Credit Risk prerequisites

Fundamentals of Options and Black Scholes

0.1 Options

- A **call option** gives its owner the right to buy stock at a specified **exercise price** on or before a specified maturity date. For example if the option price for Google is set at $\$78$ for December 2008 - we say that for $\$78$ you can acquire an option to buy one share of Google at a price of $\$370$ on or before December 2008. **Value of a call option goes down as the exercise price goes up and call option value increases as option maturity time is increased**.

- Suppose we get a Google call option at a strike of $\$430$. Then, when the maturity period comes, we will not exercise the option if the stock price at the maturity period is less than $\$430$ - in this case the call option will expire worthless. If the stock price is more than $\$430$ then our call option value will be the payoff - $(\text{stock price} - \$430)$ (obviously, we will also adjust into the payoff, the price we paid to get the option in the first place).

- A **put option** gives the owner the right to sell stock at a specified exercise price as per a certain maturity date. For example, for $\$48$ one can buy an option to sell a Google stock for a price of $\$430$ on or before March 2009.

- In the put case, if at maturity, the stock price turns out to be more than the strike price, we will not exercise the put option and instead sell the share in the market. However if the stock price is below the strike, then our put option value would be the payoff - $(\$430 - \text{stock price})$

- **writing a call** - if you sell a call option, then you are essentially promising to deliver shares if asked to do so by the call buyer. If the stock price at maturity is below the call strike, the buyer will not exercise the call and the writer would have zero liability (and would make a profit equal to the option price). But if the stock price rises above the strike, then the call buyer will exercise the option and the call writer (seller) will have to give up the shares and take a loss. **Seller's loss would be the buyer's gain**. Selling a put could be explained similarly as well.

- A crucial point to note is that an **option is riskier than the underlying stock.** This is because if you own a stock you are only exposed to the risk of the stock movements, but if you own an option - you might eventually hold the stock (after exercising the option), plus, options are typically purchased on credit so you bear that additional credit risk as well. So basically, for a call, buying a call option is the same thing as buying the stock but financing part of the purchase by borrowing.

- Note that the value of an option increases with volatility of the share price and the time to maturity as well.

0.2 Constructing an option equivalent

Recall that a standard way in which we value any asset is to discount its expected future cash flows and discount them at the appropriate opportunity cost of capital. We find that this approach is not possible for valuing options since the risk of an option keeps changing as the stock price changes, hence having a proper stable measure of opporunity cost of capital is not possible. An option strategy can be effectively replicated by - **a strategy wherein we buy or sell a 'delta' amount of shares and at the same time borrow or lend money at the risk free rate**. Consider the following situation:

- We pick a date when Google's price is trading at $\$430$ and it can do two things - either it can **move up** to 573 or **move down** to $\$322$.
- The **six month** risk free rate is considered to be 1.5%, at which we can borrow money.
- Now that the above situation is set, we will compare the above situation of **buying a Google stock and borrowing money for six months** and prove that this situation is equal to having a **six month call option**.
- Note that for the **call option strategy**, is the price falls, the option is worthless (0) and if the price of stock moves up, the option value would be $= \$573 - \$430 = \$143$.
- Now let us take the buying stock and borrowing case. Suppose we borrow $\$181$ from the bank and buy $4/7$ Google shares.
- At the end of six months, if the stock price falls to $\$322$, our stock portfolio value would be $(4/7) \times \$322 = \$184$. Also, we would have to repay the borrowed sum plus interest $= \$184$. Cumulatively we are left with 0.
- Alternatively, if the stock price rises to $\$573$, our stock portfolio value would be $(4/7) \times \$573 = \$327$. Also, again we would repay loan plus interest and cumulatively be left with $\$573(4/7) - 184 = \$143$.
- Now note how this strategy gives the exact same payoffs as does the call option.

- So finally the value of the call would be the present value of this entire strategy we recreated:

$$value_{call} = \$430 \times (4/7) - 181 = \$64 \tag{1}$$

The ultimate point is that we **borrowed money and bought shares** in such a way that the payoffs exactly matched the call option payoffs. This is known as a **replicating portfolio** and the number of shares ($4/7$ in our case) needed to replicate one call option is called the **hedge ratio** or the **option delta**. Note that the **spread of our option values** are - $(\$143 - 0)$. Also, the spread of the possible stock prices are - $(573 - 322)$. The ratio of the option value spread and the stock price spread is precisely the option delta.

$$\text{option delta} = \frac{\text{spread of option prices}}{\text{spread of stock prices}} = \frac{143 - 0}{573 - 322} = \frac{4}{7} \tag{2}$$

0.3 Risk neutral valuation

We notice that the valuation principle described above for a call option is the true value of the call option and if it does not follow this value, then there would be **arbitrage opportunities**. Also we noted that the price we arrived at, did not at all factor in the various different risk attitudes of investors. So we note that the option price does not depend on people's attitudes towards risk. Hence, here we figure out an alternate way to price an option - **we pretend that all investors are indifferent about risk, find out the expected future value of options in such a world and then simply discount it back using the risk free rate to get the current price**. Here is a small explanation:

- Let us take the Google example again. If investors are indifferent towards risk, then expected return on the stock is the risk free rate. In our example the six month expected return would be, as specified before, 1.5%.
- We earlier saw that the price (currently at $\$430$) could either rise to $\$573$ (a 33%) increase. Or it could fall to $\$322$ (a fall of 25%) - note that these percentages relfect **returns**.
- In this risk neutral world, we can assume that there is some probability by which the above stated returns happen. Since we already know the expected return to be 1.5% and we know the possible return values, we can solve for this **risk neutral probability** as follows (note that p is the probability of rise):

$$\text{expected return (1.5\%)} = [p \times 33\%] + [(1 - p) \times (-25\%)] = 0.45 \tag{3}$$

- Note that this is not the true probability, but the risk neutral probability. Now instead of doing all those computations in the previous step, the risk neutral probability can be generally calculated as:

$$p = \frac{\text{interest rate} - \text{downside \% change}}{\text{upside \% change} - \text{downside \% change}} \tag{4}$$

$$p = \frac{0.015 - (-0.25)}{0.33 - (-0.25)} = 0.45 \tag{5}$$

- Now let us see how we incorporate this into a call option strategy. If stock prices rises, our call was worth $\$143$ as shown before and if the stock price falls, it is worth 0. So, the expected value of the call based on this risk neutral probability of stock movements is:

$$[\text{prob of rise} \times \$143] + [(1 - \text{prob of rise}) \times 0] \tag{6}$$

$$= [0.45 \times 143] + 0 = \$65 \tag{7}$$

- As a final step, we discount that above value using the risk free rate as follows:

$$\frac{\text{expected future value}}{1 + \text{interest rate}} = \frac{65}{0.015} = \$64 \tag{8}$$

- Note that this is the exact same value we got from the previous option delta formulation as well.

0.4 The binomial model

The binomial model is actually a pricing model that we have already looked at - we assume that the future period stock price can either go a certain percentage up or a certain percentage down. Based on this, we create a replicating portfolio with a **delta** amount of shares and some borrowed amount. We then find that the present value of this replicating portfolio is infact the call option price. The only difference in a proper binomial model to the method we just described is that in the binomial model, we **subdivide the time period into many more time periods**. So if we are looking at a 6 month horizon for valuation over two possible stock price values, we now look at 2 sub time periods of 3 months, where in each sub time period the stock price moves a certain percentage up or down. Essentially we will employ the same valuation principle, except that this time we will carry it out twice, over each 3 month sub period - Note that this is the **two step binomial model**.

0.4.1 General binomial model

In this general approach, if we now divide the maturity horizon of say 6 months into many many small intervals of sub time periods, where in each small time period the stock behaves the same way - it goes up by a certain percentge and goes down by a certain percentage. In the previous cases we have been just taking arbitrary percentage values for up and down movements, but in practise, the general formula for up (u) and down (d) percentage movements is characterized by the standard deviation in stock returns:

$$1 + \text{upside change} = u = e^{\sigma\sqrt{h}} \tag{9}$$

$$1 + \text{downside change} = d = 1/u \tag{10}$$

Note that here h the length of the time interval that we subdivided our maturity horizon into (it is typically denoted in terms of a fraction of a year). σ is the standard deviation of continuously compounded stock returns.

0.5 Black Scholes

In the previous section we presented a situation wherein we divide the option maturity period into extremely small chunks (with each chunk representing a binomial model). Now with such fine subdivisions, by the time we reach the maturity period, we would have an array of various possible stock price changes - we find that in this situation the stock price changes follow a **lognormal distribution**. The **Black Scholes** formula for the value of a call option is:

$$\text{value of call} = [\text{delta} \times \text{share price}] - [\text{bank loan}] \tag{11}$$

$$\text{value of call} = [N(d_1) \times P] - [N(d_2) \times PV(EX)] \tag{12}$$

We see that on a very basic level, the Black Scholes formula is still the option value formula we got in the starting by constructing a replicating portfolio and finding its present value. The only difference is that we are describing the variables like 'delta' and 'bank loan' in terms of volatility and associated time periods.

- EX is the exercise price of the option. $PV(EX)$ is just the present value of this exercise price, discounted at the risk free rate. Also, P is the share price currently.
- The variable d_1 is computed as:

$$d_1 = \frac{\log[P/PV(EX)]}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2} \tag{13}$$

- The variable d_2 is computed as:

$$d_2 = d_1 - \sigma\sqrt{t} \tag{14}$$

- $N(d)$ is the cumulative normal probability density function
- t is the number of periods till exercise date (fraction of years till maturity of the option).
- σ is the standard deviation of continuously compounded rate of return on the stock.

References

[1] Brealey, Myers and Allen - Principles of Corporate Finance