MAT 2377
Probability and Statistics for Engineers

Chapter 5
Point and Interval Estimation

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Contents

Statistical Inference

The main goals of **statistics** is to make **inference** about a **population** based on a random sample from the population.

Examples:

- Can we assess the reliability of a product's manufacturing process by randomly selecting a sample of the final product and determining how many of them are compliant according to some quality assessment scheme?

- Can we determine who will win an election by polling a small sample of respondents?

Statistical inference about **Unknown Parameters** is divided into:

- Point estimation
- Interval estimation (Confidence Interval)
- Hypothesis testing

Point estimation

We seek to estimate an unknown **parameter** θ using a single quantity, called **point estimate** $\hat{\theta}$.

θ is parameter of interest and it can be a character of the population like mean, median, variance, standard deviation, ...

$\hat{\theta}$ is the point estimate of θ and it is obtained using a **statistic**, which is simply a function of a random sample. This estimate depends on data and it is random, so it has a sampling distribution.

Example: consider a process that manufactures gear wheels (in some standard gauge). Let X be the random variable that records the weight of a randomly selected gear wheel. What is the population mean $\mu_X = \mathrm{E}[X]$?.

Solution: in the absence of $f(x)$, we can estimate $\mu = X$ with the help of a random sample $X_1, \ldots, X_n$ of gear wheel weight measurements, *via* the sample mean statistic:

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n}, \quad \text{which is } \approx \mathcal{N}\left(\mu, \sigma^2/n\right) \text{ according to C.L.T.}$$

Statistic

Examples of statistics include:

- sample mean and sample median
- sample variance and sample standard deviation
- sample quantiles (median, quartiles, percentiles)
- test statistics ($t-$statistics, χ^2-statistics, $F-$statistics, etc.)
- order statistics (sample maximum and minimum, sample range, etc.)
- sample moments and functions thereof (skewness, kurtosis, etc.)

Some point estimators

Unknown population parameters θ	Sample estimates (Statistic) $\hat{\theta}$
Mean μ	$\bar{X} = \Sigma_{i=1}^{n} X_i/n$
Variance σ^2	$S^2 = \Sigma_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$
Standard deviation σ	$S = \sqrt{\Sigma_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}$
Probability of success p	$\hat{p} = X/n$

Then $\hat{\mu} = \bar{X}$ means that $\bar{X}$ is an estimator for μ.
Similarly, $\hat{\sigma^2} = S^2$, $\hat{\sigma} = S$, and $\hat{p} = X/n$.

Variance and Standard Error of $\bar{X}$

The **standard error** of a statistic is the **standard deviation of its sampling distribution**.

For instance, if observations $X_1, \ldots, X_n$ come from a a population with **unknown mean** μ and **known variance** σ^2, then $\mathrm{Var}(\overline{X}) = \sigma^2/n$ and the **standard error of** $\overline{X}$ is

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}.$$

If the variance of the original population is **unknown**, then it is estimated by the sample variance S^2 and the **estimated standard error of** $\overline{X}$ is

$$\hat{\sigma}_{\overline{X}} = \frac{S}{\sqrt{n}}, \quad \text{where} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Examples:

1. A sample of 20 baseball player heights (in inches) is shown below.

   `74,74,72,72,73,69,69,71,76,71,73,73,74,74,69,70,72,73,75,78.`

   Let $\overline{X}$ be the sampling mean of the heights. Then,

$$\overline{X} = \frac{X_1 + \cdots + X_{20}}{20} = 72.6$$

   and the sample variance S^2 is

$$S^2 = \frac{1}{20-1} \sum_{i=1}^{20} (X_i - 72.6)^2 \approx 5.6211.$$

The standard error of $\overline{X}$ is thus

$$\hat{\sigma}_{\overline{X}} = \frac{S}{\sqrt{20}} \approx \sqrt{\frac{5.6211}{20}} \approx 0.5301.$$

2. Consider a sample $\{X_1, \ldots, X_{100}\}$ of independent observations selected from a normal population $\mathcal{N}(\mu, \sigma^2)$ where $\sigma = 50$ is known, but μ is not. What is the best estimate of μ? What is the sampling distribution of that estimate?

   Solution: the sample mean $\overline{X} = \frac{X_1 + \cdots + X_{100}}{100}$ provides the best estimate of $\mu_X = \mu_{\overline{X}}$. The standard error of $\overline{X}$ is $\sigma_{\overline{X}} = \frac{50}{\sqrt{100}} = 5$. Since the observations are sampled independently from a normal population with mean μ and standard deviation 50, $\overline{X} \sim \mathcal{N}(\mu, 5^2) = \mathcal{N}(\mu, 25)$, according to the CLT.

C.I. for μ When σ is Known

Consider a sample $\{x_1, \ldots, x_n\}$ from a normal population with **known** variance σ^2 and **unknown** mean μ. The sample mean

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n}$$

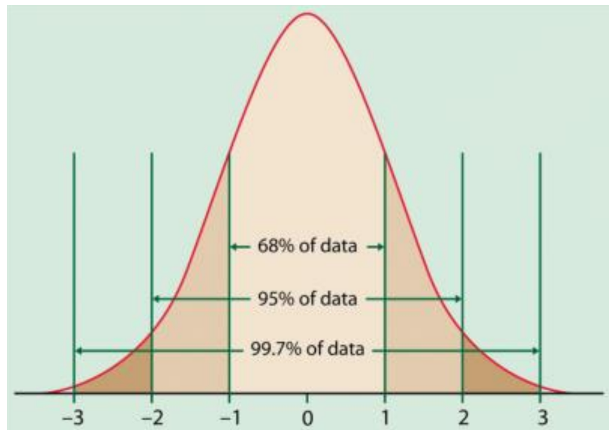
is a **point estimate** of μ.

Of course, this estimate is not exact, because $\overline{x}$ is an observed value of $\overline{X}$; it is unlikely that the observed value $\overline{x}$ should coincide with μ.

We know that $\overline{X} \sim \mathcal{N}(\mu, \sigma^2/n)$, so that

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1).$$

The 68–96–99.7 Rule



$$\text{If } Z \sim N(0,1), \text{ then}: \quad \begin{aligned} P(-1 < Z < 1) &\approx 0.683 \\ P(-2 < Z < 2) &\approx 0.955 \\ P(-3 < Z < 3) &\approx 0.997. \end{aligned}$$

Whenever we observe a sample mean $\overline{X}$ from a normal population with mean μ, we would expect the inequality

$$-k < Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < k$$

to hold approximately

$$g(k) = \begin{cases} 68.3\% \text{ of the time} & \text{if } k = 1 \\ 95.5\% \text{ of the time} & \text{if } k = 2 \\ 99.7\% \text{ of the time} & \text{if } k = 3 \end{cases}$$

Equivalently, the **symmetric** $g(k)$ **confidence interval for** μ is

$$\overline{X} - k\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + k\frac{\sigma}{\sqrt{n}} \quad \Longrightarrow \quad \overline{X} \pm k\frac{\sigma}{\sqrt{n}}.$$

Examples:

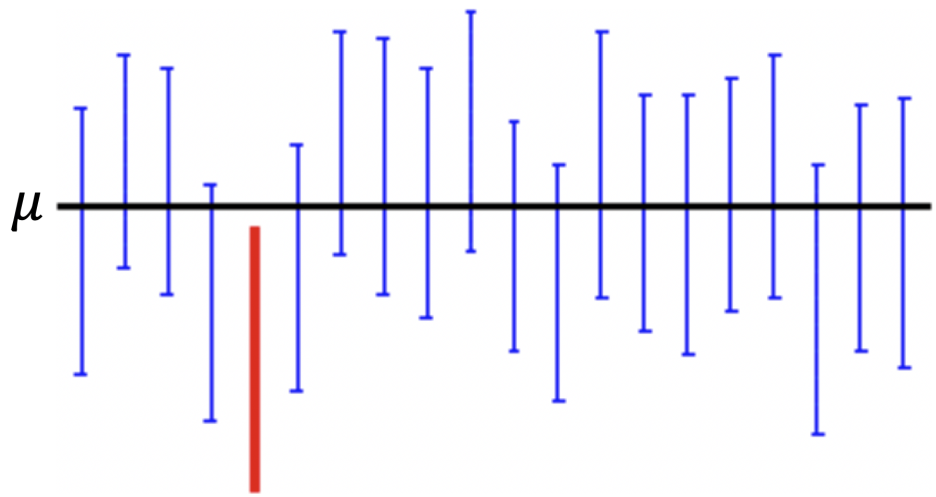
1. Consider a sample $\{X_1, \ldots, X_{64}\}$ from a normal population with standard deviation $\sigma = 72$ and unknown mean μ. The sample mean is $\overline{X} = 375.2$. Build a symmetric 68.3% confidence interval for μ.

   Solution: according to the formula, the symmetric 68.3% confidence interval $(k = 1)$ for μ in this situation is

$$375.2 \pm 1 \cdot \frac{72}{\sqrt{64}} \implies (375.2 - 9, 375.2 + 9) = (366.2, 384.2).$$

   IMPORTANT: this does not say that we're 68.3% sure that the true μ is between 366.2 and 384.2. What it says is that when a sample of size 64 is taken from a normal population $\mathcal{N}(\mu, 72^2)$ and a symmetric 68.3% confidence interval for μ is built, μ will fall between the endpoints of the interval about 68.3% of the time.

Interpretation of a 95% C.I.



A 95% C.I. indicates that we would expect 19 out of 20 samples from the same population to produce confidence intervals that contain the population parameter of interest, on average.

2. Build a symmetric 95.5% confidence interval for μ.

   Solution: the same formula applies, with $k = 2$.

$$375.2 \pm 2 \cdot \frac{72}{\sqrt{64}} \implies (375.2 - 18, 375.2 + 18) = (357.2, 393.2).$$

3. Build a symmetric 99.7% confidence interval for μ.

   Solution: the same formula applies, with $k = 3$.

$$375.2 \pm 3 \cdot \frac{72}{\sqrt{64}} \implies (375.2 - 27, 375.2 + 27) = (348.2, 402.2).$$

C.I. for μ When σ is Known (reprise)

Another approach to C.I. building is to specify the proportion of the area under $\phi(z)$ of interest, and then to determine the critical values (the endpoints) of the interval.

Let $\{X_1, \ldots, X_n\}$ be drawn from $N(\mu, \sigma^2)$. Recall that $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$. For a **symmetric** 95% **confidence interval**, we need to find $z^* > 0$ such that $P(-z^* < Z < z^*) \approx 0.95$.

But the LHS can be re-written as

$$\begin{aligned} P(-z^* < Z < z^*) &= \Phi(z^*) - \Phi(-z^*) \\ &= \Phi(z^*) - (1 - \Phi(z^*)) = 2\Phi(z^*) - 1 \end{aligned}$$

So we are looking for z^* such that

$$0.95 = 2\Phi(z^*) - 1 \Longrightarrow \Phi(z^*) = \frac{0.95+1}{2} = 0.975.$$

From the normal table, we see that $\Phi(1.96) \approx 0.9750$, so that

$$P(-1.96 < Z < 1.96) = P\left(-1.96 < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right) \approx 0.95.$$

In other words, the inequality

$$-1.96 < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < 1.96$$

holds with probability 0.95 (with the interpretation provided in Example 1).

Equivalently,

$$\boxed{\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}} \implies \overline{X} \pm 1.96\frac{\sigma}{\sqrt{n}}}$$

is the **symmetric** 95% **confidence interval for** μ **when** σ **is known**.

A similar argument shows that

$$\boxed{\overline{X} - 2.575\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 2.575\frac{\sigma}{\sqrt{n}} \implies \overline{X} \pm 2.575\frac{\sigma}{\sqrt{n}}}$$

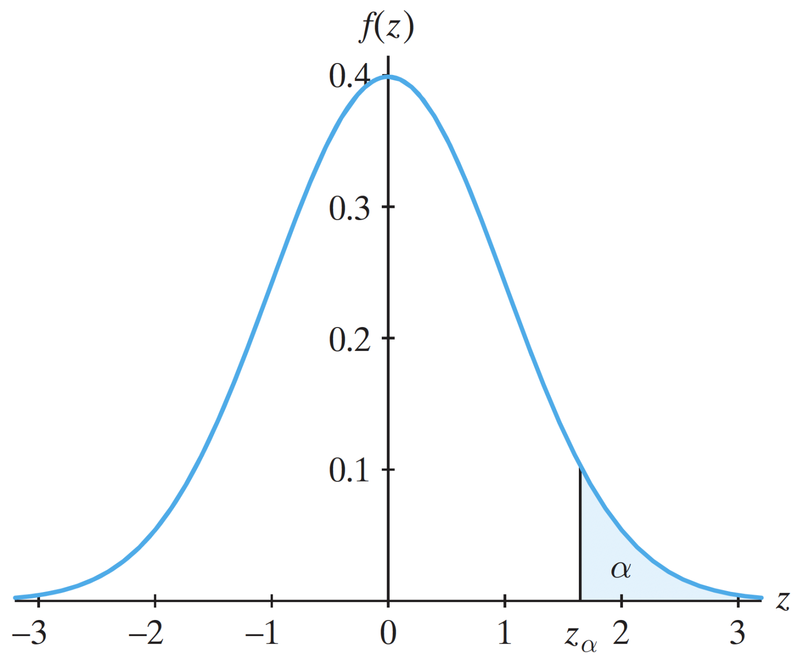
is the **symmetric** 99% **confidence interval for** μ **when** σ **is known**.

Critical Z-values

The **confidence level** $1-\alpha$ is usually expressed in terms of a **small** α, e.g. $\alpha = 0.05 \Rightarrow 1-\alpha = 0.95$ confidence level.

For $\alpha = 0.01, 0.02, \ldots, 0.98, 0.99$, the corresponding z_α are called the **percentiles** of the standard normal distribution. In general,

$$P(Z > z_\alpha) = \alpha \implies z_\alpha \text{ is the } 100(1-\alpha) \text{ percentile.}$$



$$P(Z > z_\alpha) = \alpha$$

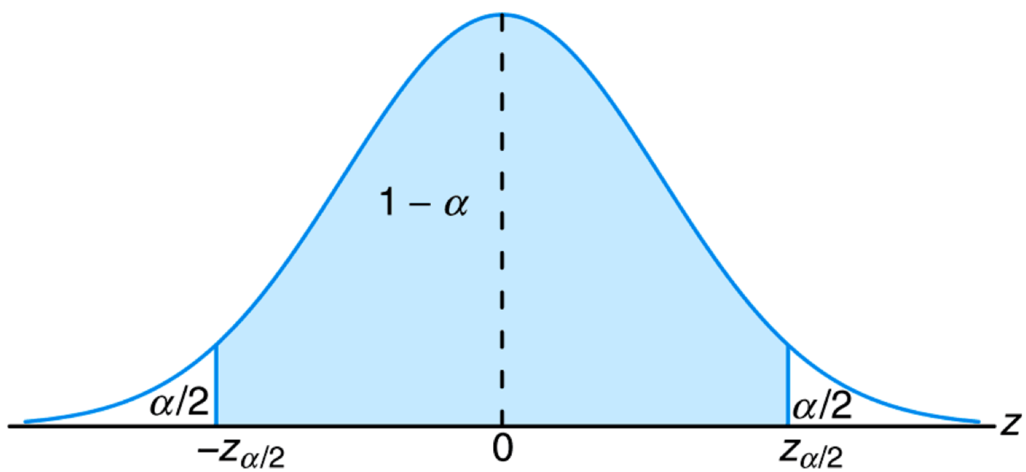
$$P(Z > z) = 1 - \Phi(z) = \Phi(-z)$$

For $2-$**sided confidence intervals**, the appropriate numbers are found by solving $P(|Z| > z^*) = \alpha$ for z^*. By the properties of $\mathcal{N}(0,1)$,

$$\alpha = P(|Z| > z^*) = 1 - P(-z^* < Z < z^*) = 1 - (2\Phi(z^*) - 1) = 2(1 - \Phi(z^*)),$$

so that

$$\Phi(z^*) = 1 - \alpha/2 \implies z^* = z_{\alpha/2}.$$

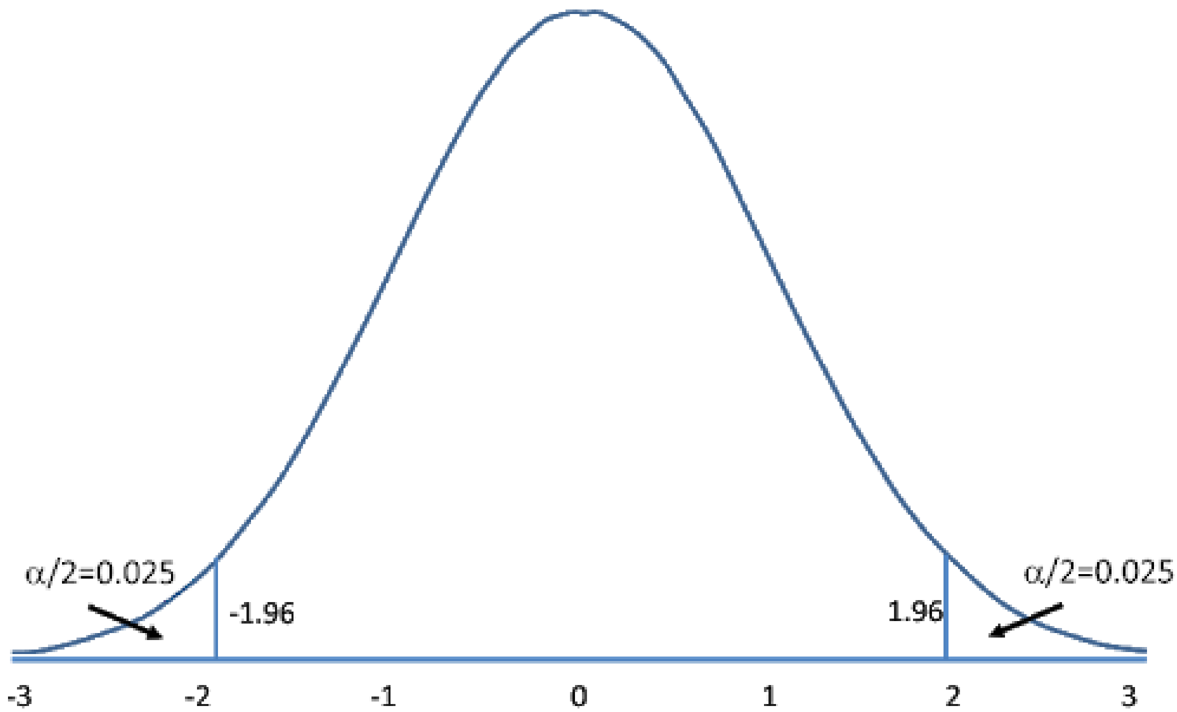


For instance,

$$P(|Z| > z_{0.025}) = 0.05 \implies z_{0.025} = 1.96$$
$$P(|Z| > z_{0.005}) = 0.01 \implies z_{0.005} = 2.575.$$

The symmetric $100(1-\alpha)\%$ confidence interval can generally be written as

$$\boxed{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \implies \overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$$


α/2=0.025
-1.96
1.96
α/2=0.025
-3
-2
-1
0
1
2
3

For a given confidence level α, **shorter confidence intervals are better** in relation to estimating the mean:

- estimates become better when the sample size n increases;
- estimates become better when σ decreases.

If $\alpha_1 > \alpha_2$, the $100(1-\alpha_1)\%$ C.I. is smaller than the $100(1-\alpha_2)\%$ C.I. (i.e. a 95% C.I. is always shorter than a 99% C.I.)

If the sample comes from a normal population, then the C.I. is **exact**. Otherwise, if n is large, we may use the CLT and get an **approximate** C.I.

Examples:

1. A sample of 9 observations from a normal population with known standard deviation $\sigma = 5$ yields a sample mean $\overline{X} = 19.93$. Provide a 95% and a 99% C.I. for the unknown population mean μ based on this sample.

   Solution: the estimate of μ is $\overline{X} = 19.93$. The $100(1-\alpha)\%$ confidence intervals are
$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

$$95\% : \overline{X} \pm z_{0.025}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 1.96\frac{5}{\sqrt{9}} \Rightarrow 19.93 \pm 3.27 \text{ or } (16.66, 23.20)$$

$$99\% : \overline{X} \pm z_{0.005}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 2.575\frac{5}{\sqrt{9}} \Rightarrow 19.93 \pm 4.29 \text{ or } (15.64, 24.22)$$

2. A sample of 25 observations from a normal population with known standard deviation $\sigma = 5$ yields a sample mean $\overline{X} = 19.93$. Provide a 95% and a 99% C.I. for the unknown population mean μ based on this sample.

   Solution: the estimate of μ is $\overline{X} = 19.93$. The $100(1-\alpha)\%$ confidence intervals are

$$95\% : \overline{X} \pm z_{0.025}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 1.96\frac{5}{\sqrt{25}} \Rightarrow 19.93 \pm 1.96 \text{ or } (17.97, 21.89)$$

$$99\% : \overline{X} \pm z_{0.005}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 2.575\frac{5}{\sqrt{25}} \Rightarrow 19.93 \pm 2.58 \text{ or } (17.35, 22.51)$$

3. A sample of 25 observations from a normal population with known standard deviation $\sigma = 10$ yields a sample mean $\overline{X} = 19.93$. Provide a 95% and a 99% C.I. for the unknown population mean μ based on this sample.

   Solution: the estimate of μ is $\overline{X} = 19.93$. The $100(1-\alpha)\%$ confidence intervals are

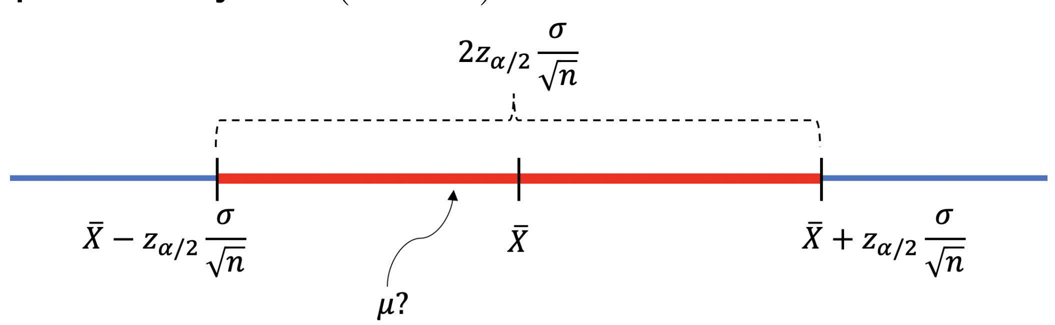
   $$95\% : \overline{X} \pm z_{0.025}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 1.96\frac{10}{\sqrt{25}} \Rightarrow 19.93 \pm 3.92 \text{ or } (16.01, 23.85)$$

   $$99\% : \overline{X} \pm z_{0.005}\frac{\sigma}{\sqrt{n}} \Rightarrow 19.93 \pm 2.575\frac{10}{\sqrt{25}} \Rightarrow 19.93 \pm 5.15 \text{ or } (14.78, 25.08)$$

   Note how the confidence intervals are affected by α, n, and σ.

Choice of Sample Size

The **error** we commit by estimating μ *via* the sample mean $\overline{X}$ is smaller than $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$, with probability $100(1-\alpha)\%$.



If we want to control the error, the only thing we can really do is control the sample size:

$$\boxed{E > z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \implies n > \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2.}$$

Examples:

1. A sample $\{X_1, \dots, X_n\}$ is selected from a normal population with standard deviation $\sigma = 100$. What sample size should be used to insure that the error on the population estimate is at most $E = 10$, at a confidence level $\alpha = 0.05$?

   Solution: as long as

$$n > \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{z_{0.025} \cdot 100}{10}\right)^2 = (19.6)^2 = 384.16,$$

   then the error committed by using $\overline{X}$ to estimate μ will be at most 10, with 95% probability.

2. Repeat the first example, but with $\sigma = 10$.

   Solution: we need

$$n > \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{z_{0.025} \cdot 10}{10}\right)^2 = (1.96)^2 = 3.8416.$$

3. Repeat the first example, but with $E = 1$.

   Solution: we need

$$n > \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{z_{0.025} \cdot 100}{1}\right)^2 = (196)^2 = 38416.$$

4. Repeat the first example, but with $\alpha = 0.01$.

   Solution: we need

$$n > \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{z_{0.005} \cdot 100}{10}\right)^2 = (25.75)^2 = 663.0625.$$

The relationship between α, σ, E, and n is not always intuitive!

C.I. for μ When σ is Known

So far, we have been in the fortunate situation of sampling from a population with known variance σ^2.

What do we do when the population variance is **unknown**?

We estimate σ using the **sample variance**

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

(remember that the true population mean μ is also unknown... that's what we're trying to find!) and the **sample standard deviation** $S = \sqrt{S^2}$.

If σ is known, we know from the CLT that $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $\mathcal{N}(0,1)$.
If σ is unknown, it can be shown that $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ follows approximately $t(n-1)$, the **Student $T-$distribution with $n-1$ degrees of freedom**.

Consequently, for a confidence level α,

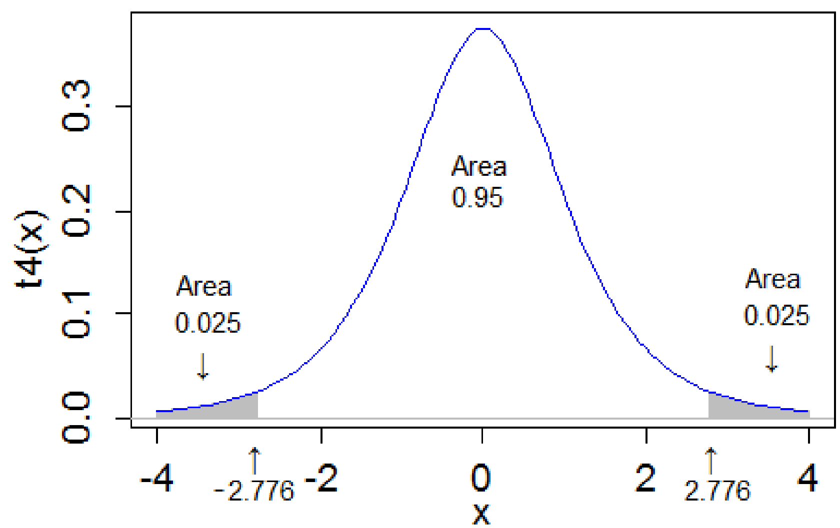
$$P\left(-t_{\alpha/2}(n-1) < \frac{\overline{X}-\mu}{S/\sqrt{n}} < t_{\alpha/2}(n-1)\right) \approx 1-\alpha,$$

where $t_{\alpha/2}(n-1)$ is the $100(1-\alpha/2)^{\text{th}}$ percentile of $t(n-1)$ (these can be read from the table). Equality is reached if the underlying population is normal. Therefore

$$\boxed{100(1-\alpha)\% \text{ C.I. for } \mu : \overline{X} \pm t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}.}$$

For instance, if $\alpha = 0.05$ and $\{X_1, X_2, X_3, X_4, X_5\}$ are samples from a normal distribution with unknown mean μ and unknown variance σ^2, then

$$t_{0.025}(5-1) = 2.776 \quad \text{and} \quad P\left(-2.776 < \frac{\overline{X}-\mu}{S/\sqrt{5}} < 2.776\right) = 0.95.$$



Examples:

1. For a given year, 9 measurements of ozone concentration are obtained:

   `3.5 5.1 6.6 6.0 4.2 4.4 5.3 5.6 4.4`

   Assume that the measured ozone concentrations follow a normal distribution with variance $\sigma^2 = 1.21$, build a 95% for the population mean μ. Note that $\overline{X} = 5.01$ and that $S = 0.97$.

   Solution: since we know the variance, we need to use the standard normal percentile $z_{\alpha/2} = z_{0.025} = 1.96$:

$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 5.01 \pm 1.96\frac{\sqrt{1.21}}{\sqrt{9}} = 5.01 \pm 0.72 \text{ or } (4.29, 5.73).$$

2. Same thing, but assume that the variance of the underlying population is unknown.

   Solution: since we do not know the variance, we need to use the Student percentile $t_{\alpha/2}(n-1) = t_{0.025}(8) = 2.306$ (make sure you understand how to get this value from the table):

$$\overline{X} \pm t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}} = 5.01 \pm 2.306\frac{0.97}{\sqrt{9}} \text{ or } (4.26, 5.76).$$

   The 95% C.I. when we know the variance is **tighter** (smaller), which is natural as we are more confident about our results when we have more information.

C.I. for a Proportion

If $X \sim \mathcal{B}(n,p)$ (number of successes in n trials), then the point estimator for p is $\hat{P} = \frac{X}{n}$.

Recall that $\mathrm{E}[X] = np$ and $\mathrm{Var}[X] = np(1-p)$.

We can standardize any random variable:

$$Z = \frac{X-\mu}{\sigma} = \frac{n\hat{P} - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately $\mathcal{N}(0,1)$.

Thus, for sufficiently large n,

$$P\left(-z_{\alpha/2} < \frac{\hat{P}-p}{\sqrt{\frac{p(1-p)}{n}}} < z_{\alpha/2}\right) \approx 1-\alpha.$$

Using the previous approach, an **approximate** $100(1-\alpha)\%$ C.I. for p is:

$$\hat{P} - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} < p < \hat{P} + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}},$$

but this is not really useful because we don't actually know p! Instead:

$$\boxed{\hat{P} - z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} < p < \hat{P} + z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}.}$$

Examples:

1. Two candidates (A and B) are running for office. A poll is conducted: 1000 voters are selected randomly and asked for their preference: 52% support A, while 48% support their rival, B. Provide a 95% C.I. for the support of each candidate.

   Solution: we use $\alpha = 0.05$ and $\hat{P} = 0.52$. The 95% C.I. for A is

$$\hat{P} \pm z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} = 0.52 \pm 1.96\sqrt{\frac{0.52 \cdot 0.48}{1000}} \approx 0.52 \pm 0.031.$$

   The 95% C.I. for B is 0.48 ± 0.031.

2. On the strength of this polling result, a newspaper prints the following headline: "Candidate A Leads Candidate B!" Is the headline warranted?

   Solution: although there is a $4-$point gap in the poll numbers, the true support for candidate A is in the $48.9\% - 55.1\%$ range, and, the true support for candidate B is in the $44.9\% - 51.1\%$ range, with probability 95% (that is to say, 19 times out of 20).

   Since there is overlap in the confidence intervals, the race is more likely to be a dead heat.

Two-samples: difference between means

Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ_X and variance σ_X^2, and $Y_1, \ldots, Y_m$ be another random sample, independent of X, from a population with mean μ_Y and variance σ_Y^2. If $\overline{X}$ and $\overline{Y}$ are the respective sample means, then

$$Z = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

has standard normal distribution as $n, m \to \infty$.

KNOWN Variances: Therefore, when variances are known, a $100(1-\alpha)\%$ confidence interval for $\mu_X - \mu_Y$ is

$$\boxed{(\overline{X}-\overline{Y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}} < \mu_X - \mu_Y < (\overline{X}-\overline{Y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}}$$

POOLED Variance: Similarly, when variances are unknown but equal, a $100(1-\alpha)\%$ confidence interval for $\mu_X - \mu_Y$ is

$$\boxed{(\overline{X}-\overline{Y}) - t_{\alpha/2}(df)\, S_p \sqrt{\frac{1}{n}+\frac{1}{m}} < \mu_X - \mu_Y < (\overline{X}-\overline{Y}) + t_{\alpha/2}(df)\, S_p \sqrt{\frac{1}{n}+\frac{1}{m}}}$$

where $df = n + m - 2$ is degree of freedom and

$$S_p^2 = \frac{(n-1)\, S_X^2 + (m-1)\, S_Y^2}{n+m-2}$$

Summary

Sample: $\{X_1, \ldots, X_n\}$.
Objective: Estimate μ with confidence level $100(1-\alpha)$.

- If population is **normal** with **known** variance σ^2, the **EXACT** $100(1-\alpha)\%$ C.I. is

$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

- If population is **non-normal** with **known** variance σ^2 and n is '**large enough**', the **approximate** $100(1-\alpha)\%$ C.I. is

$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Summary

- If population is **normal** with **unknown** variance, the **EXACT** $100(1-\alpha)\%$ C.I. is

$$\overline{X} \pm t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}.$$

- If population is **non-normal** with **unknown** variance and n is '**large enough**', the **approximate** $100(1-\alpha)\%$ C.I. is

$$\overline{X} \pm z_{\alpha/2}\frac{S}{\sqrt{n}}.$$

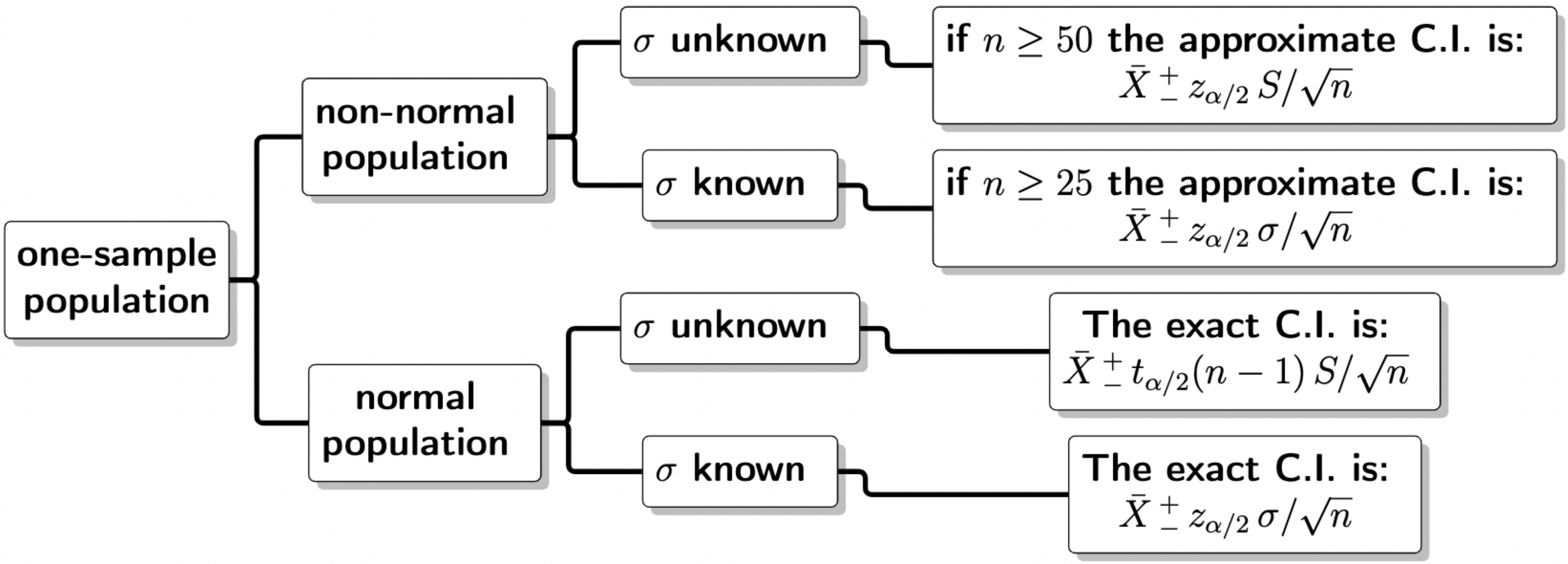


Figure 12: Confidence interval for the mean of a population