The Monty Hall Problem

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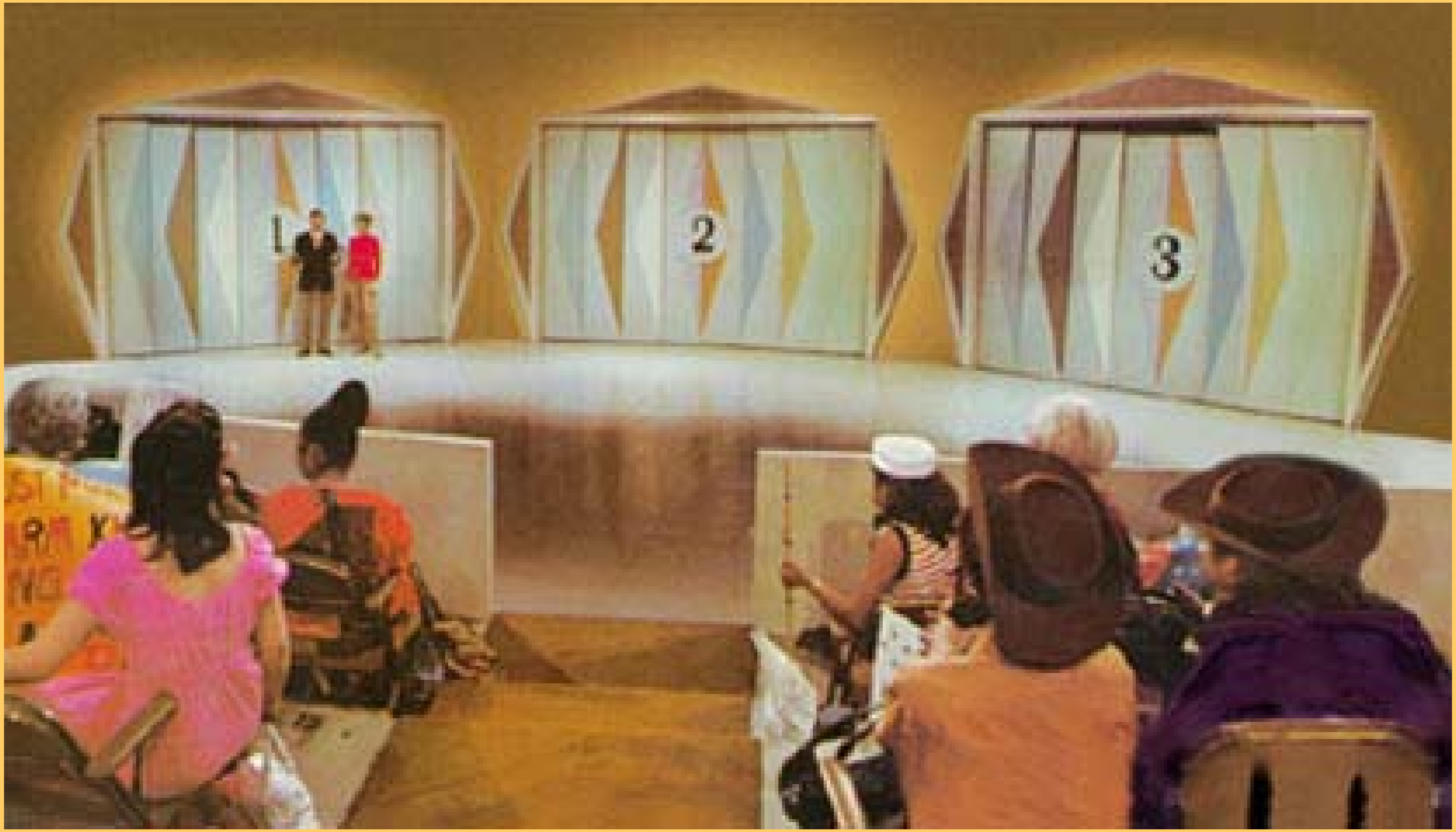
About *Let's Make a Deal*

- *Let's Make a Deal* was a game show hosted by Monty Hall and Carol Merril. It originally ran from 1963 to 1977 on network TV.
- The highlight of the show was the "Big Deal," where contestants would trade previous winnings for the chance to choose one of three doors and take whatever was behind it--maybe a car, maybe livestock.
- *Let's Make a Deal* inspired a probability problem that can confuse and anger the best mathematicians, even Paul Erdös.

Suppose you're a contestant on Let's Make a Deal.

You are asked to choose one of three doors. The grand prize is behind one of the doors; The other doors hide silly consolation gifts which Monty called “zonks”.



You choose a door.

Monty, who knows what’s behind each of the doors, reveals a zonk behind one of the other doors.
He then gives you the option of switching doors or sticking with your original choice.

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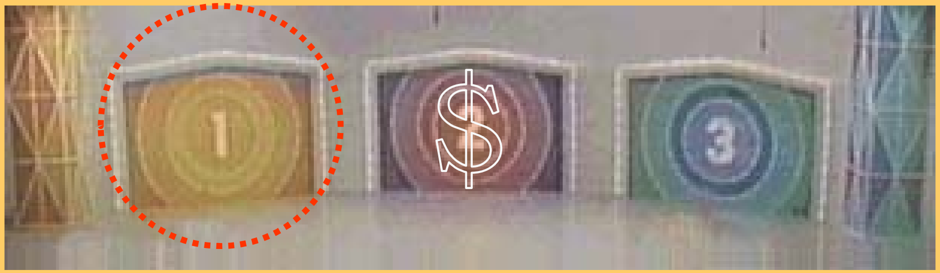
The question is: **should you switch?**

The answer is **yes**, you should switch!

Assuming that Monty always gives you a chance to switch, you double your odds of winning by switching doors.

We will see why, first by enumerating the possible cases, then by directly computing the probability of winning with each strategy.

Each door has a 1 in 3 chance of hiding the grand prize.
Suppose we begin by choosing door #1.



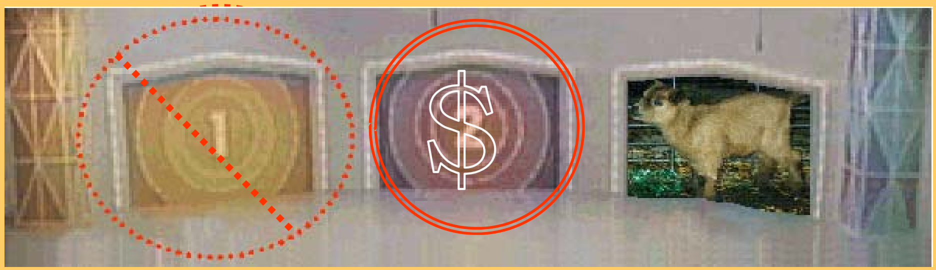
Each door has a 1 in 3 chance of hiding the grand prize. Suppose we begin by choosing door #1.

In this case Monty may open either door #2 or #3

In both of these cases, Monty is forced to reveal the only other zonk.

So what happens when you switch?

In this case you were right the first time.
You lose!



In both of these cases, you switch to the correct door.
You win!

To prove this result without listing all the cases, we need the notion of conditional probability.

Conditional probability gives us a way to determine how the occurrence of one event affects the probability of another.

Here, if we've chosen door #1 and Monty has opened door #2, we'd like to know the probability that the prize is behind door #1 and the probability that the prize is behind door #3 *given this additional information.*

We can determine these probabilities using the rule

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

In words: The probability of event A given event B is the probability of both A and B divided by the probability of B.

In the following argument:

- Assume that:
  - we originally chose door #1.
  - Monty opened door #2.
- Notation
  - Let "#1" denote the event that the prize is behind door #1, and similarly for doors #2 and #3.
  - Let "opened #2" denote the event that Monty has opened door #2.
- Our aim is to compute $p(\#1 \mid \text{opened } \#2)$ and $p(\#3 \mid \text{opened } \#2)$.

$$p(\#1 \mid \text{opened } \#2) = \frac{p(\#1 \cap \text{opened } \#2)}{p(\text{opened } \#2)}$$

$$p(\#3 \mid \text{opened } \#2) = \frac{p(\#3 \cap \text{opened } \#2)}{p(\text{opened } \#2)}$$

Rules:

1. $p(A \mid B) = \dfrac{p(A \cap B)}{p(B)}$

2. $p(A \cap B) = p(B) \times p(A \mid B)$

$$p(\#1 \cap \text{opened } \#2) = p(\text{opened } \#2 \mid \#1) \times p(\#1)$$

(By rule 2.)

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(If the prize is behind door #1, Monty can open either #2 or #3.)

$$p(\#3 \cap \text{opened } \#2) = p(\text{opened } \#2 \mid \#3) \times p(\#3)$$

(By rule 2.)

$$= 1 \times \frac{1}{3} = \frac{1}{3}$$

(If the prize is behind door #3, Monty **must** open door #2.)

$$p\left(\#1 \mid \text{opened } \#2\right) = \frac{p(\#1 \cap \text{opened } \#2)}{p\left(\text{opened } \#2\right)} = \frac{1/6}{p\left(\text{opened } \#2\right)}$$

$$p\left(\#3 \mid \text{opened } \#2\right) = \frac{p\left(\#3 \cap \text{opened } \#2\right)}{p\left(\text{opened } \#2\right)} = \frac{1/3}{p\left(\text{opened } \#2\right)}$$

$$p\left(\text{opened } \#2\right) = p\left(\text{opened } \#2 \cap \#1\right) + p\left(\text{opened } \#2 \cap \#2\right) + p\left(\text{opened } \#2 \cap \#3\right)$$

$$= \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$$

So:

$$p\left(\#1 \mid \text{opened } \#2\right) = \frac{1/6}{1/2} = \frac{1}{3}$$ and $$p\left(\#3 \mid \text{opened } \#2\right) = \frac{1/3}{1/2} = \frac{2}{3}$$

Conclusions:

- Switching increases your chances of winning to 2/3.
- A similar result holds for *n* doors.
- This strategy works only if we assume that Monty behaves predictably, offering a chance to switch every time.
- On *Let's Make a Deal*, Monty would play mind games with contestants, sometimes offering them money not to open the selected door.
- Play the game and check out the statistics at http://math.ucsd.edu/~crypto/Monty/monty.html
- *Let's Make a Deal* graphics courtesy of letsmakeadeal.com