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Data Representation

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Outline

- Number Systems
- Fixed Point Representation
- Arithmetic operations
- Floating Point Representation
- BCD, ASCII

Number Systems

- There are four main types of number systems: *binary*, *octal*, *hexadecimal*, and *decimal* (commonly used by humans).
- In general, a number system of **base**, or **radix**, *r* is a system that uses distinct symbols for *r* digits.
- A number *x* in base *r* , noted as $(x)r$, has the following general form:

  $$(x)_r = x_n \, x_{n-1} \; \ldots \; x_1 \, x_0 \,.\, x_{-1} \; x_{-2} \; \ldots \; x_{-m} \;,$$

  where $x \in \{0, 1, \ldots, r-1\}$ and $\in \{n, n-1, \ldots, -m\}$.
- To **<u>convert</u>** a number in the form above **<u>from base *r* to the decimal base</u>** (with radix 10), the following formula is used

$$(x)_r = (x_n \times r^n + x_{n-1} \times r^{n-1} + \cdots + x_0 \times r^0 + x_{-1} \times r^{-1} + \cdots + x_{-m} \times r^{-m})_{10}$$

$$r \rightarrow 10$$

Binary Number System

- The **binary** system uses **2** as its radix.
- It employs two binary digits, (bits): 0 and 1
- Represents any number using the positional notation.
- The most significant bit (MSB) is the leftmost bit of a binary number.
- The least significant bit (LSB) is the rightmost bit of a binary number.

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Positional Notation

- The value of a digit depends on its placement within a number.
- In base 10, the positional values are (starting from the decimal point to the left): $1(10^0)$, $10(10^1)$, $100(10^2)$, $1000(10^3)$, etc.
- In base-2, the positional values are $1(2^0)$, $2(2^1)$, $4(2^2)$, $8(2^3)$…

Number … in whatever base	*Decimal value of the given number*
Decimal: $\mathbf{2\,017_{10}}$	$= 2\text{x}10^3 + 0\text{x}10^2 + 1\text{x}10^1 + 7\text{x}10^0$ $= 2{,}000 + 0 + 10 + 7 =$ **2 017**

Binary:

msb

2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**

lsb

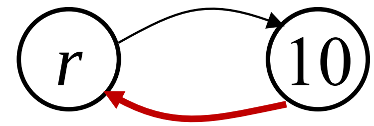
$\mathbf{11111100001_2} = 1\text{x}2^{10}+1\text{x}2^9+1\text{x}2^8+1\text{x}2^7+1\text{x}2^6+1\text{x}2^5+0\text{x}2^4+0\text{x}2^3+0\text{x}2^2+0\text{x}2^1+1\text{x}2^0 =$

$2 \rightarrow 10$ $= 1{,}024 + 512 + 256 + 128 + 64 + 32 + 1 = \mathbf{2017}$

Powers of 2

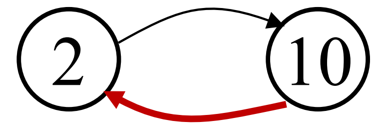
N	2^N	*Comments*
0	1	
1	2	
2	4	
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1,024	"Kilo" as 2^{10} is the closest power of 2 to 1,000 (decimal)
11	2,048	
…	…	…
15	32,768	2^{15} Hz often used as clock crystal frequency in digital watches
…	…	…
20	1,048,576	"Mega" as 2^{20} is the closest power of 2 to 1,000,000 (decimal)
…	…	…
30	1,073,741,824	"Giga" as 2^{30} is the closest power of 2 to 1,000,000,000(decimal)
…	…	…
40	…	"Terra" as 2^{40} is the closest power of 2 to 10^{12} (decimal)

Conversion



- Converting a decimal number to its equivalent representation in radix *r* is carried out by separating the number into its integer and fraction parts. Then convert each part separately.
- The conversion of the decimal integer part into a base *r* representation is done by successive divisions by *r* and accumulation of the remainders.
- The conversion of the decimal fraction part to radix *r* representation is accomplished by successive multiplications by *r* and accumulating the integer digits so obtained.

Decimal-to-Binary Conversion



2008	/2		
1004	0	lsb	2^0
502	0		2^1
251	0		2^2
125	1		2^3
62	1		2^4
31	0		2^5
15	1		2^6
7	1		2^7
3	1		2^8
1	1		2^9
0	1	msb	2^{10}

$1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0_2$

2^{10} . . 2^7 . . . 2^3 . . 2^0

- <u>Repeated Division by 2</u>
  1. Divide the number to be converted by 2.The remainder, 0 or 1, is the LSB of the binary value.
  2. Divide the quotient from Step 1 by 2. The remainder, 0 or 1, is the next most significant bit.
  3. Continue to execute Step 2 until the quotient is 0. The last remainder is the MSB.

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Negative Powers of 2

$N < 0$	2^N
-1	$2^{-1} = 0.5$
-2	$2^{-2} = 0.25$
-3	$2^{-3} = 0.125$
-4	$2^{-4} = 0.0625$
-5	$2^{-5} = 0.03125$
-6	$2^{-6} = 0.015625$
-7	$2^{-7} = 0.0078125$
-8	$2^{-8} = 0.00390625$
-9	$2^{-9} = 0.001953125$
-10	$2^{-10} = 0.0009765625$
…	

$2 \rightarrow 10$

- Fractional Binary Numbers

  Binary numbers less than 1

Binary	*Decimal value*
0.101101	$= 1 \times 2^{-1} + 1 \times 2^{-3} + 1 \times 2^{-4} + 1 \times 2^{-6} = \mathbf{0.703125_{10}}$

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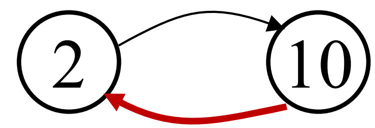
Fractional-Decimal – to – Fractional-Binary Conversion



1. Multiply the decimal fraction by 2. The integer part (0 or 1) is the first bit to the right of the binary point.
2. Discard the integer part from Step 1 and repeat Step 1 until the fraction repeats or terminates.
3. In real applications one might have to use a given number of bits, imposed by the number of bits of the registers that carry the result. Conversion error can be calculated

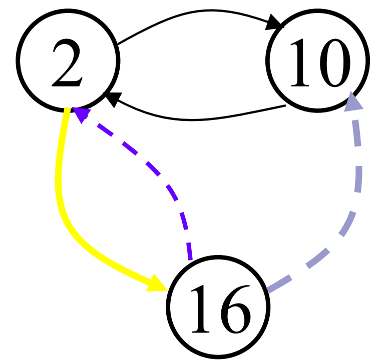
	2 x	0.703125_{10}		weight
msb	1	.40625	0.5	2^{-1}
	0	.8125	0	2^{-2}
	1	.625	0.125	2^{-3}
	1	.25	0.0625	2^{-4}
	0	.5	0	2^{-5}
lsb	1	.0	0.015625	2^{-6}
			0.703125	

$$0.101101 = 1\text{x}2^{-1} + 0\text{x}2^{-2} + 1\text{x}2^{-3} + 1\text{x}2^{-4} + 0\text{x}2^{-5} + 1\text{x}2^{-6} =$$

$$= 0.5 + 0.125 + 0.0625 + 0.015625 =$$

Hexadecimal Number System

Binary	Decimal	**Hexadecimal**
0000	0	**0**
0001	1	**1**
0010	2	**2**
0011	3	**3**
0100	4	**4**
0101	5	**5**
0110	6	**6**
0111	7	**7**
1000	8	**8**
1001	9	**9**
1010	10	**A**
1011	11	**B**
1100	12	**C**
1101	13	**D**
1110	14	**E**
1111	15	**F**



Binary:

$\mathbf{11111011010_2}$

Binary → Hex conversion

$\underbrace{\mathbf{111}}_{7_{10}}\ \underbrace{\mathbf{1101}}_{13_{10}}\ \underbrace{\mathbf{1010}}_{10_{10}}$ ← Nibbles' values

← expressed in *Decimal*

Hexadecimal:

$\mathbf{7DA_{16}} = 7\text{x}16^2 + 13\text{x}16^1 + 10\text{x}16^0$

Hex->decimal $= \mathbf{2010_{10}}$

5$mffpi$A3fmwgerfi$vitviwirxih$f}$e$wmrkpi$li|ehigmq ep$hmkmx

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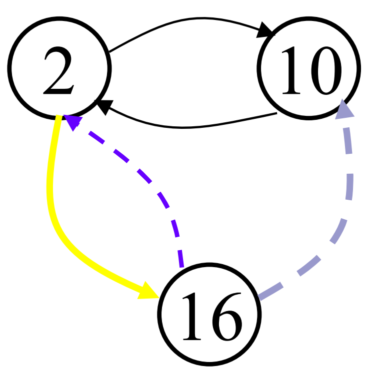
Converting from base 2 to base 8: partition the base 2 into groups of three and pad the outer to the left with 0s



Binary (base 2)	Octal (base 8)	Decimal (base 10)	Hexadecimal (base 16)
0	0	0	0
1	1	1	1
10	2	2	2
11	3	3	3
100	4	4	4
101	5	5	5
110	6	6	6
111	7	7	7
1000	10	8	8
1001	11	9	9
1010	12	10	A
1011	13	11	B
1100	14	12	C
1101	15	13	D
1110	16	14	E
1111	17	15	F

Binary, Octal, and Hexadecimal Numbers

- The conversion from binary to octal is easily accomplished by partitioning the binary number into groups of three bits each.
- The corresponding octal digit is then assigned to each group of bits.
- The string of octal digits obtained gives the octal equivalent of the binary number.
- Converting from binary to hexadecimal is similar except that the bits are divided in groups of 4 bits each this time.

Example

$$\overbrace{1}^{1}\,\overbrace{010}^{2}\,\overbrace{111}^{7}\,\overbrace{101}^{5}\,\overbrace{100}^{4}\,\overbrace{011}^{3} \quad \textbf{Octal}$$

$$1\,0\,1\,0\,1\,1\,1\,1\,0\,1\,1\,0\,0\,0\,1\,1 \quad \textbf{Binary}$$

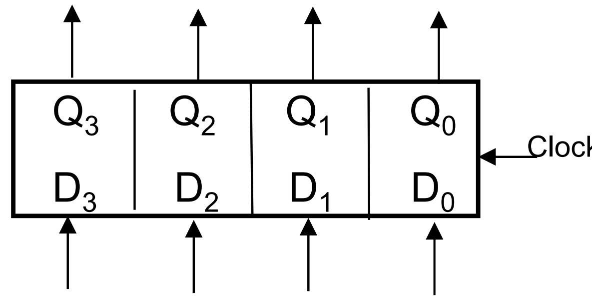
$$\underbrace{1010}_{A}\,\underbrace{1111}_{F}\,\underbrace{0110}_{6}\,\underbrace{0011}_{3} \quad \textbf{Hexadecimal}$$

Binary Number Representations

- In computer systems, binary numbers are represented as unsigned or signed numbers.
- **Unsigned** numbers are numbers that are always assumed to be non-negative (i.e., ≥ 0).
- **Signed** numbers are numbers that can be either non-negative or negative.
- The sign of a signed number is determined by the value of a pre-specified bit, known as the **sign bit**, in the number's representation.
- The sign bit is usually taken to be the left-most bit in the number's representation.
- The common convention in a binary system is that the sign bit is 0 for a non-negative number and 1 for a negative number.

Unsigned numbers

- the weight of the MSB (Most Significant Bit) is 2^{n-1}
- the range of representable numbers is $[0, 2^n-1]$

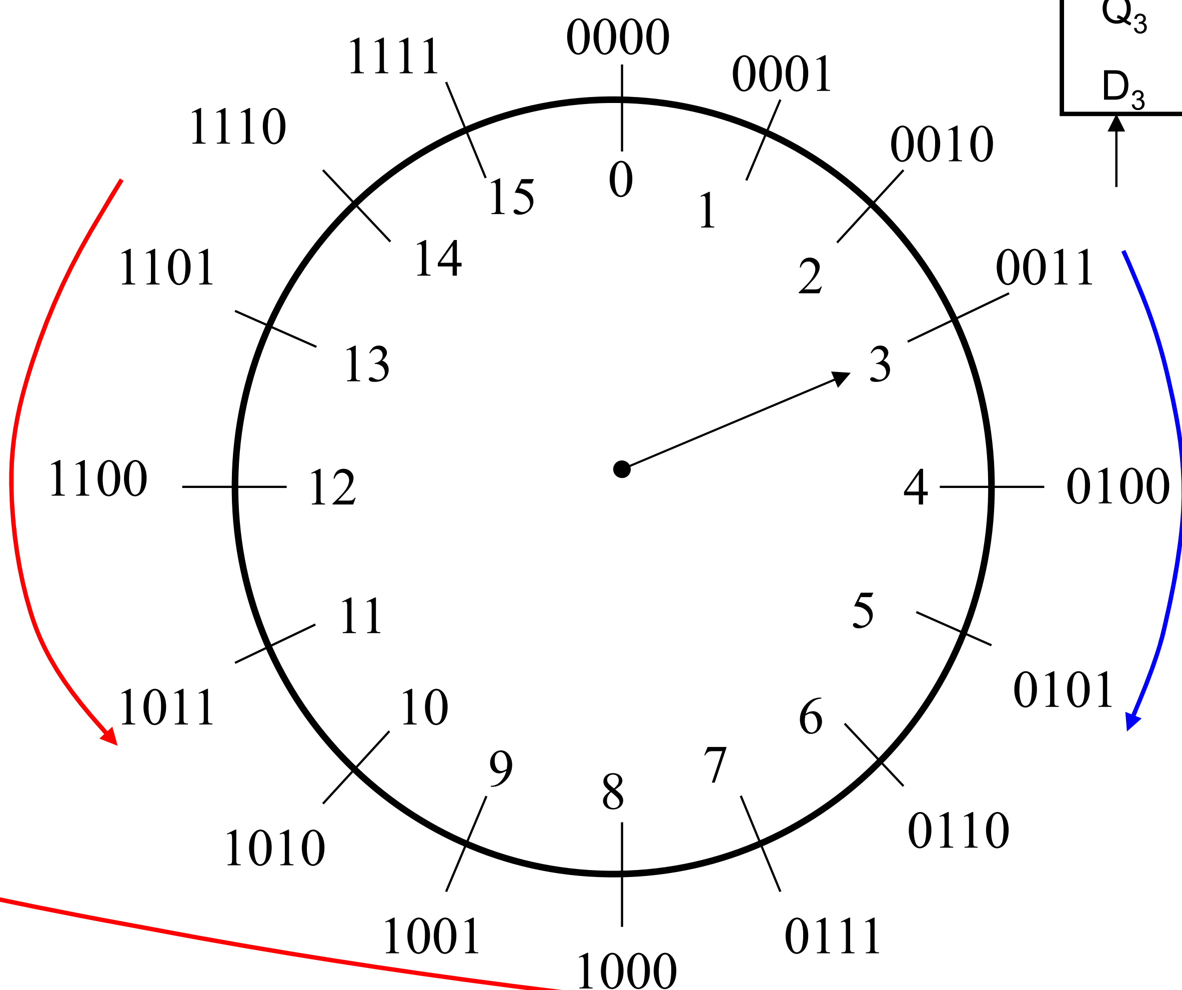


Subtraction

Computer format

6	0110
- 3	- 0011
3	0011

8	1000
- 9	- 1001
-1	1 1111



Addition

3	0011
+ 4	+0100
7	0111

8	1000
+ 9	+1001
17	1 0001

Overflow: = the result of an operation is out of the domain $[0, 2^n-1]$

= **carry**/**borrow** (bit 2^n) is set

Complements

- In digital computers, complements are mainly used to represent negative numbers.
- There are two types of complements for each base *r* system:
- The $(r-1)$'s complement, and
- the *r*'s complement.
- For instance, for the binary system (base 2), there are the 1's complement and the 2's complement.

$(r-1)$'s Complement = diminished radix complement

- Given a number N in base r with n digits, the $(r-1)$'s complement of N is defined as $[(r^n - 1) - N]_r$
- The $(r-1)$'s complement of a number may also be derived by subtracting each digit from $(r-1)$.

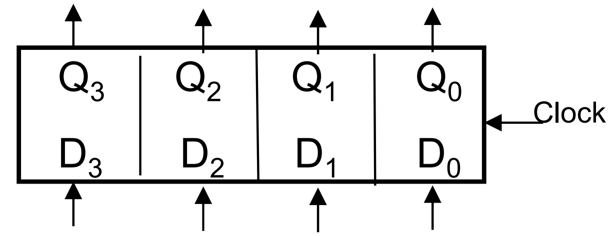
r's Complement

- The r's complement of an n-digit number N in base r is defined by

$$\begin{cases} (r^n - N)_r & \text{if } N \# 0, \\ 0 & \text{if } N = 0 \end{cases}$$

- The r's complement can also be computed by adding 1 to the $(r-1)$'s complement.
- Another way of computing the r's complement is by leaving all least significant 0's unchanged, subtract the first non-zero least significant digit from r, and then subtracting all higher significant digits from $(r-1)$.
- **Remark**: If the original number N contains a radix point, it should be removed temporarily to form the complement wanted. Then the radix point is restored to the complemented number in the **same** relative position.

2's Complement of a Number

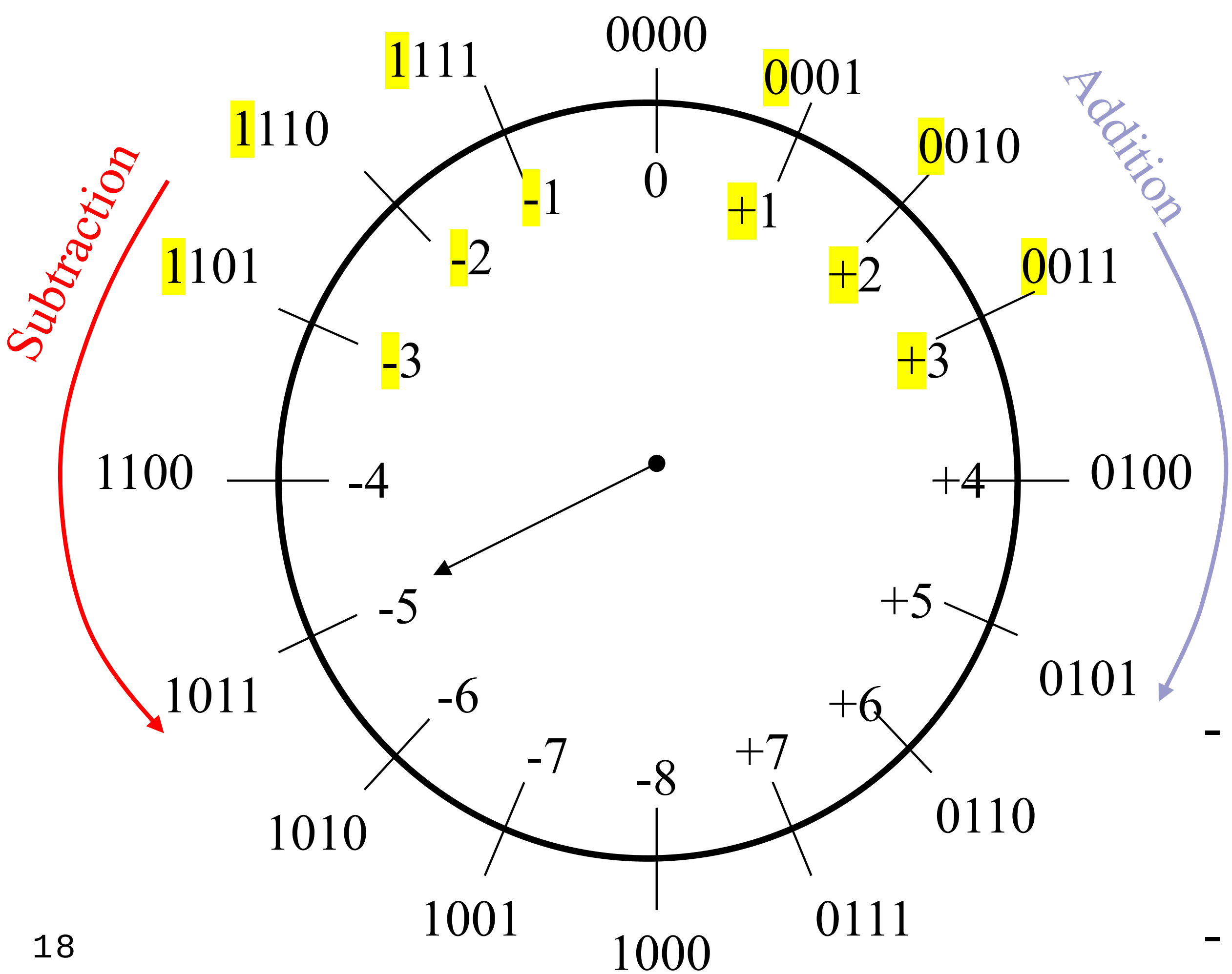


(2's complement of a number P) = $2^n - P$

Since a rotation of 2^n brings back to point 0,

doing (2's complement of a number ***P***) = – ***P*** (*negate P*)

=> One can find **negative *P*** by complementing ***P***



To find

(2's complement of *P*) =

$= 2^n - P =$

$= [(2^n - 1) - P] + 1 =$

= [1's complement of P] +1

$+5_{10} = 0101_2$

1010 = *1's complement of 5*

+1

1011 = -5_{10}

OR

- Examine the bits of ***P*** from right to left and copy all bits that are 0 and the first bit that is 1, then,
- complement the rest of the bits!

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Signed numbers

- When a signed number is negative, the sign bit assumes a value of 1 while the rest of the number may be represented in one of three possible ways:

1) Signed magnitude representation
2) Signed 1's complement representation
3) Signed 2's complement representation

- The **signed magnitude representation (#1)** of a negative number consists of the sign bit followed by the **magnitude** of the number.
- In the other two representations, the number is represented in either the 1's or 2's complement of the number's **positive signed** value.
- Example: represent -14 with 8 bits, including the sign bit:

1) In signed magnitude form, −14 is represented as: 10001110 .
2) In signed 1's complement form, −14 is represented as: 11110001.
3) In signed 2's complement form, −14 is represented as: 11110010.

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3) 2's Complement **Representation** of Signed Numbers

- The MSB (Most Significant Bit) = the sign bit
- **Positive number** = the *sign bit is* **0** and it is written in front of the number's *magnitude* (represented by its true binary value).

IMPORTANT NOTE: Positive numbers have the same representation in *sign-magnitude, 1's complement or 2's complement format*!

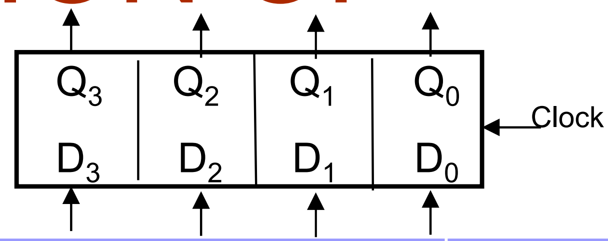
- **Negative number** = obtained by finding the 2's complement of the corresponding positive number (represented as explained above, i.e., including **0** as sign bit) that we want to negate. The result of the complementation will have the sign bit automatically changed to **1**!
- The *decimal equivalent* of a signed binary number that is represented in *2's complement format* is computed the same as for an unsigned number, except
  - the weight of the MSB (i. e., the sign bit:) is -2^{n-1} instead of $+2^{n-1}$

    Two ways to find the decimal equivalent of a signed number represented in 2's complement:

    a) e.g., for n = 5 bits: 11010 = $-2^4 + 2^3 + 2^1 = -16 + 8 + 2 = -6$, **or:**

    b) looking at say N = **1**1010 (and knowing that it is a sign number in 2's complement **representation**) one can see that it is a negative number N = $-\boldsymbol{x}$. To find its magnitude $|N| = \boldsymbol{x}$, one should negate $N = -\boldsymbol{x}$, by finding its 2's complement:
    $|N| = \boldsymbol{x} = -(-\boldsymbol{x})$ = 2's compl. ($-\boldsymbol{x}$) = 2's compl. (**1**1010) = 00110 = 6 => $N = -x = -6$
  - the range of representable numbers is $[-2^{n-1}, 2^{n-1}-1]$

BINARY REPRESENTATION OF SIGNED NUMBERS



	# > 0	+5	# < 0	-5
1) sign-magnitude representation	0 followed by magnitude	**0**101	1 followed by magnitude	**1**101
2) 1's complement representation	0 followed by magnitude	**0**101	1's complement of the corresponding positive #	**1**010
3) 2's complement representation	0 followed by magnitude	**0**101	2's complement of the corresponding positive #	**1**011

Finding the complement of a number is equivalent with finding its negative, e.g.

2's complement of 5 = 2's complement of 0101 = 1011 = -5

2's complement of -5 = 2's complement of 1011 = 0101 = +5

Potential ambiguities of terminology

- One should be cautious when using the term *2's complement*, as it can mean either a ***number format*** or a ***mathematical operator***.

  For example, 0111 represents decimal +7 in *two's-complement notation*, but the *two's complement of 7* in a 4-bit register is actually the "1001" bit string, which is the *two's complement representation* of **−7**.

- The statement "convert x to two's complement" may be ambiguous, since it could describe either:
  - the process of **representing x in two's-complement notation** without changing its value, or
  - the **calculation of the 2's complement**, which is the arithmetic negative of x if ***2's complement representation*** is used.

Converting from two's complement representation

- A 2's-complement number system encodes positive and negative numbers in a binary number representation. The weight of each bit is a power of two, except for the most significant bit, whose weight is the negative of the corresponding power of 2.
- The value w of a signed N-bit integer is given by the following formula:

$$w = -a_{N-1} \cdot 2^{N-1} + \sum_{i=0}^{N-2} a_i \cdot 2^i$$

- The most significant bit (a_{N-1}) determines the sign of the number and is sometimes called the sign bit. Unlike in *sign-magnitude representation*, the sign bit also has the weight $-(2^{N-1})$ shown above. Using N bits, all integers from $-(2^{N-1})$ to $2^{N-1} - 1$ can be represented.

(From Wikipedia, the free encyclopedia)

Addition and Subtraction of Signed Numbers Using 2’s Complement Representation

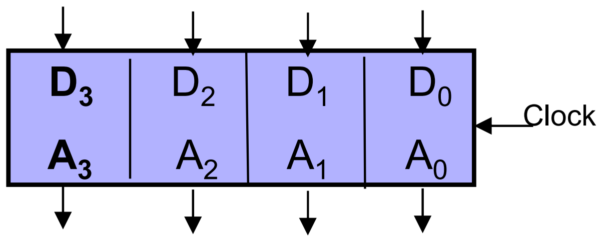
Addition

+3	0011
+ (+4)	+0100
+7	0111

+6	0110
+ (-3)	+1101
+3	1 0011

-2	1110
+ (-6)	+ 1010
-8	1 1000

+4	0100
+ (-7)	+ 1001
-3	1101



Subtraction

add to the minuend the negate of the subtrahend (i.e., the subtrahend’s 2’s complement)

+ 6	110
- (+3)	- 011
+ 6	0110
+(-3)	+ 1101
+ 3	1 0011

- 2	- 010
- (+6)	- 110
- 2	1110
+(-6)	+ 1010
- 8	1 1000

“paper” format

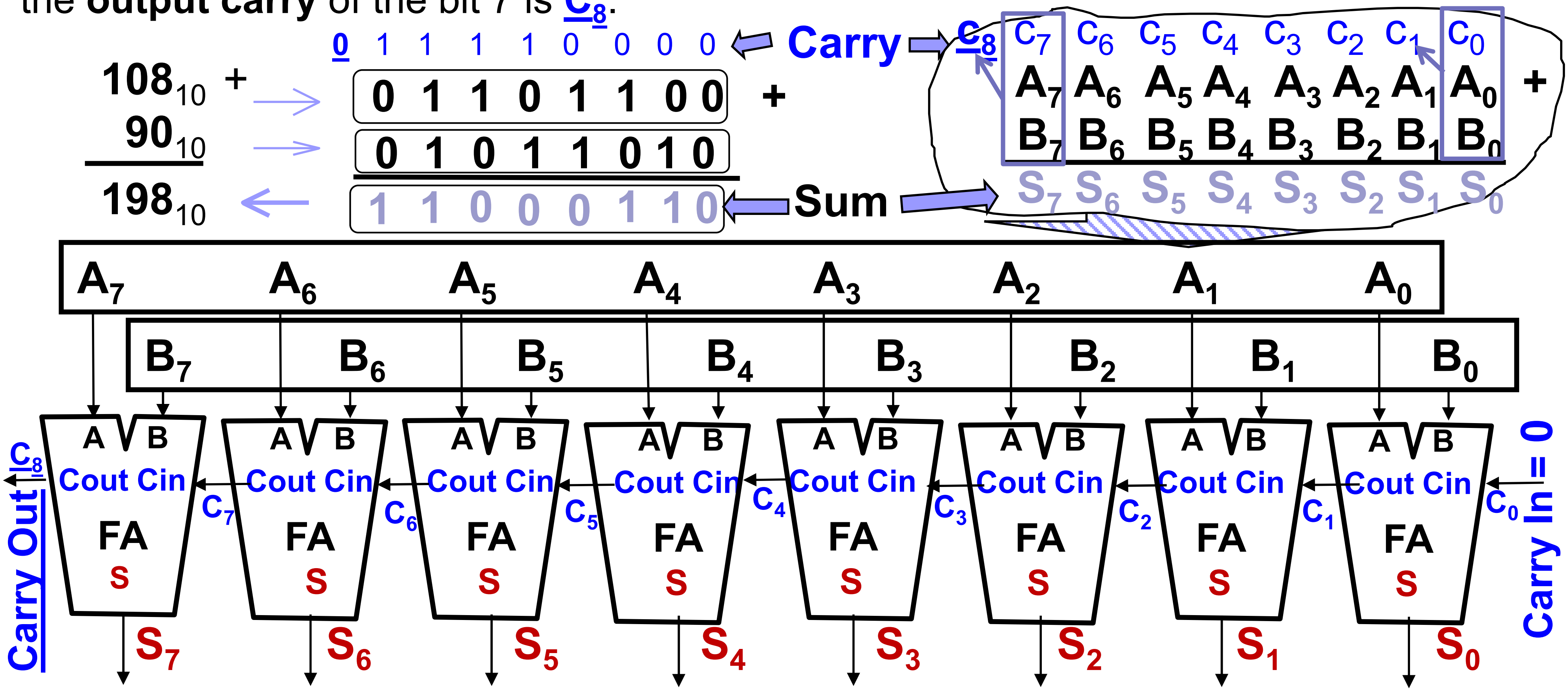
Computer format

- **Carry doesn’t mean overflow in 2’s compl.!**

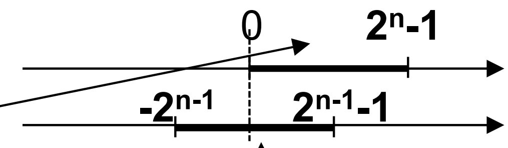
Adding multi-bit numbers:

A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

An adder is a combinational circuit made of full adders FA – see block diagram below. In this example the added numbers are $A_7 \ldots A_2A_1A_0$ and $B_7 \ldots B_2B_1B_0$, which are stored in two 8-bit registers A and B.
It can be used for signed and unsigned numbers.
The sum is represented by $S_7 \ldots S_2S_1S_0$, & is to be fed into the input of another register. The input carry to the *i*-th bit is c_i, with i = 0, 1, 2, …7; the **output carry** of the bit 7 is C_8.



Detection of Addition Overflow



- **Overflow = the result is out of the domain**
- An overflow in an addition operation can be detected
  - For **unsigned numbers** [0, 2^n-1]: from the end carry-out of the most significant bit. A value of 1 means an overflow has occurred.
  - For **2's Complement Representation of Signed #'s [-2^{n-1}, 2^{n-1}-1] an overflow is signalized :**
    1. if the addends have the same sign, but the sign of the sum is different than the addends' sign ($OFL = \bar{a}_3\, \bar{b}_3\, s_3 + a_3\, b_3\, \bar{s}_3$), or
    2. if the carry into the sign bit position and the carry out of the sign bit position are different,.

a_3 (sign)	a_2	a_1	a_0
1	1	0	1
b_3 (sign)	b_2	b_1	b_0
1	0	1	0

A +
B
S

carry: c_4 c_3 c_2 c_1 c_0

a_3 (sign)	a_2	a_1	a_0
b_3 (sign)	b_2	b_1	b_0
s_3 (sign)	s_2	s_1	s_0

$$OFL = c_n \oplus c_{n-1} = c_4 \oplus c_3$$

Carry: 1 0 0 0

-3 1101+

-6 1010

-9 < -8 →OFL! 0111 = +7?

Carry 0 1 0 0

+5 0101+

+ +6 0110

+11>+7 1011 = -5?

Addition Overflow

- For **unsigned numbers**, an overflow in an addition operation is detected from the end carry out of the most significant bit. A value of 1 means an overflow has occurred.
- An overflow in a subtraction operation cannot possibly exist.
- For **signed numbers**, an overflow in an addition or a subtraction operation can be detected by observing the carry into the sign bit position and the carry out of the sign bit position. If the two carries are different, then an overflow condition is produced.

Additions / Subtractions with Signed Numbers Using Two's Complement Representation (1)

The **2's complement** **representation** of a negative number is the 2's complement of its absolute value.

- **Add -176_{10} to -204_{10} in 2's Complement representation**

- How many bits are needed for operands and result to avoid overflow in 2's complement representation?

  Both -176 and -204 can be represented with at least 9 bits (1 sign bit + 8 bits for magnitude)

  The result is -176 - 204 = -380 $\subset [-512, +511] = [-2^9, 2^9-1]$ → n-1 = 9 → n = **10**

or: -380= –(**0**1 0111 1100) → (in 2's Complement representation) = 10 1000 0100

=> 10 bits are needed, including the sign bit

- Find 2's complement representation of the operands:

  Start from +176 = 00 1011 0000 → -176 (1's Complement) = 11 0100 1111

  -176 (in 2's Complement representation) = 11 0100 1111+1 = 11 0101 0000

  Then, from +204 = **0**0 1100 1100 → -204 (1's Complement) = **1**1 0011 0011

  -204 (in 2's Complement representation) = **1**1 0011 0011+1 = **1**1 0011 0100

Additions / Subtractions with Signed Numbers Using Two's Complement Representation (2)

	C_{10}	C_9	C_8	C_7	C_6	C_5	C_4	C_3	C_2	C_1	C_0
Carry	1	1	0	1	1	1	0	0	0	0	0
(-176)		**1**	1	0	1	0	1	0	0	0	0
+(-204)		**1**	1	0	0	1	1	0	1	0	0
		1	0	1	0	0	0	0	1	0	0

What is 10 1000 0100 (the result)? Let's convert it to signed decimal!
The msb is **1** => the result is a negative number (say **–x** = 10 1000 0100).
But what is the magnitude of this negative number (i.e. **x** = **|–x|**)?
To find **x**, let's do –(-x),
i.e. let's do complement **1**0 1000 0100 to get x = –(**1**0 1000 0100)

01 0111 1011 (1's Complement)
\+ 1

01 0111 1100 = 380_{10} (decimal) = x = the magnitude of the negative number -x
Then the answer in decimal is $10\ 1000\ 0100_2$ = -x = -380_{10}

Additions / Subtractions with Signed Numbers Using Two's Complement Representation (3)

Two numbers are given in 2's complement representation, employing 5 bits which include the sign bit.

X = 01010 and Y = 10101

1. Calculate the sum (S = X+Y) and the difference (D=X-Y) of these numbers using additions and 2's complementation only; indicate if overflow occurs, and explain how a circuit can detect these situations.
2. Convert in decimal and write each intermediate and final result, to check the correctness of your assertions.

SOLUTION

Since both X AND Y ARE IN 2'S COMPLEMENT REPRESENTATION, their value in decimal is

X = $(01010)_2$ = $+10_{10}$ and

Y = $(10101)_2$ = - |Y|; |Y| = - Y = 2's complement of Y = $(01011)_2$ = $+11_{10}$ => **Y** = - |Y| = **-11_{10}**

To calculate D we need –Y, which we just calculated above: - Y = 2's complement of Y = $(01011)_2$ = $+11_{10}$

	S=X+Y	2's complement representation					
Base 10		Cy→	0	0	0	0	0
$X=10_{10}$	X		0	1	0	1	0
$Y=-11_{10}$	+Y		1	0	1	0	1
	S		**1**	**1**	**1**	**1**	**1**

S = $(1111)_2$ = - |S|;

|S| = - S = 2's complement of S = $(00001)_2$ = $+1_{10}$

=> **S** = - |S| = **-1_{10}**

	D=X-Y	2's complement representation					
Base 10		Cy→	0	1	0	1	0
$X = 10_{10}$	X		0	1	0	1	0
$-Y=-(-11_{10})$	-Y		0	1	0	1	1
	D		**1**	**0**	**1**	**0**	**1**

D = $(10101)_2$ = - |D|

|D| = 2's complement of D = + $(01011)_2$ = $+11_{10}$

⇒ D = - |D| = - 11_{10} !!!???

⇒ **Overflow**: adding 2 positive numbers (10_{10} and 11_{10}) should give a positive number not -11_{10} !!!???

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Addition/Subtraction Overflow Detection (1)

- **Add -176 (decimal) to -204 (decimal) using Two's Complement using only 9-bit representation?**
- **Derive the logical expressions to detect overflow in terms of the last two carry bits or the sign bit of added numbers ?**

- The range of sign numbers in 2's complement representation with 9 bits is [-256,+255], so -380 is out of range.
- So, it is expected to get an overflow

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Addition Overflow Detection (2)

$$\begin{array}{llcccccccccc}
 & & C_9 & C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 \\
 & \text{Carry} & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{SgnA} & (-176) & & \boxed{1} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \quad + \\
\text{SgnB} & (-204) & & \boxed{1} & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\hline
\text{SgnS} & & & \boxed{0} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}$$

Remember: A, B and S are stored in 9 bit registers not 10 like in the previous question

Overflow Detection Expressions

Overflow is detected if

1. if (SgnA=SgnB) ≠SgnS

   $(\overline{SgnA}.\overline{SgnB}.SignS + SgnA.SgnB.\overline{SgnS})$

2. or if the carry bits TO (C_8) & FROM (C_9) the sign bit are different $(\overline{C_9} . C_8 + C_9 . \overline{C_8}) = C_9 \oplus C_8$

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Floating-Point Representation

- A floating-point is always interpreted to represent a number in the following form:

$$m \times r^e$$

- Only the **mantissa** *m* and the **exponent** *e* are physically stored in registers (including their signs).
- The radix and radix-point position in the mantissa are supposed to be known (fixed for a given hardware).
- A floating-point number is said to be **normalized** if the most significant digit (excluding the **sign bit**) is non-zero
- +1001.110 with an 8-bit mantissa and a 6-bit exponent:

Mantissa	Exponent
01001110	000100

IEEE Standard for Floating-Point Arithmetic

IEEE Computer Society

Developed by the
Microprocessor Standards Committee

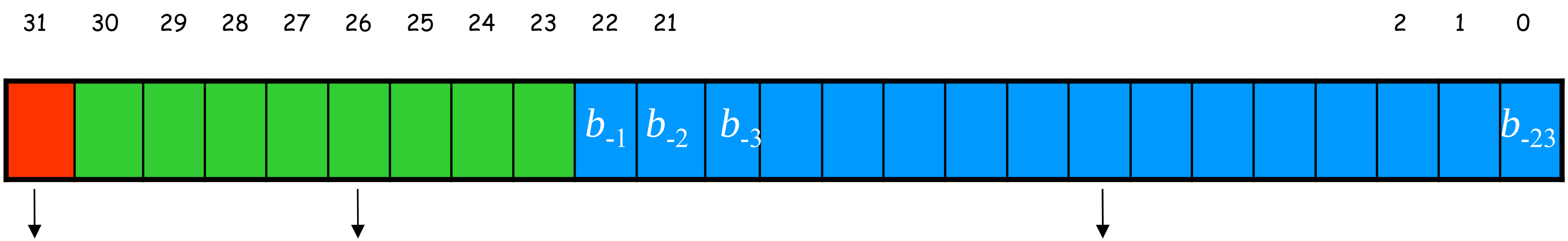
IEEE
3 Park Avenue
New York, NY 10016-5997
USA

IEEE Std 754™-2019

IEEE 754 Standard

- Most common standard for representing floating point numbers
- Single precision: 32 bits, consisting of...
  - Sign bit (1 bit)
  - Exponent (8 bits)
  - Mantissa or significand (23 bits)
- Double precision: 64 bits, consisting of…
  - Sign bit (1 bit)
  - Exponent (11 bits)
  - Mantissa (52 bits)

IEEE 754 Standard (32 bits) - Single Precision



Sign bit S

(1 bit)

Exponent E

(8 bits)

Mantissa *m*

(23 bits)

$$value = (-1)^{sign}(1 + \sum_{i=1}^{23} b_{-i}2^{-i}) \times 2^{(e-127)}$$

Range: [-2^{128}, 2^{128}] → [-10^{12}, 10^{12}]

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IEEE 754 Standard (32 bits) - Single Precision

Sign bit:

- positive, negative

Exponent:

- using unsigned number to represent signed number
- biased by adding 127 to the actual value (offset binary)

Mantissa:

- normalized, if 0<exponent<255, the first bit of mantissa (not shown) is 1
- de-normalized, if exponent = 0, mantissa is not =0,
- +/- 0, exponent = 0 , mantissa = 0
- +/- infinity, exponent = 255, mantissa = 0
- NaN, exponent = 255, mantissa is not 0

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Converting to Floating Point

- Express 36.5625_{10} as a 32-bit floating point number?

Express original value in binary

$$36.5625_{10} = 100100.1001_2$$

Normalize

$$100100.1001_2 = 1.\overbrace{001001001}^{M\,=\,\text{mantissa}}{}_2 \times 2^5$$

Put S, E, and M together to form 32-bit binary result

$$0\ 10000100\ 00100100100000000000000_2$$

S E = 5+127 M

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Converting from Floating Point (1)

- **What decimal value is represented by the following 32-bit floating point number?**

$$C17B0000_{16}$$

Converting from Floating Point (2)

- Express in binary and find S, E, and M

$C17B0000_{16} =$

$\underbrace{1}_{S}\ \underbrace{10000010}_{E}\ \underbrace{11110110000000000000000}_{M}\,{}_2$

1 = negative
0 = positive

Converting from Floating Point (3)

- Find "real" exponent, *n*

$n = E - 127$

$= 1000\ 0010_2 - 127 = (2^7 + 2^1)_{10} - 127$

$= 130 - 127$

$= 3$

- Put S, M, and n together to form binary result
- Don't forget the implied "1." on the left of the mantissa of the normalized

$-1.1111011_2 \times 2^n =$

$-1.1111011_2 \times 2^3 =$

-1111.1011_2

Converting <u>from</u> Floating Point (4)

- Express result in decimal

`-1111.1011`$_2$

-15

2^{-1} = 0.5
2^{-3} = 0.125
2^{-4} = 0.0625
0.6875

Answer: -15.6875

ASCII

American Standard Code for Information Interchange

*	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	**NUL**	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	**TAB**	**LF**	VT	FF	**CR**	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2		!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	

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Binary-Coded Decimal (BCD)

Decimal:	0	1	2	3	4	5	6	7	8	9
BCD:	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001
$ASCII_{16}$ for Decimal	30	31	32	33	34	35	36	37	38	39

Example:

Decimal	127
BCD	0001 0010 0111
$ASCII_{16}$ for decimal	31 32 37
$ASCII_2$ for decimal	0011 0001 0011 0010 0011 0111
Binary	0111 1111
Hex	7F
$ASCII_{16}$ for Hex	37 46
$ASCII_2$ for Hex	0011 0111 0100 0110

Decimal number	Binary-coded decimal (BCD) number	
0	0000	Code for one decimal digit
1	0001	
2	0010	
3	0011	
4	0100	
5	0101	
6	0110	
7	0111	
8	1000	
9	1001	
10	0001 0000	
20	0010 0000	
50	0101 0000	
99	1001 1001	
248	0010 0100 1000	

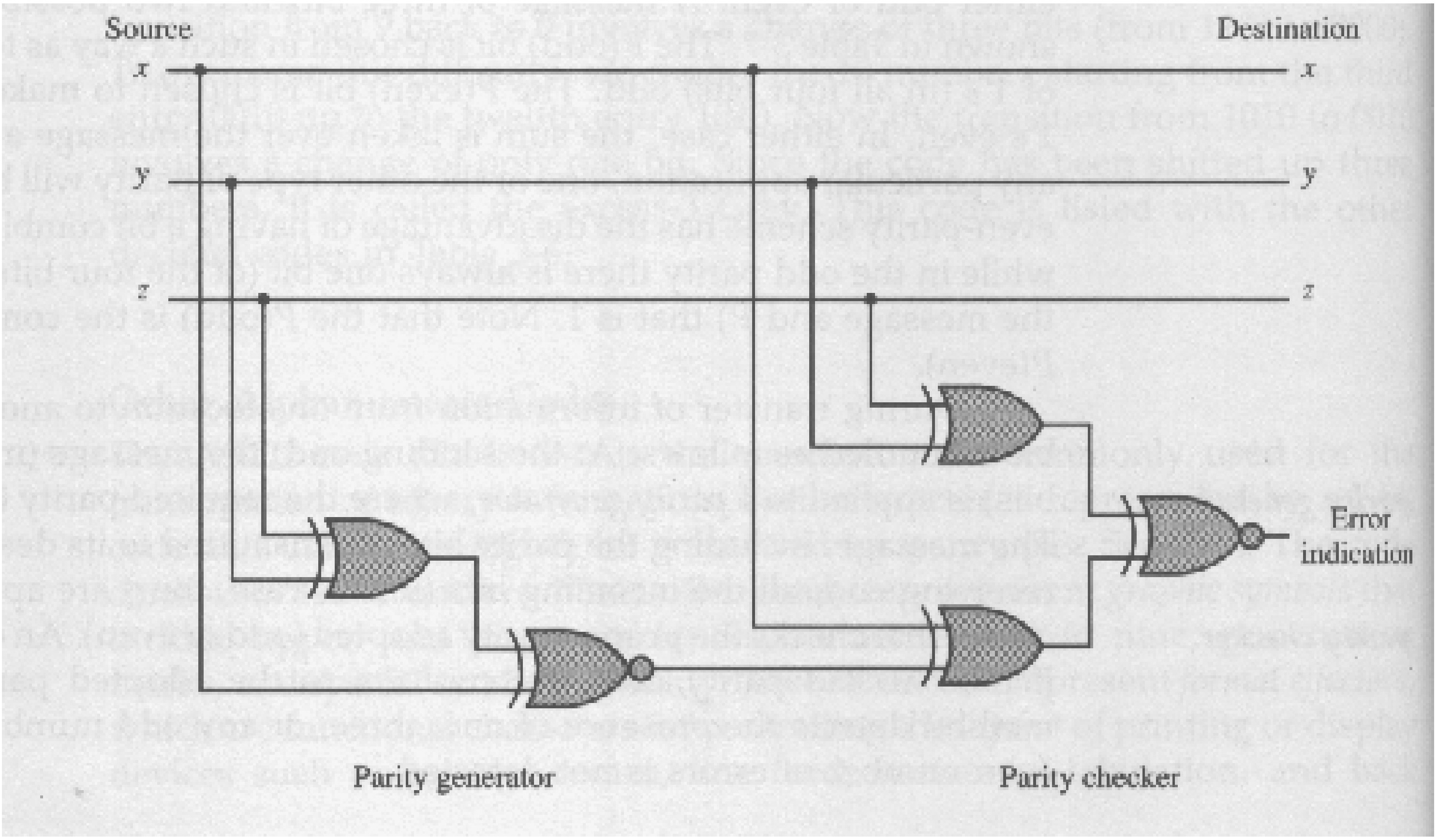
Error Detection Codes

- Binary information transmitted through a communication medium is subject to external noise that can change bits from 0 to 1 and vice versa.
- An error detection code is a binary code that detects digital errors during transmission.
- The most common error detection code is the **parity code**.
- A parity code is an extra bit added to the binary message to make the total of the number of 1's either odd (odd-parity code) or even (even-parity code).
- A message of three bits and two possible parity bits is shown below:

Message *xyz*	*P*(odd)	*P*(even)
000	1	0
001	0	1
010	0	1
011	1	0
100	0	1
101	1	0
110	1	0
111	0	1

Parity Code

- At the sending end, the message is applied to a parity generator to generate the required parity bit.
- At the receiving end, all the incoming bits (including the parity bit) are applied to a parity checker to check for the existence of an error.



ANNEX: Subtraction with pencil and paper of unsigned numbers

2^3	2^2	2^1	2^0		$2^3\ 2^2\ 2^1\ 2^0$	Step 0:
0	1	1	0	-	$A_3\ A_2\ A_1\ A_0$	$D_0 = A_0 - B_0 = 0 - 1$!?
0	0	1	1		$B_3\ B_2\ B_1\ B_0$	=> we need to borrow from
					$D_3\ D_2\ D_1\ D_0$	$A_1 = 1 \times 2^1 = 2 = b_1$
			~~1~~+1	b (borrow)	$b_4\ b_3\ b_2\ b_1$	
0	1	0	0	-	$A_3\ A_2\ A_1\ A_0$	so $D_0 = b_1 + A_0 - B_0 = 2 + 0 - 1 = 1$
0	0	1	~~1~~		$B_3\ B_2\ B_1\ B_0$	
			1		$D_3\ D_2\ D_1\ D_0$	
		0	~~1~~+1	b (borrow)	$b_4\ b_3\ b_2\ b_1$	Step 1:
0	1	0	0	-	$A_3\ A_2\ A_1\ A_0$	$D_1 = A_1 - B_1 = 0 - 1$!?
0	0	1	~~1~~		$B_3\ B_2\ B_1\ B_0$	=> we need to borrow from $A_2 = 2 = b_2$
			1		$D_3\ D_2\ D_1\ D_0$	
	0	~~1~~+1	~~1~~+1	b (borrow)	$b_4\ b_3\ b_2\ b_1$	
0	0	0	0	-	$A_3\ A_2\ A_1\ A_0$	so $D_1 = b_2 + A_1 - B_1 = 2 + 0 - 1 = 1$
0	0	~~1~~	~~1~~		$B_3\ B_2\ B_1\ B_0$	
		1	1		$D_3\ D_2\ D_1\ D_0$	
	0	~~1~~+1	~~1~~+1	b (borrow)	$b_4\ b_3\ b_2\ b_1$	Step 2:
0	0	0	0	-	$A_3\ A_2\ A_1\ A_0$	$D_2 = A_2 - B_2 = 0 - 0 = 0$ no need to borrow
0	0	1	~~1~~		$B_3\ B_2\ B_1\ B_0$	$b_3 = 0$
	0	1	1		$D_3\ D_2\ D_1\ D_0$	
0	1	1	0	b (borrow)	$b_4\ b_3\ b_2\ b_1$	Step 3:
0	0	0	0	-	$A_3\ A_2\ A_1\ A_0$	$D_3 = A_3 - B_3 = 0 - 0 = 0$ no need to borrow
0	0	1	1		$B_3\ B_2\ B_1\ B_0$	$b_4 = 0$
0	0	1	1		$D_3\ D_2\ D_1\ D_0$	
0	1	1	0	b (borrow)	$b_4\ b_3\ b_2\ b_1$	Step 4:
0	0	0	0	-	$A_3\ A_2\ A_1\ A_0$	$b_4 = 0$ => no overflow
0	0	1	1		$B_3\ B_2\ B_1\ B_0$	
0	0	1	1		$D_3\ D_2\ D_1\ D_0$	