ASSET PRICING

LECTURE NOTES: READING MATERIAL OF ASSET PRICING COURSE AT MSE INSTRUCTED BY DR. PARTHAJIT KAYAL

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Asset Pricing

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Preface

The concepts of Finance form the bedrock upon which the modern economy and businesses function. Apart from that, Finance is deeply integrated into our personal lives as well in the form of - savings, investments and so forth. Our view is that from a practical, as well as academic standpoint, it is vital to understand the fundamentals that govern various concepts in this field. It might not require technical expertise to know that one would ideally want to maximize their returns on investments, while taking on minimal risk. However a certain level of technical expertise is undoubtedly required to be able to achieve that with higher degrees of accuracy than the average layman. The underlying principles behind stock returns, risk and investments are infused with rigorous mathematical foundations and statistics. Our motivation in this series of notes is to take you through these foundations. The idea is to build the mathematics that forms the very core of Finance. Some of these principles would indeed demystify this field riddled with complicated jargon and concepts. At the outset we state that some level of prior knowledge about basic probability and statistics is a prerequisite for a more thorough understanding.

This series of notes forms the curriculum content for the Asset pricing course module offerred at MSE as part of the PGDM program. These notes have been compiled from various sources, including lecture notes, books, websites and journals. Our idea is to compile knowledge from various sources and present it for the greater good - to enhance learning outcomes for students, professionals and others.

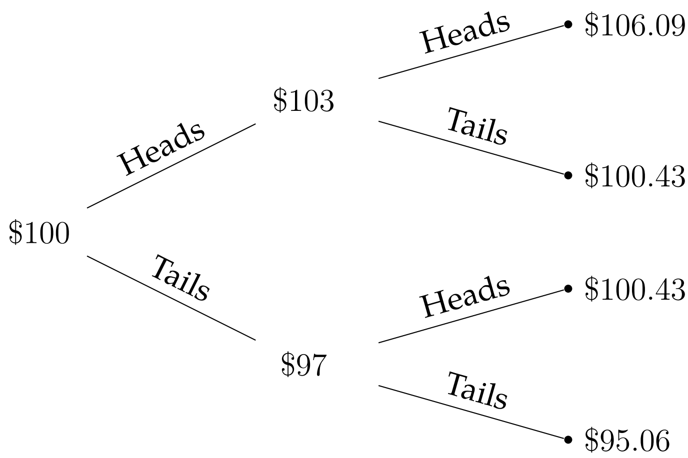
Contents

Chapter 1

Fundamentals of Risk Aversion: 1

1.1 Stock behaviour

Upon observation, the prices of stocks and commodities seem to follow a **random walk**. Suppose that we are given $\$100$ to play a game that entails tossing a coin everyday. With heads we would win 3% of our initial investment and with tails we would lose 2.5% of our initial investment. Now on at the end of the first day our investment could be either $\$103$ or $\$97$ and at the end of the second day it could be any of the 4 given outcomes at the end of the tree diagram.



Now this process is essentially what is known as a **random walk with a positive drift** of 0.25% per day, where the drift is nothing but the expected gains or losses given by:

$$\frac{1}{2} \times 3 + \frac{1}{2} \times (-2.5) = 0.25\% \tag{1.1}$$

To be a bit more precise, this essentially means that successive changes in stock value are independent of the previous value it took.

1.2 Forms of market efficiency

From the above concept of stock prices following a random walk we can say that all the information of past prices are actually reflected in today's market price of the stock, therefore we can say that patterns might not be exhibited and that price changes in one period are independent of price changes in the next. Now basically there are three forms of **market efficiency** which is the degree to which market prices reflect all available information. Broadly there are three levels of market efficiency which relate to different degrees of information being reflected in stock prices and are as follows :

- **Weak market efficiency** - In this level, prices reflect the information contained in the record of past prices. With this level of efficiency, it will be impossible to gain superior profits by studying past prices of stocks since the prices would follow a random walk.
- **Semistrong market efficiency** - This level of efficiency tells us that that current prices reflect information from not only past prices but also from all other sources of public information like Internet and financial press. In this case, prices adjust immediately as a response to public information such that earning's data and so forth.
- **Strong market efficiency** - This level suggests that prices contain all information that can possibly be gathered by indepth analysis of the company and the economy. In this case there could be no superior investors who could beat the market consistently.

1.3 Decision making with CAPM

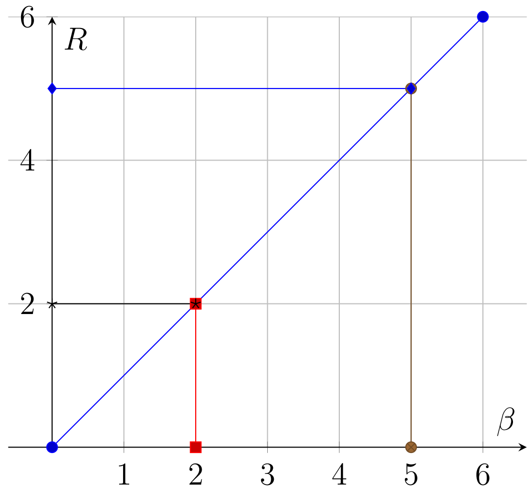
Recall that the **Capital asset pricing model** is given by the relationship of returns of an asset with its associated market risk (systematic risk) after diversifying away its idiosyncratic risk:

$$E(\tilde{r}_i) = r_f + \beta_i(E(r_m) - r_f) \tag{1.2}$$

Note that in the above equation the 'tilde' denotes that the returns are essentially random variables and that we are only using the CAPM to get an estimate of the possible returns of this asset, which may or not may not match the actual value. Two additional points are noted for clarity :

- β_i represents the market risk or the sensitivity of stock i with the market returns.
- $\beta(E(r_m) - r_f)$ represents the premium that the investor gets for taking on a quantum of risk more than that of the risk free asset.

Now we can consider one of the major weaknesses of CAPM as its limitation to model only market risk as the primary explaining factor for variability in the returns of a particular asset. Essentially no other risk is considered. A key point to note is that decision making using CAPM should inherently contain a word of caution. In the sense that any return it predicts could be different than the actual return on account of factors other than market risk. In the below figure we can we that the expected return for asset A represented by the **red line** is clearly lesser than the expected returns for asset B represented by the **black line**, as estimated by the CAPM. Now this might naively lead us to believe that asset B gives better returns. We say naive in this aspect because it is entirely possible that the actual return might turn out to be vastly different from this estimated value of returns.



The difference between actual and CAPM estimated returns is as follows:

$$\alpha_i = E(r_i) - E(\tilde{r}_i) = E(r_i) - [r_f + \beta_i(E(r_m) - r_f)] \tag{1.3}$$

Suppose it turns out in reality that $\alpha_A > 0$ and $\alpha_B < 0$, in that case we would say that asset A has certainly performed better than asset B, since asset A has delivered a return that is more than the **return expected given its level of systematic risk**. Similarly, we can say that asset B has underperformed given its systematic risk based expectation. It is important to look at the market risk adjusted return behaviour of assets as well.

1.4 Risk Aversion

The concept of risk aversion is used to study investor or consumer choice under uncertainty. We essentially parameterize objects of uncertain choice in terms of mean and variance of returns and then we map the preference of these uncertain choices to **utility functions** of consumers. For example if an uncertain gamble x is strictly prefered to uncertain gamble y then we would have:

$$if\ x \succ y \rightarrow U(x) > U(y) \tag{1.4}$$

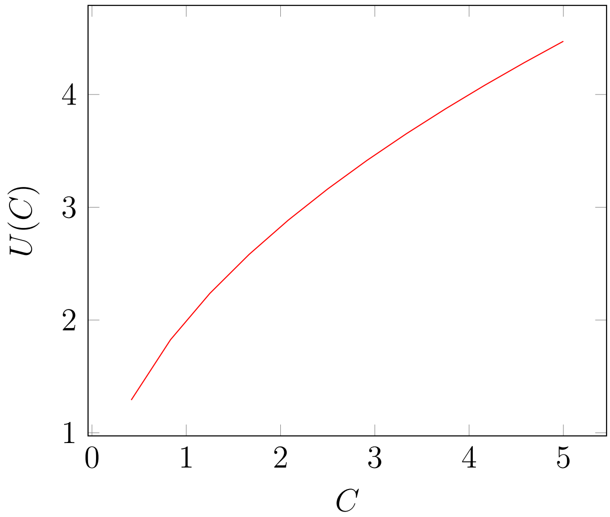
An important extension of this definition of risk aversion is - the desire on part of the consumer to obtain a smooth consumption over time and across states of nature. Consider the following consumption utlity function (*figure shown in next page*) which is defined by:

$$U(C) = 2\sqrt{C} \tag{1.5}$$

Clearly we know from theory about consumption utility functions that they are increasing, but at a decreasing rate (showing diminishing marginal utility). We can state the respective **first order** and **second order** conditions as:

$$MU(C) > 0 \tag{1.6}$$

$$\frac{d(MU(C))}{dC} < 0 \tag{1.7}$$



Now consider a situation of a classic gamble wherein under equal probabilities we might obtain a **good state** and win $\$10,000$ or obtain a **bad state** and win $\$100$. In this situation the expected utility from the gamble can be generalized as:

$$E[U(C)] = U(a) \times P(C = a) + U(b) \times P(C = b) \tag{1.8}$$

The above equation means that the expected utility from the gamble G which has two outcomes a and b is nothing but a linear combination of the utilities derived from the two separate outcomes, weighted by their respective probabilities of occurrence. In our example this would turn out to be:

$$(2 \times \sqrt{100} \times 0.5) + (2 \times \sqrt{10000} \times 0.5) = 110 \tag{1.9}$$

A point to note is that the 110 denotes a **measure of utility** and has nothing to do with actual money. In essence it is the utility we derive from playing this gamble. Now if we take the inverse of this utility function we would essentially get - **the amount of money for which this consumer obtains the same measure of utility as he does from playing the gamble**. This is termed as the consumer's **certainty equivalent wealth**. In terms of utility, the certainty equivalent is that level of wealth which would give the consumer **the same measure of utility as would the expected utility from the gamble**. We can state this mathematically as:

$$U(C*) = E[U(C)] \tag{1.10}$$

Where $C*$ denotes certainty equivalent wealth. In our case we have $U(C*) = 110$ as per the definition. Therefore we can compute:

$$2\sqrt{C*} = 110 \rightarrow \sqrt{C*} = 55 \rightarrow C* = \$3,025 \tag{1.11}$$

Now we would also calculate the expected wealth of the gamble's outcomes, which is simply the expected value of the outcomes:

$$E(C) = a \times P(C = a) + b \times P(C = b) \tag{1.12}$$

In the case of our example this would turn out to be:

$$100 \times 0.5 + 1000 \times 0.5 = \$5,050 \tag{1.13}$$

Finally, we note that the **risk premium** is defined as the maximum amount the consumer would be willing to give up to avoid playing the gamble. In other words, it is the difference between the **actuarial value of the gamble** and the **certainty equivalent wealth** and is given by:

$$\pi = E(C) - C* = \$5,050 - \$3,025 = \$2,025 \tag{1.14}$$

1.5 Generalization part 1

In this first kind of generalization of the above principles, we consider the power utility function given by:

$$U(C) = \frac{C^{1-\tau}}{1-\tau} \tag{1.15}$$

In the above function, note that the parameter 'tau' is defined to be bounded between 0 and 1. Additionally, another condition is for C to be greater than 0. Now before we move any further, we need to first check if it is infact a utility function and to check that it must satisfy the following conditions:

- The function should be strictly increasing.
- It should be strictly concave.

To prove the **first point**, we would take the first derivative and obtain a positive slope which would essentially signal an upward sloping function.

$$U'(C) = \frac{d(U(C))}{dC} = \frac{(1-\tau)C^{1-\tau-1}}{1-\tau} = C^{-\tau} = \frac{1}{C^{\tau}} \tag{1.16}$$

For the above first derivative we can say that values of τ between 0 and 1 and $C > 0$ that:

$$U'(C) > 0 \tag{1.17}$$

Now to prove the **second point** we would look at the second derivative and see if it is less than zero, indicating diminishing marginal utility.

$$U''(C) = \frac{d(C^{-\tau})}{dC} = \tau C^{-\tau-1} < 0 \tag{1.18}$$

The above function would clearly be less than 0 for the set of values τ might take, implying **strict concavity** and hence we can conclude that since the above two conditions are now satisfied, that this is infact a utility function. In the following sections, three cases are presented which cover the various risk classifications of consumers.

1.6 Case 1

In this case we take the parameter τ to be 0. As a result our utility function would become:

$$U(C) = \frac{C^{1-\tau}}{1-\tau} = C \tag{1.19}$$

Recall that our definition for certainty equivalent wealth implies:

$$U(C*) = E[U(C)] \tag{1.20}$$

Take the **LHS** of the above equation and we would get the utility measure of the certainty equivalent wealth to be just the level of consumption itself:

$$U(C*) = C* \tag{1.21}$$

Now for the **RHS** we would compute the expected utlity of the gamble as follows:

$$U(a) \times P(C = a) + U(b) \times P(C = b) \tag{1.22}$$

Since we proved before that in this case the utilty measure is the value of the consumption level itself, then could rewrite the above equation as:

$$a \times P(C = a) + b \times P(C = b) = E(C) \tag{1.23}$$

An interesting point to note here is that the expected utility from the gamble is the same as the actuarial value of the gamble. Hence the risk premium in this case would be:

$$\pi = E(C) - C* = C * -C* = 0 \tag{1.24}$$

This is the case of **Risk neutraility** wherein the consumer is indifferent between taking on the gamble and not taking on the gamble, thereby having a risk premium of zero.

1.6.1 Case 2

In this case we would take $\tau = 1$. Following would be the utility function:

$$U(C) = \frac{C^{1-\tau}}{e-\tau} = \frac{C^0}{0} \tag{1.25}$$

Note that the above function would essentially become undefined with $\tau = 1$. Let us see what we get with the first derivative:

$$U'(C) = \frac{(1-\tau)C^{1-\tau-1}}{1-\tau} = c^{-\tau} \tag{1.26}$$

$$\lim_{\tau \to 1} U'(C) = \lim_{\tau \to 1} C^{-\tau} = C^{-1} = \frac{1}{C} \tag{1.27}$$

Now since the antiderivative of the above expression gives us $\ln(C)$ we can essentially reformulate the original utility function to be of the form:

$$U(C) = \ln(C) = \log_e(C) \tag{1.28}$$

Recall that the utility derived from certainty equivalent wealth is the same as the measure of utility obtained from the expected utility of the gamble, hence we can write:

$$U(C*) = E[U(C)] \tag{1.29}$$

$$\ln(C*) = U(a) \times P(C = a) + U(b) \times P(C = b) \tag{1.30}$$

$$= \ln(a) \times 0.5 + \ln(b) \times 0.5 = 0.5\ln(ab) = \ln(ab)^{0.5} \tag{1.31}$$

$$C* = (ab)^{0.5} \tag{1.32}$$

Noting the actuarial value of the gamble as:

$$E(C) = 0.5(a + b) \tag{1.33}$$

We can write the risk premium in this case as:

$$E(C) - C* = 0.5(a + b) - (ab)^{0.5} \tag{1.34}$$

1.6.2 Case 3

Finally in the third case we would confine τ to be between 0 and 1. Again in order to compute the certainty equivalent wealth we would equate the utility measure under the certainty equivalent wealth to the expected utility of the gamble.

$$U(C*) = E[U(C)] \tag{1.35}$$

$$LHS = \frac{C*^{1-\tau}}{1-\tau} \tag{1.36}$$

$$RHS = U(a) \times P(C = a) + U(b) \times P(C = b) \tag{1.37}$$

$$RHS = 0.5(U(a) + U(b)) = \frac{1}{2(1-\tau)}(a^{1-\tau} + b^{1-\tau}) \tag{1.38}$$

Now equating the LHS and RHS (cutting out the common $(1 - \tau)$) to obtain:

$$C*^{1-\tau} = 0.5(a^{1-\tau} + b^{1-\tau}) \tag{1.39}$$

$$C* = [0.5(a^{1-\tau} + b^{1-\tau})]^{\frac{1}{1-\tau}} \tag{1.40}$$

Remember that the risk premium is given by:

$$\pi = E[C] - C* \tag{1.41}$$

This case essentially demonstrates the condition for **risk aversion**.

Chapter 2

Fundamentals of Risk aversion: 2

2.1 General case of risk aversion measures

In the previous note, we essentially looked at a typical concave utility power function and derived the risk premium mearues for three cases that varied depending on the value of our utility function parameter τ. Additionally we learnt that as this parameter increases in value from 0 to 1 the degree of risk aversion increases, as with each increase in τ the risk premium also increases. Note that the earlier examples contained an explicitly mentioned value of probability for the occurrence of a **good state** wherein we win around $\$10000$ and a **bad state** wherein we lose $\$100$. In subsequent sections we shall look at a more broad scenario when the probability of occurrence of the events could be anything - we essentially let p take on any value between 0 and 1. This is being done to arrive at generalized measures of risk premiums across different situations. Recall that the original utility function we are dealing with is:

$$U(C) = \frac{C^{1-\tau}}{1-\tau} \tag{2.1}$$

It would also be helpful to recall the definition of **certainty equivalent wealth** which is the amount of money that gives the same measure of utility to an individual that he might get from playing a gamble. If C^* is the certainty equivalent wealth, then we have:

$$U(C^*) = E[U(C)] \tag{2.2}$$

Finally we recall the definition of **risk premium** which is the difference between the actuarial value of the gamble and the certainty equivalent wealth.

$$\pi = E[C] - C^* \tag{2.3}$$

2.1.1 Case 1: Risk neutrality

Here we set $\tau = 0$ and as a result our utility function value would resolve to:

$$U(C) = \frac{C^{1-0}}{1-0} = C \tag{2.4}$$

Therefore if we wanted the utlity measure for the certainty equivalent wealth we would have:

$$U(C^*) = C^* \tag{2.5}$$

In order to obtain C^* we must equate the utility derived from C^* to the expected value of the gamble:

$$U(C^*) = E[U(C)] \tag{2.6}$$

Resolving the **LHS** first, we have:

$$LHS = U(C^*) = C^* \tag{2.7}$$

Resolving the **RHS** we have the following. Please recall here that a refers to the realisation of the event that $C = a$ and b is the realisation of the event that $C = b$. They are essentially the prizes in our gamble.

$$RHS = E[U(C)] = U(a)\underbrace{Prob(C = a)}_{1-p} + U(b)\underbrace{Prob(C = b)}_{p} \tag{2.8}$$

$$= a(1 - p) + bp \tag{2.9}$$

Now we will begin to calculate the actuarial value of the gamble which is nothing but the expected value of the value of prizes in the gamble.

$$E[C] = a \times Prob(C = a) + b \times Prob(C = b) = a(1 - p) + bp \tag{2.10}$$

Therefore we have from equations 8 and 9 that:

$$E[U(C)] = E[C] = C^* \tag{2.11}$$

Writing out the corresponding value of risk premium we would see that it essentially resolves to 0:

$$\pi = E[C] - C^* = 0 \tag{2.12}$$

2.1.2 Case 2

Here we set $\tau = 1$. Recall from earlier discussions that if we simply take the utility function in its original form then the function would resolve to an undefined form in this case:

$$U(C) = \frac{C^{1-1}}{1-1} = \frac{C^0}{0} \tag{2.13}$$

To get past this problem, we take the first derivative of this function and then take its antiderivative to get the log transformation of the original utility function, which would consequently, not resolve to an undefined form and hence we can then move on with regular computations.

$$U'(C) = \frac{(1-\tau)C^{1-\tau-1}}{1-\tau} = C^{-\tau} \tag{2.14}$$

$$lim_{\tau\to 1}U'(C) = \lim_{\tau\to 1} C^{-\tau} = C^{-1} = \frac{1}{C} \tag{2.15}$$

Taking the antiderivative would give us $\ln(C)$ and hence our original utility function is now a log function. Note that this would not change the fundamental behaviour of this function from the original one since **log transformations are monotonic transformations** that preserve the characteristic properties of the original function.

$$U(C) = \ln(C) \tag{2.16}$$

Getting back to the task at hand, we can formulate the equation for finding the certainty equivalent wealth as follows:

$$U(C^*) = E[U(C)] \tag{2.17}$$

$$\ln C^* = U(a)Prob(C = a) + U(b)Prob(C = b) = \ln(a)(1-p) + \ln(b)p \tag{2.18}$$

Expanding the above equation we would get:

$$= \ln(a) - p\ln(a) + p\ln(b) = \ln(a) + p[\ln(b) - \ln(a)] = \ln(a) + p\ln\left(\frac{b}{a}\right) \tag{2.19}$$

$$= \ln a\left(\frac{b}{a}\right)^p \tag{2.20}$$

Taking antilog on both sides we would get the certainty equivalent wealth as:

$$C^* = a\left(\frac{b}{a}\right)^p \tag{2.21}$$

The actuarial value of the gamble is given by:

$$E[C] = a(1-p) + bp \tag{2.22}$$

Finally we can obtain the corresponding risk premium measure by:

$$E[C] - C^* \tag{2.23}$$

2.1.3 Case 3

This is the case wherein our paramter value τ lies between 0 and 1. Laying down the equation to find certainty equivalent wealth we have:

$$U(C^*) = E[U(C)] \tag{2.24}$$

Then our **LHS** would be:

$$LHS = \frac{C^{*(1-\tau)}}{1-\tau} \tag{2.25}$$

Subsequently, we evaluate the **RHS** which is basically the expected utility of the gamble:

$$E[U(C)] = U(a)Prob(C = a) + U(b)Prob(C = b) = U(a)(1-p) + U(b)p \tag{2.26}$$

$$= \frac{a^{1-\tau}}{1-\tau}(1-p) + \frac{b^{1-\tau}}{1-\tau}p = \frac{a^{1-\tau}}{1-\tau} - p\frac{a^{1-\tau}}{1-\tau} + \frac{b^{1-\tau}}{1-\tau} \tag{2.27}$$

$$= \frac{a^{1-\tau}}{1-\tau} + p\left(\frac{b^{1-\tau}}{1-\tau} - \frac{a^{1-\tau}}{1-\tau}\right) \tag{2.28}$$

$$= \frac{1}{1-\tau}\left[a^{1-\tau} + p(b^{1-\tau} - a^{1-\tau})\right] \tag{2.29}$$

$$= \frac{a^{1-\tau}}{1-\tau}\left[1 + p\left(\left(\frac{b}{a}\right)^{1-\tau} - 1\right)\right] \tag{2.30}$$

Now we can let $(b/a) = k$ and as a result of this we will get:

$$RHS = \frac{a^{1-\tau}}{1-\tau}\left[1 + p(k^{1-\tau} - 1)\right] \tag{2.31}$$

Note that we did all this solving to ultimately be able to equate the above expression with $U(C^*)$ that will help us find the certainty equivalent wealth. With this in mind we can equate:

$$\frac{C^{*(1-\tau)}}{1-\tau} = \frac{a^{1-\tau}}{1-\tau}\left[1 + p(k^{1-\tau} - 1)\right] \tag{2.32}$$

$$C^{*(1-\tau)} = a^{1-\tau}\left[1 + p(k^{1-\tau} - 1)\right] \tag{2.33}$$

Note that the $(1-\tau)$ term gets cancelled and thats how we get the above equation. Now to get an explicit expression for certainty equivalent wealth we will just take the $(1-\tau)^{th}$ root on both sides to get:

$$C^* = a[1 + p(k^{1-\tau} - 1)]^{1/(1-\tau)} \tag{2.34}$$

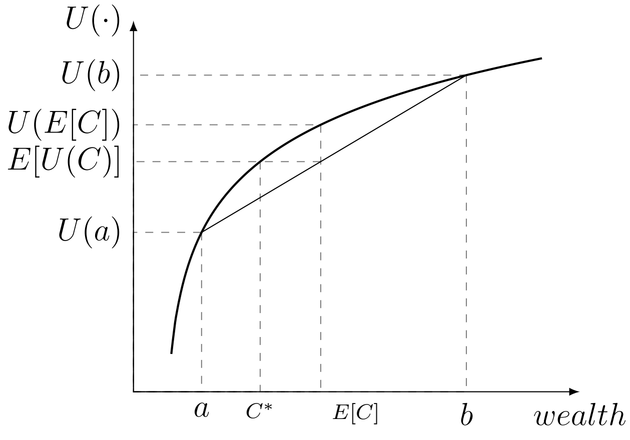
Again, recalling the actuarial value of our gamble to be:

$$E[C] = a \times Prob(C = a) + b \times Prob(C = b) = a(1-p) + bp \tag{2.35}$$

Finally the risk premium is, as in previous cases, given by:

$$\pi = E[C] - C^* \tag{2.36}$$

The figure shown below illustrates the utility function of a typically **risk averse** investor. For this investor we can see that the level of utility from playing the gamble $(E[U(C)])$ is clearly less than the utility derived from the actuarial value of the gamble which is given by $(U(E[C]))$.



2.1.4 Special cases

In case $p = 0$, then we would have $U(C^*) = E(U)$. Expanding this further we get:

$$U(C^*) = U(a)(1-0) + U(b)(0) = U(a) \tag{2.37}$$

$$\rightarrow C^* = a \tag{2.38}$$

Now computing the actuarial value we get:

$$E(C) = a(1-0) + b(0) = a \tag{2.39}$$

Therefore the risk premium would be:

$$E(C) - C^* = 0 \tag{2.40}$$

Now we take another case wherein $p = 1$.

$$U(C^*) = U(a)(0) + U(b)(1) = U(b) \tag{2.41}$$

$$\rightarrow C^* = b \tag{2.42}$$

Now computing the actuarial value we get:

$$E(C) = a(1-1) + b(1) = b \tag{2.43}$$

Therefore the risk premium would be:

$$E(C) - C^* = 0 \tag{2.44}$$

Chapter 3

Shortfall probability: 1

3.1 A primer on stock Returns

Let price of an asset at time t be P_t. With this in mind, we can depict **one period** returns as follows:

$$1 + R_t = \frac{P_t}{P_{t-1}} \tag{3.1}$$

$$P_t = P_{t-1}(1 + R_t) \tag{3.2}$$

Consequently, the **simple net return** is given by:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{3.3}$$

If we factor in **dividends** as well then our **simple rate of return** would come out to be as follows. Assume P_0 as the price initially and P_1 as the price at time period 1 which also happens to be the time period when we receive dividends.

$$ROR = \frac{P_1 + Div_1 - P_0}{P_0} = \frac{P_1 - P_0}{P_0} + \frac{Div_1}{P_0} \tag{3.4}$$

For sake of convenience in further derivations, we shall consider dividends to be 0. Now let us assume a situation. Suppose we are at time 0 and posses an initial wealth of \$$100$. One period later, that is at time 1 lets suppose our money doubles to \$$200$. In this case the return would be:

$$R_1 = \frac{P_1 - P_0}{P_0} = \frac{200 - 100}{100} = 1.00 = 100\% \tag{3.5}$$

We got a 100% return as on time period 1. Now lets go further. Suppose that another time period goes by but this time, our money halves in value. So from \$$200$ in time period 1 we now find ourselves in time period 2 with only \$$100$. **Note that we started out in time period** 0 **with** \$$100$ **and ended up with** \$$100$ **in time period** 2. It is obvious that our **net return over two time periods** is 0 since we are ending up the same amount of money that we started with. Now comes the tricky bit. We often tend to make the naive error of simply adding single period returns over a span of time, which gives us the **wrong value**. Let us see:

$$R_{total} = R_1 + R_2 = 100\% + (-50\%) = 50\% \tag{3.6}$$

This falsely implies that our net return was 50%! By this logic we would have had $\$150$ by time period 2. But that is not the case in reality. In reality, returns are **compounded** or we can simply say, multiplied across time periods.

$$(1 + R_{total}) = (1 + R_1)(1 + R_2) \tag{3.7}$$

As is often the case, when we have a series of multiplicands, computations can be simplified by applying **log operators**. This is because the **log** over a continuous product results in continuous summations. Sums are generally easier to deal with.

$$\ln(1 + R_{total}) = \ln[(1 + R_1)(1 + R_2)] \tag{3.8}$$

$$\ln(1 + R_{total}) = \ln(1 + R_1) + \ln(1 + R_2) \tag{3.9}$$

As a general rule, we usually denote small letters signifying **log values** and capital letters signifying **absolute values**. Therefore the appropriate notations should be:

$$r_1 = \ln(1 + R_1) \tag{3.10}$$

$$r_2 = \ln(1 + R_2) \tag{3.11}$$

With this, we can actually write the **total log returns** as the summation of the **individual log returns**.

$$r_{total} = r_1 + r_2 \tag{3.12}$$

Now as per the above example, we can plug in the appropriate values to obtain the following:

$$r_1 = \ln(1 + R_1) = \ln(1 + 1.00) = \ln(2) = 0.6931 \tag{3.13}$$

$$r_2 = \ln(1 + R_2) = \ln(1 - 0.5) = \ln(2) = -0.6931 \tag{3.14}$$

$$r_{total} = r_1 + r_2 = 0 \tag{3.15}$$

Now we are getting the correct answer of 0 total returns. **NOTE**: When it comes to absolute returns, writing them in ratio form (example: 0.5) is the same thing as writing it in percentage form (example: 50%). Please note that **percentages are nothing but ratios, multiplied by 100**. To generalize with respect to appropriate notations, here is how we can present the same relationships in a slightly different manner.

$$r_1 = \ln(1 + R_1) = \ln\left(1 + \frac{P_1 - P_0}{P_0}\right) = \ln\left(1 + \frac{P_1}{P_0} - 1\right) = \ln\left(\frac{P_1}{P_0}\right) \tag{3.16}$$

With a similar simplification, we can show that:

$$r_2 = \ln\left(\frac{P_2}{P_1}\right) \tag{3.17}$$

$$r_1 + r_2 = \ln\left(\frac{P_1}{P_0}\right) + \ln\left(\frac{P_2}{P_1}\right) = \ln\left(\frac{P_1}{P_0} \times \frac{P_2}{P_1}\right) = \ln\left(\frac{P_2}{P_0}\right) \tag{3.18}$$

Therefore with the above equation we can generalize the **total return over T time periods** as:

$$r_{total} = \ln\left(\frac{P_T}{P_0}\right) = r_1 + r_2 + \cdots + r_T \tag{3.19}$$

Further, we let $r(0, T)$ denote the multiperiod log returns over the time period $[0, T]$ and it is given by:

$$r_1 + r_2 + \cdots + r_T = \sum_{t=1}^{T} r_t \tag{3.20}$$

3.1.1 Extending the concept to risk free returns

First of all we note that risk free returns are assumed to be generated typically from safe securities, carrying minimial, mostly in the form of **sovereign bonds**. Before we go ahead, here are some key notations:

- r_f: Risk free interest rate.
- B_1: Bond value at time t_1. It can also be given by:

  $$B_1 = B_0 e^{r_f} \tag{3.21}$$

  NOTE: When interest rate is raised to the power of an exponent, it is a case of **continuous compounding**. What this means is that while computing the value of a bond at time period 1, we essentially look at the expected continuous stream of compounded returns that the bond would give us. The value is essentially the future value of continuously compounded returns.

- Similarly, we can write value of bond at time t_1 as:

  $$B_2 = B_1 e^{r_f} = (B_0 e^{r_f}) e^{r_f} = B_0 e^{2r_f} \tag{3.22}$$

Now that we have these general principles laid down, the risk free return, in log terms, can be found out over 2 years as:

$$r_{total} = \ln\left(\frac{B_2}{B_0}\right) = \ln\left(\frac{B_0 e^{2r_f}}{B_0}\right) = 2r_f \tag{3.23}$$

NOTE: in the above equation the last step resolves to $2r_f$ because of the **log-exponent** conversion rule:

$$\ln(e^x) = x \tag{3.24}$$

Following this general rule, we can arrive at the total risk free returns over T years as:

$$r_{total} = T \times r_f \tag{3.25}$$

3.2 Shortfall Probability

Previously we studied various complicated formulae and derivations regarding the concepts of risk aversion and risk premiums. Let us now remember them again. We note that when an investor is investing in a portfolio of **risky assets**, he typically demands an extra return for the extra risk he is willing to take - this is the **risk premium** that the investor demands. Suppose an investor holds a certain portfolio over a time period T. Naturally, this investor would want a higher return than the risk free assets. However, since there is an inherent undertaking of risk in the risky portfolio, there is a chance that the risky assets don't perform well and the investor ends up with an even lower return than what a perfectly safe risk free asset would have given him. This is the situation of the investor suffering a **shortfall**. We further note that the probability that the risky portfolio over time period T would give a lower return than a risk free asset over the same time period, is known as the **shortfall probability**. As intuitive as it sounds, it is simply the probability that the summation of returns all time periods till T is less than the total risk free return over T time periods.

$$Prob\{\sum_{t=1}^{T} r_t < T \times r_f\} \tag{3.26}$$

We note that returns are essentially **random variables** and because log transformations are monotonic, we say the log returns are also random variables. Here is an important assumption we make:

r_t: log return from stock portfolio at year t, where $r_t \sim iidN(\mu, \sigma^2)$

This is akin to the **stationarity assumption** when dealing with time series of returns, wherein we assume that the distribution of random variable returns stays the same across time, with time invariant mean and variance. Now let us suppose that we sample various returns and compute a statistic that is the sum of all the returns, we can then compute the mean and variance of such a statistic as follows:

$$E(\sum_{t=1}^{T}) = E(r_1 + r_2 + \cdots + r_T) = E(r_1) + E(r_2) + \cdots + E(r_T) = \mu T \tag{3.27}$$

$$Var(\sum_{t=1}^{T}) = Var(r_1) + Var(r_2) + \cdots + Var(r_T) + 2\{Cov\ terms\} = \sigma^2 T \tag{3.28}$$

Note that the covariance terms will be 0 since we have assumed stochastic independence among the random variable returns. Hence we can say that the statistic of the sum of the random variables follows a normal distribution with:

$$\sum_{t=1}^{T} r_t \sim N(\mu T, \sigma^2 T) \tag{3.29}$$

Recall shortfall probability and an alternative form of expressing it:

$$Prob\{\sum_{t=1}^{T} r_t < Tr_f\} = Prob\{\sum_{t=1}^{T} r_t - Tr_f < 0\} \tag{3.30}$$

In the above equation we simply took the right hand side of the inequality to the left hand side. Further let us denote:

$$X = \sum_{t=1}^{T} r_t - Tr_f \tag{3.31}$$

Intuitively, we can think of X as the excess return from the time period T returns of the risky portfolio, over the time period T benchmark risk free returns. With this we can rewrite shortfall probability as:

$$Prob\{X < 0\} \tag{3.32}$$

We can now find the mean and variance of this X as follows:

$$E(X) = E(\sum_{t=1}^{T} r_t - Tr_f) = \mu T - Tr_f = (\mu - r_f)T \tag{3.33}$$

$$V(X) = Var(\sum_{t=1}^{T} r_t - Tr_f) = V(\sum_{t=1}^{T} r_t) + V(Tr_f) - 2Cov(\sum_{t=1}^{T} r_t, Tr_f) = \sigma^2 T \tag{3.34}$$

NOTE: Risk free returns are taken as a constant. Risk in finance means probability, and when we have something thats risk free, its essentially a non-stocastic variable. Since Tr_f is a constant its mean is the value of the constant itself and its variance and associated covariance is 0. Putting all this together, we can obtain the distribution of X as:

$$X \sim N\{(\mu - r_f)T, \sigma^2 T\} \tag{3.35}$$

We know from basic statistics that it is much easier to deal with **standard normal** variables rather than normally distributed variables, for the ease of computations, among various other reasons. If we simply subtract the mean from X and divide by the standard deviation we would get the standard normal counterpart of X as follows:

$$Prob(X < 0) = Prob\left[\frac{X - (\mu - r_f)T}{\sigma\sqrt{T}} < \frac{0 - (\mu - r_f)T}{\sigma\sqrt{T}}\right] \tag{3.36}$$

$$= Prob\left[Z < -\frac{\mu - r_f}{\sigma}T\right] \tag{3.37}$$

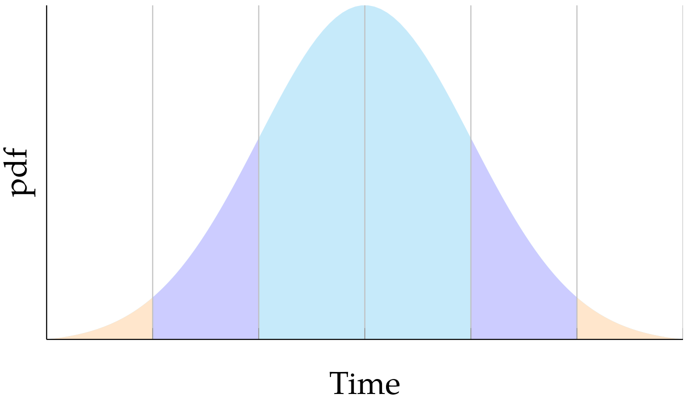
First of all we note that Z follows a standard normal distribution and the above equation merely resolve to finding the CDF of Z at a particular value. Secondly, we note the presence of **sharpe ratio** in the above expression. Recall that the Sharpe Ratio is nothing but the ratio of the risk premium and standard deviation.

$$Z \sim N(0, 1) \tag{3.38}$$

Finally we can write our probability measure as:

$$Prob\{X < 0\} = Prob\{Z < -SR\sqrt{T}\} = \Phi(-SR\sqrt{T}) \tag{3.39}$$

NOTE: the Φ is standard notation for the CDF of a standard normal variable. Further note that the Sharpe Ratio needs to be positive as nobody will invest in the portfolio of risky assets unless they get higher risk adjusted returns than the risk free assets. Finally we note that as $T \to \infty$ the above probability measure of **shortfall probability** will tend to 0.



We can see from the above figure that as time tends to infinity, the probability function asymptotes to 0. As is evident from the 'orange' shaded region in the normal curve, the shortfall probability in such a case would be negligible. While ending this section, note that the central assumption here was that of a **non stochastic** benchmark asset - the risk free asset. In subsequent sections we will make things more dynamic by allowing for stochastic benchmark measures.

Chapter 4

Shortfall probability: 2

4.1 Stochastic benchmark

In the earlier sections we saw how to compute the **shortfall probability** of a series of asset returns over a period of time, against a benchmark risk free asset. Typically we look at the risk free asset because when investing in a risky portfolio, every average investor is sensible enough to seek higher returns than this benchmark asset for taking on an extra level of risk. We also saw how these concepts have deep mathematical and theoretical underpinnings in the form of **the theory of risk aversion** and **probability theory**. As the previous sections ended, we were able to ascertain that the shortfall measure converges in distribution to the **standard normal distribution** and that the CDF values for measuring shortfall probability can be computed by using the **sharpe ratio**. Now we will build on these concepts further by making things quite dyanamic and rather interesting. We will now allow the benchmark return to be **stochastic** in nature. This could be any portfolio. In ideal situation we might even consider to be the market portfolio. But the central idea to grasp is that whatever it may be, it inherently has aspects of randomness associated with it. What we mean by randomness being associated with it, is that these benchmark returns, unlike the risk free returns, have finite variance.

4.1.1 Building the foundations

Before going further into developing the mathematics, some key notations need to be pointed out.

- **Benchmark return**: This refers to the benchmark portfolio with finite mean and variance. The subscript t signifies the returns at time t. Note that as in the previous cases, the distribution is assumed to be the same across time.

$$r_{bm,t} \sim N(\mu_{bm}, \sigma_{bm}^2) \tag{4.1}$$

- **Stock returns**: These refer to the stock we are holding, for which we want to measure the shortfall probability against the benchmark. Here also, the subscript t signifies the returns at time t.

$$r_{s,t} \sim N(\mu_s, \sigma_s^2) \tag{4.2}$$

- Let rho denote the correlation between $r_{s,t}$ and $r_{bm,t}$.
- The random variables of $r_{s,t}$ and $r_{bm,t}$ are considered jointly iid.

Now we know from the previous definitions that shortfall probability is given by:

$$SP = Prob(\sum_{t=1}^{T} r_{s,t} < \sum_{t=1}^{T} r_{bm,t}) \tag{4.3}$$

Now the excess log return for a particular time period is given by:

$$r_{e,t} = r_{s,t} - r_{bm,t} \tag{4.4}$$

Often we are more interested in the expected behaviour of random variables rather than isolated measurements, so we can essentially apply the expectations operator on both sides to get the expected excess returns as:

$$E(r_{e,t}) = E(r_{s,t}) - E(r_{bm,t}) = \mu_s - \mu_{bm} = \mu_e \tag{4.5}$$

Note that we are getting these expressions because of time invariance of the mean in these distributions. In a similar manner, we can obtain a measure of the variance as well:

$$Var(r_{e,t}) = Var(r_{s,t} - r_{bm,t}) = Var(r_{s,t}) + Var(r_{bm,t}) - 2Cov(r_{s,t}, r_{bm,t}) \tag{4.6}$$

$$\sigma_e^2 = \sigma_s^2 + \sigma_{bm}^2 - 2\rho\sigma_s\sigma_{bm} \tag{4.7}$$

4.1.2 Distribution of overall returns

Going further, we can say that it is generally in our interest to oberve the behaviour of a **stochastic process** (which is nothing but a series of random variables) rather than an individual realisation of a random variable return in a particular time period. This motivation also comes about because of the inherent practicality in investor judging their returns over a long time period rather than single period returns. With this in mind, we now attempt to define the distribution of the overall excess returns.

$$\sum_{t=1}^{T} r_{e,t} \tag{4.8}$$

Now since $r_{s,t}$ and $r_{bm,t}$ are iid normal, the distribution of the overall excess returns can be characterised as:

$$\sum_{t=1}^{T} r_{e,t} = N(\mu_e T, \sigma_e^2 T) \tag{4.9}$$

Recall from earlier definitions that the shortfall probability after T years is given by:

$$Prob(\sum^{T} r_{s,t} < \sum^{T} r_{bm,t}) = Prob(\sum^{T}(r_{s,t} - r_{bm,t}) < 0) = Prob(\sum^{T} r_{e,t} < 0) \tag{4.10}$$

Now we will subtract out the mean and divide by the variance to obtain the standard normal variable and its associated converging distribution (which happens to be the standard normal).

$$Prob\left[\frac{\sum^T r_{e,t} - \mu_e T}{\sigma_e^2\sqrt{T}} < \frac{0 - \mu_e T}{\sigma_e^2\sqrt{T}}\right] = Prob\left[Z < -\frac{\mu_e\sqrt{T}}{\sigma_e^2}\right] \tag{4.11}$$

$$= Prob(Z < -GSR\sqrt{T}) = \Phi(-GSR\sqrt{T}) \tag{4.12}$$

Note that here GSR is known as the **generalized sharpe ratio** and its expression is evaluated as:

$$GSR = \frac{\mu_e}{\sigma_e} = \frac{E(r_{s,t} - r_{bm,t})}{SD(r_{s,t} - r_{bm,t})} \tag{4.13}$$

This generalized Sharpe ratio essentially resolves to the commonly observed Sharpe ratio if we take the benchmark asset to be the risk free asset. Essentially when $r_{bm} = r_f$ then $GSR = SR$.

4.1.3 Computing with an example

Consider the following parameters given to us:

- $\mu_s = 10\%$.
- $r_f = 5\% = \mu_{bm}$.
- $\sigma_s = 20\%$.
- $\sigma_{bm} = 10\%$.
- $\rho = 0.5$.

Now the excess return mean and variance would be given by:

$$\mu_e = \mu_s - \mu_{bm} = 0.05 \tag{4.14}$$

$$\sigma_e^2 = \sigma_s^2 + \sigma_{bm}^2 - 2\rho\sigma_s^2\sigma_{bm}^2 = 0.2^2 + 0.1^2 - 2(0.5)(0.2)(0.1) = 0.03 \tag{4.15}$$

Therefore the standard deviation of the excess returns is:

$$\sigma_e = \sqrt{0.03} = 0.1732 \tag{4.16}$$

Finally we compute the GSR as:

$$GSR = \frac{\mu_e}{\sigma_e} = \frac{0.05}{0.1732} = 0.2887 \tag{4.17}$$

We can also compute the Sharpe ratio (SR) by plugging in corresponding values of the risk free asset. Note that it has a variance and covariance as 0.

$$SR = \frac{0.1 - 0.05}{0.2} = 0.25 \tag{4.18}$$

From these two computations we find that $GSR > SR$. From this fact we can write:

$$(GSR > SR) \rightarrow (-GSR < -SR) \rightarrow (-GSR\sqrt{T} < -SR\sqrt{T}) \tag{4.19}$$

$$\rightarrow \Phi(-GSR\sqrt{T}) < \Phi(-SR\sqrt{T}) \tag{4.20}$$

Hence with this we get the result that **shortfall probability is lower in the case of a stochastic benchmark as compared to the risk free benchmark**.

4.1.4 Generalizing the relationship

We know the expressions of GSR and SR are as follows:

$$GSR = \frac{\mu_e}{\sigma_e} \tag{4.21}$$

$$SR = \frac{\mu_s - r_f}{\sigma_s} \tag{4.22}$$

Now the condition for GSR to be generally greater than SR is given by:

$$if\ r_f = r_{bm} \rightarrow GSR > SR \iff \sigma_e^2 < \sigma_s^2 \tag{4.23}$$

What the above statement essentially means is that if we replace r_f with r_{bm} in the SR expression, then the only condition in which GSR would be greater than SR is that the variance of excess returns must be lesser than the variance of the stock returns. This condition can be further expanded as:

$$\sigma_e^2 = \sigma_s^2 + \sigma_{bm}^2 - 2\rho\sigma_s\sigma_{bm} < \sigma_s^2 \tag{4.24}$$

$$\rightarrow -2\rho\sigma_s\sigma_{bm} + \sigma_{bm}^2 < 0 \tag{4.25}$$

$$\rightarrow \sigma_{bm}^2 < 2\rho\sigma_s\sigma_{bm} \tag{4.26}$$

$$\rightarrow \rho > \frac{1}{2}\left(\frac{\sigma_{bm}}{\sigma_s}\right) \tag{4.27}$$

We can say that in a general case, this is the condition under which the shortfall probability with respect to the benchmark would be lower than the risk free asset shortfall probability.

Chapter 5

Funding ratio shortfall

5.1 Funding ratio shortfall

On a fundamental level, funding ratio can be thought of as a debt equity ratio of sorts. We can consider the **assets** as the investment portfolio or the level of equity holding. Also consider the **liability** to be the committed payment at the benchmark rate (could be bonds). A few notations at this point would be useful:

- $r_{s,t}$: Log returns from the asset portfolio. This can be considered as the inflows. This is essentially treated as an asset.

- $r_{b,t}$: Log returns of the benchmark bond. These are treated as outflows and liabilities.

- Stock bond ratio can be defined as follows. Note that S represents the value of assets and B represents the value of liabilities. Further note that this measure can be greater or lesser than 1.

$$V_T = \frac{\$\ value\ of\ S_T}{\$\ value\ of\ B_T} \tag{5.1}$$

- $T =$ some future date. $t = 0$ is the present.

- We assume that in the initial state $B_0 = S_0$ and hence $V_0 = 1$.

- Lastly, assume that we have some sense of an ideal funding ratio that we might desire. Represent it as θ where $0 < \theta \leq 1$.

- We let the **Multinomial log returns** of our stock holding be represented as:

$$r_s(0, T) = r_{s,1} + r_{s,2} + \cdots + r_{s,T} = \log(S_T/S_0) \tag{5.2}$$

Now that we have something akin to a **benchmark** in the form of a **desired funding ratio** we are now on fertile ground to start computing the shortfall probability, with the shortfall here representing an actual realisation of a funding ratio being lesser than our desired measure for the same. The FRS probability would be given by:

$$FRSP = Prob(V_T \leq \theta) \tag{5.3}$$

We know from basic corporate finance theory that the innate valuation of an asset or a bond essentially comes from its future value over continuously compounded returns. So we can now write expressions for the value of our asset holding at the end of T time periods and the value of our bond liabilities at the end of T as:

$$S_T = S_0 \exp\{\sum^T r_{s,t}\} \tag{5.4}$$

$$B_T = B_0 \exp\{\sum^T r_{b,t}\} \tag{5.5}$$

Now with this we can express our funding ratio in terms of the above valuations:

$$V_T = \frac{S_T}{B_T} = \frac{S_0 \exp\{\sum^T r_{s,t}\}}{B_0 \exp\{\sum^T r_{b,t}\}} \tag{5.6}$$

$$= V_0 \exp\{\sum^T r_{s,t} - \sum^T r_{b,t}\} = V_0 \exp\{\sum^T r_{e,t}\} \tag{5.7}$$

Where, as before our standard expression for excess returns is $r_{e,t} = r_{s,t} - r_{b,t}$. Also recall that since our V_0 was assumed to be 1, the above expression resolves to:

$$V_T = \exp\{\sum^T r_{e,t}\} \tag{5.8}$$

With this we can now rewrite the condition for the **funding ratio shortfall probability** as follows:

$$FRSP = Prob(V_T \leq \theta) = Prob(\exp\{\sum^T r_{e,t} < \theta\}) \tag{5.9}$$

Now using the log-exponent conversion principle, we can essentially take log on both sides of the inequality in the last expression and obtain:

$$FRSP = Prob(r_{e,t} \leq \log\theta) \tag{5.10}$$

5.1.1 Forming the FRS distribution

Recall from earlier discussions about the distribution of excess log returns. We can reiterate the following key characteristics about this random variable.

- $r_{e,t} \sim iidN(\mu_e, \sigma_e^2)$
- $\sum^T r_{e,t} \sim N(\mu_e T, \sigma_e^2 T)$
- $\dfrac{\sum^T r_{e,t} - \mu_e T}{\sigma_e \sqrt{T}} \sim N(0, 1)$

Simply substituting these expressions in the $FRSP$ expression we had derived in equation 37 we would get:

$$FRSP = Prob\left[\frac{\sum^{T} r_{e,t} - \mu_e T}{\sigma_e \sqrt{T}} \leq \frac{\log\theta - \mu_e T}{\sigma_e \sqrt{T}}\right] \tag{5.11}$$

$$= Prob\left[Z \leq -\frac{\mu_e T - \log\theta}{\sigma_e \sqrt{T}}\right] = \Phi\left[-\frac{\mu_e T - \log\theta}{\sigma_e \sqrt{T}}\right] = \Phi(-f(T)) \tag{5.12}$$

The last expression gives us a CDF wise measure for the $FRSP$, where we define $f(T)$ as follows:

$$f(T) = \frac{\mu_e T - \log\theta}{\sigma_e \sqrt{T}} \tag{5.13}$$

Now note that if our desired funding ratio shortfall is the same as the one assumed initially, that is if $\theta = 1$ then we would have $\log\theta = \log 1 = 0$. As a result of this the $FRSP$ measure can be given as:

$$FRSP = \Phi\left(-\frac{\mu_e \sqrt{T}}{\sigma_e}\right) = \Phi(-GSR\sqrt{T}) \tag{5.14}$$

Now for a given $0 < \theta \leq 1$ we will see what happens if $T \to 0$. We would essentially have:

$$f(T) = \frac{\mu_e T - \log\theta}{\sigma_e T} = \frac{\mu_e \sqrt{T}}{\sigma_e} - \frac{\log\theta}{\sigma_e \sqrt{T}} \tag{5.15}$$

$$\to 0 - (-\infty) = +\infty \tag{5.16}$$

Note that this evaluation happens since $\log\theta < 0$ (remember that log takes on negative values with arguments betwee 0 and 1). With this the associated funding shortfall probability would be given by:

$$\Phi(-f(T)) \to \Phi(-\infty) \to 0 \tag{5.17}$$

Hence we can say that $FRSP \to 0$ as $T \to 0$. In another case we can compute the limiting behaviour of $FRSP$ as $T \to \infty$. We have:

$$f(T) = \underbrace{\frac{\mu_e \sqrt{T}}{\sigma_e}}_{\to\infty} - \underbrace{\frac{\log\theta}{\sigma_e \sqrt{T}}}_{\to 0} \to \infty \tag{5.18}$$

With this we can clearly see that the $FRSP \to 0$ as $T \to \infty$ due to the following condition:

$$\Phi(-f(T)) \to \Phi(-\infty) \to 0 \tag{5.19}$$

5.1.2 Optimal conditions

We are interested in finding, based on a measure of T, the optimal or maximum value of $FRSP$. Note that the condition that maximizes $FRSP$ is essentially the same condition that minimizes the function $f(T)$. We can write this as:

$$\max_T \Phi(-f(T)) \tag{5.20}$$

$$\min_T f(T) \tag{5.21}$$

Recall that the condition for finding the optimal solution to this would essentially be to take the first order derivative with respect to T and equate it to 0. We do this as follows:

$$f'(T) = \left(\frac{\mu_e}{\sigma_e}\right)\frac{1}{2}T^{-1/2} - \left(\frac{\log\theta}{\sigma_e}\right)\left(-\frac{1}{2}T^{-3/2}\right) = 0 \tag{5.22}$$

We can factor out the common $(1/2)$ from the above expression to get:

$$\left(\frac{\mu_e}{\sigma_e}\right)T^{-1/2} - \left(\frac{\log\theta}{\sigma_e}\right)T^{-3/2} = 0 \tag{5.23}$$

$$\rightarrow \frac{\mu_e}{\sigma_e} = -\frac{\log\theta}{\sigma_e}\cdot\frac{1}{T} \tag{5.24}$$

$$T^* = -\frac{\log\theta}{\sigma_e}\cdot\frac{\sigma_e}{\mu_e} = -\frac{\log\theta}{\mu_e} > 0 \tag{5.25}$$

Note again that the above expression resolves to 0 due to the fact that $\log\theta$ values between 0 and 1 are negative, so the overall expression would assume a positive value. Finally we can write the optimum value of T as:

$$T^* = -\frac{\log\theta}{\mu_e} \tag{5.26}$$

5.2 Percentiles

Now we will have a look at the Funding ratio shortfall probability from the perspective of **percentiles**. Recall that a percentile is the value in a distribution below which a certain percentage or proportion of values fall. For example the 50^{th} percentile of a distribution would correspond to the particular realisation below which 50% of the observed realisations lie. Now we will try to determine the shortfall probability that our actual realisation of funding ratio would lie below a certain percentile value of the funding ratio distribution. For example, we would like to answer the question - **what is the probability that our funding ratio below the 5^{th} percentile of the funding ratio distribution ?** Now let us we denote the 5^{th} percentile of V_T as $C(\theta)$ or simply C. Further we note that before we go on about finding this probability we need to find the percentile itself. Let us denote the following probability now:

$$Prob\{V_T \leq q^{th}\ percentile\ of\ V_T\} = q \tag{5.27}$$

Taking log on both sides of the expression within the probability operator we get:

$$Prob\{\log V_T \leq \log(q^{th}\ percentile\ of\ V_T)\} = q \tag{5.28}$$

Recall from equation 10 we evaluated the log of V_T as follows:

$$\log V_T = \sum^{T} r_{e,t} \sim N(\mu_e T, \sigma_e^2 T) \tag{5.29}$$

Now let us, as usual, transform the normal probability function of the funding ratio shortfall as a standard normal variable.

$$Prob\left[\frac{\log V_T - \mu_e T}{\sigma_e\sqrt{T}} \leq \frac{\log(q^{th}\ percentile\ of\ V_T) - \mu_e T}{\sigma_e\sqrt{T}}\right] = q \tag{5.30}$$

Since with this we essentially have a standard normal variable, we can rewrite the above expression in much simpler terms as:

$$Prob\{Z \leq q^{th}\ percentile\ of\ Z\} = q \tag{5.31}$$

Now from equations 30 and 31 we can effectively get:

$$\frac{\log(q^{th}\ percentile\ of\ V_T) - \mu_e T}{\sigma_e T} = q^{th}\ percentile\ of\ Z \tag{5.32}$$

Note that in the earlier definition we stressed upon the fact that the percentile is a **value of a realisation/observation** rather than a percentage (which is often confused). With this knowledge, if we take the inverse of the probality function, naturally we would get the percentile value. This could be denoted as $N^{-1}(q)$ or as $\Phi^{-1}(q)$. Now since the standard normal distribution is quite well known to us, we can easily calculate this percentile value as:

$$N(-1.645) = 0.05 \tag{5.33}$$

$$5^{th}\ percentile\ of\ Z = -1.645 \tag{5.34}$$

A subtle point to note that is that the percentile would assume a negative value for q falling between 0 and 0.5. Note carefuly that we can actually rewrite the equation 32 as follows:

$$\log(q^{th}\ percentile\ of\ V_T) = \mu_e T + (q^{th}\ percentile\ of\ Z)\sigma_e T \tag{5.35}$$

We can now take the exponent on both sides to get of the log operator and hence compute the requisite percentile of V_T. Also note that we can express the 'percentile of Z' in the form of the inverse distribution function.

$$q^{th}\ percentile\ of\ V_T = \exp\{\underbrace{\mu_e T + (\sigma_e T \times N^{-1}(q))}_{g(T)}\} = \exp\{g(T)\} \tag{5.36}$$

Now we will attempt to minimize $g(T)$ so as to obtain a percentile value for which the shortfall probability is maximum. To do this, we naturally take the first order condition and set that equal to 0 and then move on to compute the optimum T that gives us the desired percentile.

$$g'(T) = \mu_e + \sigma_e \frac{1}{2} T^{-1/2} N^{-1}(q) = 0 \tag{5.37}$$

$$T^{-1/2} = -\frac{\mu_e}{\sigma_e} \frac{2}{N^{-1}(q)} \tag{5.38}$$

$$T^* = \left[-\frac{N^{-1}(q)}{2} \frac{\sigma_e}{\mu_e} \right]^2 \tag{5.39}$$

The above expression is the value of T for which we can maximize the shortfall probability for a given percentile. This is like solving for a local maxima of sorts, wherein we might be interested in certain percentile ranges as opposed to finding a global maxima for the shortfall probability.