{-ip^0\left(x^0-y^0\right)}}{2\omega_{\vec{p}}} + \theta(y^0-x^0)\frac{e^{+ip^0\left(x^0-y^0\right)}}{2\omega_{\vec{p}}}\right] =$

$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \frac{dE}{(2\pi)}\frac{ie^{-iE\left(x^0-y^0\right)}}{E-|\vec{p}|^2-m^2+i\varepsilon} \underset{=}{\text{change integral and limit}}$

$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\frac{ie^{-\eta(p,\,x-y)}}{\eta(p,\,p)-m^2+i\varepsilon}$

■

2. Path Integral Quantization

(a) <u>Claim:</u> $\mathcal{Z}[J] = \exp\left[-\frac{1}{2}\int d^4x \int d^4y J(x)\tilde{D}_F(x-y)J(y)\right]\left(\mathcal{Z}[J]|_{J=0}\right)$

where $\tilde{D}_F(x-y)$ is a Green's function of $\left(-\partial_\nu\partial^\nu - m^2 + i\varepsilon\right)$.
<u>Proof:</u>

i. $\mathcal{Z}[J] \equiv \mathcal{N}\int\mathcal{D}\phi\exp\left\{i\int d^4x\left[\mathcal{L}+J\phi+\frac{1}{2}i\varepsilon\phi^2\right]\right\}$

ii. Manipulate $\int d^4x\left[\mathcal{L}+J\phi+\frac{1}{2}i\varepsilon\phi^2\right]$:

A. $\int d^4x\left[\mathcal{L}+J\phi\right] = \int d^4x\left[\frac{1}{2}\left(\partial_\nu\phi\right)\left(\partial^\nu\phi\right) - \frac{1}{2}m^2\phi^2 + J\phi + \frac{1}{2}i\varepsilon\phi^2\right] =$
$= \int d^4x\left[\frac{1}{2}\partial_\nu\left(\phi\left(\partial^\nu\phi\right)\right) - \frac{1}{2}\phi\left(\partial_\nu\partial^\nu\phi\right) - \frac{1}{2}m^2\phi^2 + J\phi + \frac{1}{2}i\varepsilon\phi^2\right] =$

B. Use Stokes' Thereom to ascertain that $\int d^4x\left[\frac{1}{2}\partial_\nu\left(\phi\left(\partial^\nu\phi\right)\right)\right] = 0$ assuming that ϕ on the boundary of integration is zero.

C. Thus we get:
$= \int d^4x\left[\frac{1}{2}\phi\left(-\partial_\nu\partial^\nu - m^2 + i\varepsilon\right)\phi + J\phi\right]$

D. Define a differential operator $KG := -\partial_\nu\partial^\nu - m^2 + i\varepsilon$, so that we have:
$= \int d^4x\left[\frac{1}{2}\phi KG\phi + J\phi\right]$

E. Next, define $\tilde{\phi}(x) := \phi(x) - i\int d^4y\tilde{D}_F(x-y)J(y)$ where $\tilde{D}_F(x-y)$ is a Green's function of KG, that is, we assume the following equation holds:
$KG\,\tilde{D}_F(x-y) = i\delta(x-y)$.
Plug in $\phi(x) = \tilde{\phi}(x) + i\int d^4y\tilde{D}_F(x-y)J(y)$ into the above to get:

F. $= \int d^4x\left\{\frac{1}{2}\left[\tilde{\phi}(x) + i\int d^4y\tilde{D}_F(x-y)J(y)\right]KG\left[\tilde{\phi}(x) + i\int d^4y\tilde{D}_F(x-y)J(y)\right]\right\} +$
$+ \int d^4xJ(x)\left[\tilde{\phi}(x) + i\int d^4y\tilde{D}_F(x-y)J(y)\right] =$
$= \int d^4x\left[\frac{1}{2}\tilde{\phi}(x)KG\tilde{\phi}(x)\right] +$
$+ \int d^4x\left\{\frac{1}{2}\left[\tilde{\phi}(x)\right]KG\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]\right\} +$
$+ \int d^4x\left\{\frac{1}{2}\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]KG\left[\tilde{\phi}(x)\right]\right\} +$
$+ \int d^4x\left\{\frac{1}{2}\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]KG\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]\right\} +$
$+ \int d^4xJ(x)\tilde{\phi}(x) + i\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y) =$

- <u>Claim:</u> $\int d^4x\left\{\frac{1}{2}\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]KG\left[\tilde{\phi}(x)\right]\right\} = \int d^4x\left\{\frac{1}{2}\left[KGi\int d^4y\tilde{D}_F(x-y)J(y)\right]\left[\tilde{\phi}(x)\right]\right\}$
  <u>Proof:</u>
  The only thing we need to reverse is the derivative, all the other terms can be applied just as well to the right or to the left.
  We write $\int d^4x\left\{\frac{1}{2}\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]\partial_\nu\partial^\nu\tilde{\phi}(x)\right\} =$
  $= \int d^4x\partial_\nu\left\{\frac{1}{2}\left[i\int d^4y\tilde{D}_F(x-y)J(y)\right]\partial^\nu\tilde{\phi}(x)\right\} -$
  $- \int d^4x\left\{\frac{1}{2}\left[i\partial_\nu\int d^4y\tilde{D}_F(x-y)J(y)\right]\partial^\nu\tilde{\phi}(x)\right\}$
  Now use Stokes' theorem and assume that $\partial^\nu\tilde{\phi}(x) \to 0$ on the boundary of spacetime, so that the first term is zero.

So we get:
$= -\int d^4x \left\{\frac{1}{2}\left[i\partial_\nu \int d^4y\tilde{D}_F(x-y)J(y)\right]\partial^\nu\tilde{\phi}(x)\right\} =$
$= -\int d^4x\partial^\nu \left\{\frac{1}{2}\left[i\partial_\nu \int d^4y\tilde{D}_F(x-y)J(y)\right]\tilde{\phi}(x)\right\} +$
$+\int d^4x \left\{\frac{1}{2}\left[i\partial^\nu\partial_\nu \int d^4y\tilde{D}_F(x-y)J(y)\right]\tilde{\phi}(x)\right\}$
Now we use Stokes' theorem again, assuming even more reasonably that $\tilde{\phi}(x) \to 0$ on the boundary of spacetime so that again the first term is zero. So we get:
$= \int d^4x \left\{\frac{1}{2}\left[i\partial^\nu\partial_\nu \int d^4y\tilde{D}_F(x-y)J(y)\right]\tilde{\phi}(x)\right\}$
Thus we have proven that we can move the KG operator to the other term in an integral. ■

G. Now use the fact that $KG\,\tilde{D}_F(x-y) = i\delta(x-y)$, and the above claim, and so we get:
$= \int d^4x\left[\frac{1}{2}\tilde{\phi}(x)KG\tilde{\phi}(x)\right] +$
$-\frac{1}{2}\int d^4xJ(x)\tilde{\phi}(x) -$
$-\frac{1}{2}\int d^4x\left\{J(x)\tilde{\phi}(x)\right\} -$
$-\frac{1}{2}i\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y) +$
$+\int d^4xJ(x)\tilde{\phi}(x) + i\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y) =$
$= \int d^4x\left[\frac{1}{2}\tilde{\phi}(x)KG\tilde{\phi}(x)\right] + \frac{1}{2}i\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)$

iii. Thus we have:
$\mathcal{Z}[J] \equiv \mathcal{N}\int\mathcal{D}\phi\exp\left\{i\int d^4x\left[\frac{1}{2}\tilde{\phi}(x)KG\tilde{\phi}(x)\right] - \frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right\}$

iv. Make a change of variable in the functional integral $\phi \mapsto \tilde{\phi}$. Since we are shifting by a constant, namely, $i\int d^4y\tilde{D}_F(x-y)J(y)$, we have that $\mathcal{D}\phi = \mathcal{D}\tilde{\phi}$, thus:
$= \mathcal{N}\int\mathcal{D}\tilde{\phi}\exp\left\{i\int d^4x\left[\frac{1}{2}\tilde{\phi}(x)KG\tilde{\phi}(x)\right] - \frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right\} =$
$= \exp\left[-\frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right]\mathcal{N}\int\mathcal{D}\tilde{\phi}\exp\left[i\frac{1}{2}\int d^4x\tilde{\phi}(x)KG\tilde{\phi}(x)\right] =$
$= \exp\left[-\frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right]\left(\mathcal{Z}[J]|_{J=0}\right)$

(b) <u>Claim:</u> $\left\langle 0 \,\middle|\, T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right) \,\middle|\, 0\right\rangle = \tilde{D}_F(x-y)$
<u>Proof:</u>

i. As shown in the lecture, $\left\langle 0 \,\middle|\, T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right) \,\middle|\, 0\right\rangle = \frac{\left.\left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right]\mathcal{Z}[J]\right\}\right|_{J=0}}{\mathcal{Z}[J]|_{J=0}}$

ii. $\frac{\left.\left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right]\mathcal{Z}[J]\right\}\right|_{J=0}}{\mathcal{Z}[J]|_{J=0}} =$
$= \frac{\left.\left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right]\exp\left[-\frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right]\left(\mathcal{Z}[J]|_{J=0}\right)\right\}\right|_{J=0}}{\mathcal{Z}[J]|_{J=0}} =$
$= \left.\left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[-i\frac{\delta}{\delta J(x_2)}\right]\exp\left[-\frac{1}{2}\int d^4x\int d^4yJ(x)\tilde{D}_F(x-y)J(y)\right]\right\}\right|_{J=0} =$
$= \left.\left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\exp[\dots]\left[+\frac{1}{2}i\int d^4x\int d^4y\frac{\delta}{\delta J(x_2)}J(x)\tilde{D}_F(x-y)J(y)\right]\right\}\right|_{J=0} =$

$= \left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\exp\left[\dots\right]\left[+\frac{1}{2}i\int d^4y\tilde{D}_F\left(x_2-y\right)J\left(y\right)+\frac{1}{2}i\int d^4xJ\left(x\right)\tilde{D}_F\left(x-x_2\right)\right]\right\}\Big|_{J=0} =$
$= \left\{\left[-i\frac{\delta}{\delta J(x_1)}\right]\exp\left[\dots\right]\left[i\int d^4x\tilde{D}_F\left(x-x_2\right)J\left(x\right)\right]\right\}\Big|_{J=0} =$
$= \left\{\exp\left[\dots\right]\left[-i\frac{\delta}{\delta J(x_1)}\right]\left[i\int d^4x\tilde{D}_F\left(x-x_2\right)J\left(x\right)\right]\right\}\Big|_{J=0} +$
$+ \left\{\exp\left[\dots\right]\left[i\int d^4x\tilde{D}_F\left(x-x_1\right)J\left(x\right)\right]\left[i\int d^4x\tilde{D}_F\left(x-x_2\right)J\left(x\right)\right]\right\}\Big|_{J=0} =$

iii. Since we are setting $J=0$, the second term is just zero. The first one is:
$= \left\{\exp\left[\dots\right]\tilde{D}_F\left(x_1-x_2\right)\right\}\Big|_{J=0} =$
$= \tilde{D}_F\left(x_1-x_2\right)$
■

(c) Claim: $\tilde{D}_F\left(x_1-x_2\right) = \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x_1-x_2)}}{p^2-m^2+i\varepsilon}$
Proof:

i. Start from the defining equation of $\left(-\partial_\nu\partial^\nu-m^2+i\varepsilon\right)\tilde{D}_F\left(x-y\right) = i\delta\left(x-y\right)$.
ii. Whatever $\tilde{D}_F\left(x-y\right)$, write its Fourier transform: $\tilde{D}_F\left(x-y\right) = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x_1-x_2)}D_F\left(p\right)$.
iii. Write $\delta\left(x-y\right) = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x_1-x_2)}$.
iv. Thus we get:
$\left(-\partial_\nu\partial^\nu-m^2+i\varepsilon\right)\int\frac{d^4p}{(2\pi)^4}e^{-ip(x_1-x_2)}D_F\left(p\right) = i\int\frac{d^4p}{(2\pi)^4}e^{-ip(x_1-x_2)}$
Thus:
$\int\frac{d^4p}{(2\pi)^4}\left(p^2-m^2+i\varepsilon\right)e^{-ip(x_1-x_2)}D_F\left(p\right) = \int\frac{d^4p}{(2\pi)^4}ie^{-ip(x_1-x_2)}$
Thus it must hold that:
$D_F\left(p\right) = \frac{i}{p^2-m^2+i\varepsilon}$
■
v. Note that using the first question, we can now say that $\lim_{\varepsilon\to 0}\tilde{D}_F\left(x-y\right) = \theta\left(x^0-y^0\right)\int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,x-y)}}{2\omega_{\vec{p}}}+\theta\left(y^0-x^0\right)\int_{\mathbb{R}^3}\frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-i\eta(p,\,y-x)}}{2\omega_{\vec{p}}}$.

3. Let $m\in\mathbb{N}\backslash\{0,\,1,\,2\}$. Let $g\in\mathbb{R}$.
Let an action be given as $S = \int dt\left(\frac{1}{2}\dot{x}^2-\frac{1}{2}\omega^2x^2-\frac{g}{m!}x^m\right)$. (The mass of the "particle" is set to 1).

- First, let us work in Euclidean time so that $S_E = \int dt\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2+\frac{g}{m!}x^m\right)$. The reason why we make this switch will become apparent in what follows. Thus all quantities below are assumed to be Wick rotated to imaginary time and the subscript $_E$ is omitted but implied.
- Observe that in the Euclidean action, the "source" term enters with a minus sign.
- Next, let us work out a few details about the *simple* harmonic oscillator (without the $\frac{g}{m!}x^m$ term) which we have worked out in the previous exercise sheet about a *free* non-Euclidean particle:

- Define $\eta(t) := x(t) - x_{cl}(t)$ where $x_{cl}(t)$ satisfies the classical Euclidean equation of motion with source:
  $-\ddot{x_{cl}}(t) + \omega^2 x_{cl}(t) = J(t)$
  with boundary conditions $x_{cl}(t_a) = x_a = x_{cl}(t_b) = x_b = 0$ and $\eta(t)$ being "quantum fulctuations", which also satisfy $\eta(t_a) = \eta(t_b) = 0$.
  Claim: $S_{SHO}[x,\, J] = S_{SHO}[x_{cl},\, J] + S_{SHO}[\eta,\, 0]$
  Proof:
  * $S_{SHO}[x,\, J] \equiv \int_{t_a}^{t_b} dt \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 - J(t)\, x\right] \overset{\text{plug in x}}{=}$
    $= \int_{t_a}^{t_b} dt \left\{\frac{1}{2}\left[\dot{x_{cl}}(t) + \dot{\eta}(t)\right]^2 + \frac{1}{2}\omega^2\left[x_{cl}(t) + \eta(t)\right]^2 - J(t)\left[x_{cl}(t) + \eta(t)\right]\right\} =$
    $= \int_{t_a}^{t_b} dt \left\{\frac{1}{2}\left[\dot{x_{cl}}(t)^2 + \dot{\eta}(t)^2 + 2\dot{x_{cl}}(t)\,\dot{\eta}(t)\right] + \frac{1}{2}\omega^2\left[x_{cl}(t)^2 + \eta(t)^2 + 2x_{cl}(t)\,\eta(t)\right]\right\} +$
    $+ \int_{t_a}^{t_b} dt \left\{-J(t)\left[x_{cl}(t) + \eta(t)\right]\right\} =$
    $= \int_{t_a}^{t_b} dt \left\{\frac{1}{2}\dot{x_{cl}}(t)^2 + \frac{1}{2}\omega^2 x_{cl}(t)^2 - J(t)\, x_{cl}(t) + \frac{1}{2}\dot{\eta}(t)^2 + \frac{1}{2}\omega^2\eta(t)^2 + \dot{x_{cl}}(t)\,\dot{\eta}(t)\right\} +$
    $+ \int_{t_a}^{t_b} dt \left\{\omega^2 x_{cl}(t)\,\eta(t) - J(t)\,\eta(t)\right\} =$
    $= S_{SHO}[x_{cl},\, J] + S_{SHO}[\eta,\, 0] + \int_{t_a}^{t_b} dt \left\{\dot{x_{cl}}(t)\,\dot{\eta}(t) + \omega^2 x_{cl}(t)\,\eta(t) - J(t)\,\eta(t)\right\} =$
    $= S_{SHO}[x_{cl},\, J] + S_{SHO}[\eta,\, 0] + \left[\dot{x_{cl}}(t)\,\eta(t)\right]\Big|_{t_a}^{t_b} +$
    $+ \int_{t_a}^{t_b} dt \left\{-\ddot{x_{cl}}(t)\,\eta(t) + \omega^2 x_{cl}(t)\,\eta(t) - J(t)\,\eta(t)\right\} =$
  * Now use the fact that $\eta(t_a) = \eta(t_b) = 0$ to ascertain that $\left[\dot{x_{cl}}(t)\,\eta(t)\right]\Big|_{t_a}^{t_b} = 0$ and the fact that $-\ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) = J(t)$ by definition of the classical path $x_{cl}(t)$, and so we are left with:
    $= S_{SHO}[x_{cl},\, J] + S_{SHO}[\eta,\, 0]$
    ■
- Claim: $\mathcal{Z}_{SHO}[J] = \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}}\exp\left\{-S_{SHO}[x_{cl},\, J]\right\}$
  Proof:
  * $\mathcal{Z}_{SHO}[J] \equiv \int_{x_a=0}^{x_b=0}\mathcal{D}x\,\exp\left\{-S_{SHO}[x,\, J]\right\} \overset{(a)}{=}$
    $= \int_{x_a=0}^{x_b=0}\mathcal{D}x\,\exp\left\{-\left\{S_{SHO}[x_{cl},\, J] + S_{SHO}[\eta,\, 0]\right\}\right\} =$
  * But $x_{cl}(t)$ is a constant path (determined by the equations of motion) which doesn't vary during the integration process, so we can pull it out of the integral:
    $= \exp\left\{-S_{SHO}[x_{cl},\, J]\right\}\times\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}}\mathcal{D}x\,\exp\left\{-S_{SHO}[\eta,\, 0]\right\} =$
  * Now we make a "change of variable" $x(t) \mapsto x(t) - x_{cl}(t) = \eta(t)$. Since "$\mathcal{D}x_{cl} = 0$", $\mathcal{D}x = \mathcal{D}\eta$. (this needs more mathematical rigor) So we get:
    $= \exp\left\{-S_{SHO}[x_{cl},\, J]\right\}\times\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}}\mathcal{D}\eta\,\exp\left\{-S_{SHO}[\eta,\, 0]\right\} =$
    $= \exp\left\{-S_{SHO}[x_{cl},\, J]\right\}\times\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}}\mathcal{D}\eta\,\exp\left\{-\int_{t_a}^{t_b} dt\left[\frac{1}{2}\dot{\eta}(t)^2 + \frac{1}{2}\omega^2\eta(t)^2\right]\right\}$
  * How to compute $\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}}\mathcal{D}\eta\,\exp\left\{-\int_{t_a}^{t_b} dt\left[\frac{1}{2}\dot{\eta}(t)^2 + \frac{1}{2}\omega^2\eta(t)^2\right]\right\}$?
    We have been asked already to compute this expression in

HW1Q3, but we didn't finish the job then because it was too complicated. Here we present an alternative solution (from Arkady Vainshtein):

* $\int_{t_a}^{t_b} dt \left[\frac{1}{2}\dot{\eta}(t)^2 + \frac{1}{2}\omega^2\eta(t)^2\right] =$
  $= \int_{t_a}^{t_b} dt \left[\frac{1}{2}\dot{\eta}(t)\dot{\eta}(t) + \frac{1}{2}\omega^2\eta(t)^2\right] =$
  $= \underbrace{\frac{1}{2}\dot{\eta}(t)\eta(t)\Big|_{t_a}^{t_b}}_{\text{zero by B.C. for}\eta} + \frac{1}{2}\int_{t_a}^{t_b} dt \left[-\ddot{\eta}(t)\eta(t) + \omega^2\eta(t)^2\right] =$
  $= +\frac{1}{2}\int_{t_a}^{t_b} dt\eta(t)\left[-\frac{d^2}{dt^2} + \omega^2\right]\eta(t)$
* Let $\{y_n(t)\}_{n\in\mathbb{N}}$ be a complete orthonormal set of eigenvectors of $-\frac{d^2}{dt^2}+\omega^2$ which vanish at t_a and at t_b: $\int_{t_a}^{t_b} y_n(t)\, y_l(t)\, dt = \delta_{nl}$. We know the eigenvalues have to be $\left(\frac{\pi n}{t_b-t_a}\right)^2 + \omega^2 =: \lambda_n$ (Plug in the eigenvectors–simply sines and cosines–and impose the boundary conditions).
* Write $\eta(t) = \sum_{n\in\mathbb{N}} c_n y_n(t)$ where $c_n \in \mathbb{R}$ are the expansion coefficients of $\eta(t)$.
* Then $\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}} \mathcal{D}\eta \exp\left\{-\int_{t_a}^{t_b} dt\left[\frac{1}{2}\dot{\eta}(t)^2 + \frac{1}{2}\omega^2\eta(t)^2\right]\right\} =$
  $= \mathcal{N}\int dc_0 \int dc_1 \ldots \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b} dt \sum_{n\in\mathbb{N}} c_n y_n(t)\left[-\frac{d^2}{dt^2}+\omega^2\right]\sum_{l\in\mathbb{N}} c_l y_l(t)\right\} =$
  where $\mathcal{N}$ is some normalization coefficient which we will determine soon. In this step we are integrating over all functions η by integrating over all coefficients in its expansion. In a way this is the most concrete perspective on the path integral.
  $= \mathcal{N}\int dc_0 \int dc_1 \ldots \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b} dt \sum_{(n,l)\in\mathbb{N}^2} c_n c_l y_n(t)\left[-\frac{d^2}{dt^2}+\omega^2\right] y_l(t)\right\} =$
  $= \mathcal{N}\int dc_0 \int dc_1 \ldots \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b} dt \sum_{(n,l)\in\mathbb{N}^2} c_n c_l y_n(t)\lambda_l y_l(t)\right\} =$
  $= \mathcal{N}\int dc_0 \int dc_1 \ldots \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b} dt \sum_{(n,l)\in\mathbb{N}^2} c_n c_l y_n(t)\lambda_l y_l(t)\right\} =$
  $= \mathcal{N}\int dc_0 \int dc_1 \ldots \exp\left\{-\frac{1}{2}\sum_{n\in\mathbb{N}} c_n^2\lambda_n\right\} =$
  $= \mathcal{N}\prod_{n\in\mathbb{N}}\left[\int dc_n \exp\left(-\frac{1}{2}c_n^2\lambda_n\right)\right] =$
* But $\int_{\mathbb{R}} dc_n \exp\left(-\frac{1}{2}c_n^2\lambda_n\right)$ is easy as pie to compute: it's a mere Gaussian integral. From HW1Q1 we have $\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$ and so $\int_{\mathbb{R}} dc_n \exp\left(-\frac{1}{2}c_n^2\lambda_n\right) = \sqrt{\frac{\pi}{\lambda_n/2}}$.
* So we are left with:
  $= \mathcal{N}\prod_{n\in\mathbb{N}}\left(\sqrt{2\pi}\lambda_n^{-1/2}\right) =$
  $= \mathcal{N}\left(\prod_{n\in\mathbb{N}}\sqrt{2\pi}\right)\prod_{n\in\mathbb{N}}\left[\left(\left(\frac{\pi n}{t_b-t_a}\right)^2+\omega^2\right)^{-1/2}\right] =$

$$= \underbrace{\mathcal{N}\left(\prod_{n\in\mathbb{N}}\sqrt{2\pi}\left(\frac{t_b-t_a}{\pi n}\right)\right)}_{\text{free theory}}\underbrace{\left\{\prod_{n\in\mathbb{N}}\left[\left(1+\left(\frac{\omega\left(t_b-t_a\right)}{\pi n}\right)^2\right)^{-1/2}\right]\right\}}_{\text{addition because } \omega\neq 0} =$$

- We can use this decomposition now to *define* $\mathcal{N}$ such that it gives us the expected result for the free theory.
- The expected result for the free theory is:
  $\mathcal{N}\left(\prod_{n\in\mathbb{N}}\sqrt{2\pi}\left(\frac{t_b-t_a}{\pi n}\right)\right) \overset{!}{=} \left\langle x=0\,\middle|\,e^{-\frac{\hat{p}^2}{2}(t_b-t_a)}\,\middle|\,x=0\right\rangle =$
  $= \left\langle x=0\,\middle|\,e^{-\frac{\hat{p}^2}{2}(t_b-t_a)}\mathbb{1}\,\middle|\,x=0\right\rangle =$
  $= \left\langle x=0\,\middle|\,e^{-\frac{\hat{p}^2}{2}(t_b-t_a)}\int_{\mathbb{R}}dp\,|p\rangle\langle p|\,\middle|\,x=0\right\rangle =$
  $= \int_{\mathbb{R}}dpe^{-\frac{p^2}{2}(t_b-t_a)}\left|\langle x=0\,|\,p\rangle\right|^2 =$
  $= \int_{\mathbb{R}}dpe^{-\frac{p^2}{2}(t_b-t_a)}\left|\langle x=0\,|\,p\rangle\right|^2$
  $= \int_{\mathbb{R}}dpe^{-\frac{p^2}{2}(t_b-t_a)}\left|\frac{e^{\frac{i}{\hbar}p\cdot 0}}{\sqrt{2\pi}}\right|^2 =$
  $= \frac{1}{2\pi}\int_{\mathbb{R}}dpe^{-\frac{p^2}{2}(t_b-t_a)} =$
  $= \frac{1}{2\pi}\sqrt{\frac{\pi}{(t_b-t_a)/2}} = \sqrt{\frac{1}{2\pi(t_b-t_a)}}$
- What about the other, nasty looking part, $\prod_{n\in\mathbb{N}}\left[\left(1+\left(\frac{\omega(t_b-t_a)}{\pi n}\right)^2\right)^{-1/2}\right]$?
- Claim: $\prod_{n\in\mathbb{N}}\left[\left(1+\left(\frac{\omega(t_b-t_a)}{\pi n}\right)^2\right)\right] = \frac{sinh(\omega(t_b-t_a))}{\omega(t_b-t_a)}$
  Proof: ... (took from table of integrals infinite series and products by Gradshtein and Ryzhik)
- Thus we find all together that $\int_{\{\eta:\,\eta(t_a)=\eta(t_b)=0\}}\mathcal{D}\eta\,\exp\left\{-\int_{t_a}^{t_b}dt\left[\frac{1}{2}\dot{\eta}\left(t\right)^2+\frac{1}{2}\omega^2\eta\left(t\right)^2\right]\right\} =$
  $\sqrt{\frac{1}{2\pi(t_b-t_a)}}\left(\frac{sinh(\omega(t_b-t_a))}{\omega(t_b-t_a)}\right)^{-1/2} = \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}}$
  ■

– Claim: $S_{SHO}\left[x_{cl},\,J\right] = -\frac{1}{2}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\,J\left(t\right)G\left(t,\,t'\right)J\left(t'\right)$ where $G\left(t,\,t'\right)$ is the Green's function of $-\frac{d^2}{dt^2}+\omega^2$.
  Proof:
  - We start from $S_{SHO}\left[x_{cl},\,J\right] \equiv \int_{t_a}^{t_b}dt\left[\frac{1}{2}\dot{x_{cl}}^2+\frac{1}{2}\omega^2x_{cl}^2-Jx_{cl}\right] =$
    - We integrate $\dot{x_{cl}}^2 \equiv \dot{x_{cl}}\dot{x_{cl}}$ by parts:
      $$= \underbrace{\frac{1}{2}mx_{cl}\dot{x_{cl}}\Big|_{t_a}^{t_b}}_{\text{zero as } x_{cl} \text{ is zero on the boundaries}} + \int_{t_a}^{t_b}dt\left[-\frac{1}{2}\ddot{x_{cl}}x_{cl}+\frac{1}{2}\omega^2x_{cl}^2-Jx_{cl}\right] =$$
      $= \int_{t_a}^{t_b}dtx_{cl}\left[-\frac{1}{2}\ddot{x_{cl}}+\frac{1}{2}\omega^2x_{cl}-J\right] =$
    - Use the classical equation of motion which says $-\ddot{x_{cl}}+\omega^2x_{cl}=J$:
      $= \int_{t_a}^{t_b}dtx_{cl}\left[\frac{1}{2}J-J\right] = -\frac{1}{2}\int_{t_a}^{t_b}dtx_{cl}J$

- $\cdot$ Use the relation $x_{cl}(t)=\int_{t_a}^{t_b}dt'G(t,\,t')\,J(t')$, which is *the* defining property of the Green's function of $-\frac{d^2}{dt^2}+\omega^2$:
  $=-\frac{1}{2}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'G(t,\,t')\,J(t')\,J(t)=$
  $=-\frac{1}{2}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'G(t,\,t')\,J(t')\,J(t)$
  $\blacksquare$

- Next, we have computed $G(t,\,t')$ in HW2Q3 and found that $G(t,\,t')=\frac{1}{2\omega}e^{-\omega|t-t'|}$.
- All together we have found then that

$$\boxed{\mathcal{Z}_{SHO}[J]=\sqrt{\frac{\omega}{2\pi}\frac{1}{sinh\left(\omega\left(t_b-t_a\right)\right)}}\exp\left\{\frac{1}{2}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\frac{1}{2\omega}e^{-\omega|t-t'|}J(t')\,J(t)\right\}}.$$

- For later it will be helpful to compute the following quantity:
  $\lim_{(t_a,\,t_b)\to(\infty,\,-\infty)}\mathcal{Z}_{SHO}[0]=\lim_{(t_a,\,t_b)\to(\infty,\,-\infty)}\sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}}=$
  $=\lim_{(t_a,\,t_b)\to(\infty,\,-\infty)}\sqrt{\frac{\omega}{\pi}\frac{e^{-\omega(t_b-t_a)}}{1-e^{-2\omega(t_b-t_a)}}}\approx$
  $\approx\sqrt{\frac{\omega}{\pi}}\lim_{(t_a,\,t_b)\to(\infty,\,-\infty)}e^{-\frac{\omega}{2}(t_b-t_a)}\left(1+\frac{1}{2}\mathcal{O}\left(e^{-2\omega(t_b-t_a)}\right)\right)$
- Now we may finally begin to deal with the *anharmonic* oscillator:

(a) Claim: $\mathcal{Z}[J]=\mathcal{Z}_{SHO}[J]-\frac{g}{m!}\int dt'\frac{\delta^m}{\delta J(t')^m}\mathcal{Z}_{SHO}[J]+\mathcal{O}\left(g^2\right)$
Proof:

i. $\mathcal{Z}[J]\equiv\mathcal{N}\int\mathcal{D}x\exp\left\{-\int dt\left[\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2+\frac{g}{m!}x^m\right)+Jx\right]\right\}=$

$$=\mathcal{N}\int\mathcal{D}x\exp\left\{-\underbrace{\int dt\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2\right)}_{S_{SHO}[x]}-\underbrace{\int dt\left(\frac{g}{m!}x^m\right)}_{S_{int}[x]}+\int dtJx\right\}=$$

$=\mathcal{N}\int\mathcal{D}x\exp\left\{-S_{int}[x]\right\}\exp\left\{-S_{SHO}[x]+\int dtJx\right\}=$
$=\mathcal{N}\int\mathcal{D}x\exp\left(-\frac{g}{m!}\int dtx^m\right)\exp\left\{-S_{SHO}[x]+\int dtJx\right\}=$
$=\mathcal{N}\int\mathcal{D}x\left(1-\frac{g}{m!}\int dtx^m+\mathcal{O}\left(g^2\right)\right)\exp\left\{-S_{SHO}[x]+\int dtJx\right\}=$
$=\mathcal{N}\int\mathcal{D}x\exp\left\{-S_{SHO}[x]+\int dtJx\right\}-\frac{g}{m!}\int dt'\mathcal{N}\int\mathcal{D}x\,x(t')^m\exp\left\{-S_{SHO}[x]+\int dtJx\right\}+\mathcal{O}\left(g^2\right)=$
$=\mathcal{Z}_{SHO}[J]-\frac{g}{m!}\int dt'\frac{\delta^m}{\delta J(t')^m}\mathcal{Z}_{SHO}[J]+\mathcal{O}\left(g^2\right)$
$\blacksquare$

- In order to compute the *ground state energy* we will employ a trick:
  - Let $\hat{H}$ be the quantum mechanical operator associated with the above given Lagrangian L.
  - Assume $\hat{H}$ has some discrete spectrum: $\hat{H}|n\rangle=E_n|n\rangle$.
  - Then
    $\left\langle x_f\middle|e^{-\hat{H}T}\middle|x_i\right\rangle=$
    $=\left\langle x_f\middle|e^{-\hat{H}T}\mathbb{1}\middle|x_i\right\rangle=$
    $=\left\langle x_f\middle|e^{-\hat{H}T}\sum_n|n\rangle\langle n|\middle|x_i\right\rangle=$

$= \left\langle x_f \,\middle|\, \sum_n e^{-E_n T} |n\rangle \langle n| \,\middle|\, x_i \right\rangle =$
$= \sum_n e^{-E_n T} \langle x_f \,|\, n\rangle \langle n \,|\, x_i\rangle$

– Next, pick $x_f = x_i = 0$. Then denote the wave function $\langle x = 0 \,|\, n\rangle$ by $\psi_n(0)$.

– Then $\lim_{T\to\infty} \left\langle x = 0 \,\middle|\, e^{-\hat{H}T} \,\middle|\, x = 0 \right\rangle = \lim_{T\to\infty} \sum_n e^{-E_n T} |\psi_n(0)|^2 \approx \lim_{T\to\infty} e^{-E_0 T} |\psi_0(0)|^2$ where $E_0 := \min\{E_n | n\}$, assuming such a minimum indeed exists. The preceding approximate equal sign is true in the sense that for large values of T, $e^{-E_0 T}$ is the dominant term in the sum.

– Thus we have obtained the following formula:
$\lim_{T\to\infty} e^{-E_0 T} |\psi_0(0)|^2 \approx \lim_{T\to\infty} \mathcal{N} \int \mathcal{D}x \, \exp\{-S[x]\}$
where the path integral is on paths that start and end at $x = 0$.
More explicitly,
$\lim_{T\to\infty} e^{-E_0 T} |\psi_0(0)|^2 \approx \mathcal{N} \int_{x(\pm\infty) \overset{!}{=} 0} \mathcal{D}x \, \exp\left\{-\int_{-\infty}^{\infty} dt \, L\right\}$

(b) Claim: If $m = 4$ the ground state of the energy is given by $E_0 = \frac{1}{2}\omega + \frac{g}{32\omega^2} + \mathcal{O}\left(g^2\right)$
Proof:

i. So we would like to compute now $\mathcal{N} \int_{x(\pm\infty) \overset{!}{=} 0} \mathcal{D}x \, \exp\left\{-\int_{-\infty}^{\infty} dt \, L\right\}$, as, by the above, that would give us the ground state energy and its corresponding ground state wave function evaluated at $x = 0$. This is by definition $\mathcal{Z}[J]|_{J=0}$ evaluated at $(t_a, t_b) = (-\infty, \infty)$.

ii. $\mathcal{Z}[J]|_{J=0} = \mathcal{Z}_{SHO}[0] - \left.\left\{ \frac{g}{4!} \int_{\mathbb{R}} dt' \frac{\delta^4}{\delta J(t')^4} \mathcal{Z}_{SHO}[J] \right\}\right|_{J=0} + \mathcal{O}\left(g^2\right)$.
For $\mathcal{Z}_{SHO}[0]$ we have already computed the value above to be $\sqrt{\frac{\omega}{\pi}} \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} e^{-\frac{\omega}{2}(t_b - t_a)} \left(1 + \frac{1}{2}\mathcal{O}\left(e^{-2\omega(t_b - t_a)}\right)\right)$.

iii. Claim: $\left.\left\{ \int_{\mathbb{R}} dt' \frac{\delta^4}{\delta J(t')^4} \mathcal{Z}_{SHO}[J] \right\}\right|_{J=0} = 3 \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} \times$
$\times \sqrt{\frac{\omega}{2\pi sinh(\omega(t_b - t_a))}} \int_{t_a}^{t_b} dt' \left[G(t' - t')\right]^2$ (this is like Wick's theorem for QM)
Proof:
$\left.\left\{ \int_{t_a}^{t_b} dt' \frac{\delta^4}{\delta J(t')^4} \mathcal{Z}_{SHO}[J] \right\}\right|_{J=0} =$
$= \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} \times$
$\times \left.\left\{ \int_{t_a}^{t_b} dt'' \frac{\delta^4}{\delta J(t'')^4} \sqrt{\frac{\omega}{2\pi} \frac{1}{sinh(\omega(t_b - t_a))}} \exp\left\{ \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' G(t,\, t') J(t') J(t) \right\} \right\}\right|_{J=0} =$
$= \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} \sqrt{\frac{\omega}{2\pi} \frac{1}{sinh(\omega(t_b - t_a))}} \times$
$\times \left.\left\{ \int_{t_a}^{t_b} dt'' \frac{\delta^3}{\delta J(t'')^3} \exp\{\dots\} \left\{ \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' G(t,\, t') \left[\delta(t'' - t') J(t) + \delta(t'' - t) J(t')\right] \right\} \right\}\right|_{J=0} =$
$= \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} \sqrt{\frac{\omega}{2\pi} \frac{1}{sinh(\omega(t_b - t_a))}} \times$
$\times \left.\left\{ \int_{t_a}^{t_b} dt'' \frac{\delta^3}{\delta J(t'')^3} \exp\{\dots\} \int_{t_a}^{t_b} dt G(t,\, t'') J(t) \right\}\right|_{J=0} =$
Use the product rule for the 3rd derivative: $(fg)''' = \sum_{k=0}^{3} \frac{3!}{k!(3-k)!} f^{(3-k)} g^{(k)} =$

$f'''g + 3f''g' + 3f'g'' + fg'''$
$= \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}} \times$
$\times \left\{\int_{t_a}^{t_b} dt'' \left(\frac{\delta^3}{\delta J(t'')^3} \exp\{\dots\}\right) \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right\}\Big|_{J=0} +$
$+ \left\{\int_{t_a}^{t_b} dt'' \exp\{\dots\} \frac{\delta^3}{\delta J(t'')^3} \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right\}\Big|_{J=0} +$
$+3 \left\{\int_{t_a}^{t_b} dt'' \left(\frac{\delta^2}{\delta J(t'')^2} \exp\{\dots\}\right) \frac{\delta}{\delta J(t'')} \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right\}\Big|_{J=0} +$
$+3 \left\{\int_{t_a}^{t_b} dt'' \left(\frac{\delta}{\delta J(t'')} \exp\{\dots\}\right) \frac{\delta^2}{\delta J(t'')^2} \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right\}\Big|_{J=0} =$
We can already get rid of the first term as it will vanish when setting $J = 0$. The term $\frac{\delta^3}{\delta J(t'')^3} \int_{t_a}^{t_b} dt G(t,\, t'') J(t) = 0$ because there aren't enough powers of J, likewise $\frac{\delta^2}{\delta J(t'')^2} \int_{t_a}^{t_b} dt G(t,\, t'') J(t) = 0$. Thus we are left with:
$= \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}} \times$
$\times 3 \left\{\int_{t_a}^{t_b} dt'' \left(\frac{\delta^2}{\delta J(t'')^2} \exp\{\dots\}\right) \frac{\delta}{\delta J(t'')} \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right\}\Big|_{J=0} =$
$= \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}} \times$
$\times 3 \left\{\int_{t_a}^{t_b} dt'' \left(\frac{\delta}{\delta J(t'')} \exp\{\dots\} \int_{t_a}^{t_b} dt G(t,\, t'') J(t)\right) G(t'',\, t'')\right\}\Big|_{J=0} =$
$= \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}} \times$
$\times 3 \left\{\int_{t_a}^{t_b} dt'' \left(\left(\frac{\delta}{\delta J(t'')} \exp\{\dots\}\right) \int_{t_a}^{t_b} dt G(t,\, t'') J(t) + \exp\{\dots\} G(t'',\, t'')\right) G(t'',\, t'')\right\}\Big|_{J=0} =$
$= \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} \sqrt{\frac{\omega}{2\pi}\frac{1}{sinh(\omega(t_b-t_a))}} 3 \int_{t_a}^{t_b} dt'' G(t'',\, t'')^2$
■
In general it is (probably?) possible to prove the full Wick's theorem for quantum mechanics in this fashion.

iv. But what is $\int_{t_a}^{t_b} dt' \left[G(t'-t')\right]^2$?
$\int_{t_a}^{t_b} dt' \left[G(t'-t')\right]^2 = \int_{t_a}^{t_b} dt' \frac{1}{4\omega^2} = \frac{t_b-t_a}{4\omega^2}$

v. Thus we find
$\mathcal{Z}[J]|_{J=0} =$
$= \sqrt{\frac{\omega}{\pi}} \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} e^{-\frac{\omega}{2}(t_b-t_a)} \left(1 + \frac{1}{2}\mathcal{O}\left(e^{-2\omega(t_b-t_a)}\right)\right) \left(1 - \frac{g}{4!} 3 \frac{t_b-t_a}{4\omega^2} + \mathcal{O}\left(g^2\right)\right) \approx$
$\approx \sqrt{\frac{\omega}{\pi}} \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} e^{-\frac{\omega}{2}(t_b-t_a)} \left(1 + \frac{1}{2}\mathcal{O}\left(e^{-2\omega(t_b-t_a)}\right)\right) \left(\exp\left\{-g\frac{t_b-t_a}{32\omega^2}\right\} + \mathcal{O}\left(g^2\right)\right) =$
$= \sqrt{\frac{\omega}{\pi}} \lim_{(t_a,\,t_b)\to(\infty,\,-\infty)} e^{-\left(\frac{\omega}{2}+\frac{g}{32\omega^2}\right)(t_b-t_a)} \left(1 + \frac{1}{2}\mathcal{O}\left(e^{-2\omega(t_b-t_a)}\right)\right) +$
$\mathcal{O}\left(g^2\right)$

vi. By comparing this result to $\lim_{T\to\infty} e^{-E_0 T} |\psi_0(0)|^2$, to which it is equal by the above, we find that $\boxed{E_0 = \dfrac{\omega}{2} + \dfrac{g}{32\omega^2} + \mathcal{O}\left(g^2\right)}$, and that $|\psi_0(0)|^2 = \sqrt{\frac{\omega}{\pi}} + \mathcal{O}\left(g^2\right)$.
■

(c) Claim: If $m = 3$ the ground state of the energy is given by $E_0 = \frac{1}{2}\omega - \frac{11}{288}\frac{g^2}{\omega^4} + \mathcal{O}\left(g^4\right)$
Proof:

i. We save the best for the last: because $m = 3$, we have three functional derivatives in the term of order $\mathcal{O}(g)$ for $\mathcal{Z}[J]$. Three derivatives will always be zero because "we will always have an uncontracted term vanishing on the vacuum" (in the language of QFT's Wick's theorem) or rather because we will always be left with a term linear in J which will vanish when setting $J = 0$.

ii. Thus to see any effects we must go to $\mathcal{O}\left(g^2\right)$:

$\mathcal{Z}[J] = \mathcal{Z}_{SHO}[J] + \frac{1}{2}\left(-\frac{g}{3!}\int dt' \frac{\delta^3}{\delta J(t')^3}\mathcal{Z}_{SHO}[J]\right)^2 + \mathcal{O}\left(g^4\right)$

iii. <u>Claim:</u> $\left(\int dt' \frac{\delta^3}{\delta J(t')^3}\mathcal{Z}_{SHO}[J]\right)^2 =$

$= \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} \frac{\omega}{2\pi sinh(\omega(t_b-t_a))}\int dt' \int dt'' \left[9G(t'', t'')\, G(t',\, t'')\, G(t',\, t') + 36G(t',\, t'')^3\right]$

<u>Proof:</u> ... (later)

iv. Then we get two integrals:

A. $\int_{t_a}^{t_b} dt' \int_{t_a}^{t_b} dt'' \left[36G(t',\, t'')^3\right] = 36\frac{(t_b-t_a)^2}{(2\omega)^3}$

B. $\int dt' \int dt'' \left[G(t'', t'')\, G(t',\, t'')\, G(t',\, t')\right] = 9\frac{1}{(2\omega)^3}\int dt' \int dt'' e^{-\omega|t'-t''|}$

C. $\int_{t_a}^{t_b} dt'' e^{-\omega|t'-t''|} \overset{\tau:=t''-t'}{=} \int_{t_a-t'}^{t_b-t'} d\tau e^{-\omega|\tau|} \overset{t'\in[t_a,\, t_b]}{=} \int_{t_a-t'}^{0} d\tau e^{-\omega|\tau|} + \int_{0}^{t_b-t'} d\tau e^{-\omega|\tau|} =$

$= \int_{t_a-t'}^{0} d\tau e^{\omega\tau} + \int_{0}^{t_b-t'} d\tau e^{-\omega\tau} =$

$= \frac{1}{\omega}\left[1 - e^{\omega(t_a-t')} + 1 - e^{-\omega(t_b-t')}\right]$

D. $\int_{t_a}^{t_b} dt' \frac{1}{\omega}\left[1 - e^{\omega(t_a-t')} + 1 - e^{-\omega(t_b-t')}\right] =$

$= \frac{2}{\omega^2}\left[-1 + e^{-\omega(t_b-t_a)} + \omega(t_b - t_a)\right]$

v. Thus we get $\mathcal{Z}[0] = \lim_{(t_a,\, t_b)\to(\infty,\, -\infty)}\{\mathcal{Z}_{SHO}[0] +$

$+\frac{1}{2}\frac{g^2}{36}\frac{\omega}{2\pi sinh(\omega(t_b-t_a))}\left\{36\frac{(t_b-t_a)^2}{(2\omega)^3} + 9\frac{1}{(2\omega)^3}\frac{2}{\omega^2}\left[-1 + e^{-\omega(t_b-t_a)} + \omega(t_b - t_a)\right]\right\} +$

$\mathcal{O}\left(g^4\right)\} =$

$= \sqrt{\frac{\omega}{\pi}}\lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} e^{-\frac{\omega}{2}(t_b-t_a)} +$

$+\left(1 + \sqrt{\frac{\omega}{\pi}}e^{-\frac{\omega}{2}(t_b-t_a)}\frac{1}{2}\frac{g^2}{36}\left\{36\frac{(t_b-t_a)^2}{(2\omega)^3} + 9\frac{1}{(2\omega)^3}\frac{2}{\omega^2}\left[-1 + e^{-\omega(t_b-t_a)} + \omega(t_b - t_a)\right]\right\} + \mathcal{O}\left(g^4\right)\right) \approx$

$= \sqrt{\frac{\omega}{\pi}}\lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} e^{-\frac{\omega}{2}(t_b-t_a)} +$

$+\left(1 + \frac{1}{2}g^2 e^{-\frac{\omega}{2}(t_b-t_a)}\frac{1}{36}\sqrt{\frac{\omega}{\pi}}\left\{36\frac{(t_b-t_a)^2}{(2\omega)^3} + 9\frac{1}{(2\omega)^3}\frac{2}{\omega^2}\left[-1 + e^{-\omega(t_b-t_a)} + \omega(t_b - t_a)\right]\right\} + \mathcal{O}\left(g^4\right)\right)$

$= \sqrt{\frac{\omega}{\pi}}\lim_{(t_a,\, t_b)\to(\infty,\, -\infty)} e^{-\frac{\omega}{2}(t_b-t_a)} +$

$+\exp\left(\frac{g^2}{2}e^{-\frac{\omega}{2}(t_b-t_a)}\frac{1}{36}\sqrt{\frac{\omega}{\pi}}\left\{36\frac{(t_b-t_a)^2}{(2\omega)^3} + 9\frac{1}{(2\omega)^3}\frac{2}{\omega^2}\left[-1 + e^{-\omega(t_b-t_a)} + \omega(t_b - t_a)\right]\right\} + \mathcal{O}\left(g^4\right)\right)$

???