Macroeconomics B Notes

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2019

Contents

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Dynamic Optimization in Discrete Time

Optimization Issues: solving large problems (like that below is challenging using traditional methods and so we must rely on dynamic programming)

- Example: consider the problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\text{subject to: } c_t + k_{t+1} \leq A k_t^{\alpha} \theta_t \ \ \forall t, \quad k_0 \text{ given}, \quad \ln(\theta_t) \sim_{iid} N(0, \sigma^2)$$

  where c_t is consumption, k_t is capital, θ_t is a productivity shock

  – Issue: This problem involves maximization over an infinite sequence of controls, subjects to an infinite sequence of constraints, and where each period's optimal decisions depend on the history of shocks and past decisions - making the problem more difficult

  – Solution: we are looking for a solution of the form $\{c_t^*(\theta^t)\}_{t=0}^{\infty}$ and $\{k_t^*(\theta^t)\}_{t=0}^{\infty}$ where $\theta^t = \{\theta_0, \theta_1, \ldots, \theta_t\}$ is the history of shocks up to tS

Dynamic Programming: breaks up optimization problems into a series of small and more tractable problems where the smaller problems are casted in a recursive way. Here the solution to such problems (under certain regularity conditions) can be used to recover the solution to the original problem

- Bellman's Optimality Principle: an optimal policy has the property that whwatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.
  – Note: based on this principle the solution to the dynamic programming problem is a valid solution to the original problem
- Horizon Types: can involve either finite horizon or infinite horizon problems
- State Variables: set of variables summarizing the state of the economy at each point in time
- Control Variables: the set of choice variables at each point in time
- Value Function: the optimal value of the original problem, given the states
- Policy Function: the optimal value for the control variables, given the states

Finite Horizon Problem: consider a decision-maker solving the deterministic problem

$$\max_{\{u_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t r(x_t, u_t) + W(x_T)$$

$$\text{subject to: } x_{t+1} = g(x_t, u_t) \ \forall t, \quad x_0 \text{ is given}$$

Here we assume a convex set, to ensure a maximum exists, and that the state space is discrete (ie $\mathcal{X} = \{x^1, x^2, \ldots, x^n\}$). Our variables include:

$$\beta \in (0,1): \text{ the discount factor}$$

$$x \in \mathcal{X}: \text{ the state variable}$$

$$u \in \mathcal{U}: \text{ the control variable}$$

$$r(x_t, u_t): \text{ concave return function}$$

$$g(x_t, u_t): \text{ the law of motion for the state}$$

$$W(x_T): \text{ the terminal condition at } x_t$$

- Recursive Transformation: based on the original problem being time separable we can write it as a series of recursive functions. The original problem asks us to find a sequence of T controls (i.e. find $\{u_t^*\}_{t=0}^{T-1}$) while the recursive formulation asks us to solve T problems with one control each (i.e. find the series of $U_t^*(x)$ over the set t). We make the following Transformation

$$\max_{\{u_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t r(x_t, u_t) + W(x_T)$$

$$\text{subject to: } x_{t+1} = g(x_t, u_t)\ \forall t, \quad x_0 \text{ is given}$$

$$\Downarrow$$

$$V_t(x) = \max_u \{r(x,u) + \beta V_{t+1}(g(x,u))\}$$

- Value Function $V_t(x)$: is the greatest feasible payoff from time t forward if the state at time t is x_t. Here the value function is

$$V_t(x_t) = \max_{\{u_s\}_{s=t}^{T-1}} \sum_{s=t}^{T-1} \beta^s r(x_s, u_s) + W(x_T)$$

$$\text{subject to: } x_{s+1} = g(x_s, u_s)\ \forall s, \quad x_t \text{ is given}$$

  Note that this is different from our objective function in that the time starts at t and not 0

  - Bellman Condition: given the structure of the problem, the value function satisfies the Bellman Equation where $V_t(x) = \max_u \{r(x,u) + \beta V_{t+1}(g(x,u))\}$ with the terminal condition $V_T(x) = W(x)$

    * Intuition: the value function at time t is going to be the maximum utility given next period's value function, today's return, and the terminal condition. Note that the Bellman Equation describes the value function, at a given point in time, as a function of itself in another point in time

- Policy Function $U_t(x)$: the policy function describes the utility that, given the state, is the maximum of the value function. Given $V_t(x)$, we define the policy function by

$$U_t(x) = \arg\max_u \{r(x,u) + \beta V_{t+1}(g(x,u))\}$$

- Solution: the Bellman Equation and the Policy Function are the building blocks for constructing the optimal solution
  - Step 1 - Backward Induction: compute $\{V_t(x)\}_{t=0}^T$ and $\{U_t(x)\}_{t=0}^{T-1}$ by backward induction using the Bellman Equation
    - Part 1: compute $V_t(x) = W(x)\ \forall x \in \mathfrak{X}$
    - Part 2: for all $t = T-1, T-2, \ldots, 0$ compute $V_t(x) = \max_u\{r(x,u) + \beta V_{t+1}(g(x,u))\}$ and $U_t(x) = \arg\max_u\{r(x,u) + \beta V_{t+1}(g(x,u))\}$ for all $x \in \mathcal{X}$
    - Example

$$\boxed{t = T}$$

$$\forall x^i,\ V_T(x^i) = W(x^i) \to \text{ final condition}$$

$$\boxed{t = T-1}$$

$$\forall x^i,\ V_{T-1}(x^i) = \max_u \left\{r(x^i,u) + \beta V_T(x^i)\right\}$$

$$U_{T-1}(x^i) = \arg\max_u \left\{r(x^i,u) + \beta V_T(x^i)\right\}$$

$$\ldots$$

$$\text{repeat these steps until we reach } \boxed{t = 0}$$

Note that because the state variable x is discrete, we know how many times we need to iterate backwards until $t = 0$

  - Step 2 - Iterating Forward: given x_0 compute $\{u_t^*\}_{t=1}^{T-1}$ and $\{x_t^*\}_{t=1}^T$ by iterating forward on

$$\{U_t(x)\}_{t=0}^{T-1}$$

* <u>Method</u>: we iterate forward using x

$$
\begin{gathered}
u_0^* = U_0(x_0^*), \quad x_1^* = g(x_0^*, u_0^*), \\
u_1^* = U_1(x_1^*), \quad x_2^* = g(x_1^*, u_1^*), \\
\dots \\
u_t^* = U_t(x_t^*), \quad x_{t+1}^* = g(x_t^*, u_t^*), \\
\dots \\
u_{T-1}^* = U_{T-1}(x_{T-1}^*), \quad x_T^* = g(x_{T-1}^*, u_{T-1}^*),
\end{gathered}
$$

* <u>Intuition</u>: the law of motion will indicate what the future value of x_t is given the control variable u, allowing us to derive the solution. Note that the control variable u is able to be calculated due to having previously found the value of u via backward induction in step 1

<u>Infinite Horizon Problem</u>: consider the problem

$$
\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)
$$
$$
\text{subject to : } \; x_{t+1} = g(x_t, u_t), \quad x_0 \text{is given}
$$

We want to compute the solution $\{u_t^*, x_{t+1}^*\}_{t=0}^{\infty}$

- <u>Assumptions</u>: in addition to the previous finite horizon problem assumptions we assume that $r(\cdot)$ is bounded, also note that now the bellman equation is a functional equation in V
- <u>No Backward Induction</u>: since there is no terminal condition to start with (ie V is time-invariant) we cannot use backward induction
- <u>Conditions for a Solution</u>: the following four conditions must hold for the infinite horizon problem to be solved
  - <u>Condition 1</u>: the Bellman Equation has a unique strictly concave solution V^*
  - <u>Condition 2</u>: let V^0 be any bounded and continuous VF and let V^j satisfy

$$
V^j(x) = \max_u \left\{ r(x,u) + \beta V^{j-1}(g(x,u)) \right\} \quad \text{for } j \geq 1
$$

    Then $V^* = \lim_{j \to \infty} V^j$
  - <u>Condition 3</u>:There is a unique and time-invariant policy function U^* such that the solution to the original problem can be obtained by iterating on $u_t^* = U^*(x_t)$

- Condition 4: off the corners, the limiting value function V^* is differentiable with

$$V^{*'}(x) = r_1(x, U(x)) + \beta V^{*'}(g(x, U(x)))g_1(x, U(x))$$

Note that the second condition implies that if you have any function V_0 that is bounded and continuous then if you iterate on j you will eventually find the solution V^* (ie a steady state). The third condition implies that from finding V^* you can use u_* to then backwards iterate to find U^*

- Guess and Verify Method: also known as the method of undetermined coefficients, this relies on the uniqueness of the solution to the Bellman Equation (condition 1) and can be solved analytically with pen and paper (though does not always work)
  - Step 1: guess that the true value function has a particular parametric form $V^G(x; A)$ where A is a vector of unknown parameters
  - Step 2: compute $\widehat{V}(x; A) = \max_u\{r(x, u) + \beta V^G(g(x, u); A)\}$
    - Note: you can solve for the control variable that is implied by the policy function by using the FOC (provided in condition 4). This then allows you to solve for the optimized $\widehat{V}(x; A)$
  - Step 3: if there exists parameters A^* such that $\widehat{V}(x; A^*) = V^G(x; A^*)$ then $V^G(x; A^*)$ is the solution
    - Note: compare the derived optimized value function $\widehat{V}(x; A)$ from step 2 to $V^G(x; A)$
- Value Function Iteration Method:this method always works because by solution method condition 2 we have $V^* = \lim_{j\to\infty} V^j$, however this method can be very slow particularly for problems with more than one state
  - Step 1: start from any bounded and continuous guess V^0
  - Step 2: iterate on $V^j(x) = \max_u\{r(x, u) + \beta V^{j-1}(g(x, u))\}$ until convergence (ie until $||V^j - V^{j-1}|| < \varepsilon$)
- Policy Function Iteration Method
  - Step 1: pick a feasible policy function $U^0(x)$ and compute the value associated with operating forever with that policy by finding a function g determined only by given variables/parameters
    - Note: $V^0(x) = \frac{r(x,U^0(x))}{1-\beta}$
  - Step 2: for $j \geq 1$ generate a new policy by solving $U^j(x) = \arg\max_u\{r(x, u) + \beta V^{j-1}(g(x, u))\}$
  - Step 3: compute the value V^j associated with operating with U^j forever
  - Step 4: iterate until convergence of V^j
- Example: consider a social planner solving

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$\text{subject to}: \; c_t + k_{t+1} = Ak_t^{\alpha}, \quad k_0 \text{ is given}$$

where $\alpha \in (0,1)$ and $\beta \in (0,1)$

– Simplification: rearranging the constraint and substituting

$$c_t$$

into the objective function we have

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^{\alpha} - k_{t+1})$$
$$\text{subject to : } \; k_0 \text{ is given}$$

Here the State Variable is $t \to k^t$ (i.e. today's capital) and the control variable is $t \to k_{t+1}$ (i.e. tomorrow's capital). Note that indexes have been dropped since it is an infinite horizon problem with k being the state and k' being the control. The Bellman Equation given this simplication is

$$V(k) = \max_{k^{'}}\{\ln(Ak^{\alpha} - k^{'}) + \beta V(k^{'})\}$$

– Guess and Verify Solution: here we guess the functional form $V^G(k; E, F) = E + F\ln(k)$ where E and F are unknown constants. We then plug in $V^G(x; E, F)$ into the RHS of the Bellman equation yielding

$$\widehat{V}(k; E, F) = \max_{k} \Big\{ \ln(Ak^{\alpha} - k^{'}) + \beta V^G(k^{'}; E, F)\Big\}$$

Taking first order conditions results in the optimal policy function $k^{'}(k) = \frac{\beta F}{1+\beta F} Ak^{\alpha}$. Substituting the optimal policy function into the objective function yields

$$\widehat{V}(k; E, F) = \underbrace{\left[\ln\Big(\frac{A}{1+\beta F}\Big) + \beta F \ln\Big(\frac{\beta F}{1+\beta F}A\Big) + \beta E\right]}_{E^{'}} + \underbrace{\alpha(1+\beta F)\ln(k)}_{F^{'}}$$
$$\Updownarrow$$
$$\widehat{V}(k; E^{'}, F^{'}) = E^{'} + F^{'}\ln(k)$$

When grouping the constants for $\widehat{V}(k; E', F')$ it has the same functional form as $V^G(k; E, F)$. Finally, we pin down the unknown coefficients by solving the system of Equations

$$E^* = \left[\ln\Big(\frac{A}{1+\beta F}\Big) + \beta F \ln\Big(\frac{\beta F}{1+\beta F}A\Big) + \beta E\right]$$
$$F^* = \alpha(1+\beta F)\ln(k)$$

This leaves us with the final solution $V^* = E^* + F^* \ln(k)$ and the optimal policy function $k^{'*}(k) = \frac{\beta F^*}{1+\beta F^*} A k^\alpha$

- $*$ Solution Write Out: here the Bellman Equation is $V(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k^{'}) + \beta V(k^{'}) \right\}$
  - Step 1: my guess is $V^G(k) = F\ln(k) + E$ where F and E are our unknown constants
  - Step 2: using my guess we have

$$\widehat{V}(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta(Ak^\alpha - k') \right\}$$

Taking first order conditions we have the following

$$-\frac{1}{Ak^\alpha - k'} + \beta F \frac{1}{k'} = 0$$
$$\Downarrow$$
$$k^{'} = \frac{\beta F}{1+\beta F} A k^\alpha \tag{$\star$}$$

Plugging $\star$ into the objective function, we obtain the optimized value function $\widehat{V}(k)$

$$\widehat{V}(k; E, F) = \underbrace{\left[\ln\left(\frac{A}{1+\beta F}\right) + \beta F \ln\left(\frac{\beta F}{1+\beta F} A\right) + \beta E \right]}_{E'} + \underbrace{\alpha(1+\beta F)}_{F'} \ln(k)$$

  - Step 3: to evaluate the parameters (E^*, F^*) we solve the system of equations implied by the functional form

$$\widehat{V}(k) = E^{'} + F^{'} \ln(k) V^G(k) \qquad\qquad = E + F\ln(k)$$

This yields

$$E^{'} = E \Rightarrow E = \ln\left(\frac{A}{1+\beta F}\right) + \beta F \ln\left(\frac{\beta F}{1+\beta F} A\right) + \beta E$$
$$F^{'} = F \Rightarrow F = \alpha(1+\beta F)$$

Note that by $F = \alpha(1+\beta F)$ and $F^{'} = F$ we have $F^* = \frac{\alpha}{1-\alpha\beta}$ which gives us the policy function with one unknown $k^{'*}(k) = \frac{\beta F^*}{1+\beta F^*} A k^\alpha$

Stochastic Problems: consider the stochastic problem

$$\max_{\{u_t\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^\infty$$
$$\text{subject to}: \; x_{t+1} = g(x_t.U_t, \varepsilon_{t+1}), \quad \text{given } x_t$$

- iid Shocks: ε_t is a sequence of iid shocks with the cumulative distribution function $\Pr[\varepsilon_t \leq e] = F(e)$
  - Timing: ε_{t+1} is realized at $t+1$ after u_t has been chosen, therefore x_t is a sufficient statistic at

$$t$$

  - iid Assumption: this is important for recursive formulation for if shocks were persistent instead then the state vector should also incorporate past shock realizations
- Bellman Equation: $V(x) = \max_u\{r(x,u)+\beta\mathbb{E}[V(g(x,u,\varepsilon))|x]\}$ where $\mathbb{E}[V(g(x,u,\varepsilon))|x] = \int V(g(x,u,\varepsilon))dF(\varepsilon)$
- Solution Methods: the previous solution methods still apply
  - Integral Valuation: the integral in expectation can be approximated by quadrature, if closed form solutions are infeasible

Dynamic Optimisation in Continuous Time

Continuous vs Discrete Time: in discrete time the intervals between periods are $\Delta > 0$ (e.g. the interval between t and $t+1$ is 1), in continuous time the intervals between periods are $\Delta \to 0$

- Continuous Variables Issue: while state variables are often naturally continuous, when using computational methods we can only evaluation the value function on a finite number of points. Since continuous variables are not countable this becomes an issue
  - Solution: we can address this problem via using (1) discretization of the state space, or, (2) polynomial approximation to the value function

Discretization Method: consider a problem with continuous state and control variables $\mathfrak{x} \in \mathbb{R}$, discretization just replaces $\mathfrak{x}$ and $\mathfrak{u}$ by the finite grids $\widehat{\mathfrak{x}} = \{x^1, \ldots, x^n\}$ and $\widehat{\mathfrak{u}} = \{u^1, \ldots, u^n\}$. Here we discretize for a vector of unknown capital levels $k = [0, k_1, \ldots, \overline{k}]$, we need to decide how big of a grid should we create when discretizing

- Value Function Impact: now the value function becomes a finite list of numbers, $V = [V^1, \ldots, V^n]^T$
- Advantage: the maximization step is much simpler than under the original bellman equation, which is a key advantage of discretization methods
- Disadvantage: there is a "curse of dimensionality" in muiltidimensional state spaces where we can have either N points for a one-dimensional state space vs N^k points for a k-dimensional state space, which increases the number of states.
  - Grid Decision: requires some a-priori information about the state space, which is sometimes difficult to obtain (e.g. upper and lower bounds). Typically, you choose a steady state level of capital and then select the grid up to that steady state
- Optimal Growth Illustration: suppose $V(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta V(k')\}$ where in this case $\mathfrak{x} = \mathfrak{u}$
  - Discretization: if we discretize $\mathfrak{x}$ then the Bellman equation becomes $V_i = \max_j \{\ln(Ak_i^\alpha - k_j) + \beta V_j\}$ for all $i = 1, \ldots, n$ where i indexes the different possible current levels of capital and j indexes the different possible levels of future capital
  - Value Function Iteration Illustration: suppose we applied value function iteration and therefore iterate on the mapping $V_i^s = \max\left\{\ln(Ak_i^\alpha - k_1) + \beta V_1^{s-1}, \ldots, \ln(Ak_i^\alpha - k_n) + \beta V_n^{s-1}\right\}$ for all $i = 1, \ldots, n$ where s indexes the iteration step
    - Step 1: start with a guess

$$v^0_{n\times 1} = \begin{bmatrix} v_1^0 \\ v_2^0 \\ \cdots \\ v_n^0 \end{bmatrix}, \quad \text{where} \quad v_i^0 \equiv v^0(k_i)$$

Note that our guess on V^0 has a dimension of $N \times 1$ which is the vector of all values of v_0 given different levels of capital. Therefore you have $N \times 1$ guesses

* Step 2: find V^1 by solving $V_i^1 = \max_j \left\{ \ln(Ak_i^\alpha - k_j) + \beta V_j^0 \right\}$ where

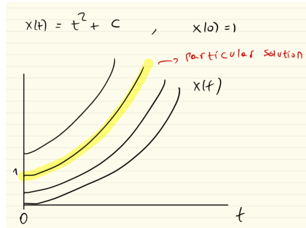
$$V_i^1 = \max \begin{bmatrix} \ln(Ak_i^\alpha - k_i) + \beta V_1^0 \\ \ln(Ak_i^\alpha - k_2) + \beta V_2^0 \\ \dots \\ \ln(Ak_i^\alpha - k_n) + \beta V_n^0 \end{bmatrix} \quad \forall i, \quad \text{which reduces to} \quad V_{n\times 1}^1 = \begin{bmatrix} V_1^1 \\ V_2^1 \\ \dots \\ V_n^1 \end{bmatrix}$$

  · Note: Where V_j^0 is the maximum element contained in the $N \times 1$ vector of $v_{n\times 1}^0$'s. Note that this involves just picking the maximum possible element in the v_i^1 vector containing every possible level of capital for today (ie period i)

* Step 3: check for convergence, setting the tolerance level to the euclidean norm
  · Tolerance Level(ε): $||v^1 - v^0|| < \varepsilon$ where $\varepsilon = \sqrt{\sum_{i=1}^n (v_i^1 - v_i^0)}$
* Step 4: compute v^s by solving $v_i^s = \max \left\{ \ln(Ak_i^\alpha - k_j) + \beta v_j^{s-1} \right\} \quad \forall i$
* Step 5: check convergence $||v^s - v^{s-1}|| < \varepsilon$, if convergence holds then stop else go back to step 4 and iterate again

Ordinary Differential Equations (ODEs): a "differential" equation" is one where the unknown is a function (instead of a variable) and the equation includes one or more of the derivatives of the function, an "ordinary differential equation" equation is one for which the unknown is a function of only one variable (typically time)

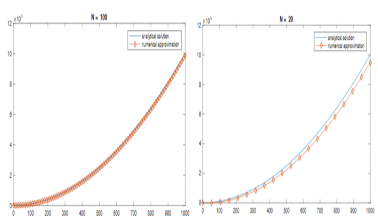
- Partial ODEs: where the unknown is a function of more than one variable
- First-Order ODE Form: $\dot{x}(t) = F(t, x(t))$, where $\dot{x}(t) \equiv dx(t)/dt$ and $t \in [t_a, t_b]$
  - Unknown: here the unknown is a function $x(t)$ with $x : [t_a, t_b] \to \mathbb{R}$ and can be found (without uniqueness) by integrating and combining with a initial condition
  - Uniqueness: the solution is not unique with the form having infinitely many solutions indexed by an integrating constant C. However, generally the constant C can be uniquely determined by requiring the solution to pass through a given point on the $x(t)$-plane, providing us with a general and particular solution
  - Example: $\dot{x}(t) = 2t$ with $t \in [0, \infty)$
    * Solution without Initial Condition: the general solution is $x(t) = t^2 + C$
    * Solution with Initial Condition: suppose we impose the initial condition $x(0) = 1$, therefore $C = 1$ and the particular solution is $x(t)t^2 + 1$



- Finite-Difference Methods for Solution: here we approximate derivates using finite-differences to approximate the solution to the ODE. This involves finding $\dot{x}(t) = F(t, x(t))$, $x(t_a) = x_a$, and $t \in [t_a, t_b]$. There are a wide range of methods depending on how the derivatives are approximated
  - Euler's Method: given the equation implied by the ODE
    * Step 1: specify a grid for t, i.e. $t_0 = t_a < t_1 < t_2 < \cdots < t_N = t_b$
    * Step 2: we can approximate the ODE using the difference equation

$$\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} = F(t_i, x(t_i))$$

      over $i = 0, \ldots, N-1$, where $x(t_0) = x_a$ is fixed by the initial condition
      · Note: the difference equation is obtained by rearranging the equation in step 1, we know all elements on the RHS of the difference equation which provides us with an explicit solution for $x(t_1)$
  - Euler's Method Illustration:
    * Step 1: discretize $[t_a, t_b]$
    * Step 2: iterate forward on $x(t_{i_1}) = x(t_i) + (t_{i+1} - t_1)F(t_i, x(t_i)) \quad \forall i$
      · i=0: $x(t_1) = x(t_0) + (t_1 - t_0)F(t_0, x(t_0))$
      · i=1: … (etc)
  - Euler's Method Numerical vs Analytical Solution Equivalency: $\dot{x}(t) = 2t, \;\; x(0) = 1, \;\; t \in [0, 1000]$



- ODE Systems: consider the system of two equations in two unknowns

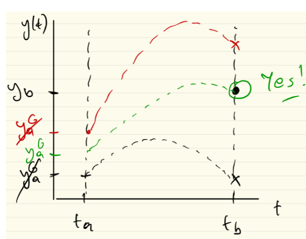
$$\dot{x}(t) = f(t, x(t), y(t))$$
$$\dot{y}(t) = g(t, x(t), y(t))$$

where $t \in [t_a, t_b]$. The general solution typically depends on two arbitrary constants, say A and B, and can be written as

$$x(t) = \phi_1(t; A, B)$$
$$y(t) = \phi_2(t; A, B)$$

Here the two arbitrary constants A and B can be pinned down via two conditions on the solution which causes two types of problems emerge. In other words, the function must be solved such that both the initial and boundary conditions hold. Note that, like single-equation ODEs, exact solutions ODE systems can only be obtained under special cases

- Initial Value Problem(IVP): where you have conditions on the initial time (a) values - here $x(t_a) = x_a$, $y(t_a) = y_a$ can be solved numerically by applying Euler's Methods
  - Step 1: discretize the domain
  - Step 2: iterate forward starting from $x(t_a) = x_a$ and $y(t_a) = y_a$
- Boundary Value Problem (BVP): where you have a condition on the value at a time period (b) that different from the initial time value (a) - here $x(t_a) = x_a$, $y(t_b) = y_b$ can be solved by applying a shooting algorithm
  - Step 1: make an initial guess for $y(t_a)$ called y_a^G
  - Step 2: solve the ODE system by applying Euler's method given $x(t_a) = x_a$ and $y(t_a) = y_a^G$
  - Step 3: if $y(t_b)$ is 'close enough' to y_b then stop, else update y_a^G and go back to step 2



· Note: the solution must cause the value of $y(t)$ to equal the a-priori $y(a)$ at time t_a and the a-priori $y(b)$ at time t_b

- Higher-Order ODEs: techniques for solving ODE systems also apply to analysing higher-order Equations. Here we can express the second order ODE equation as a first order ODE system
  - Example: consider the second-order ODE

$$\ddot{x}(t) = f(t, \dot{(t)}, x(t))$$

where $\ddot{x}(t) \equiv \partial^2 x(t)/\partial t^2$. Define the new variables $y = \dot{x}$ and you are left with the two-equation

ODE system

$$\dot{y}(t) = f(t, y(t), x(t))$$
$$\dot{x}(t) = y(t)$$

The Maximum Principle: the typical continuous time optimization problem is

$$\max_{x(t),y(t)} \int_0^{t_1} f(t, x(t), y(t)) \quad dt$$

subject to

$$\dot{x}(t) = g\big(t, x(t), y(t)\big), \quad x(t) \in \mathcal{X} \ \forall t, \quad y(t) \in \mathcal{Y} \ \forall t, \quad x(0) = x_0$$

There are two main issues with conventional solution methods to this problem; (1) we are choosing over infinitely dimensional objects such as the function $x : [0, t_1] \to \mathcal{X}$, (2) the constaints include a differential equation rather than a set of equalities/inequalities. In order to overcome these issues and find a solution, we apply the maximum principle theorem

- Notation and Assumptions: x is the state variable, y is the control variable, $\mathcal{X} \subset \mathbb{R}$ and $\mathcal{Y} \subset \mathbb{R}$ are nonempty and convex, f and g are continuously differentiable in their arguments, we define a Hamiltonian, and for simplicity we assume $t_1 < \infty$
  - Hamiltonian: we define the hamiltonian

$$H(t, x(t), y(t), \mu(t)) \equiv f(t, x(t), y(t)) + \mu(t)g(t, x(t), y(t))$$

    where $\mu(t)$ is a continuously differentiable function called the costate variable

- Maximum Principle Theorem: suppose that the aforementioned continuous time problem has an interior continuous solution $(\widehat{x}(t), \widehat{y}(t))$. Then there exists a continuously differentiable function $\mu(t)$ such that

$$H_y(t, \widehat{x}(t), \widehat{y}(t), \mu(t)) = 0 \ \ \forall t \in [0, t_1]$$
$$\dot{\mu}(t) = -H_x(t, \widehat{x}(t), \widehat{y}(t), \mu(t)) \ \ \forall t \in [0, t_1]$$
$$\mu(t_1) = 0$$

  - ODE Usage: by $H_y(t, \widehat{x}(t), \widehat{y}(t), \mu(t)) = 0$ we can solve for y as a function of x thereby writting $y = Y(t, x\mu)$. Plugging $y = Y(t, x\mu)$ into $\dot{\mu}(t) = -H_x(t, \widehat{x}(t), \widehat{y}(t), \mu(t))$ and $\dot{x}(t) = g\big(t, x(t), y(t)\big)$

and combining the law of motion we are left with

$$\dot{x} = g\big(t, x, Y(t, x, \mu)\big)$$
$$\dot{\mu} = -H_x\big(t, x, Y(t, x, \mu), \mu\big)$$
$$x(0) = x_0, \quad \mu(t_1) = 0$$

which is an ODE system in x and μ that can be solved numerically by applying the shooting algorithm

- Sufficient Condition Requirements: the maximum principle only provides necessary conditions for an optimum, sufficient conditions for a maximum rely on concavity properties of the objective and on convexity of the feasible set
- Technique Generalization: the techniques generalize to multidimensional problems (where x and y are vectors), infinite horizon problems (ie $t_1 = \infty$), and to including terminal conditions on the final state (e.g. $x(t_1) = x_1$)
- Index: the variable t may represent time or any other index

Heuristic Proof of The Maximum Principle: to solve problem (x) it would be natural to start trying "some sort" of lagrangian optimization such as constructing the function

$$\mathcal{L} = \int_0^{t_1} \{F(t, x(t), y(t)) + \mu(t)[g(t, x(t), y(t)) - \dot{x}(t)]\}$$

where $\mu(t)$ is a continuously differentiable function from the hamiltonian. Supposing that $(\widehat{x}(t), \widehat{y}(t))$ is an interior solution to the lagrange problem, then $(\widehat{x}, \widehat{y})$ should maximize $\mathcal{L}$.

- Lemma: the function $\mathcal{L}$ can be written as

$$\mathcal{L} = \int_0^{t_1} \{f(t, x(t), y(t)) + \mu(t)g(t, x(t), y(t)) + \dot{\mu}(t)x(t)\} \quad dt - \mu(t_1)x(t_1) + \mu(0)x(0)$$

  - Proof: note that

$$\int_0^{t_1} \tfrac{\partial(\mu(t)x(t))}{\partial t} \quad dt = \int_0^{t_1} \dot{\mu}(t)x(t) \quad dt + \int_0^{t_1} \mu(t)\dot{x}(t) \quad dt$$

    If we integrate both sides over $[0, t_1]$ then we have

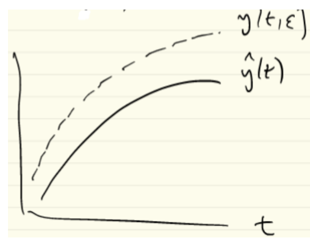
$$\mu(t_1)x(t_1) - \mu(0)x(0) = \int_0^{t_1} \dot{\mu}(t)x(t) \quad dt + \int_0^{t_1} \mu(t)\dot{x}(t) \quad dt$$

This can be rearranged as

$$\mu(t)\dot{x}(t) \;\; dt = \mu(t_1)x(t_1) - \mu(0)x(0) - \int_0^{t_1} \dot{\mu}(t)x(t) \;\; dt$$

Plugging this into our lagrangian yields the above lemma

- <u>Proof of Function Features</u>: consider a variation/perturbation around the optimal path $(\widehat{x}(t), \widehat{y}(t))$. Specifically, take an arbitrary function $P_y(t)$, let $\varepsilon \in \mathbb{R}$, and define $y(t,\varepsilon) = \widehat{y}(t) + \varepsilon P_y(t)$. Here$y(t,\varepsilon)$ is the solution plus a deviation from the solution $(\widehat{y}(t) + \varepsilon P_y(t))$ Graphically



Similarly, let $x(t,\varepsilon) = \widehat{x}(t) + \varepsilon P_x(t)$ where $P_x(t)$ denotes the corresponding perturbation to $\widehat{x}(t)$ when $\widehat{y}(t)$ is varied according to $P_y(t)$. Note that the sum of the perturbations is the variation around the optimum. Define the 'perturbed' lagrangian function as

$$\mathcal{L}(\varepsilon) = \int_0^{t_1} \{f(t, x(t,\varepsilon), y(t,\varepsilon)) + \mu(t)g(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\mu}(t)x(t,\varepsilon)\} \quad dt - \mu(t_1)x(t_1,\varepsilon) + \mu(0)x_0$$

Recall that $\mathcal{L}$ is maximized at $\widehat{x}, \widehat{y}$, where the marginal gain from increasing ε (ie the perturbation) is 0, and therefore we must have

$$\tfrac{\partial \mathcal{L}(\varepsilon)}{\partial \varepsilon}|(\varepsilon, x, y) = (0, \widehat{x}, \widehat{y}) = 0$$

Using the definitions of $x(t,\varepsilon)$, $y(t,\varepsilon)$, and $\mathcal{L}$ the above equation can be expressed as:

$$\begin{aligned} 0 = &\int_0^{t_1} \Big\{ \overbrace{f_x(t, \widehat{x}(t), \widehat{y}(t)) + \mu(t)g_x(t, \widehat{x}(t), \widehat{y}(t))}^{H_x(t,\widehat{x}(t),\widehat{y}(t),\mu(t))} + \dot{\mu}(t) \Big\} P_x(t) \quad dt \\ &+ \int_0^{t_1} \underbrace{\Big\{ f_y(t, \widehat{x}(t), \widehat{y}(t)) + \mu(t)g_y(t, \widehat{x}(t), \widehat{y}(t)) \Big\}}_{H_y(\cdot)} P_y(t) \quad dt - \mu(t_1)P_x(t_1) \end{aligned}$$

  - <u>Note</u>: this expression can only hold for all feasible perturbation paths (P_x, P_y) if each integrand vanishes and $\mu(t_1) = 0$ such that $P_y(t)$ and $P_x(t)$ equal 0, which are the conditions of the maximum principle

Enforcement Frictions I

Overview: the analysis of contractual arrangements and enforcement frictions aims to rectify conflicts of interest that arise from violations of two assumptions - (1) both parties were perfectly committed to the rules of the exchange, (2) both parties in the exchange shared the same information. Dropping either of these assumptions creates conflicts of interest

- Assumption 1 Violation: here an agent who is not committed to the rules of the exchange would deviate from such rules when she finds it more convenient
- Assumption 2 Violation: here an agent who possesses private information would manipulate such information to her advantage

Contract Theory: the study of the economic aspects of contract design where a contract is a set of rules that facilitates cooperation among individuals with conflicting objectives. The goal of contract theory is twofold with the normative goal of how to draw up better contracts and the positive goal of analysing why real life contracts have various forms and designs. We can arbitrarily classify contract theory models by two classifications - (1) according to the type of private information, (2) according to the duration of the contractual relationship

- Type of Private Information: here we assume that the uninformed party designs the contract and address two types of information issues (adverse selection and moral hazard)
  - Adverse Selection: asymmetric information about the characteristics of the informed party (e.g. life insurance where the insured's state of health is unknown by the insurer)
  - Moral Hazard: asymmetric information about the actions of the informed party (e.g. CEO compensation where the CEO's actions are imperfectly observed by the Board of Directors)
- Duration of Contractual Relationship: here we address two types of contract periods (static and dynamic)
  - Static Contracts: relationship lasts for one period
  - Dynamic Contracts: relationship lasts for more than one period (gives rise to enforcement/commitment issues)

Principal-Agent Paradigm: since contracts involve two parties we need to address how the parties bargain over the terms of the exchange. We bypass the difficulties inherent to the bargaining process by allocating all bargaining power to one of the parties (the Principal) who will offer a take-it or leave-it contract to the Agent. The Principal-Agent Paradigm is the norm in contract theory

- w.l.o.g: the set of Pareto optimal contracts can be characterised by maximizing the utility of one agent subject to a given level of utility of the other agent. This outcome is a result of the Principal-Agent analogy

Enforcement Frictions: occur when agents are free to walk away from the contract at any time, note that for most models of enforcement frictions we assume that all information is public

- One-Sided Limited Commitment: occurs when the principal is committed but the agent is not
  - Applications: long term labour contracts (principal is the firm and agent is the worker), international lending (principal is the lending country and agent is the borrowing country), firm dynamics (principal is the bank, agent is the entrepreneur seeking to finance a project)

One-Sided No Commitment Model: the economy lasts for $T = \infty$ periods and consists of a village with a large number of households. Household preferences are

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ and u is concave and has the usual properties. Each HH receives a stochastic endowment stream $\{y_t\}_{t=0}^{\infty}$ where $y_t \in \{\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_S\}, \overline{y}_{s+1} > \overline{y}_s$. HHs can only trade with a risk-neutral moneylender who can borrow or lend at the risk free rate $R = \beta^{-1}$. The moneylender maximizes the expected present value of profits from making contracts:

$$P = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (y_t - c_t)$$

where $(y_t - c_t)$ is the net transfer from HH at time t

- Assumptions: y_t is iid across time and HHs with $\Pr(y_t = \overline{y}_s) = \Pi_S$, due to concavity HHs value insurance against endowment realizations, y_t is non-storage and therefore self-insruance is precluded, a moneylender (ie the planner) is the only person who has access to a risk-free loan market outside the village, HHs cannot trade with one another
- Contract: a contract between the moneylender and the HH is a sequence of history dependent functions $\{c_t\}_{t=0}^{\infty}$ with $c_t = f_t(y^t)$. The contract specifies that in each period the HH contributes her endowment realization y_t in exchange for consumption c_t
  - Contractual Friction: the HH can walk away from the contract. If the HH walks away then the HH consumers their endowment and lives in autarky evermore. The HH cannot return from autarky
  - One-Sided No Commitment: only the HH can break the contract, the moneylender is fully committed to keeping their promise
  - Information is Symmetric: the moneylender can observe y_t
- Optimal Contract Solution: the optimal contract can be analysed using recursive tools, this relies on

knowing the continuation utility under autarky (v_{aut}) where:

$$v_{aut} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(y_t) = \frac{1}{1-\beta} \sum_{s=1}^{S} \Pi_s u(\overline{y}_s)$$

– Participation Contraint: since autarky is the outside option of the agents, the optimal contract must satisfy the participation constaints:

$$u(\overbrace{f_t(y^t)}^{c_t}) + \underbrace{\beta \mathbb{E}_t \sum_{j=1}^{\infty} \beta^{j-1} u(\overbrace{f_{t+j}(y^{t+j})}^{c_{t+j}})}_{w_t(y^t)\text{: continuation utility at t}} \geq u(y_t) + \beta v_{aut}$$

for all t, y^t A contract that satisfies the participation contraint is said to be self-enforcing. Participation constraints can be written as

$$u(f_t(y^t)) + \beta w_t(y^t) \geq u(y_t) + \beta v_{aut}$$

for all t, y^t where

$$w_t(y^t) \equiv \mathbb{E}_t \sum_{j=1}^{\infty} \beta^{j-1} u(f_{t+j}(y^{t+j}))$$

is the continuation utility of the agent at time t, given endowment history y^t

– Moneylender Problem: the money lender solves

$$P(v) = \max_{\{c_s, w_s\}} \sum_{s=1}^{S} \Pi_s[\overline{y}_s - c_s + \beta P(w_s)]$$

subject to

$$\sum_{s=1}^{S} \Pi_s[u(c_s) + \beta w_s] \geq v \rightarrow \text{ Promise Keeping Constraint}$$

$$u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v_{aut}, \quad s = 1, \ldots, S \rightarrow \text{ Participation Constraint}$$

Here $w_s \in [v_{aut}, \overline{v}]$ where $\overline{v}$ is a large number. We have one state variable, promised utility v. The Promise Keeping Constraint renders the planning problem time-consistent.

* Promised Utility w: encodes all history dependence in the contract, allowing one to summarize large dimensional shocks and rewards into a single variable
* Characterisation: it can be shown that the constraint set is convex and P is stictly concave (as well as differentiable). Given this we have the following lemmas that essentially state that agents have incentives to walk away when income realisations are large enough

· Lemma 1: for each v, there exists a threshold $\underline{y}(v)$ such that the PC binds binds if $\overline{y}_s \geq \overline{y}(v)$

· Lemma 2: the threshold $\overline{y}(v)$ is increasing in v

· Proof: we write the lagrangian as

$$\mathcal{L} = \sum_{s=1}^{S} \Pi_s[\overline{y}_s - c_s + \beta P(w_s)]$$
$$+\varepsilon \left\{ \sum_{s=1}^{S} \Pi_s[u(c_s) + \beta w_s - v \right\}$$
$$+\sum_{s=1}^{S} \eta_s \left\{ u(c_s) + \beta w_s - u(\overline{y}_s) - \beta v_{aut} \right\}$$

where ε and η are the lagrangian multipliers on the Promise Keeping Constraint and Participation Contraint respectively. Taking first order conditions and applying the envelope theorem yields the following:

$$\frac{\partial \mathcal{L}}{\partial C_s} = 0 \rightarrow -\Pi_s + \varepsilon \Pi_s u^{'}(C_s) + \eta_s u^{'}(C_s) = 0 \tag{(1)}$$
$$\frac{\partial \mathcal{L}}{\partial w_s} = 0 \rightarrow \eta_s + \varepsilon \Pi_s = -\Pi_s P^{'}(w_s) \tag{(2)}$$
$$\text{Envelope Theorem :} \quad \rightarrow P'(v) = -\varepsilon < 0 \tag{(3)}$$

Rearranging equations (1) and (2) for $\varepsilon\Pi_s + \eta_s$ and solving them simultaneously yields

$$u^{'}(C_s) = -P^{'}(w_s)^{-1} \tag{(4)}$$

It is clear to see that the LHS of this equation falls with C_s due to concavity of $u(\cdot)$ and the RHS of this equation falls with w_s due to concavity of $P(\cdot)$. This implies that C_s and w_s are positively related.

Combining (3) and (2) yields the law of motion for continuation Utility

$$P^{'}(w_s) = P^{'}(v) - \frac{\eta_s}{\Pi_s} \tag{(5)}$$

Lemma 1 Proof

Take as given that $\overline{y}_s$ subject to PC is not binding (i.e. $\eta_s = 0$), therefore by (5) we have

$$P^{'}(w_s) = P^{'}(v) \Rightarrow w_s = v \text{ by concavity}$$

Using this fact into (4) we have

$$u^{'}(C_s) = -P^{'}(v)^{-1}$$

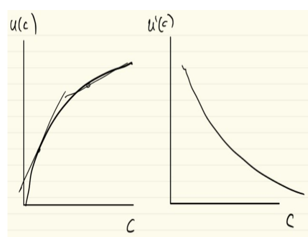
Which can be rewritten as $C_s = g_1(v)$, $g_1^{'} > 0$. This allows us to rewrite the PC as

$$u(C_s) + \beta w_s > u(\overline{y}_S) + \beta v_{aut}$$
$$\Downarrow$$
$$u(g_1(v)) + \beta v > u(\overline{y}) + \beta v_{aut} \equiv \overline{y}(v)$$
$$\Downarrow$$
$$u^{-1}(u(g_1(v))_{\beta}(v - v_{aut})) \equiv \overline{y}(v) > \overline{y_s}$$

Thus, given the definition of $\overline{y}(v)$, the PC is not binding if $\overline{y}_s < \overline{y}(v)$

<u>Lemma 2 Proof</u>

Given the definition of $\overline{y}(v)$ Q.E.D.



- <u>Properties of PC Binding vs Not Binding</u>: suppose PC is not binding, then $w_s = v$ and $C_s = g_1(v)$ where $g_1^{'} > 0$. Suppose PC is binding, then $w_s = l(\overline{y}_s) > v$ and $C_s = g_2(\overline{y}_s) \in (g_1(v), \overline{y}_s)$ where $l^{'}, g_2^{'} \geq 0$
  * <u>Note</u>: w_s and C_s never decrease, they are either constant (if PC is not binding) or they increase (if PC is binding)
  * <u>Not Binding Proof</u>: shown in the proof of the lemma
  * <u>Binding Proof</u>: assume that PC is binding
    · <u>Step 1</u>: show that w_s and c_s are only functions of $\overline{y}_s$

    By FOC (4) we have $u^1(C_s) = -P^{'}(w_s)^{-1}$ and given the face that PC is binding we have $u(C_s) + \beta w_s = u(\overline{y}_s) + \beta v_{aut}$. Note that these two equations form a system of 2 equations in 2 unknowns (C_s, w_s). Since promised v does not show up in this system we can write its solution as $w_s = \updownarrow(\overline{y}_s)$ and $C_s = g_2(\overline{y}_s)$

    · <u>Step 2</u>: show that $g_2^{'} > 0$ and $\updownarrow^{'} > 0$

    By $u(C_s) + \beta w_s = u(\overline{y}_s) + \beta v_{aut}$ we know that $[u(C_s) + \beta w_s]$ is non-decreasing in $\overline{y}_s$. Also, by $u^1(C_s) = -P^{'}(w_s)^{-1}$ (as shown previously), C_s is non-decreasing in w_s and hence C_s

should be non-decreasing in $\overline{y}_s$

- Step 3: show that $w_s > v$

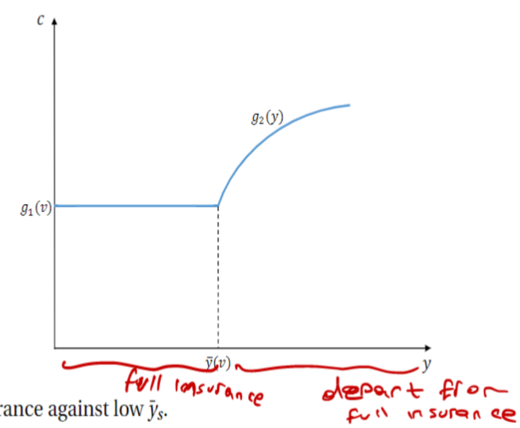
  By $P^{'}(w_s) = P^{'}(v) - \frac{\eta_s}{\Pi_s}$, if the PC binds then $\eta_s > 0$ and so $P^{'}(w_s) < P^{'}(v) \Rightarrow w_s > v$ via concavity

- Step 4: show that $C_s \in (g_1(v), \overline{y}_s)$

  A binding PC implies that $u(C_s) + \beta w_s = u(\overline{y}_s) + \beta v_{aut}$. Therefore, $c_s < \overline{y}_s$. Because $w_s > v$ we have

$$-P^{'}(w_s)^{-1} < -P^{'}(v)^{-1}$$
$$\Downarrow$$
$$u^{'}(C_s) < u^{'}(g_1(v)) \text{ by } (4)$$
$$\Downarrow$$
$$C_s > g_1(v)$$

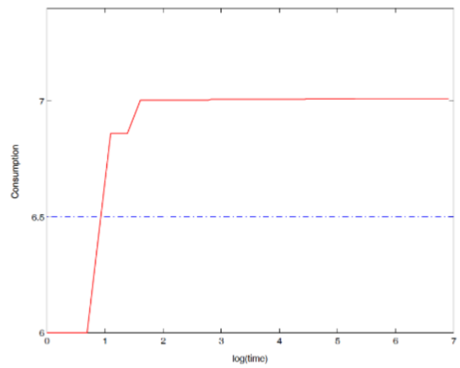
– Consumption under Oprtimal Contract: graphically, given v, consumption under the optimal contract is



Full insurance against low $\bar{y}_s$.

$cov(c_s, \bar{y}_s) > 0$ for large $\bar{y}_s$.

– Long Run Dynamics: c_t increases over time until $\overline{y}_S$ is realized



In graph: $c_t \to g_1(\bar{v})$, where $\bar{v}$ is given by:

$$u(g_1(\bar{v})) + \beta\bar{v} = u(\bar{y}_S) + \beta v_{aut}.$$

All agents eventually become fully insured (no fluctuations in consumptions). With many HHs, all agents eventually get to consume $g_1(\overline{v})$: $\lim_{t\to\infty} \Pr(c_t = g_1(\overline{v})) = 1$. Overall, we have "fanning in" of the cross-sectional distribution of consumption and full risk sharing in the long run

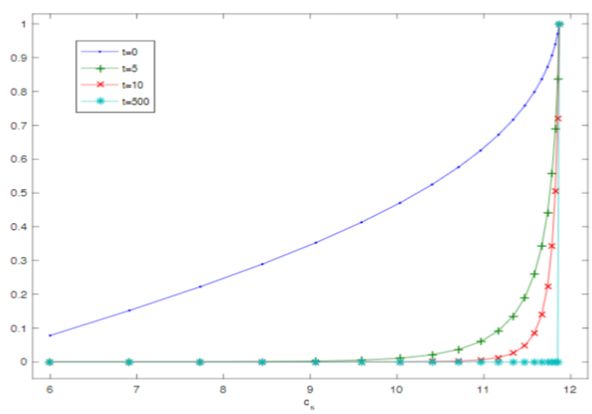


Figure: Consumption cdf.

- Connection to Labour Contracts: early work on one-sided limited commitment models aimed at analysing the characteristics of labour contracts. In our model this can be mapped as C being the wage offered within a firm, y shock being wages offered in the spot market, and the enforcement friction being labour mobility. Since workers are risk-averse, they value insurance against fluctuations in the spot market for wages (i.e. implicit insurance premium in contractual wages). Firms find contractual arrangement beneficial because in the short run they can pay wages below those in the spot market (due to the implicity insurance premium), which is embedded by $C_s < \overline{y}_s$ when PC binds in our model