Applying Polynomial Regression to Dyadic Data

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Outline

- Intro and Motivation
- Model
- Re-creation of Previous Results
- Our Application

Introduction

- Family researchers comparing the attitudes, behaviors, and opinions of pairs
- Dyadic data
  - Inter-individual reporting e.g. comparison of husband's report of his support toward his wife to the wife's report of husband's support
  - Intra-individual reporting e.g. comparison of an observer's report of husband's hostility to an observer's report of the wife's hostility

Goal:

- Compare observed reports of a dyadic structure to self-reported data

Motivation

- Difference scores most common method used to analyze dyadic data
- Difference scores allow researchers to see how well 2 variables "fit" together
- Common types: algebraic, absolute, and squared difference

$$Z = \beta_0 + \beta_1(X - Y) + \epsilon \qquad (1)$$
$$Z = \beta_0 + \beta_1|X - Y| + \epsilon \qquad (2)$$
$$Z = \beta_0 + \beta_1(X - Y)^2 + \epsilon \qquad (3)$$

Motivation

- Any alternatives?
- Edwards and Parry suggested Polynomial Regression (Edwards and Parry 1993)
  e.g.

$$Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 X^2 + \beta_4 Y^2 + \beta_5 XY + \epsilon \qquad (4)$$

- Take the Squared Difference and expand it

$$(X - Y)^2 = X^2 + Y^2 - 2XY \qquad (5)$$

- Expand on the ideas from Simple Linear Regression

- Theoretical model:

$$Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 X^2 + \beta_4 Y^2 + \beta_5 XY + \epsilon \qquad (6)$$

- Fitted model:

$$\hat{z} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 \qquad (7)$$

We can fit a model like this in R using the `lm()` function:

```r
# Z - the response variable
# X - the first explanatory variable
# Y - the second explanatory variable
QuadFit <- lm(z ~ x + y + I(x^2) + I(x*y) + I(y^2), data=d)
```

Stationarity Points

- What are they?
    - Points where the slope is zero no matter which direction you take the derivative - local max/min
- Values of our explanatory variables provide the "best" fit for the response
- How do you derive the stationary points?
    1. Take the derivatives of Equation 7 with respect to x and y
    2. Set the derivatives equal to zero
    3. Solve for x and y in terms of the coefficients to find the stationarity points
    4. Refer to these points as x_0 and y_0

The stationary points are:

$$x_0 = \frac{b_2 b_4 - 2 b_5 b_1}{4 b_5 b_3 - b_4^2} \tag{8}$$

$$y_0 = \frac{b_1 b_4 + 2 b_2 b_3}{4 b_5 b_3 - b_4^2} \tag{9}$$

These points represent the values in our predictors that will optimize our response (local minimum or maximum depending on the surface shape).

Predicted Response at Stationarity Points

We then get the local maximum (or minimum) predicted response ($\hat{z}_0$) by plugging in the Stationary Points to Eq 7:

$$\hat{z}_0 = b_0 + b_1 x_0 + b_2 y_0 + b_3 x_0^2 + b_4 x_0 y_0 + b_5 y_0^2 \qquad (10)$$

Principal Axes

- Measure the amount of "bend" in two directions at the stationary points
- Make interpretations of the model easier by rotating the axes by removing all the cross-product terms (Ex: PCA)

In the Creative Component,

- Brought together a complete picture of how to derive the Principal Axes.
    - Khuri and Cornell (1996) in *Response Surfaces: Designs and Analyses* lay out some pieces of how derive them but never completely laid out the complete process

Basic idea in the derivation of principal axes:

- Derive canonical equations to get surface in what is known as canonical form
- Transform canonical equations back onto original variables
- These are the Principal Axes

$$y = -P_{21}x + P_{20} \tag{11}$$

where $P_{20} = y_0 + P_{21}x_0$ and $P_{21} = \frac{b_5 - b_3 - \sqrt{(b_5 - b_3)^2 + b_4^2}}{b_4}$.
Using similar algebra we can find the second principal axes:

$$y = -P_{11}x + P_{10} \tag{12}$$

where $P_{10} = y_0 + P_{11}x_0$ and $P_{11} = \frac{b_5 - b_3 + \sqrt{(b_5 - b_3)^2 + b_4^2}}{b_4}$.

Interpretations:

1. In a concave surface:
   - First principal axis (Eq 11): lowest downward curvature where two explanatory variables create a maximized surface that decreases the least
   - Second principal axis (Eq 12): greatest downward curvature where two explanatory variables create the greatest decrease in the response
2. In a convex surface:
   - Flip the interpretations of the two Principal Axes

An example of this methodology being applied is found in Phillips et al. (2012), *Congruence research in behavioral medicine: methodological review and demonstration of alternative methodology*.
Interested in seeing what created good situations in which patients followed through with their doctors orders
The variables used in this example are:

$Z_{1,i}$ – patient-reported adherence a month after dr. visit for patient i: $0 - 6$
$Z_{2,i}$ – physician's perceived agreement with patient i on illness treatment: $0 -$
 X_i – physician's rating of patient i's health: $1 - 5$
 Y_i – physician's est of how patient i would rate of their own health: $1 - 5$

Here in Table 1, some summary statistics are presented of the response and explanatory variables used.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	NA's	n
Z_1	-2.48	-0.20	0.30	-0.00	0.53	0.54	225.00	177.00
Z_2	1.00	3.25	4.00	3.88	4.62	5.00	151.00	251.00
X	-2.00	-1.00	0.00	0.28	1.00	2.00	112.00	290.00
Y	-2.00	-1.00	0.00	0.18	1.00	2.00	111.00	291.00

Table: Summary statistics of response variables(Z_1 and Z_2) and explanatry variable X and Y.

Normality Checks

Before we get into whether or not we should use polynomial regression, they first check some statistics for obvious non-normality:

1. High Leverage
2. Influential Points

No points were of major concern but one point consistently had a higher leverage and Cook's D value.

Criterion Checks

The models being used in the criteria are listed below (note that these may have been listed previously but are re-listed here for ease of reading):

$$\text{Null Model: } Z = \beta_0 + \epsilon \tag{13}$$

$$\text{Diff Model: } Z = \beta_0 + \beta_1(X - Y) + \epsilon \tag{14}$$

$$\text{Uncon Diff Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \epsilon \tag{15}$$

$$\text{Abs Diff Model: } Z = \beta_0 + \beta_1|X - Y| + \epsilon \tag{16}$$

$$\text{Uncon Abs Diff Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 W + \beta_4 WX + \beta_5 WY + \epsilon \tag{17}$$

$$\text{Fitted Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 X^2 + \beta_4 Y^2 + \beta_5 XY + \epsilon \tag{18}$$

where W is a dummy variable equal to 0 when $X > Y$, 1 when $X < Y$, and randomly assigned 1 or 0 when $X = Y$, and $\epsilon \sim N(0, \sigma_e^2)$

Criterion Checks

The four criterion are:

1. Check whether the unconstrained model (Eq 15 and 17) explained a significant amount of the variance in the data (compared to Eq 13).
2. Check whether the regression coefficients for the unconstrained model are in expected directions and whether they are significant to the model.
3. Check whether the unconstrained models (Eq 15 and 17) is significantly better than their constrained models (Eq 14 and 16).
4. Check whether a higher order model i.e. polynomial regression (Eq 18) is significantly better than the constrained model (Eq 14).

Criterion Checks

1. First criterion checks the basic linear model assumption i.e. is the model better than a basic mean(intercept) only model (Eq 21)?
2. Second criterion checks whether the coefficients are lined up in the way we expect them too and whether they add anything to the model
3. Third criterion checks if there is benefit to using simple polynomial models compared to simpler difference models
4. Fourth criterion checks whether a simple or complex polynomial model should be used

Table: This table is a replication of the Table 2 in Phillips et al. (2012) where $*p < 0.05; **p < 0.01; ***p < .001$

Outcome	Constrained Model		Unconstrained Model						F_c	F_h
	$(X-Y)$	R^2	X	Y				R^2		
Patient Adherence	-0.35^{***}	0.1^{***}	-0.33^{**}	0.39^{***}				0.1^{**}	0.84	0.83
Physician perception of patient agreement	-0.05	0.001	0.1	0.27^{**}				0.18^{***}	54.89^{***}	17.39^{**}
	$|X-Y|$	R^2	X	Y	W	WX	WY	R^2		
Patient Adherence	-0.41^{***}	0.1^{***}	-0.46^{**}	0.46^{**}	0.14	0.57^{*}	-0.45	0.16^{**}	2.19	2.18^{*}
Physician perception of patient agreement	-0.32^{**}	0.04^{**}	-0.06	0.44^{**}	0.1	0.51^{**}	-0.57^{*}	0.21^{***}	13.22^{***}	7.45

Criterion Checks

1. Criterion 1: Unconstrained model for both outcomes explained a significant amount of variation in the data (R^2 of 0.1,0.18 for responses Z_1 and Z_2 respectively).

2. Criterion 2: Only satisfied for the outcome dealing with patient adherence(coefficients of -.33 and .39) as the regression coefficients for physician perceptions (.1 and .27) were not going in opposite directions and were not both significant

3. Criterion 3: Met for the second outcome i.e. the physician perceptions ($F_c = 54.89$) and not for patient adherence ($F_c = 0.84$)

4. Criterion 4: Satisfied for the outcome concerning physician perceptions of patient-agreement ($F_h = 17.39$) and not for patient adherence ($F_h = 0.83$).

Consider using polynomial regression for the outcome dealing with physician perceptions while it looks as if the model using patient adherence as its outcome will be sufficiently explained using an algebraic difference score model

Criterion Checks

- Repeat above looking at the absolute difference
- Conclude the absolute-value difference score model was not an adequate fit for either of the outcome variables
- Difference score is an adequate model for the outcome concerning patient adherence
- Polynomial regression would be the best model for the data using the physician perception outcome

Then they test what kind of polynomial regression:

- Quadratic model vs linear model with a p-value of 0.0065404: Quadratic model explains the data better
- Cubic vs quadratic (p-value 0.673799): Cubic does not explain the data significantly better

So the fitted model(Eq 18) via OLS:

$$\hat{z} = 3.82 + 0.13x + 0.18y - 0.15x^2 + 0.43xy - 0.22y^2$$

Stationary points are at $(-2.48, -2)$

The first and second principal axis are Equations 19 and 20, respectively:

$$y = 0.08 + 0.84x \qquad (19)$$
$$y = -4.95 - 1.19x \qquad (20)$$

The surface slice plots in of both principal axes in Figure 1.
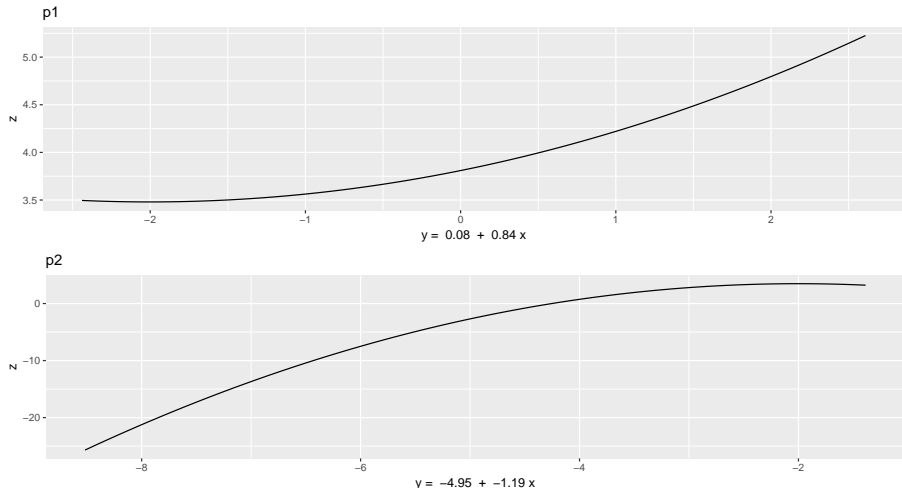

Figure: This Figure shows slices of the response surface taken at the two principal axes.

To look at the predicted surface plot, we use `Plotly`.
We can follow the link to the `Plotly` website and look at and manipulate the predicted surface.

Conclusions

- First principal axis is the axis that makes the most sense: it represents the path in which the values of patients preferences for shared decision-making and the values of patients experiences of shared decision-making along their entire scales that optimize patient satisfaction

- When doctor and patient agree with what is going on, the patient is most likely to follow the orders and vice-versa

- Second principal axis doesn't have any real interpretations because it lies outside of the scope of the data i.e. it doesn't lie on our predictive surface

Our Example

Overall goal:

- Interested in understanding the congruence between survey reports and observer reports

Background on data:

- Interested in understanding the congruence between survey reports and observer reports
- Comes from the Iowa Midlife Transitions Project (MTP); this longitudinal study conducted between 1991 and 2001 on families from eight counties in rural Iowa.
- Study was designed to look at the effects of the 1960's farm crisis on rural Midwestern families
- Families in MTP either participated in the Iowa Youth and Families Project (IFYP) or the Iowa Single Parent Project (ISPP).

IFYP:

- Started in 1989 to follow families with at least 2 children and married parents;
- Requirement: one of the children be in seventh grade in '89 while the other sibling was in a four year range in age of the child.
- 78% participation rate of all possible participants in the IFYP

ISPP:

- Started in '91
- Focus on recently divorced mothers with a minimum of two children, one of which was a ninth grader in the starting year and the other child was in a 4 year age range of the other child.
- 99% participation rate.

Data

Our focus was on the husband and wife relationship variables. Our variables of interest are divided into two sets:
Set 1:

Z : Observed Relationship Quality of couple i at Time Wave 0: $1 - 9$

Y_1 : Wife(i)'s Report of Husband's Hostility at Time Wave 0: $1 - 5$

X_1 : Husband(i)'s Report of Husband's Hostility at Time Wave 0: $1 - 5$

Set 2:

Z : Observed Relationship Quality of couple i at Time Wave 0: $1 - 9$

Y_2 : Wife(i)'s Report of Relationship Instability at Time Wave 0: $1 - 4$

X_2 : Husband(i)'s Report of Relationship Instability at Time Wave 0: $1 - 4$

Here in Table 5, some summary statistics are presented of the response and explanatory variables used.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	NA's	n
Z	-4.00	-1.00	1.00	0.76	2.00	4.00	148.00	407.00
X_1	-2.40	-1.80	-1.20	-1.17	-0.60	1.80	123.00	432.00
Y_1	-2.60	-2.05	-1.60	-1.52	-1.00	3.00	115.00	440.00
X_2	-1.50	0.85	1.50	1.04	1.50	1.50	115.00	440.00
Y_2	-1.50	0.90	1.50	1.08	1.50	1.50	123.00	432.00

Table: Summary statistics of response variable (Z) and explanotry variables (X_1, Y_1 and X_2, Y_2).

Normality Checks

Just like before we checked some statistics for obvious non-normality:

1. High Leverage
2. Influential Points

No points were of major concern.

Criterion Checks

The models being used in the criteria are listed below (note that these may have been listed previously but are re-listed here for ease of reading):

$$\text{Null Model: } Z = \beta_0 + \epsilon \tag{21}$$

$$\text{Diff Model: } Z = \beta_0 + \beta_1(X - Y) + \epsilon \tag{22}$$

$$\text{Uncon Diff Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \epsilon \tag{23}$$

$$\text{Abs Diff Model: } Z = \beta_0 + \beta_1|X - Y| + \epsilon \tag{24}$$

$$\text{Uncon Abs Diff Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 W + \beta_4 WX + \beta_5 WY + \epsilon \tag{25}$$

$$\text{Fitted Model: } Z = \beta_0 + \beta_1 X + \beta_2 Y + \beta_3 X^2 + \beta_4 Y^2 + \beta_5 XY + \epsilon \tag{26}$$

where W is a dummy variable equal to 0 when $X > Y$, 1 when $X < Y$, and randomly assigned 1 or 0 when $X = Y$, and $\epsilon \sim N(0, \sigma_e^2)$

Criterion Checks

The four criterion are:

1. Check whether the unconstrained model (Eq 23 and 25) explained a significant amount of the variance in the data (compared to Eq 21).

2. Check whether the regression coefficients for the unconstrained model are in expected directions and whether they are significant to the model.

3. Check whether the unconstrained models (Eq 23 and 25) is significantly better than their constrained models (Eq 22 and 24).

4. Check whether a higher order model i.e. polynomial regression (Eq 26) is significantly better than the constrained model (Eq 22).

Criterion Checks

1. First criterion checks the basic linear model assumption i.e. is the model better than a basic mean(intercept) only model (Eq 21)?
2. Second criterion checks whether the coefficients are lined up in the way we expect them too and whether they add anything to the model
3. Third criterion checks if there is benefit to using simple polynomial models compared to simpler difference models
4. Fourth criterion checks whether a simple or complex polynomial model should be used

Table: This table follows the same format as Table 2. Here we see that the where
$*p < 0.05; **p < 0.01; ***p < .001$

Outcome	Constrained Model		Unconstrained Model						F_c	F_h
	$(X - Y)$	R^2	X	Y				R^2		
Relationship Quality w/ Hostility	0.03	0	−0.54***	−0.66***				0.22***	112.54***	28.47***
Relationship Quality w/ Instability	−0.09	0	0.35*	0.35***				0.11***	50.41***	14.48***
	$\lvert X - Y \rvert$	R^2	X	Y	W	WX	WY	R^2		
Relationship Quality w/ Hostility	−0.44**	0.02**	−0.42*	−0.74**	−0.09	−0.23	0.04	0.22***	25.81***	13.12***
Relationship Quality w/ Instability	−0.75***	0.05***	−0.1	0.74***	−0.4	0.59	−0.24	0.12***	7.95***	5.36***

Criterion Checks

1. Criterion 1: Unconstrained model explained a significant amount of variation in the data ($R^2 = .22$, pval of $2.3632091 \times 10^{-23}$ and $R^2 = .11$, pval of $5.6530671 \times 10^{-12}$, respectively) in both sets

2. Criterion 2: Regression coefficients for both models were consistent with each other which is what we would expect. For both of the unconstrained algebraic models, the regression coefficients are significant, but the regression coefficients for the unconstrained absolute difference models were not all significant

3. Criterion 3 was both met by both examples/models (112.54^{***}, 50.41^{***}

4. Criterion 4: was both met by both examples/models 28.47^{***}, 14.48^{***})

Both models suggest using something greater than a simple difference model

Criterion Checks

- Repeat above looking at the absolute difference
- Conclude the absolute-value difference score model was not an adequate fit for the model considering Relationship Quality with Instability as a predictor

Criterion Checks

What kind of polynomial regression:

- Model with Hostility as a predictor: Use a linear model between to the two predictors and the response (p-val 0.636177 i.e. $H0 : \beta_3 = \beta_4 = \beta_5 = 0$ in Eq 26)
- Model with Instability as a predictor: has moderate evidence of using a quadratic model for the relationship (p-val 0.073613 i.e. $H0 : \beta_3 = \beta_4 = \beta_5 = 0$ in Eq 26) and no evidence for using a cubic model on the relationship (p-val 0.0945018)

The fitted quadratic polynomial using perceived relationship instability as a predictor for relationship quality is:

$$\hat{z} = -0.52 + 0.25x + 0.23y + 0.02x^2 + 0.09xy + 0.37y^2 \qquad (27)$$

The estimates of the model and their standard errors in Table 5

	b_0	b_1	b_2	b_3	b_4	b_5
Estimate	-0.52	0.25	0.23	0.02	0.09	0.37
Std. Error	0.26	0.19	0.21	0.21	0.21	0.20
$Pr(> |t|)$	0.05	0.19	0.28	0.94	0.68	0.06

Table: Model estimates and SE for polynomial regression model using relationship instability ratings as a explanatory variables.

Stationary points are at $(-10.19, 0.92)$
Likewise we can look at the principal axes:

$$y = 82.92 + 8.05x \tag{28}$$
$$y = -0.34 - 0.12x \tag{29}$$

The surface slice plots of both principal axes are in Figure 2.
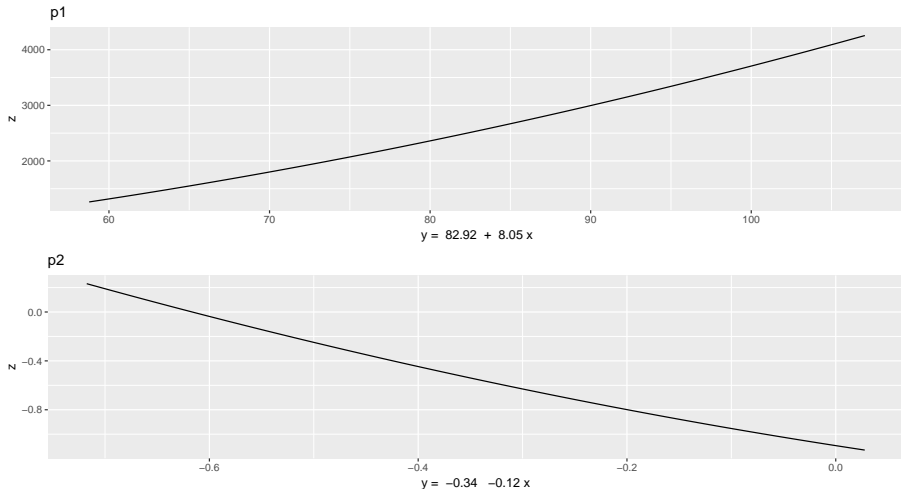

Figure: This Figure shows slices of the response surface taken at the two principal axes.

To look at the predicted surface plot, we use `Plotly`.
We can follow the link to the `Plotly` website and look at and manipulate
the predicted surface.

Conclusions

- Convex surface so interpretations of the principal axis are flipped;
  - First principal axis, Equation 28, represents the line in the surface where relationship quality decreases the most i.e. high level of perceived relationship instability by the wife
  - Second principal axis, Equation 29, is where the surface has the lowest downward curvature in relationship quality i.e. when RQ does not decrease that much when the wife's perceived relationship instability hangs around 0(2)
- First principal axis lies out of the bounds of data, but the second is good
- Most integral to relationship quality is the wife's perception of instability

Conclusions

Pros:

- Easily visualize the realm of possible combinations of the predictor variables and their effect on the predicted response
- Helps identify the underlying mechanism by seeing how different predictors affect the surface

Cons:

- Not super intuitive interpretations of model components
- Issues with data greatly affect inferences
- Figure 3 and 4 shows the predictor variables are very left skewed
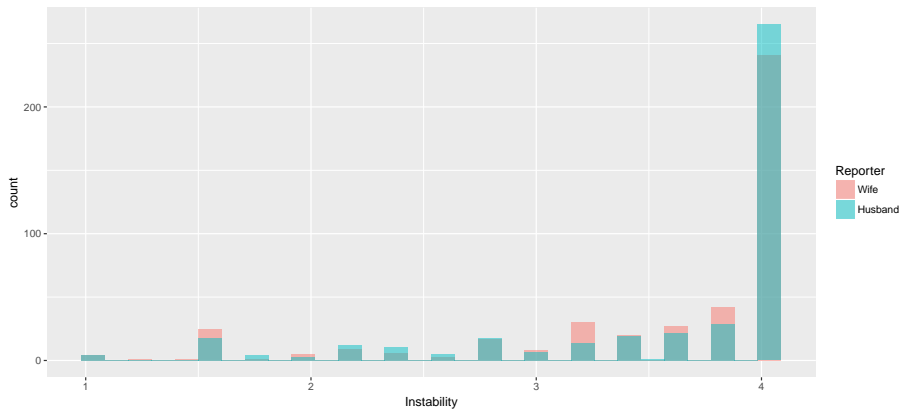
Conclusions

Figure: This Figure shows the histograms of our two predictor variables plotted on top of each other to show skewness.
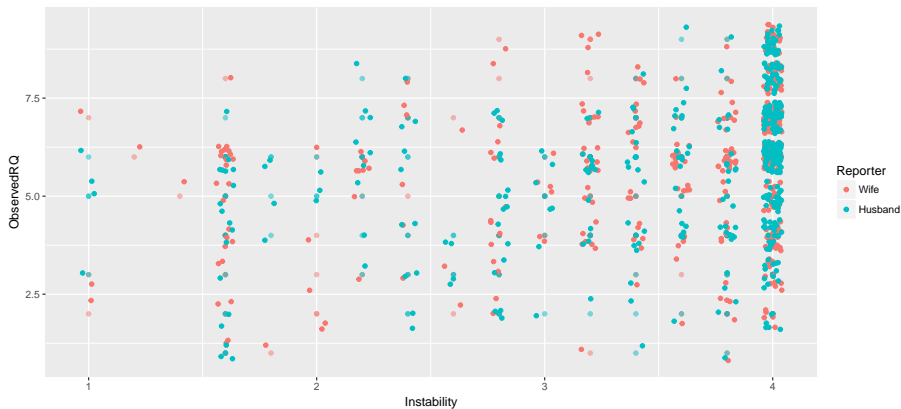
Conclusions

Figure: This Figure shows a scatterplot of Instability against Observed Relationship Quality colored by who reported the instability rating.

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