Stochastic Processes: Prerequisites of transforms

Laplace, Dirac, Fourier

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0.1 The Laplace Transform

The Laplace function is primarily a mapping of points in the t domain of a function $f(t)$ to points in the s domain. We must note that the Laplace transform **exists for functions that are of exponential order, are bounded and have converging infinite integrals**. The mathematical definition of the Laplace Transform is as follows:

$$L\{F(t)\} = f(s) = \int_0^{\infty} F(t)e^{-st}dt \tag{1}$$

A popular property of the Laplace transform is that of **linearity** and can be stated as:

$$L\{aF_1(t) + bF_2(t)\} = aL\{F_1(t)\} + b\{F_2(t)\} \tag{2}$$

Yet another important theorem associated with this transform is called the **first shift theorem** and can be defined as follows:

$$L\{e^{-bt}F(t)\} = f(s+b) \tag{3}$$

The proof of this theorem is pretty straightforward.

$$L\{e^{-bt}F(t)\} = \lim_{T\to\infty}\int_0^{T} e^{-st}e^{-bt}F(t)dt \tag{4}$$

$$= \int_0^{\infty} e^{-st}e^{-bt}F(t)dt = \int_0^{\infty} e^{-(s+b)t}F(t)dt = f(s+b) \tag{5}$$

0.1.1 Examples and Properties

To demonstrate how this transform works, we will show a simple example of transforming the function $F(t) = t$. Note that integration by parts is used.

$$L(t) = \lim_{T\to\infty}\int_0^{T} te^{-st}dt \tag{6}$$

$$\to \int_0^T te^{-st}dt = \left[-\frac{t}{s}e^{-st}\right]_0^T - \int_0^T -\frac{1}{s}e^{-st}dt \tag{7}$$

$$= -\frac{T}{s}e^{-sT} + \left[-\frac{1}{s^2}e^{-st}\right]_0^T \tag{8}$$

$$= -\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-st} + \frac{1}{s^2} \overset{T\to\infty}{\to} \frac{1}{s^2} \tag{9}$$

Therefore the Laplace transform of the function $F(t) = t$ is given by $f(s) = 1/s^2$. Now we note some general formulae regarding various Laplace transforms. The derivation of these expressions is omitted in this section.

- $L(t^n) = \dfrac{n!}{s^{n+1}}$
- $L\{te^a t\} = \dfrac{1}{(s-a)^2}$
- Before the next formula we must recall the **Euler's formula** that gives us an expression for the polar coordinates of complex numbers:

$$e^{it} = \cos(t) + i\sin(t) \tag{10}$$

  Now we note that due to the linearity property, the Lapace transform of $e_i t$ is given by:

$$L(e^{it}) = L(\cos(t)) + iL(\sin(t)) \tag{11}$$

  Where the Laplace transforms of the individual trigonometric functions are:

$$L(\cos(t)) = \frac{s}{s^2+1} \tag{12}$$

$$L(\sin(t)) = \frac{1}{s^2+1} \tag{13}$$

- $L\{tF(t)\} = -\dfrac{d}{ds}f(s)$
- A popular function whose Laplace transform is immensely useful is the **Heaviside's unit step function** which is given by:

$$H(t) = \begin{cases} 0, & \text{if } t < 0. \\ 1, & \text{if } t \geq 0. \end{cases} \tag{14}$$

  Consequently its Laplace transform is given by:

$$L(1) = \frac{1}{s} \tag{15}$$

- Laplace transform of a first order differentiable function can be writen as:

$$L\{F'(t)\} = \int_0^\infty e^{-st}F'(t)dt = -F(0) + sf(s) \tag{16}$$

- Laplace transform of a second order differentiable function is given as:

$$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0) \tag{17}$$

  In a similar manner, we can generalize the above two points to write the Laplace transform of an n times differentiable function as:

$$L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0) \tag{18}$$

- $L\{\sin(wt)\} = \dfrac{w}{s^2 + w^2}$
- $L\{\cos(wt)\} = \dfrac{s}{s^2 + w^2}$

0.1.2 Expanding a little on Heaviside function

We earlier mentioned that the Laplace transform of the Heaviside function is given by $L\{H(t)\} = 1/s$. However we are usually more interested in finding out the transform os $H(t - t_0)$ where $t_0 > 0$. Applying the Laplace transform definition to this we get:

$$l\{H(t - t_0)\} = \int_0^\infty H(t - t_0)e^{-st}dt \tag{19}$$

Note that as per the way this function is defined, for $t < t_0$ we would have $H(t - t_0) = 0$ and the transform would evaluate as follows, taking only those t such that $t > t_0$ and consequently the function evaluating to $H(t - t_0) = 1$.

$$L\{H(t - t_0)\} = \int_{t_0}^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_{t_0}^\infty = \frac{e^{-st_0}}{s} \tag{20}$$

This function assumes special relevance when it is multiplied with another function and that action of multiplying this Heaviside function is analogous to 'switching on' the other function. With this intuition we can state the **second shift theorem** defined as:

$$L\{H(t - t_0)F(t - t_0)\} = e^{-st_0} f(s) \tag{21}$$

Note that with this we can find the Laplace transform of a function that is switched on at $t = t_0$.

0.1.3 Inverse Laplace transform

Taking out the inverse of Laplace transforms usually involves a bit of solving using the **partial fractions decomposition method**. The standard definition for the inverse transform is given as:

$$if\ L\{F(t)\} = f(s) \tag{22}$$

$$then\ L^{-1}(f(s)) = F(t) \tag{23}$$

Now we take a simple example wherein the inverse transform is determined using partial fractions.

$$\rightarrow L^{-1}\left(\frac{a}{s^2-a^2}\right) \tag{24}$$

Solving the undetermined coefficients using partial fractions we get:

$$\frac{a}{s^2-a^2} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \tag{25}$$

Now we can simply apply the **linearity property** of the inverse transform operator to get:

$$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \frac{1}{2}(e^{at} - e^{-at}) \tag{26}$$

0.2 The Dirac Delta Impulse function

It is observed that there exist certain functions which might not classify as functions in the true sense. In order to classify as a function, an expression needs to be defined for all values of the variable in the specified range. Note that if this is not the case, then the expression would not be a function since it would cease to be well defined. We are usually not interested in such expressions, however we note that even if some of these expressions might not be well defined, if they have some desirable **global properties**, then such expressions indeed turn out to be rather useful. One such function is **Dirac's** δ function. The definition is as follows:

$$\delta(t) = 0,\ \forall t,\ t \neq 0 \tag{27}$$

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0) \tag{28}$$

The above is defined for any function $h(t)$ that is continuous in the interval $(-\infty, \infty)$. The Dirac-δ function can be thought of as the limiting case of a **top hat function** with unit area as it becomes infinitesimally thin and tall. First we define a function

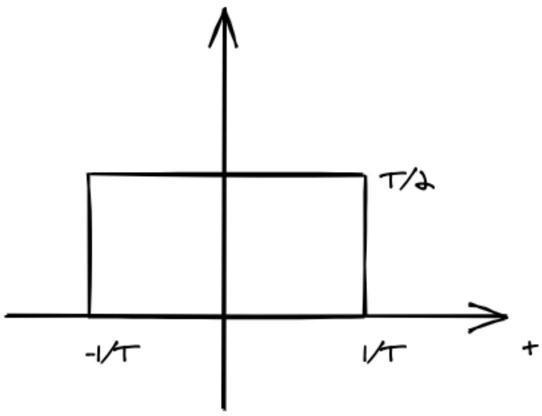

Figure 1: Top hat function

as follows:

$$T_p(t) = \begin{cases} 0, & \text{if } t \leq -1/T. \\ 0.5T, & \text{if } -1/T < t < 1/T. \\ 0, & \text{if } t \geq 1/T. \end{cases} \tag{29}$$

The Dirac Delta function then models the limiting behaviour of this function and can be written as:

$$\delta(t) = \lim_{T \to \infty} T_p(t) \tag{30}$$

The integral definition can be written as follows:

$$\int_{-\infty}^{\infty} h(t) \lim_{T \to \infty} T_p(t) dt = \lim_{T \to \infty} \int_{-\infty}^{\infty} h(t) T_p(t) dt \tag{31}$$

The value of the integral within the limits indicates the area under the cuve $h(t)T_p(t)$ and we say that this area would approach the value $h(0)$ as $T \to \infty$. Further we say that for very large value of T the interval $[-1/T, 1/T]$ will be small enough for the value of $h(t)$ to not differ from its value at the origin. With this we can express h in the form of: $h(t) = h(0) + \epsilon(t)$ where the term $\epsilon(t)$ tends to 0 as T goes to infinity. Therefore we can say that $h(t)$ tends to $h(0)$ for extremely large values of T. Note that $\delta(t)$ is not a true function since it is not defined for $t = 0$, therefore $\delta(0)$ has no value. Writing out the left and right side limits we get:

$$\int_{0^-}^{\infty} h(t)\delta(t) dt = h(0) \tag{32}$$

$$\int_{-\infty}^{0^+} h(t)\delta(t) dt = h(0) \tag{33}$$

As a limiting case of the top hat function the Dirac Delta function then looks this: We note an important property that as the interval gets smaller and smaller due

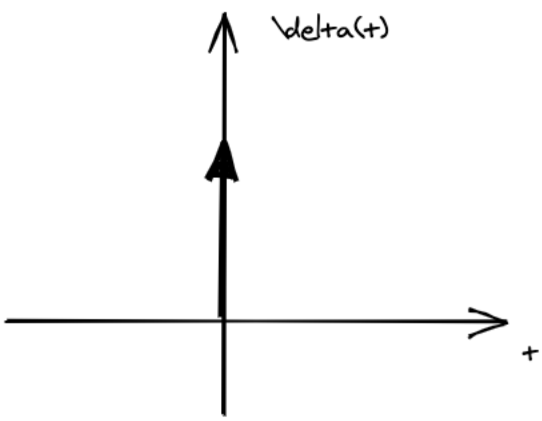

Figure 2: Dirac Delta function: limiting case of a top hat

to T becoming large, the area under the top hat function would always be unity.

Hence in the limiting case, the length of the arrow (which happens to represent the Dirac-δ function) is 1. Therefore we have with $h = 1$:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{34}$$

The Laplace transform of the Dirac Delta function is given by:

$$L\{\delta(t)\} = 1 \tag{35}$$

This essentially means that we are reducing the width of the top hat function such that it lies between 0 and $1/T$ (because in the exponential order laplace transformation we usually have limits starting from 0), and that we are increasing the height from $T/2$ to T so as to preserve the unit area.

0.2.1 Filtering property

Going further with the Dirac-δ function we say that the function $\delta(t-t_0)$ represents an impulse that is centered at time t_0. This can be thought of as a transient signal and the limiting case of a function $K(t)$ which is the displaced version of the top hat function:

$$K(t) = \begin{cases} 0, & \text{if } t \leq t_0 - 1/T. \\ 0.5T, & \text{if } t_0 - 1/T < t < t_0 + 1/T. \\ 0, & \text{if } t \geq t_0 + 1/T. \end{cases} \tag{36}$$

Now as $T \to \infty$ and utilising the definition of the Dirac-δ function we get:

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0) \tag{37}$$

We can get the Laplace transform of this Dirac Delta function, provided that $t > 0$ as:

$$L\{\delta(t - t_0)\} = e^{-st_0} \tag{38}$$

This has been called the **filtering property** since we can see clearly from the definition that the Dirac-δ function helps us pick out a particular value of a function.

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0) \tag{39}$$

0.3 Fourier Series

The central idea behind a **Fourier series** is that any given function can be expressed as a series of Sine and Cosine functions. Here we will be dealing mostly with periodic and piecewise continuous functions. Let us first start with functions defined on the closed interval $[-\pi, \pi]$ which possess one sided limits at $-\pi$ and π. We have a function that maps values such that $f : [-\pi, \pi] \to C$. We can now state the **Dirichlet theorem** as follows: If f is a member of the space of piecewise continuous functions which are 2π periodic on the closed interval $[-\pi, \pi]$, having both left and right derivatives at the end points, then we say that for each $x \in [-\pi, \pi]$ the Fourier series of f converges to:

$$\frac{f(x^-) + f(x^+)}{2} \tag{40}$$

And at both the end points ($x = \pm\pi$) the series converges to:

$$\frac{f(\pi^-) + f((-\pi)^+)}{2} \tag{41}$$

The fourier series gives us a result that **at points of discontinuity the value of Fourier series of f takes the value of the mean of one sided limits of f as the value at the discontinuous point**.

0.3.1 Fourier series formula

Remember that the whole point of a Fourier series is to express a periodic function as a series of sine and cosine functions. We see the components of such a series are typically periodic functions of period 2π given as:

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cdots, \cos(nx), \sin(nx) \tag{42}$$

These terms, together form a **trigonometric system** and the resulting series so obtained is called the **trigonometric series**:

$$a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots \tag{43}$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{44}$$

Here the a and b terms are the coefficients of the series and we say that if the coefficients are such that the series **converges**, then its sum would also have the same period as the individual components, that is 2π. Now if we have a function $f(x)$ of period 2π and can be represented by a convergent series of the form in equation 44 then we say that the **Fourier series** of $f(x)$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{45}$$

Consequently, the **Fourier coefficients** can be found using the following equations:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{46}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx \tag{47}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx \tag{48}$$

A crucial point to note is that the underlying concept behind this Fourier series is the **orthogonality** of the trigonometric system - which means that every term in the trigonometric series is orthogonal to each other, or that their inner product is zero. In terms of integrals we can write this condition as:

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx = 0 \tag{49}$$

References

[1] An introduction to Laplace transforms and Fourier series - PPG Dyke

[2] Advanced Engineering Mathematics - Kreyszig