Stochastic Processes: Fundamental tools

Calculus

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0.1 What are intervals

First note that an interval is nothing but a **subset of the real line** if it contains at least two numbers and all real numbers lying between any two of its elements. A typical example of an interval is as follows: *The set of all real numbers x such that* $-2 \leq x \leq 5$. Note that a **finite interval** is said to be **closed** if it contains both the end points, **half open** if it contains one end point and **open** if it contains no endpoints. The endpoints are known as **boundary points** whereas all the other points are called **interior points**. Some typical ways in which these intervals are written:

- $(a, b) \longrightarrow \{x|a < x < b\}$
- $[a, b] \longrightarrow \{x|a \leq x \leq b\}$
- $(a, \infty) \longrightarrow \{x|x > a\}$
- $(-\infty, b) \longrightarrow \{x|x < b\}$

0.1.1 Solving inequalities

Solving an inequality is defined as the process of **finding an interval** of numbers that satisfy an inequality in x. A typical example is presented below:

$$\longrightarrow 2x - 1 < x + 3 \tag{1}$$

$$2x < x + 4 \tag{2}$$

$$x < 4 \tag{3}$$

$$\longrightarrow (-\infty, 4) \tag{4}$$

0.1.2 Absolute value

The **absolute value** of a number x is denoted as $|x|$ and is defined by:

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases} \tag{5}$$

Noting some common properties about absolute value:

- $|-a| = |a|$
- $|ab| = |a||b|$
- $|a+b| \leq |a| + |b|$

The problem statement of solving an inequality with absolute values can be stated as: *The inequality $|a| < D$ states that the distance from a to 0 is less than D or we can say that a lies between $-D$ and D.* This is traditionally denoted as:

$$|a| < D \longrightarrow -D < a < D \tag{6}$$

To give a clear demonstration, we will compute the solution for the inequality $|2x-3| \leq 1$ and $|2x-3| \geq 1$ given as follows:

$$\textbf{Part 1: } |2x-3| \leq 1 \tag{7}$$

$$\rightarrow -1 \leq 2x - 3 \leq 1, \text{ add 3 to all sides} \tag{8}$$

$$\longrightarrow 2 \leq 2x \leq 4, \text{ now divide all sides by 2} \tag{9}$$

$$\longrightarrow 1 \leq x \leq 2, \text{ Solution set is: } \rightarrow [1, 2] \tag{10}$$

$$\textbf{Part 2: } |2x-3| \geq 1 \tag{11}$$

$$\longrightarrow 2x - 3 \geq 1, \text{ or, } -(2x-3) \geq 1 \tag{12}$$

$$\longrightarrow 2x - 3 \geq 1, \text{ or, } 2x - 3 \leq -1 \tag{13}$$

$$\longrightarrow x \geq 2, \text{ or, } x \leq 1, \text{ Solution: } \rightarrow (-\infty, 1) \cup (2, \infty) \tag{14}$$

Figure 1: Inequality solutions

0.2 Rates, Limits and Derivatives

Before even getting to limits we note the definition of **average rate of change of a function**: Given an arbitrary function $y = f(x)$ we calculate the average rate of change of y with respect to x over an interval $[x_1, x_2]$ by dividing the change in y given by $\Delta y = f(x_2) - f(x_1)$ by the change in x denoted by $\Delta x = x_2 - x_1 = h$. This is given as:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_2)}{h} \tag{15}$$

Now getting to one definition of the **limit of a function**, we say that for a function $f(x)$ defined on an open interval about x_0, except at x_0 itself, if $f(x)$ gets arbitrarily close to a number L for all x sufficiently close to x_0 we say that f approaches the limit L as x approaches x_0. This is typically expressed as:

$$\lim_{x \to x_0} f(x) = L \tag{16}$$

Noting the **Sandwich theorem** we say that if a function f is sandwiched between the values of two other functions g and h, which have the same limit L at a point c, then the value of f must also approach L at this point. Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except $x = c$ itself and also suppose the following condition:

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \tag{17}$$

Then we say that the following also holds:

$$\lim_{x \to c} f(x) = L \tag{18}$$

Now we note the **formal definition of a limit**: Let a function $f(x)$ be defined on an open interval about x_0 except at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0:

$$\lim_{x \to x_0} f(x) = L \tag{19}$$

If for every small number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x:

$$0 < |x - x_0| < \delta \longrightarrow |f(x) - L| < \epsilon \tag{20}$$

Now we shall note the definition of a **continuous function**. We say that a function $f(x)$ is continuous at an interior point $x = c$ of its domain if the following holds:

$$\lim_{x \to c} f(x) = f(c) \tag{21}$$

Further, we say that a function is **continuous** if the continuity property shown above holds everywhere in its domain.

0.2.1 Differential calculus

The definition of a derivative of a function is linked to the idea of the **slope** of the average rate of change of a function f at point $x = x_0$. The derivative of a function f with respect to x is said to be another function f' whose value at x is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{22}$$

We say that the domain of $f'(x)$ is the set of all points in the domain of f for which the limit exists. If $f'(x)$ exists we say that f is **differentiable** at x. Finally, the process of calculating the derivative of a function is called **differentiation**. Note some common differentiation rules:

- The product rule is given by:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{23}$$

- The quotient rule is given by:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{24}$$

- Second order derivatives are denoted as:

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \tag{25}$$

Next we discuss the **chain rule**. This is a rule that used to compute the derivatives of **composite functions**. If $f(u)$ is differentiable at a point $u = g(x)$ and in turn $g(x)$ is differentiable at a point x, then we composite function $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. Letting $y = f(u)$ and $u = g(x)$ we can write:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{26}$$

0.3 Basics of integrals

Integration is the process of determining the original function from its derivative f. In this regard, the **first step** is to find all possible functions that could have f as their derivatives - these functions are collectively called **antiderivatives** of f and the procedure for getting them is called **indefinite integration**. The **second step** is to hone in on one particular function corresponding to the derivative by looking at a known function value. We note that a function $F(x)$ is the **antiderivative** of $f(x)$ if:

$$F'(x) = f(x) \tag{27}$$

The set of all antiderivatives of f are got by the indefinite integral of f given by:

$$\int f(x)dx = F(x) + C \tag{28}$$

Where C is the **constant of integration**.

0.3.1 Integrals by substitution

Often a change of variable can simplify the integration we are trying to evaluate. This idea is demonstrated quite well with an example:

$$\textbf{Solve:} \quad \int (x+2)^5 dx \tag{29}$$

$$\rightarrow \text{Let } u = x+2, \text{ such that } du = d(x+2) \rightarrow du = dx \tag{30}$$

$$\rightarrow \int (x+2)^5 dx = \int u^5 du = \frac{u^6}{6} + C \tag{31}$$

$$= \frac{(x+2)^6}{6} + C \tag{32}$$

Next we come to the definition of **definite integrals**. Let $f(x)$ be a function defined on a closed interval $[a,b]$. We can think of the area under the curve to be approximated by the integral:

$$\int_a^b f(x)dx \tag{33}$$

A **Key point to note is that the value of the definite integral over an interval depends on the function and not on the letter we choose to represent the independent variable**. Basically we can use t, u, x or whatever for that matter. We say that the variable of integration is called a **dummy variable**. Note some common properties of definite integrals:

- $\int_a^a f(x)dx = 0$
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Lastly we note the **mean value theorem** that helps us find the average value of an integrable function f at some point c in the interval $[a,b]$ as:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \tag{34}$$

We will now note the **first part of the fundamental theorem of calculus**. It states that if f is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)dt$ has a derivative at every point of $[a,b]$ which is given by:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{35}$$

Now, the **second part of the fundamental theorem of calculus** states that if f is continuous at every point $[a,b]$ and F is the antiderivative of f on $[a,b]$ then:

$$\int_a^a f(x)dx = F(b) - F(a) \tag{36}$$

0.3.2 Integration by parts

Integration by parts is a technique which helps us simplify integrals of the form:

$$\int f(x)g(x)dx \tag{37}$$

In which we say that f can be differentiated repeatedly and g can be integrated repeatedly without any difficulty. For example look at the following example:

$$\int xe^x dx \tag{38}$$

In this $f(x) = x$ can be differentiated twice to get 0 and $g(x) = e^x$ can be integrated multiple times without any problem. The formula for this is given by:

$$\int udv = uv - \int vdu \tag{39}$$

Demonstrating with this a simple example:

$$\text{find } \int x \cos x dx \tag{40}$$

$$u = x,\ dv = \cos x dx \longrightarrow du = dx,\ v = \sin x \tag{41}$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C \tag{42}$$

0.4 Inverse functions

Recall that a function is rule that assigns a value from it **range** to each point in its **domain**. Some functions have a one-to-one correspondence, whereas others assign the same value to more than one point. A function that has distinct values at distinct points is called **one-to-one**. Now note that since each ouput of a one-to-one function comes from only one point, this function can effectively be **reversed** to map back to the input from the output. Therefore, the function obtained by reversing a one-to-one function is called the **inverse** of the function. It is denoted as f^{-1}. Note that if the **compose** a function and its invese, we get the **identity function** which is a function that assigns each number to itself. We say that functions f and g are an inverse pair if:

$$f(g(x)) = x,\ \text{and},\ g(f(x)) = x \tag{43}$$

Where $f = g^{-1}$ and $g = f^{-1}$. Note the steps for finding the inverse of a function with an example:

- **Question:** Find the inverse of:

$$y = \frac{1}{2}x + 1 \tag{44}$$

- **Step 1:** Solve for x in terms of y:

$$x = 2y - 2 \tag{45}$$

- **Step 2:** Interchange x and y:

$$y = 2x - 2 \tag{46}$$

- Therefore the inverse of the function $f(x) = (1/2)x + 1$ is given by:

$$f^{-1}(x) = 2x - 2 \tag{47}$$

- Lastly, we note the **derivative rule** for inverse functions as given by:

$$(f^{-1})' = \frac{1}{f'} \tag{48}$$

References

[1] Calculus and Analytic Geometry - Thomas and Finney