Theoretical Stochastic Model for Cryptocurrency Price Spreads

1 Definition of Spread

Let

$$P_{\mathrm{Bin}}(t), \qquad P_{\mathrm{Coin}}(t)$$

be the instantaneous prices for Bitcoin on Binance and Coinbase. The price spread is defined as:

$$s(t) = P_{\mathrm{Bin}}(t) - P_{\mathrm{Coin}}(t).$$

This document provides a self-contained theoretical model for the evolution of $s(t)$.

2 Stochastic Delay Model

We consider the following stochastic delay differential equation (SDDE) for the spread:

$$\frac{ds(t)}{dt} = -\kappa\big(s(t) - \mu\big) - \gamma\, s(t-\tau) + \sigma\, \xi(t), \tag{1}$$

where:

- $\kappa > 0$ is the instantaneous mean-reversion rate,
- μ is the long-run average spread,
- $\gamma \geq 0$ is the delayed feedback (arbitrage pressure acting with latency),
- $\tau > 0$ is the latency or reaction delay,
- $\sigma > 0$ is the noise amplitude,
- $\xi(t)$ is white noise.

The term $-\kappa(s(t)-\mu)$ captures short-term reversion, while $-\gamma s(t-\tau)$ models arbitrageurs correcting mispricings with finite delay.

3 Ornstein–Uhlenbeck Special Case

If $\gamma = 0$, the model reduces to the classical Ornstein–Uhlenbeck (OU) process:

$$ds(t) = -\kappa(s(t) - \mu)\, dt + \sigma\, dW_t, \tag{2}$$

with W_t a Wiener process.

The OU process is mean-reverting and Gaussian, making it a standard baseline for spread modeling.

4 Discrete-Time Approximation

For observations sampled at interval Δt, the Euler–Maruyama discretization of (**??**) is:

$$s_{n+1} = s_n - \kappa(s_n - \mu)\Delta t - \gamma s_{n-\ell}\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_n, \tag{3}$$

where $\varepsilon_n \sim \mathcal{N}(0, 1)$ and $\ell = \tau/\Delta t$ is the integer delay.

4.1 OU Discrete Form

If $\gamma = 0$, (**??**) becomes the AR(1) representation:

$$s_{n+1} = \phi s_n + (1 - \phi)\mu + \eta_n, \qquad \phi = 1 - \kappa\Delta t, \tag{4}$$

with innovations $\eta_n \sim \mathcal{N}(0, \sigma^2\Delta t)$.

5 Critical Delay and Stability

Ignoring noise, the delay-only equation:

$$\dot{s}(t) = -\gamma s(t - \tau)$$

has characteristic equation:

$$\lambda + \gamma e^{-\lambda\tau} = 0.$$

A pair of purely imaginary roots appears when:

$$\tau = \tau_c = \frac{\pi}{2\gamma}.$$

Thus:

- If $\tau < \tau_c$, the spread is stable and returns toward 0.

- If $\tau > \tau_c$, oscillations can arise due to delayed arbitrage reaction.

In the general case with both κ and γ, stability boundaries require numerical root finding for:

$$\lambda + \kappa + \gamma e^{-\lambda\tau} = 0.$$

6 Simulation Equation

To simulate the stochastic spread dynamics, use the discrete form:

$$s_{n+1} = s_n + \left[-\kappa(s_n - \mu) - \gamma s_{n-\ell}\right] \Delta t + \sigma\sqrt{\Delta t}\, \varepsilon_n.$$

Initial conditions $s_0, s_1, \ldots, s_\ell$ can be set from historical data or through a short burn-in period.

7 Summary

This standalone document provides a complete stochastic and delay-based model for cryptocurrency exchange price spreads. The model:

- incorporates mean-reversion,
- includes latency-induced feedback,
- provides a stability condition via critical delay τ_c,
- supports simulation through a discrete stochastic update rule.