A note on a proof of Zorn's Lemma from Axiom of Choice without transfinite induction

Koji Nuida

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Abstract

In this note, we give a proof of Zorn's Lemma from Axiom of Choice without transfinite induction (the essential idea in the first version of this note was the same as [3, Theorem 4.19], but the present proof is an improvement of the proof in [2]).

Throughout this note, $(X, \leq)$ denotes an arbitrary non-empty partially ordered set in which every totally ordered subset has an upper bound. Then Zorn's Lemma states that such an X always has a maximal element. In this note, we give a proof of Zorn's Lemma from Axiom of Choice (in Zermelo–Fraenkel set theory), without transfinite induction which would be used in a "natural" proof of the claim.

Assume, for the contrary, that X has no maximal elements. Let $\mathcal{W}$ denote the family of the well-ordered subsets[1] of X. For any $C \in \mathcal{W}$, we define $U_C := \{x \in X \mid y < x \text{ for any } y \in C\}$. Now $U_C \cap C = \emptyset$, and as an upper bound $x \in X$ for C is not maximal, we have $\emptyset \neq U_{\{x\}} \subseteq U_C$, therefore $U_C \neq \emptyset$. As $\mathcal{U} := \{S \subseteq X \mid S = U_C \text{ for some } C \in \mathcal{W}\}$ is a family of non-empty sets, <u>Axiom of Choice</u> yields its choice function f; that is, $f(U_C) \in U_C$ for every $C \in \mathcal{W}$. Let $\mathcal{C}_0$ denote the set of all $C \in \mathcal{W}$ satisfying the condition (i-C): $S \subseteq C$ and $U_S \not\subseteq U_C$ imply $f(U_S) \in C$. Let $\mathcal{C}$ denote the set of all $C \in \mathcal{C}_0$ satisfying the condition (ii-C): $C' \in \mathcal{C}_0$ implies $C \setminus C' \subseteq U_{C'}$.

We show that $C^* := \bigcup_{C \in \mathcal{C}} C \in \mathcal{C}$. First, $C' \in \mathcal{C}_0$ implies that $C^* \setminus C' \subseteq \bigcup_{C \in \mathcal{C}} C \setminus C' \subseteq U_{C'}$ (from (ii-C) for each $C \in \mathcal{C}$); hence (ii-C^*) holds. Secondly, when $\emptyset \neq S \subseteq C^*$, we have $S \cap C \neq \emptyset$ for some $C \in \mathcal{C}$. Now $S \setminus C \subseteq C^* \setminus C \subseteq U_C$ from (ii-C^*), implying that $\min(S \cap C)$ is the minimum element of S. Hence $C^* \in \mathcal{W}$. Moreover, when $S \subseteq C^*$ and $U_S \not\subseteq U_{C^*} = \bigcap_{C \in \mathcal{C}} U_C$, we have $U_S \not\subseteq U_C$ for some $C \in \mathcal{C}$, therefore $x \notin U_C$ for some $x \in U_S$. Now for any $y \in S$, we have $y < x$, therefore $y \notin U_C$. Hence $S \cap U_C = \emptyset$. As $S \setminus C \subseteq C^* \setminus C \subseteq U_C$ from (ii-C^*), we have $S \subseteq C$. Now from (i-C), we have $f(U_S) \in C \subseteq C^*$. Hence (i-$C^*$) holds. Summarizing, we have $C^* \in \mathcal{C}$. Let $u := f(U_{C^*})$ and $C^{**} := C^* \cup \{u\}$.

As $u = \max C^{**}$ and $C^* \in \mathcal{W}$, we have $C^{**} \in \mathcal{W}$. When $S \subseteq C^{**}$ and $U_S \not\subseteq U_{C^{**}}$, we have $u \notin S$ (as otherwise we would have $U_S = U_{\{u\}} = U_{C^{**}}$) and hence $S \subseteq C^*$ and $U_{C^*} \subseteq U_S$. Now if $U_S \subseteq U_{C^*}$, then we have $U_S = U_{C^*}$ and $f(U_S) = f(U_{C^*}) = u \in C^{**}$. On the other hand, if $U_S \not\subseteq U_{C^*}$, then we have $f(U_S) \in C^* \subseteq C^{**}$ from (i-C^*). Hence we have $f(U_S) \in C^{**}$ in any case, therefore (i-C^{**}) holds and $C^{**} \in \mathcal{C}_0$. As $C^{**} \not\subseteq C^*$, we have $C^{**} \notin \mathcal{C}$, therefore (ii-$C^{**}$) fails and $C^{**} \setminus C' \not\subseteq U_{C'}$ for some $C' \in \mathcal{C}_0$. From (ii-C^*), we have $C^* \setminus C' \subseteq U_{C'}$, therefore $u \notin C'$ and $u \notin U_{C'}$ (as otherwise $\emptyset \neq (C^{**} \setminus C') \setminus U_{C'} = (C^* \setminus C') \setminus U_{C'} = \emptyset$, a contradiction). This and the fact $u \in U_{C^*}$ imply that $U_{C^*} \not\subseteq U_{C'}$ and $C^* \cap U_{C'} = \emptyset$, therefore $C^* \subseteq C'$. Now by applying (i-C') to $C^* \subseteq C'$, it follows that $u = f(U_{C^*}) \in C'$, a contradiction.

This completes the proof of Zorn's Lemma.

Appendix: A proof using transfinite induction

In this appendix, for the sake of comparison, we describe a proof of Zorn's Lemma from Axiom of Choice using transfinite induction. First we clarify the statement of the principle for "definition by transfinite

[1] that is, any non-empty subset S has the minimum element $\min S$; note that now for any elements $x, y \in S$, one of x and y is the minimum element of $\{x, y\}$, implying that $x \leq y$ or $y \leq x$, hence S is totally ordered

recursion" (see e.g., [1, Chapter I, Theorem 9,3]):

Theorem 1. *Let $\varphi(x,y)$ be a formula (in Zermelo–Fraenkel set theory) with free variables x,y satisfying $\forall x \exists! y \varphi(x,y)$. Then there exists a formula $\Phi(x,y)$ with free variables x,y satisfying the following two conditions;*

1. $\forall x((x \in \mathbf{ON} \to \exists! y \Phi(x,y)) \wedge (\neg x \in \mathbf{ON} \to \neg \exists y \varphi(x,y)))$*;*
2. $\forall x(x \in \mathbf{ON} \to \forall y, z(y = \Phi \restriction_x \wedge \varphi(y,z) \to \Phi(x,z)))$*,*

where "$x \in \mathbf{ON}$" is an abbreviation of "x is an ordinal number" and "$\Phi \restriction_x$" is an abbreviation of the set $\{\langle a,b \rangle \mid a \in x \wedge \Phi(a,b)\}$ (with $\langle a,b \rangle$ denoting the ordered pair of a and b).

Intuitively, the theorem means that, if we would like to define a "function" Φ with domain consisting of all ordinal numbers (the whole of which is never a set) in such a way that the value of Φ at each ordinal number α is determined by a given rule from the values of Φ at ordinal numbers less than α, then there indeed exists such a "function" Φ. Note that this is a theorem of ZF set theory and does not depend on Axiom of Choice.

Now we give a proof of Zorn's Lemma from Axiom of Choice using Theorem 1 (as well as transfinite induction). Let $X \neq 0$ be a partially ordered set appeared in the statement of Zorn's Lemma. Assume, for the contrary, that X has no maximal elements. Then, for each non-empty subset C of X which is isomorphic to an ordinal number (hence is totally ordered), it follows from Axiom of Choice that there exists a distinguished upper bound b_C of C with $b_C \in X \setminus C$.

To apply Theorem 1, first we define a formula $\varphi(x,y)$ in the following manner, where we fix an element $a \in X$ throughout the proof:

- If $x = 0$ $(= \emptyset)$, then let $\varphi(x,y)$ mean that $y = a$.
- If x is a function from an ordinal number $\alpha > 0$ to X which is an isomorphism (between partially ordered sets) onto the image $\mathrm{Im}(x)$ of x, then let $\varphi(x,y)$ mean that $y = b_{\mathrm{Im}(x)}$ (note that $\mathrm{Im}(x)$ is isomorphic to the non-empty ordinal number α, therefore $b_{\mathrm{Im}(x)}$ is indeed defined).
- Otherwise, let $\varphi(x,y)$ mean that $y = 0$.

This formula $\varphi(x,y)$ satisfies the hypothesis of Theorem 1, therefore a formula $\Phi(x,y)$ as in the theorem exists. Now we have the following lemma:

Lemma 1. *Let x be an ordinal number, and let x' be the unique element satisfying $\Phi(x,x')$.*

1. *We have $x' \in X$.*
2. *If $y < x$ and $\Phi(y,y')$, then $y' < x'$ in X.*

Proof. We prove the claim by transfinite induction on x. First, if $x = 0$, then it follows from the definition of the formula φ that $x' = a$, therefore the specified conditions are satisfied. Secondly, suppose that $x > 0$. Then, by the hypothesis of the transfinite induction, the set $\Phi \restriction_x$ in the statement of Theorem 1 is an isomorphism from x to a subset of X, say, C (note that x is totally ordered). Now by the definitions of Φ and φ, it follows that $x' = b_C$, therefore the specified conditions are satisfied for x (the second condition follows from the property that $b_C \in X \setminus C$ is an upper bound of C). Hence the claim holds. $\square$

By the second property shown in Lemma 1, for each element $v \in X$, there exists at most one ordinal number x satisfying $\Phi(x,v)$. Let X' denote the subset of X defined in such a way that $v \in X'$ if and only if $v \in X$ and $\Phi(x,v)$ for some (or equivalently, a unique) ordinal number x. By the Axiom Schema of Replacement applied to the set X' and the formula $\Phi'(x,y) := \Phi(y,x)$, there exists a set Y for which we have $y \in Y$ if y is an ordinal number and the unique element y' satisfying $\Phi(y,y')$ belongs to X'. Now by the first property shown in Lemma 1, the set Y contains every ordinal number. However, this contradicts Burali–Forti Paradox (which states that there exist no sets containing all ordinal numbers). Hence X should have a maximal element, concluding the proof of Zorn's Lemma.

References

[1] K. Kunen, "Set Theory: An Introduction to Independence Proofs", Elsevier, 1980

[2] J. Lewin, "A Simple Proof of Zorn's Lemma", Amer. Math. Monthly **98**(4) (1991), 353–354

[3] H. Rubin, J. E. Rubin, "Equivalents of the Axiom of Choice, II", Second Edition, Studies in Logic and the Foundations of Mathematics vol.116, North-Holland, 1985