Quantitative Methods

Concepts of Contour Plot and Partial Derivative

2020DMB02

Chapter 1

Contour Plots and Partial Derivatives

Topics discussed are:

1. Functions of one variable

2. Functions of two variable

    - Visualization
    - Contour Plot

3. Partial Derivative

    - Approximation Formula
    - Geometrical Interpretation
    - Applications of Partial Derivatives

1.1 Functions of one variable

Function of one variable only depend on one parameters for example x.

1. Definition

2. Examples with Graph

1.1.1 Definition

A function tries to give a relationship a mathematical form. An equation is a mathematical way of looking at the relationship between concepts or items. These concepts or items are represented by what are called variables.
A function which only depend one variable is called one variable function.

1.1.2 Examples with Graph

Example 1:

$\boldsymbol{f(x) = \sin x}$

In this example we can see that the function $f(x) = \sin x$ only depend on one variable i.e. x.

Graph:

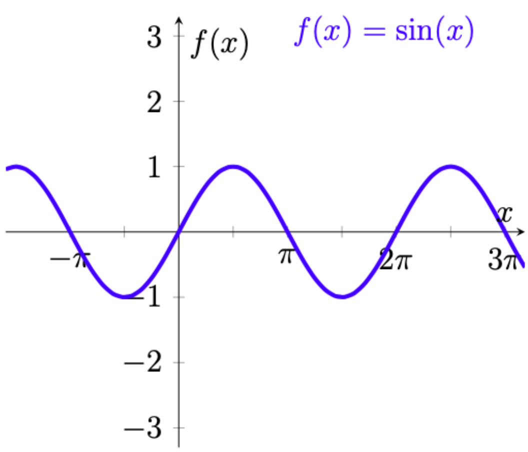

Figure 1.1: $\sin x$

Example 2:

$\boldsymbol{f(x) = x^2 - 2x - 1}$

Same as example 1 this function also depends on only one variable i.e. x.

That means if we change the value of x the value of function also changes. As $f(x)$ is a function of x.

Graph:

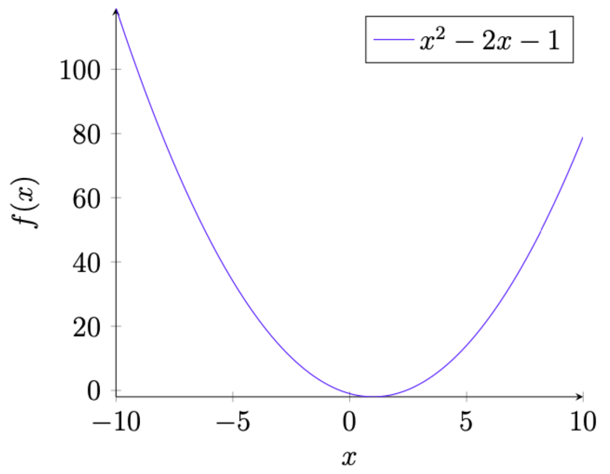

Figure 1.2: $x^2 - 2x - 1$

1.2 Functions of two variables

Function of two variables only depend on two parameters for example x and y.

1. Definition

2. Visualization

3. Contour plots

4. Examples with Graph

1.2.1 Definition

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

A function of two variables $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the figure 1.3.

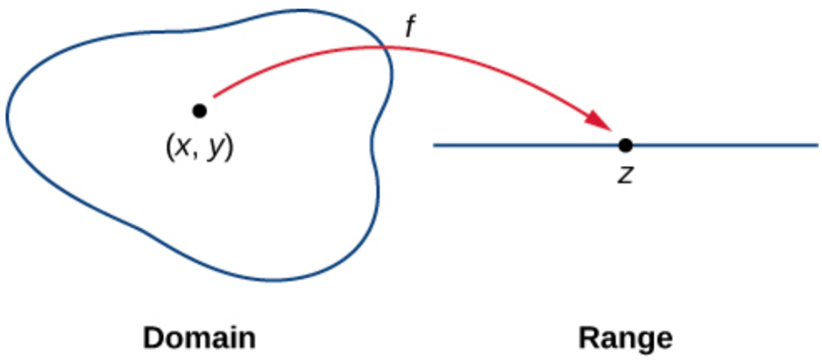

Figure 1.3: Mapping of a function

The domain of a function of two variables consists of ordered pairs (x,y).

Given:$(x, y) \rightarrow$ get number $f(x, y)$

$\rightarrow f(x, y) = (x^2 + y^2)$, **where** $(x, y) \in \mathbb{R}^2 \Leftrightarrow \mathbb{R}$ x $\mathbb{R}$

$\rightarrow f(x, y) = \sqrt{y}$ only defined if $y \geqslant 0$

$\rightarrow f(x, y) = \frac{1}{(x+y)}$ only defined when $(x + y) \neq 0$

1.2.2 Examples

Example 1:

$\boldsymbol{g(x, y) = \sqrt{9 - x^2 - y^2}}$

Solution:

For the function $g(x, y)$ to have a real value, the quantity under the square root must be non-negative:

$$9 - x^2 - y^2 \geqslant 0 \tag{1.1}$$

This inequality can be written in the form

$$x^2 + y^2 \leqslant 9 \tag{1.2}$$

Therefore, the domain of $g(x, y)$ is $\{(x, y) \in \mathbb{R}^2 \text{ and } x^2 + y^2 \leqslant 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the figure 1.4.

Graph:

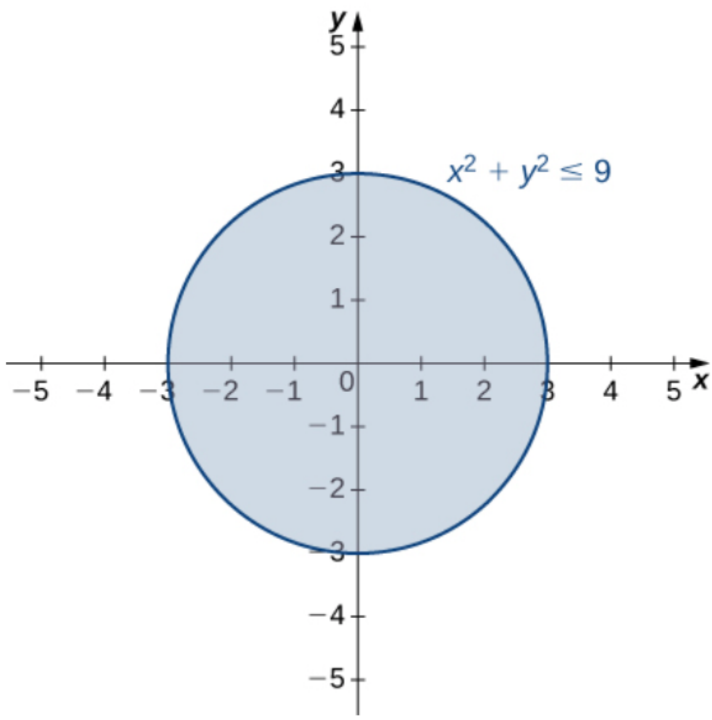

Figure 1.4: Circle with a radius 3 or less.

The domain of the function $g(x, y) = \sqrt{9 - x^2 + y^2}$ is a closed disk of radius 3.

To determine the range of $g(x, y) = \sqrt{9 - x^2 - y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2 + y^2 = 9$. It follows that $x_0^2 + y_0^2 = 9$ and

$$g(x_0, y_0) = \sqrt{9 - X_0^2 + y_0^2} = \sqrt{9 - (x_0^2 + y_0^2)} = \sqrt{9 - 9} = 0 \tag{1.3}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x, y) = c$:

$$\sqrt{9 - x^2 - y^2} = c \tag{1.4}$$

$$9 - x^2 - y^2 = c^2 \tag{1.5}$$

$$x^2 + y^2 = 9 - c^2 \tag{1.6}$$

Since $9 - c^2 > 0$, this describes a circle of radius $\sqrt{9 - c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x, y) = c$. Therefore, the range of this function can be written in interval notation as $[0, 3]$.

Example 2:

$\boldsymbol{f(x, y) = x^2 + y^2}$

Solution:

This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0$).When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. Can be recalled from vectors in space that the name of the graph of $f(x, y) = x^2 + y^2$ is a paraboloid. As shown in figure 1.5.

Graph:

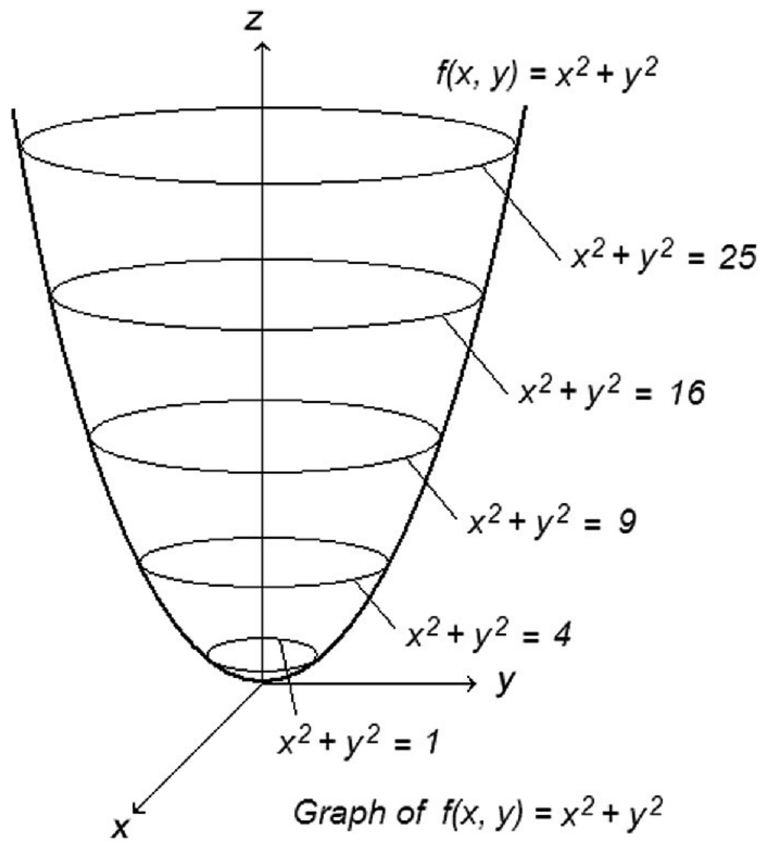

Figure 1.5: Paraboloid

A paraboloid is the graph of the given function of two variables.

Example 3:

$f(x,y) \rightarrow$ temperature at point (x,y)

Note: Weather could be defined as 4 parameters (x, y, z, t). Where, x, y, z are co-ordinates, we have considered z because temperature changes according to height and we have also considered fourth variable t which is a function of time.

As we see in this function we have four variable so by convention we can not plot this on the graph. So, for simplicity we will focus mostly on two or three variables only. For more than 3 variables it's not possible to graph with the given technology.

1.2.3 Visualization

How do we visualize $f \to$ is a function of 2 variables?

Example 1:

This example will show us how to plot a point in three dimensional.

Refer figure1.6.

$\boldsymbol{z = f(x, y)}$

Graph:

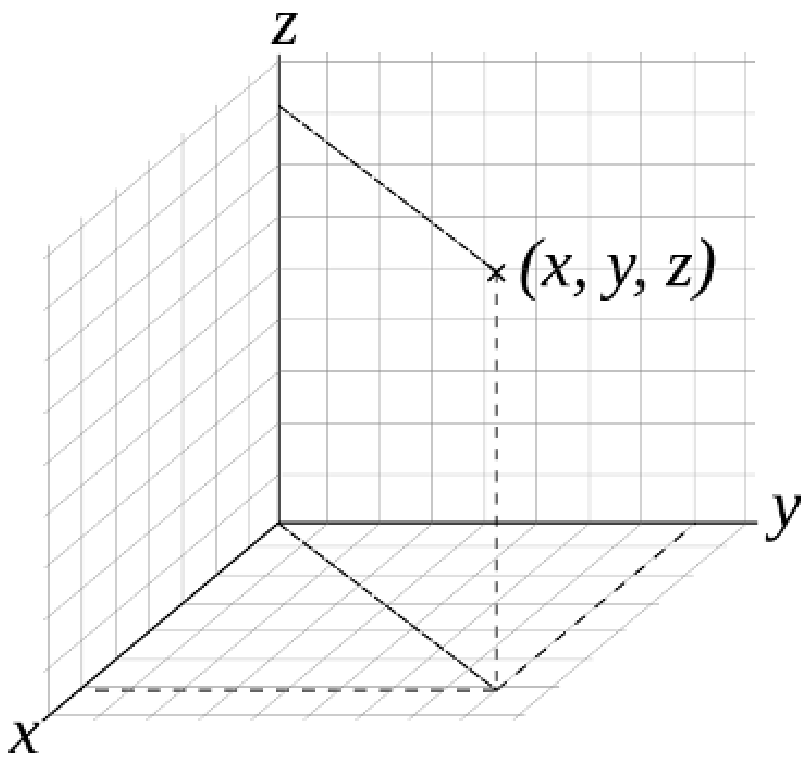

Figure 1.6: Surface $\big(x, y, f(x, y)\big)$

Example 2:

$\boldsymbol{f(x, y) = -y}$

In this example $x = 0$ so xy and xz plane will empty and the plane will be on yz plane. As shown in figure 1.7.

Graph:

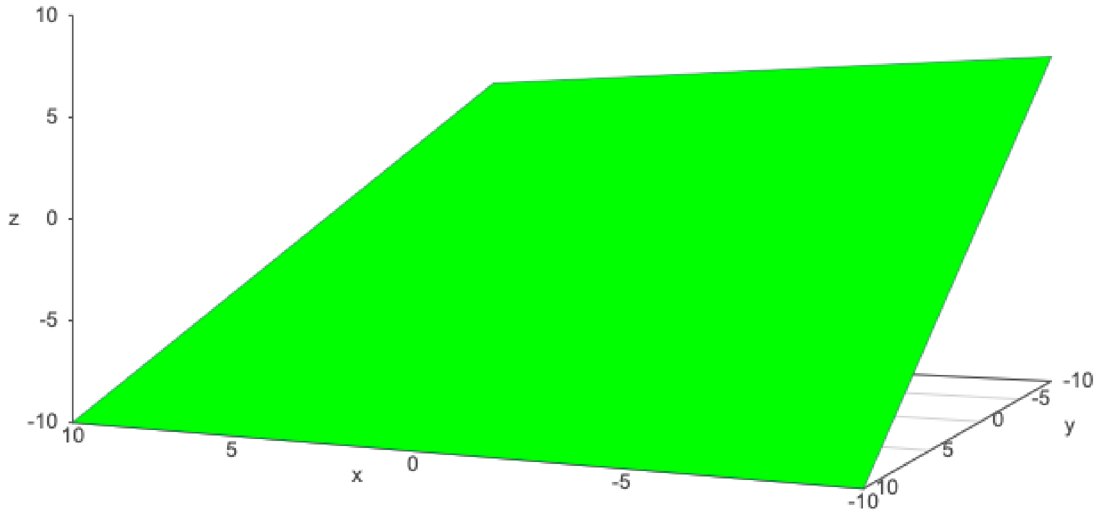

Figure 1.7: $\boldsymbol{f(x,y) = -y}$

Example 3:

$\boldsymbol{f(x,y) = 1 - x^2 - y^2}$

Graph for:

$z = 1 - x^2 - y^2$

$\rightarrow$ in yz plane: $x = 0 \Rightarrow z = 1 - y^2$

$\rightarrow$ in xz plane: $y = 0 \Rightarrow z = 1 - x^2$

$\rightarrow$ in xy plane: $z = 0 \Rightarrow 1 = x^2 + y^2$ (Unit Circle)

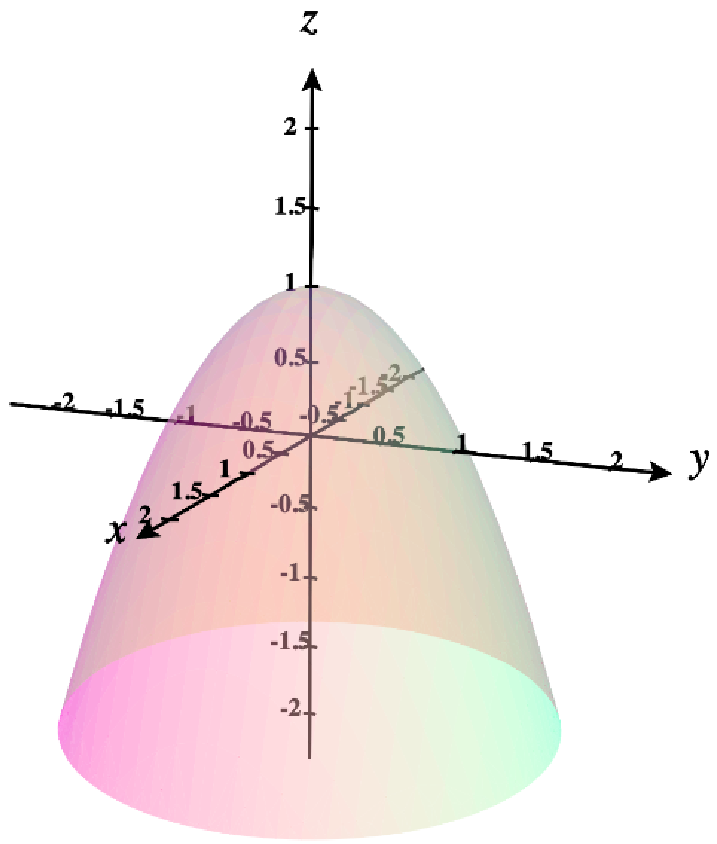

Figure 1.8: $\boldsymbol{f(x, y) = 1 - x^2 - y^2}$

1.2.4 Contour Plot

Contour Plots are a function of two variables.

These are sometimes called Level Plots and are a way to show a three-dimensional surface on a two-dimensional plane. It graphs two predictor variables X Y on the y-axis and a response variable Z as contours. These contours are sometimes called z-slices or iso-response values.

It shows all the points where $f(x, y)$ equals some fixed constants, its chosen at regular intervals.

We slice the graph by horizontal plane. For example, $z = c$.

Example 1:

Below given contour plot (figure 1.9) can be imagined as a mountain, $f = 40$ could be a peak. If we change the level curve that means we change the altitude.

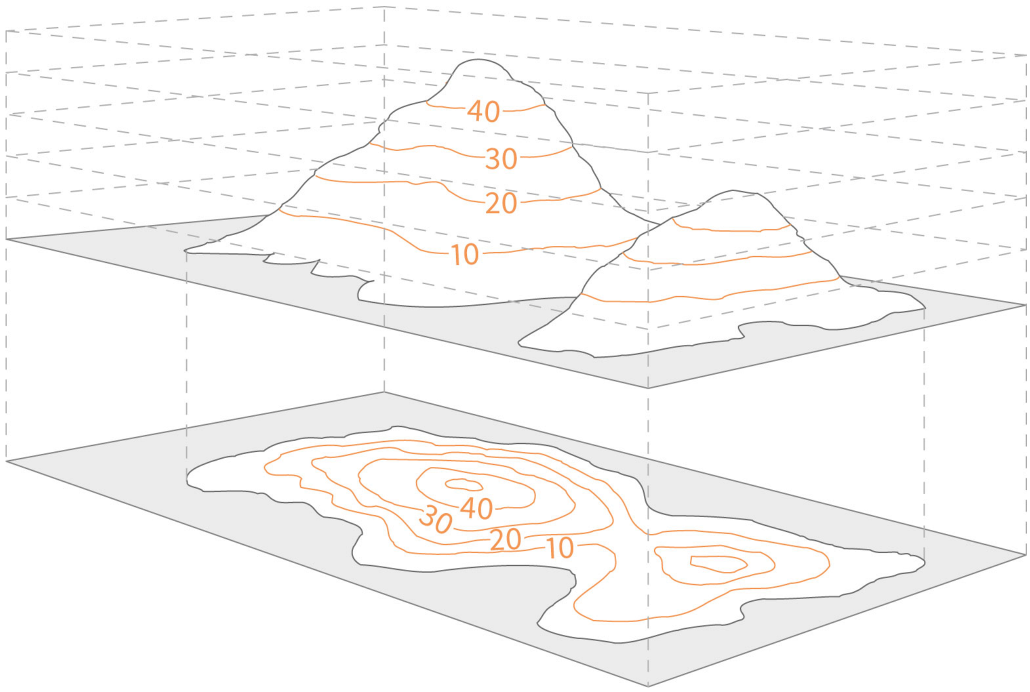

Figure 1.9: Contour Plot

It shows all the points where $f(x, y)$ equals some fixed constant. It is chosen at regular interval.

For the level curves, we slice the graph by horizontal plane, $\boldsymbol{z = c}$. If we change the level curve the value of $\boldsymbol{c}$ will also be changed.

Example 2:

From the below given graph (figure 1.10) we can clearly see how we get the level curve. If we repeat this process at different height we will get different level curves corresponding to its z value.

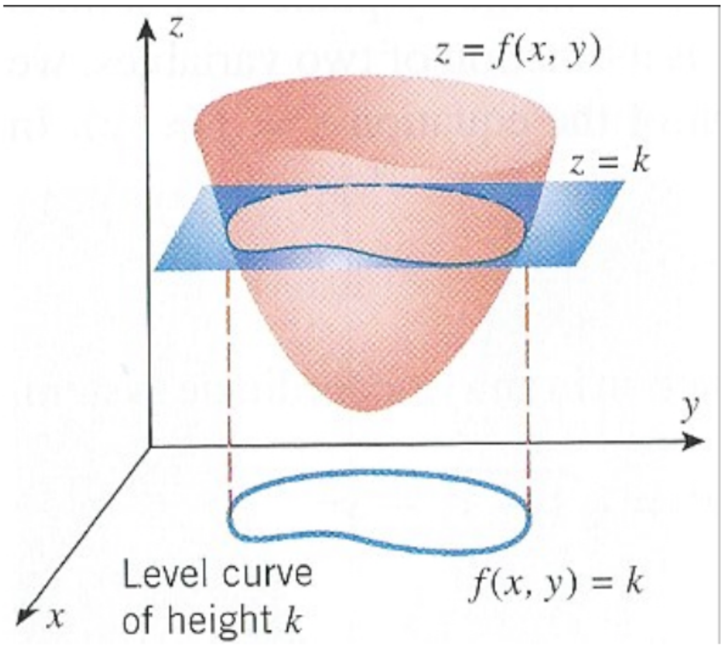

Figure 1.10: $\boldsymbol{f(x, y) = z = k}$ **in level curve** $\boldsymbol{f}$**.**

Example 3:

Contour plot of $\boldsymbol{f(x, y)}$**.** $\boldsymbol{Z = 0,\ 4,\ 8,\ 12,\ 16}$ **are the lines of constant height at** $\boldsymbol{z = 0,\ z = 4}$ **and so on in the figure 1.11**.

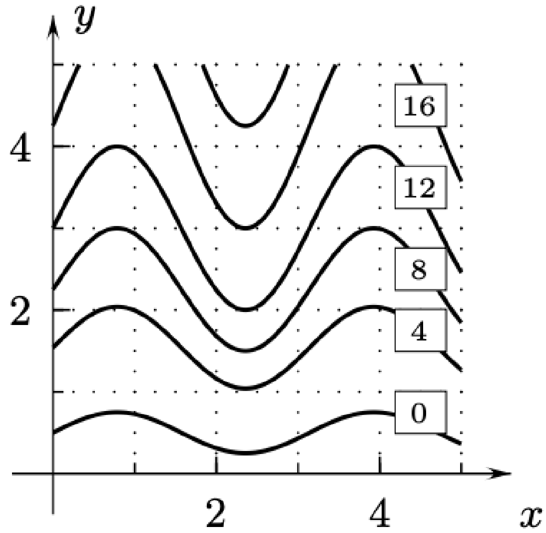

Figure 1.11: Contour lines

Example 4:

$\boldsymbol{f(x, y) = -y}$

Below given graph (figure1.12) is the contour plot of the figure 1.7 in the **section 1.2.3 example 2.**

Graph for:

$f = 1$

$\Rightarrow -y = 1$

$\Rightarrow y = -1$

Similarly, we can draw for other values of f also.

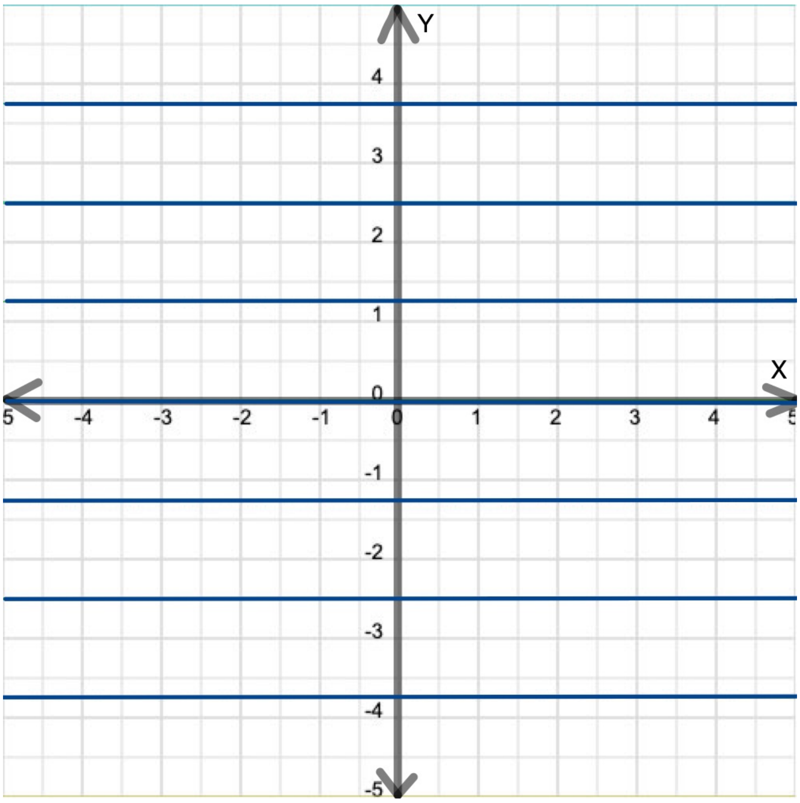

Figure 1.12: Contour Plot of, $z = -y$

Example 5:

$\boldsymbol{f(x, y) = 1 - x^2 - y^2}$

Graph for:

When,$f = 0$

$\Rightarrow 1 - (x^2 + y^2) = 0$

$\Rightarrow (x^2 + y^2) = 1$

When,$f = 1$

$\Rightarrow 1 - (x^2 + y^2) = 1$

$\Rightarrow (x^2 + y^2) = 0$

When,$f = -1$

$\Rightarrow 1 - (x^2 + y^2) = -1$

$\Rightarrow (x^2 + y^2) = 2$

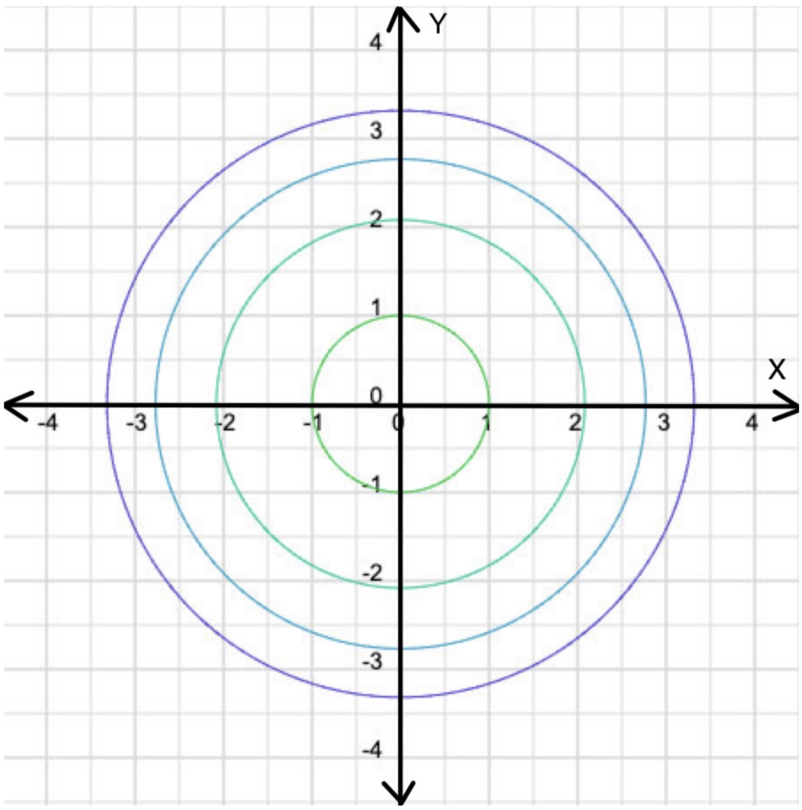

Figure 1.13: Contour Plot for $f(x, y) = 1 - x^2 - y^2$

$\rightarrow$ From the above graph (figure 1.13) we can see that level curves are getting closer to each other, it means the slope is getting steeper and steeper.

$\rightarrow$ It means if I need to change the level I need to travel shorter distance.

$\rightarrow$ The curve starts as a flat surface and gets steeper as we travel along.

Example 6:

The below graph (figure 1.14) is the contour plot of the figure mentioned in section 1.2.2, example 2.

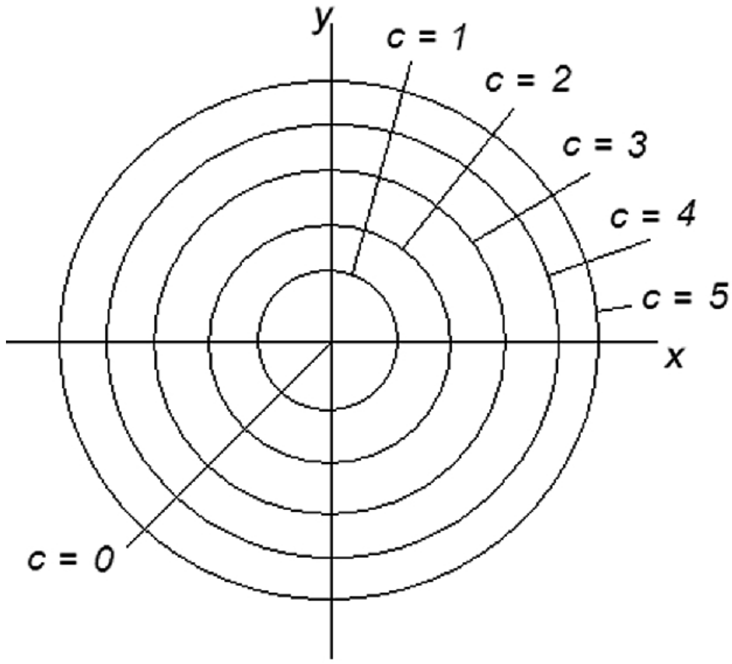

Figure 1.14: Contour Plot: $f(x, y) = x^2 + y^2$

The graph shows different values of z in the xy plane, where $z = c$ (constant).

Example 7:

In this example we will see that the **contour plot** can also tell us if the function increases or decreases, when move in any direction.

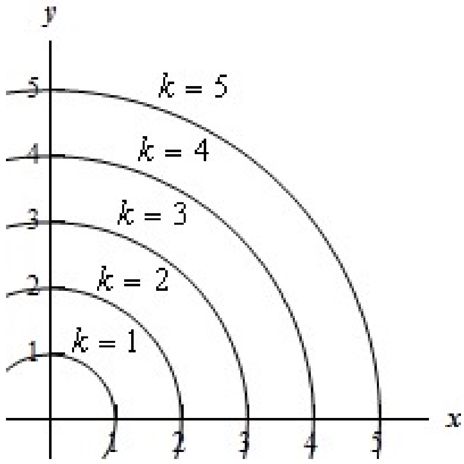

Figure 1.15: Contour Plot

In the above graph (figure 1.15) assume (x_0, y_0) is a starting point on $k = 2$. The question is I want to know if I change x or y, how does f changes?
From the above graph we can that if we increase $(\uparrow)x$ and keep y constant the $f(x,y)(\uparrow)$ increases. Similarly if decrease $(\downarrow)x$ and keep y constant the $f(x,y)(\downarrow)$ decreases.
We can also conclude the same thing for y if we increase $(\uparrow)y$ and keep x constant the $f(x,y)(\uparrow)$ increases. Similarly if decrease $(\downarrow)y$ and keep x constant the $f(x,y)(\downarrow)$ decreases.
Note: To be more precise of the change i.e. to see the rate of the change we need to use derivatives.

1.3 Partial Derivatives

A partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant.

1. Definition

2. Approximation Formula

3. Geometrical Interpretation

4. Application of Partial Derivatives (Optimization Problem)

5. Few more examples of Partial Derivative (Computation)

1.3.1 Definition

From derivatives of functions of one variable, we already know that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Let $f(x, y)$ be a function of two variables.

Then the **partial derivative** of f with respect to x is written as $\boldsymbol{\partial f/\partial x}$, or $\boldsymbol{f_x}$, and is defined as:

$$\boldsymbol{\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}}$$

The **partial derivative** of f with respect to y, written as $\partial f/\partial y$, or f_y, and is defined as:

$$\boldsymbol{\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether.

For Example:

A function for a surface that depends on two variables x and y.

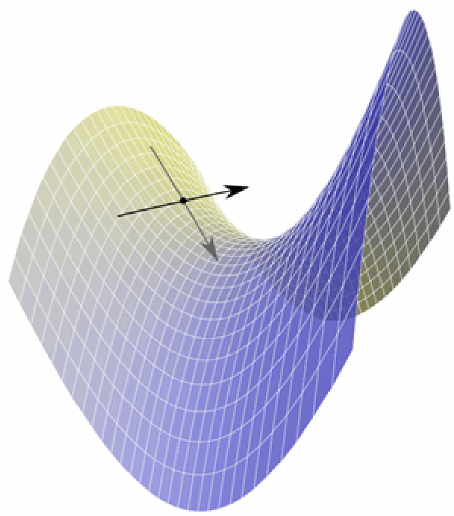

Figure 1.16: $f(x, y) = x^2 - y^2$

When we find the slope in the x direction (while keeping y fixed) we have found a partial derivative.

Or we can find the slope in the y direction (while keeping x fixed).

Here is a function of one variable (x):

$$\boldsymbol{f(x) = x^2} \tag{1.7}$$

And its derivative (using the Power Rule):

$$\boldsymbol{f'(x) = 2x} \tag{1.8}$$

But what about a function of two variables (x and y):

$$\boldsymbol{f(x, y) = x^2 + y^3} \tag{1.9}$$

To find its partial derivative with respect to x we treat y as a constant (imagine y is a number like 7 or something):

$$\boldsymbol{f'x = 2x + 0 = 2x} \tag{1.10}$$

Explanation:

→the derivative of x^2 (with respect to x) is $2x$

→we treat y as a constant, so y^3 is also a constant (imagine $y = 7$, then $7^3 = 343$ is also a constant), and the derivative of a constant is 0

To find the partial derivative with respect to y, we treat x as a constant:

$$\boldsymbol{f'y = 0 + 3y^2 = 3y^2} \tag{1.11}$$

Explanation:

$\rightarrow$we now treat x as a constant, so x^2 is also a constant, and the derivative of a constant is 0

$\rightarrow$the derivative of y^3 (with respect to y) is $3y^2$

That is all there is to it. Just remember to treat **all other variables as if they are constants.**

Function of one variable:

Differentiating $f(x)$ with respect to x.

$\Rightarrow f'(x) = \frac{df}{dx}$

$\Rightarrow \frac{df}{dx} = \frac{\triangle f}{\triangle x}$

Formula is given by:

$$\boxed{\boldsymbol{= \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}}}$$

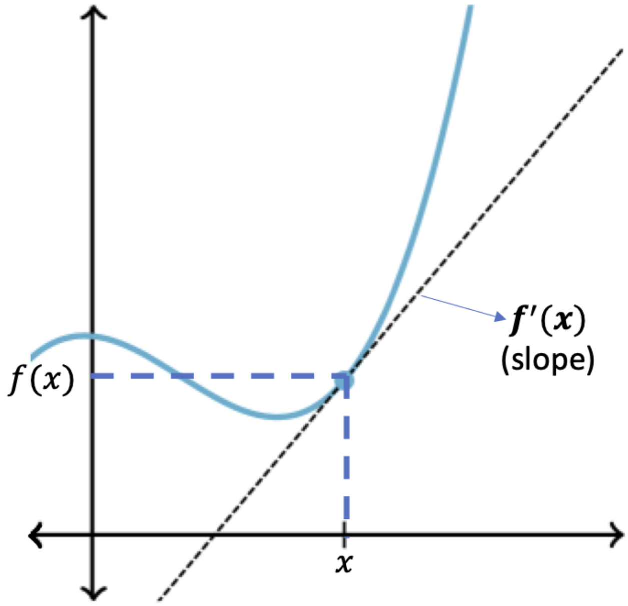

Figure 1.17: Finding slope

Note: Not every function has derivative, some functions are not regular enough to have derivatives.

1.3.2 Approximation Formula

$x_0 \rightsquigarrow f(x_0)$

If x_0 is known then find $f(x)$ for x close to x_0.

$$\boldsymbol{f(x) \approx f(x_0) + f^{'}(x_0)(x - x_0)}$$

But now how do we do the same for 2 variables $f(x, y)$**?** $\rightarrow$ We have different notation of derivative because the value of f changes differently when x is changed or when y is changed.

$\rightarrow$ Function of more than one variable does not have single derivative with respect to each variable.

$\rightarrow$ **Partial derivative** at point (x_0, y_0) with respect to x and y are given separately and are as follows:

$\rightarrow$ **Partial Derivative** w.r.t. x

$$\boldsymbol{\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x, y_0) - f(x_0, y_0)}{\triangle x}}$$

Here, we can see that y is not changing at all and it is constant at point y_0.

→ **Partial Derivative** w.r.t. y

$$\boldsymbol{\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\triangle y \to 0} \frac{f(x_0, y_0 + \triangle y) - f(x_0, y_0)}{\triangle y}}$$

Here, we can see that x is not changing at all and it is constant at point x_0.

$\boldsymbol{\partial}$→ "Partial" derivative symbol.

→ All the above are the definitions of partial derivative.

→ We say function is differentiable if theses criteria exist.

Summary:

Function of one variable, say $f(x)$	Function of two variables, say $f(x, y)$
$f'(x)$ or $\frac{df}{dx}$	$\frac{\partial f}{\partial x}$ or f_x or $f_x(x, y)$
$f'(a)$ or $(\frac{df}{dx})]_{x=a}$	$(\frac{\partial f}{\partial x})]_{x=a}$ or $f_x(a, y)$ $(\frac{\partial f}{\partial x})]_{x=a,y=b}$ or $f_x(a, b)$

1.3.3 Geometrical Interpretation

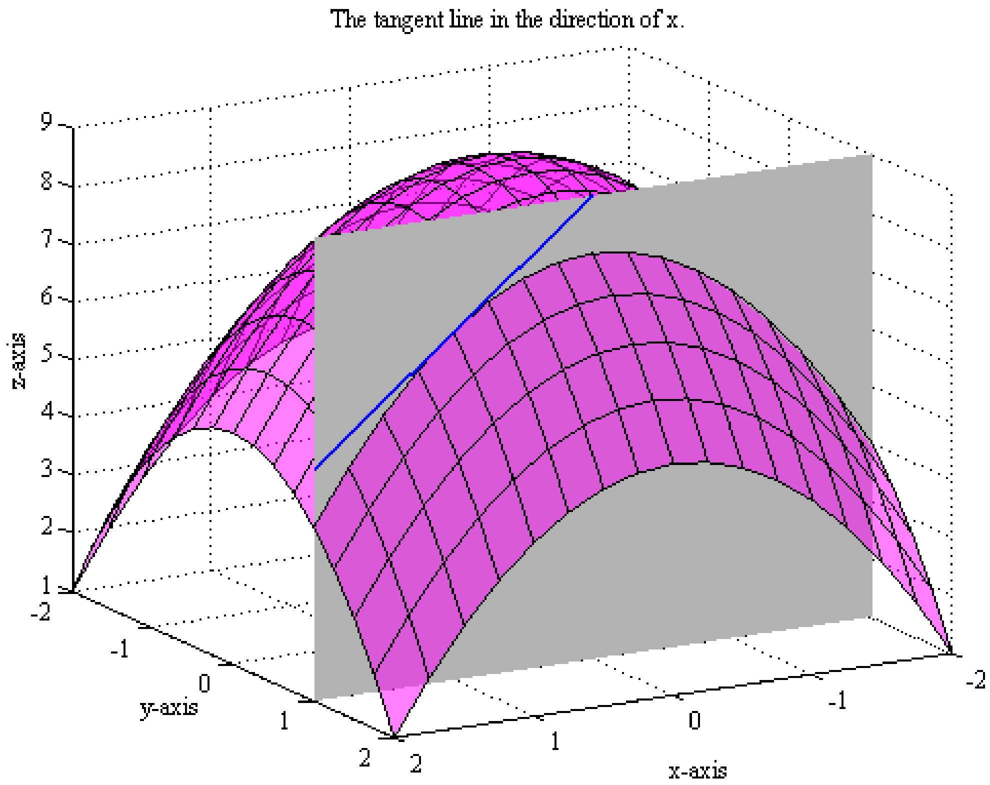

Figure 1.18: Finding partial of a function.

We want to know the rate of change, so for that we slice the graph (or slice of f) parallel to xz plane keeping y constant. Now we want to know how x changes keeping y constant (moving forward or backward parallel to $x - axis$, we want to know the rate of change if we move along the curve). So, our slope will be $\frac{\partial f}{\partial x}$. For this we fix a point (x_0, y_0) then ask ourselves, how the function changes if I change x keeping y constant.

For example:

Suppose the given function is $f(x, y) = x^2 + y^2$ and we want to know the rate of change with respect to x at x_0, y_0.

The graph and the solution are as follows:

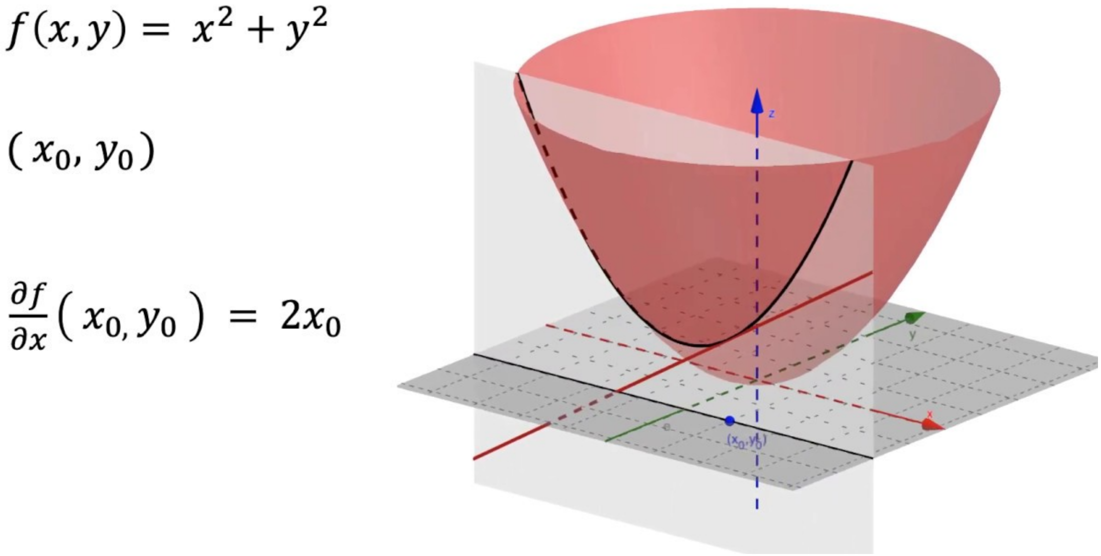

Figure 1.19: $f(x,y) = x^2 + y^2$

Interpreting Partials at a Specific Point.

Intuitive interpretation: We may interpret a partial derivative as a rate of change. For example suppose we have a function $f(x, y)$ and $f_x(2, 3) = 10$ and $f_y(2, 3) = 2$. This can then be interpreted as saying that at the point (2,3), a unit increase in x will result in an approximate increase of 10 units in the function, f , whereas a unit increase in y at the same point will result in an approximate increase of only 2units for the function, f.

Geometric Interpretation: As with the ordinary derivative, partial derivatives can be interpreted as slopes of tangent lines. The partial derivative of a function $f(x, y)$ with respect to x at the point where $(x, y) = (a, b)$ (i.e. at the point $(a, b, f(a, b))$ on the surface) for example, (see diagram) will be the slope of the tangent line to the curve obtained by holding y constant at b, at x = a. We are in effect intersecting the surface $z = f(x, y)$ with the plane y = b which is a curve, and then finding the slope of the usual tangent line at the point where x = a.

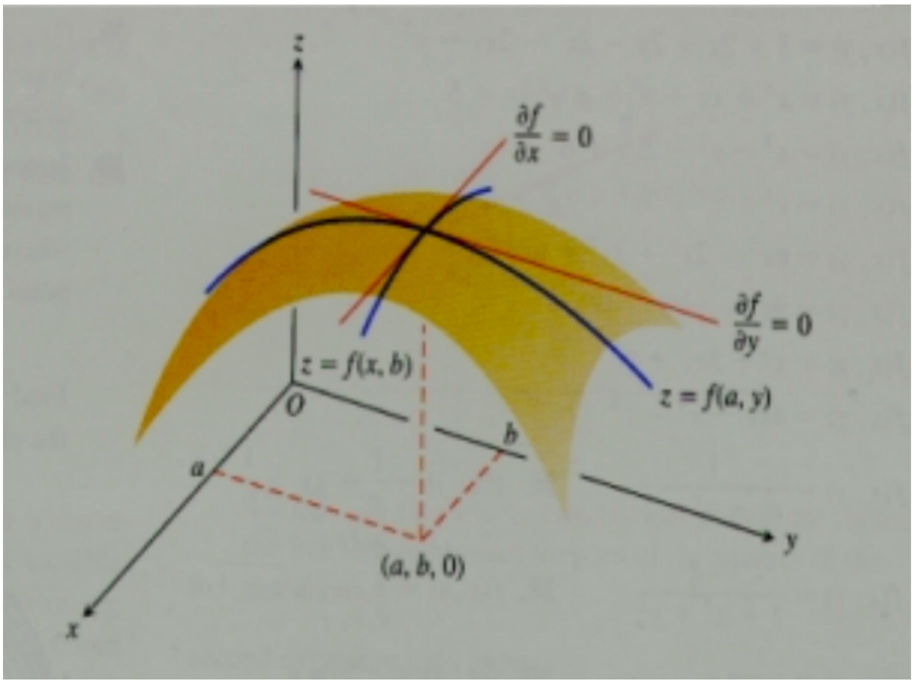

Figure 1.20: **The Geometric Interpretation of Partials**

Both of the above interpretations have their use, the intuitive is the easiest to see in applications whereas the geometric interpretation often provides the reasoning behind results.

Computation of Partial Derivatives

Example 1:

$\boldsymbol{f(x, y) = x^3 y + y^2}$

To find:

$\frac{\partial f}{\partial x} = f_x$?

$\rightarrow$ Treat y as constant and x as variable.

So we get,

$$\frac{\partial f}{\partial x} = 3x^2 y + 0 \tag{1.12}$$

$\frac{\partial f}{\partial y} = f_y$?

$\rightarrow$ Treat x as constant and y as variable.

So we get,

$$\frac{\partial f}{\partial y} = x^3 + 2y \tag{1.13}$$

Summary:

$\boldsymbol{f(x,y) \rightarrow \frac{\partial f}{\partial x}}$ is denoted by $\boldsymbol{f_x}$.

In this we keep y constant and x variable.

$\boldsymbol{f(x,y) \rightarrow \frac{\partial f}{\partial y}}$ is denoted by $\boldsymbol{f_y}$.

In this we keep x constant and y variable.

The above method is general and is also applicable when there is three or more variables.

Approximation Formula

We also have a approximation formula that tell us what happens if we vary both variables.

If we change $x \Rightarrow x \rightsquigarrow x + \triangle x$,

If we change $y \Rightarrow y \rightsquigarrow y + \triangle y$

When, $z = f(x,y)$ then it changes by the amount which is approximately:

$$\boldsymbol{\triangle z \approx f_x \triangle x + f_y \triangle y}$$

(Two effect adds up.)

Justification:

Tangent plane to the graph $f(x,y) = z$

We know that f_x, f_y are slopes of two tangent lines.

$\rightarrow$ If $\frac{\partial f}{\partial x}(x_0, y_0) = a$

Then, $l_1 = \begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$

$\rightarrow$ If $\frac{\partial f}{\partial y}(x_0, y_0) = b$

Then, $l_2 = \begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$

Both the lines l_1 and l_2 are tangent to the graph of $z = f(x, y)$. Together they determine a plane:

$$\boldsymbol{z = z_0 + a(x - x_0) + b(y - y_0)}$$

On tangent plane $\rightarrow \triangle z = f_x \triangle x + f_y \triangle y$

Near tangent plane $\rightarrow \triangle z \approx f_x \triangle x + f_y \triangle y$

Approximation formula says, if graph of f is close to it's tangent plane then:

$$\boldsymbol{z = f(x, y) \approx z_0 + a(x - x_0) + b(y - y_0)}$$

1.3.4 Application of Partial Derivatives (Optimization Problem)

Finding local minima and local maxima of a function $f(x, y)$.

At local min/max:

$$\boldsymbol{f_x = 0 \text{ and } f_y = 0} \tag{1.14}$$

At the same time **tangent plane to the graph will be horizontal.**

(It's a necessary condition.)

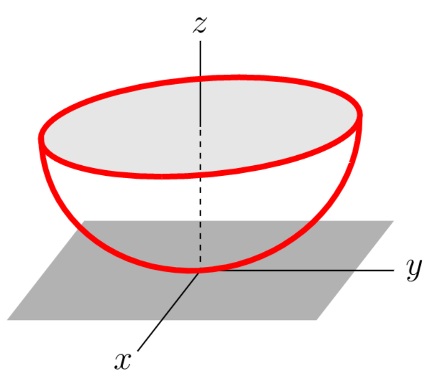

Figure 1.21: When tangent plane to the function is horizontal.

If $f_x = 0$ then f does not change at first order slice.
Similarly, if $f_y = 0$ then f does not change at first order slice.

$$\boldsymbol{\triangle z \approx f_x \triangle x + f_y \triangle y = 0}$$

Definition:

$x_0.y_0$ is critical point of f if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

Example 1:

$\boldsymbol{f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y}$

Solution:

$$f_x = 2x - 2y + 2 \tag{1.15}$$

$$f_y = -2x + 6y - 2 \tag{1.16}$$

Equating f_x and $f_y = 0$ and then solving both the equations we get
One critical point $(-1, 0)$

$$\text{So, the possibilities are: } \begin{cases} \rightarrow \textbf{local minima} \\ \rightarrow \textbf{local maxima} \\ \rightarrow \textbf{saddle point} \end{cases}$$

Saddle Point:

A **saddle point** or minimax point is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal directions are all zero (a critical point), but which is not a local extremum of the function. This only happens is two or higher variables not in one variable. **Saddle point** can be a **critical point.**

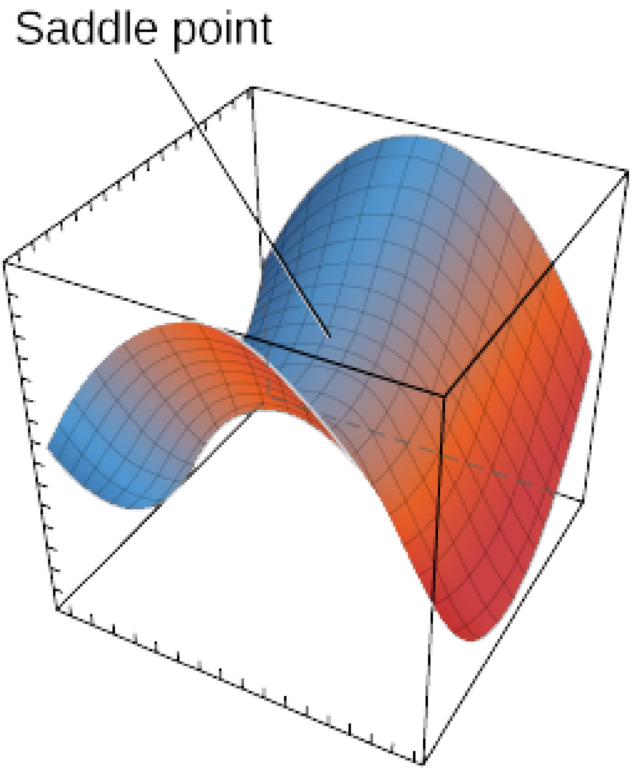

Figure 1.22: Saddle Point

Example 2:

$\boldsymbol{f(x,y) = (x-y)^2 + 2y^2 + 2x - 2y}$

Solution: By completing the square we get:

$$f(x,y) = [(x-y)+1]^2 + 2y^2 - 1 \tag{1.17}$$

From the above equation we can conclude that: $f(x,y) \geqslant -1$

As both the term are square so (-1) is min $f(-1,0)$.

Example 3: (Saddle Point)

$\boldsymbol{f(x,y) = y^2 - x^2}$

Solution:

$\rightarrow$ When we slice at $x = 0$ and $y = 0$, we get the following:

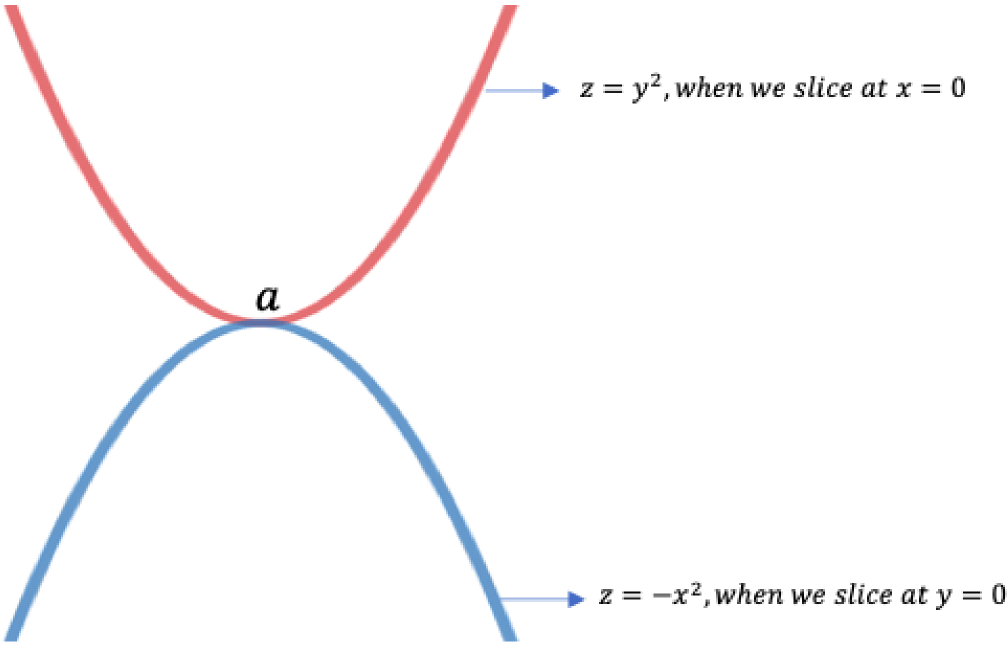

Figure 1.23: Saddle Point example

$$\text{point } \mathbf{a} \text{ has three possibilities: } \begin{cases} \rightarrow \textbf{local minima} \\ \rightarrow \textbf{local maxima} \\ \rightarrow \textbf{saddle point} \end{cases}$$

Local Minima, Local Maxima and Saddle Point

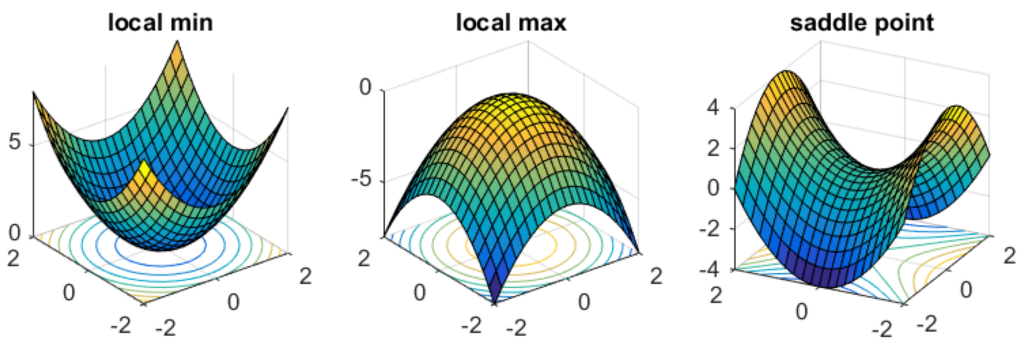

Figure 1.24: Summary

1.3.5 Few more examples of Partial Derivative (Computation)

Example 1:

$\boldsymbol{f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z}$

Solution:

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function, then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

We first calculate $\partial f/\partial x$, then we calculate the other two partial derivatives by holding the remaining variables constant. To find $\partial f/\partial x$, we first need to calculate $f(x+h, y, z)$: $f(x+h, y, z) = (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z$

$= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z$

and recall that

$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$

Next, we

$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4xz-4hz+5yz^2-12x-12h+4y-3z)-(x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z)}{h}\right]$

$$= \lim_{h\to 0}\left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{h(2x + h - 3y - 4z - 12)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{2x + h - 3y - 4z - 12}{1}\right] \tag{1.18}$$

applying the limit we get,

$$= 2x - 3y - 4z - 12 \tag{1.19}$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \tag{1.20}$$

$$= 0 - 3x + 4y - 0 + 5z^2 - 0 + 4 - 0 \tag{1.21}$$

$$= -3x + 4y + 5z^2 + 4 \tag{1.22}$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial z}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \tag{1.23}$$

$$= 0 - 0 + 0 - 4x + 10yz - 0 + 0 - 3 \tag{1.24}$$

$$= -4x + 10yz - 3 \tag{1.25}$$

Example 2:

$\boldsymbol{f(x,y,z) = \frac{x^2-4xz+y^2}{x-3yz}}$

Solution:

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^y - 4xz + y^2}{x - 3yz}\right] \tag{1.26}$$

$$= \left[\frac{\frac{\partial}{\partial x}(x^2y - 4xz + y^2)(x - 3yz) - (x^2y - 4xz + y^2)\frac{\partial}{\partial x}(x - 3yz)}{(x - 3yz)^2}\right] \tag{1.27}$$

$$= \left[\frac{(2xy - 4z)(x - 3yz) - (x^2y - 4xz + y^2)(1)}{(x - 3yz)^2}\right] \tag{1.28}$$

$$= \frac{x^2 - 6xy^2z - 4xz + 12yz^2 + 4xz - y^2}{(x - 3yz)^2} \tag{1.29}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right] \tag{1.30}$$

$$= \frac{x^3 + 2xy - 3y^2z - 12xz^2}{(x - 3yz)^2} \tag{1.31}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right] \tag{1.32}$$

$$= \frac{-4x^2 + 3x^2y^2 + 3y^3}{(x - 3yz)^2} \tag{1.33}$$