Macroeconomics B Notes

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1 Dynamic Optimisation in Continuous Time

Continuous vs Discrete Time: in discrete time the intervals between periods are $\Delta > 0$ (e.g. the interval between t and $t+1$ is 1), in continuous time the intervals between periods are $\Delta \to 0$

Continuous Variables Issue: while state variables are often naturally continuous, when using computational methods we can only evaluation the value function on a finite number of points

- Solution: we can address this problem via using (1) discretization of the state space, or, (2) polynomial approximation to the value function

Discretization Method: consider a problem with continuous state and control variables $\mathfrak{x} \in \mathbb{R}$, discretization just replaces $\mathfrak{x}$ and $\mathfrak{u}$ by the finite grids $\widehat{\mathfrak{x}} = \{x^1, \ldots, x^n\}$ and $\widehat{\mathfrak{u}} = \{u^1, \ldots, u^n\}$

- Value Function Impact: now the value function becomes a finite list of numbers, $V = [V^1, \ldots, V^n]^T$
- Advantage: the maximization step is much simpler than under the original bellman equation, which is a key advantage of discretization methods
- Disadvantage: there is a "curse of dimensionality" in muiltidimensional state spaces where we must decides whether to have N points for a one-dimensional state space vs N^k points for a k-dimensional state space
  - Grid Decision: requires some a-priori information about the state space, which is sometimes difficult to obtain (e.g. upper and lower bounds)
- Optimal Growth Illustration: suppose $V(k) = \max_{k'} \{\ln(Ak^{\alpha} - k') + \beta V(k')\}$ where in this case $\mathfrak{x} = \mathfrak{u}$

*Please see google notes

– Discretization: if we discretize $\mathfrak{k}$ then the Bellman equation becomes $V_i = \max_j \{\ln(Ak_i^\alpha - k_j) + \beta V_j\}$ for all $i = 1, \ldots, n$

– Value Function Iteration Solution: suppose we applied value function iteration and therefore iterate on the mapping $V_i^s = \max\left\{\ln(Ak_i^\alpha - k_1) + \beta V_1^{s-1}, \ldots, \ln(Ak_i^\alpha - k_n) + \beta V_n^{s-1}\right\}$ for all $i = 1, \ldots, n$ where s indexes the iteration step

* VF ITERATION

  (1) Start w/ a guess:

  $$\underset{n \times 1}{V^0} = \begin{bmatrix} V_1^0 \\ V_2^0 \\ \vdots \\ V_n^0 \end{bmatrix}, \text{ where } V_i^0 \equiv V^0(k_i)$$

  (2) Find V^1 by solving:

  $$V_i^1 = \max_j \left\{ \ln\left(Ak_i^\alpha - k_j\right) + \beta V_j^0 \right\}$$

  expanding

* $$V_i^1 = \max \begin{bmatrix} \ln(Ak_i^\alpha - k_1) + \beta V_1^0 \\ \ln(Ak_i^\alpha - k_2) + \beta V_2^0 \\ \vdots \\ \ln(Ak_i^\alpha - k_n) + \beta V_n^0 \end{bmatrix} \quad \forall i$$

  so I end up with:

  $$\underset{n \times 1}{V^1} = \begin{bmatrix} V_1^1 \\ V_2^1 \\ \vdots \\ V_n^1 \end{bmatrix}$$

3 Check for convergence:

$$\underbrace{\|V^1 - V^0\|} \leq \varepsilon \longrightarrow \text{tolerance level}$$

$$\sqrt{\sum_{i=1}^{n} (V_i^1 - V_i^0)^2} \longrightarrow \text{Euclidean Norm.}$$

4 Compute V^s by solving:

$$V_i^s = \max \left\{ \ln(A k_i^\alpha - k_j) + \beta V_j^{s-1} \right\} \quad \forall i$$

5 Check convergence

$$\|V^s - V^{s-1}\| \leq \varepsilon \longrightarrow$$

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- If $\|V^s - V^{s-1}\| \leq \varepsilon$ holds $\rightarrow$ STOP
- If $\|V^s - V^{s-1}\| > \varepsilon$ does not hold

$\Rightarrow$ Go Back to step 4

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Ordinary Differential Equations (ODEs): a "differential equation" is one where the unknown is a function (instead of a variable) and the equation includes one or more of the derivatives of the function, an "ordinary differential equation" equation is one for which the unknown is a function of only one variable (typically time)

- Partial ODEs: where the unknown is a function of more than one variable
- First-Order ODE Form: $\dot{x}(t) = F(t, x(t))$ where $\dot{x}(t) \equiv dx(t)/dt$ and $t \in [t_a, t_b]$
  - Unknown: here the unknown is a function $x(t)$ with $x : [t_a, t_b] \to \mathbb{R}$
  - Uniqueness: the solution is not unique with the form having infinitely many solutions indexed by an integrating constant C. However, generally the constant C can be uniquely determined by requiring the solution to pass through a given point on the tx-plane

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