Macroeconomics A Notes

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Contents

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Mathematical Concepts

1.1 Concepts for Consumer Theory

In this section, the concepts of consumption sets, open balls, and the properties of different types of sets will be explored. The section will also briefly provie an overview of convergence and continuous functions.

1.1.1 Consumption/Choice Set X

The set of all alternative, or complete consumption plans, that the consumer can conceive - whether achievable or not

Alternative Definition: the consumpsion set is the entire non-negative orthan, $X = \mathbb{R}^n_+$

Consumption Bundle: a consumption bundle of n goods is defined as a vector $x == (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, specifying the quantities of the n available commodities

Properties of Consumption Set: (1) $X \subseteq \mathbb{R}^n_+$, (2) X is closed, (3) X is convex, (4) $0 \in X$

1.1.2 Complements of Sets

For a given subset $A \subseteq X$, we define the complement of A in X to be the set $A^c = \{x \in X : x \notin A\}$

Alternative Definition: the complement set of set A in X is all the stuff in the universe X that is not in A

Application: typically we define either *open sets* or *closed sets* and the other (*closed* or *open*) as the complement of the first

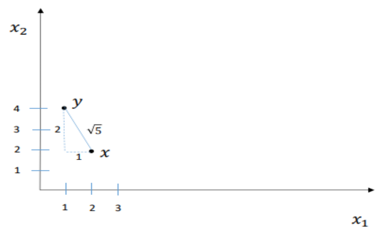
1.1.3 Metric Distance

In huge space $\mathbb{R}^n_+$, open balls (neighbourhoods) are determined using a metric distance function

Euclidean Metric: measures the distance between two points $x, y \in \mathbb{R}^n_+$ as $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

- n-Dimensional Spaces: in a two dimensional space this is equivalent to $d(x, y) == \sqrt{(x_1 - y_1)^2 + (x_2 = y_2)^2}$, while in a one dimensional space this is equivalent to $d(x, y) = \sqrt{(x_1 - y_1)^2} = x_1 - y_1$
- Triangular Inequality: for all $x, y, z \in \mathbb{R}^n_+$, $d(x, y) \leq d(x, z) + d(z, y)$

Euclidean Metric Example: the distance between $x = (2, 2)$ and $y = (1, 4)$ is $d(x, y) = \sqrt{(2-1)^2 + (2-4)^2} = \sqrt{5}$

- 

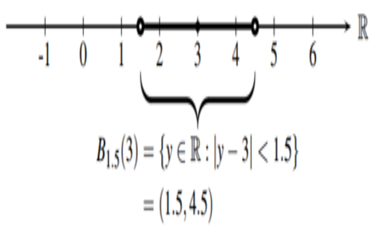
1.1.4 Open Balls (neighborhoods)

An open ball $B_\varepsilon(x) = \left\{ y \in \mathbb{R}^n_+ : d(x,y) < \varepsilon \right\}$ is the set of points that are closer in Euclidean distance to $x \in \mathbb{R}^n_+$ than some real number $\varepsilon > 0$

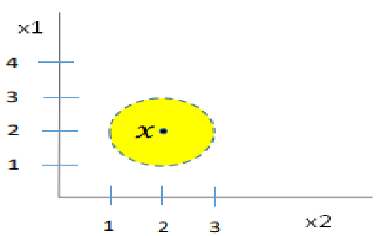
Alternative Definition: the open ball $B_\varepsilon(x)$ with radius $\varepsilon > 0$ and center $x \in \mathbb{R}^m$ is the set of points $y \in \mathbb{R}^m$ with distance less than r from x, i.e. $B_\varepsilon(x) = \{y \in \mathbb{R}^m : \ ||y - x|| < \varepsilon\}$

- Note: the theorem must contain $<$ instead of $\leq$ because otherwise the ball would contain its boundary points making it a closed instead of open ball. In otherwords, for an open ball each point in the element must be less than a selected radius on a plane

Example 1: in $\mathbb{R}^1_+$ an open ball is an open interval on the number line. For $B_{1.5}(3)$ the circles at the endpoints of the interval $(1.5, 3.5)$ indicate that 1.5 and 3.5 are not in the open ball

- 

Example 1: for $x = (2,2)$ and $\varepsilon = 1$, the open ball $B_\varepsilon((2,2)) = \left\{ y \in \mathbb{R}^n_+ : \ \sqrt{(y_1-2)^2 + (y_2-2)^2} < 1 \right\}$

- 

1.1.5 Open Set

A set $A \subseteq \mathbb{R}^n_+$ is open if and only if for each point $x \in A$ there is a radius $\varepsilon_x > 0$ such that the open ball $B_\varepsilon(x) \subseteq A$

Alternative Definition: a set $S \in \mathbb{R}^m$ is open if and only if for each $x \in S$ there exists an open ball around x that is completely contained in S

Boundary Points: open sets cannot contain their boundary points, in other words the limit is not in the set since there is no open ball around $limit_x$ that is entirely in S. This distinguishes open sets from closed sets

Theorem 1: every open ball $A = B_\varepsilon(x)$ is an open set

- Proof Outline: take an arbitrary point y in $B_\varepsilon(x)$ and prove that any arbitrary point z contained in $B_{\varepsilon'}(y)$ is also contained in $B_\varepsilon(x)$
- Proof: take an arbitrary point y in $B_\varepsilon(x)$. We need to show that there is an open ball $B_{\varepsilon'}(y)$ around y that is contained in $B_\varepsilon(x)$. To do this, we prove that taking an arbitrary point y in $B_\varepsilon(x)$ implies that $d(x, y) < \varepsilon$. Let $\varepsilon' = \varepsilon - d(x, y) > 0$, we have that $d(x, y) = \varepsilon - \varepsilon'$. Using $d(x, y) = \varepsilon - \varepsilon'$ and picking an arbitrary $z \in B_{\varepsilon'}(y)$ we can show that $d(x, z) < \varepsilon$. By triangular inequality, $d(x, z) \leq d(x, y) + d(y, z)$ and plugging $d(x, y) = \varepsilon - \varepsilon'$ into this yields $d(x, z) \leq \varepsilon - \varepsilon' + d(y, z)$. Since z is contained in $B_{\varepsilon'}(y)$, it must be that $d(y, z) < \varepsilon'$. Using $d(y, z) < \varepsilon'$ yields $d(x, z) \leq \varepsilon - \varepsilon' + d(y, z) < \varepsilon - \varepsilon' + \varepsilon' = \varepsilon$. Since every point z is contained in $B_{\varepsilon'}(y)$, $d(x, z) < \varepsilon$ implies that $d(x, y) < \varepsilon$. Therefore, $B_{\varepsilon'}(y) \subseteq B_\varepsilon(x)$.
- Logic: since ε can be infinitismal, the infinitismal open ball $B_\varepsilon(y)$ contained in the open ball $B_\varepsilon(x)$ does not contain the boundary points of $B_\varepsilon(x)$

Theorem 2: $A = \mathbb{R}^n$ is open in $\mathbb{R}^n$, $A = \mathbb{R}^n_+$ is open in $\mathbb{R}^n_+$

- Logic 1: take a point on the boundary. Since, by definition, open balls only include elements contained in $\mathbb{R}^n$ then any open ball that goes around the boundary point excludes points outside $\mathbb{R}^n$. The same is true for R^n_+ if we consider only positive natural numbers in our universe and therefore restrict the definition of open balls to only include points in $\mathbb{R}^n_+$. In otherwords, every point that is in the ball is in the universe
- Logic 2: take any point $x \in A$ and let $\varepsilon_x = \min\{x_1, x_2, \ldots, x_n\}$. Then every point in the open ball $B_{varepsilon_x}(x) \in \mathbb{R}^n_+$. For example, let $x = (3, 4) \in R^2_+$ and therefore $\varepsilon_x = \min\{3, 4\} = 3$ - every point y in the open yellow ball will be in $\mathbb{R}^2_+$ since each point y satisfies $y_i > 0$ for $i = 1, 2$.

Theorem 3: $A = \varnothing$ is open in $\mathbb{R}^n_+$

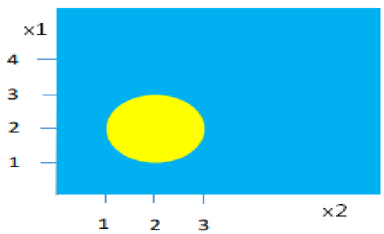
- Proof: observe that the conditions of openness of a set is satisfied if each $x \in A$ has an open neighbourhood in A. Since there is no x in $A = \varnothing$, it follows that each $x \in A$ has an open ball in A
- Both Open and Closed: the empty set $\varnothing$ is both open and closed, this is as $\forall x \in \varnothing$ x is an interior point (i.e. is open) and since the set of boundary points of $\varnothing$ is the empty set (i.e. is closed)
  - Complement: since the complement of $A = \varnothing$ is $\mathbb{R}^n_+$ and the complement of $A = \mathbb{R}^n_+$ is $\varnothing$ then both $\mathbb{R}^n_+$ and $\varnothing$ are closed

1.1.6 Closed Set

A set $A \subset \mathbb{R}^n_+$ is closed if and only if the complement $A^c = \left\{x \in \mathbb{R}^n_+ :\ x \notin A\right\}$ of A is an open set

Alternative Definition: a set $S \in \mathbb{R}^m$ is closed if, whenever $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence completely contained in S, its limit is also contained in S - in other words, a set $S \subseteq \mathbb{R}^m$ is closed if it contains all of its boundary points

Example 1: continuing from example 1 of open balls, the complement of the yellow open ball is a closed set. Formally, this is the entire positive quadrant excluding the yellow open ball, $A^c = \left\{x \in \mathbb{R}^n_+ : \ x \notin B_1((2,2))\right\}$

- 

1.1.7 DeMoran's Laws

(Law 1) the complement of the union of any family of sets is equal to the intersection of the complements of the family

(Law 2) the complement of the intersection of any family of sets is equal to the union of the complements of the family

Implications: (1) the empty set $\varnothing$ and universal set $\mathbb{R}^n_+$ are both closed and open, (2) the union of any finite family of closed(open) sets is closed(open), (3) the intersection of any family of closed(open) sets is closed(open)

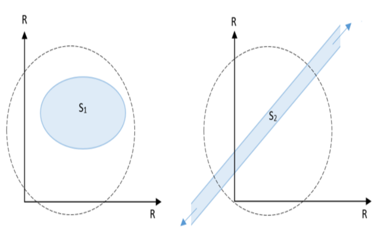
- Finite Intersection Logic: if we take the intersection of two finite open sets $A = \cap_{K \in \mathbb{N}}(1 - \frac{1}{k}, 1 + \frac{1}{k})$ then only $1 \in A$ and $A = \{1\}$ is not open
- Note: implications 2 and 3 arise by combining Law 1 and Law 2

1.1.8 Bounded Set

A set $A \subseteq \mathbb{R}^n_+$ is bounded if and only if we can put an open ball in $\mathbb{R}^n$ around the set A

Alternative Definition: a set is bounded if there is an open ball that contains it, i.e. $S \subseteq R^m$ is bounded if $\exists K > 0 : \ ||x|| < K, \ \forall x \in S$

Example: the left image is bounded, the right is not bounded

- 

1.1.9 Compact Set

A set $A \subseteq \mathbb{R}^n_+$ is bounded if and only if it is both closed and bounded

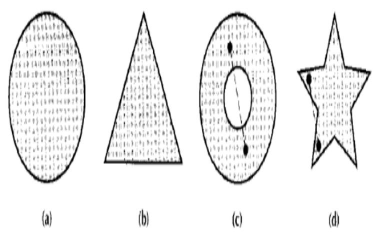
Example: $B_1 = \{(x_1, x_2) \in mathbbR^2 : p_1x_1 + p_2x_2 \leq w\}$ is closed but not bounded. However, $B_2 = \{(x_1, x_2) \in \mathbb{R}^2 : p_1x_1 + p_2x_2 \leq w, x_1 \geq 0, x_2 \geq 0\}$ is closed and bounded and therefore compact.

1.1.10 Convex Set

A set $A \subseteq \mathbb{R}^n_+$ is a convex set if and only if for all $x, y \in A$, and each $\alpha \in [0, 1]$, the point $\alpha x + (1-\alpha)y$ is also in A

Alternative Definition: a set $S \subseteq \mathbb{R}^n$ is convex if and only if for every two points of the set the line segment between the two points also belongs to the set

Example 1: parts (a) and (b) are convex sets since all points between any two points are contained in the set, parts (c) and (d) are not convex sets since they contain two points where points between them are not contained in the set

- 

Example 2: suppose we have two bundles $x = (2, 3)$ and $y = (4, 7)$ for goods x_1 and x_2. Using $\alpha = \frac{1}{3}$, is the set $x, y \in A$ convex?

- Answer: $\alpha(2, 3) + (1-\alpha)(4, 7) = \frac{1}{3}(2, 3) + \frac{2}{3}(4, 7)$. This yields $(\frac{1}{3}(2) + \frac{2}{3}(4), \frac{1}{3}(3) + \frac{2}{3}(7)) = (\frac{10}{3}, \frac{17}{3}) = (x_1', x_2')$ which is contained in A and therefore convex.

Example 3: the below figure is convex though not closed

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Example 3: the below budget sets are not convex

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$[0,10) \cup (10,20]$

1.1.11 Convergent Sequences

Occurs when for all $\varepsilon > 0$ there exists a $\mathbb{N}$ such that for all values of n greater than $\mathbb{N}$ the n^{th} *elements value in the sequence minus the limit's value* will be less than ε

Mathemtical Definition: let $(x_n)_{n\in\mathbb{N}}$ be a sequence with codomain $\mathbb{R}$. We say that (x_n) converges to x^* or that x^* is the limit of $(x_n)_{n\in\mathbb{N}}$ if: $\forall\varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : \ |x_n - x^*| < \varepsilon$

Sequence: a sequence of real numbers is an assignment of a real number to each natural number. In other words, $f : N \to X$ is a function that matches a set of natural numbers, n, to a set of numbers, x. A sequence with a limit is called convergent

- Convergence Defined by the Limit of a Sequence: let $\{x_1, x_2, x_3\}$ be a sequence of real numbers and let r be a real number. We say that r is the limit of this sequence if for any small positive number ε, there is a positive integer N such that for all $n \geq N : \ |x_n - r| < \varepsilon$, i.e. x_n is in the interval about r given ε for all n past given point N

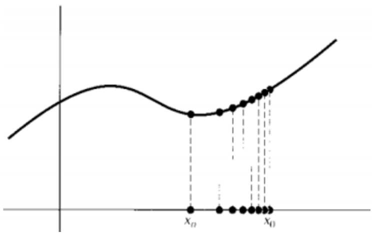
1.1.12 Continuous Functions

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous if and only if the inverse image $f^{-1}(B) = \{x \in \mathbb{R}^n_+ : f(x) \in B\}$ of each open ball B in the range $\mathbb{R}$ is also open in the doman $\mathbb{R}^n_+$

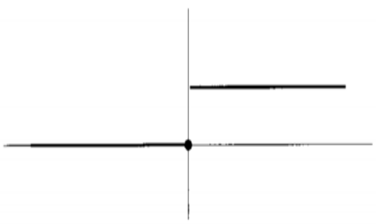
Alternative Definition 1: let $D \subseteq \mathbb{R}^k$. A function $f : D \to \mathbb{R}^m$ is continuous at $x = (x_1, x_2, \ldots, x_k) \in D$ if for every sequence $(x_n)_{n\in\mathbb{N}} \in D$ that converges to x, the sequence $(f(x_n))_{n\in\mathbb{N}} \in \mathbb{R}$ converges to $f(x)$. The function is continuous if it is continuous at x for all $x \in D$

Explanation: a function that maps nearby points into nearby points - i.e. you can draw the graph of the function without lifting your pencil from the paper. In other words, if we want to get all the $f(x)$ values to stay in some small neighborhood around $f(x_0)$ we simply need to choose a small enough neighborhood for the x values around x_0. If we can do that no matter how small the $f(x)$ neighborhood is, then f is continuous at x_0

Continuous Example: the below function is continuous as for any x_0 that we select and converges, the codomain converges to y_0

- 

Discontinuous Example: the function $f(x) = \{1 \text{ if } x > 0, 0 \text{ if } x \leq 0\}$ is not continuous as the codomain does not converge to the y_0 given by the x_0 selected. In other words, for $\varepsilon < 1$ the definition of convergence is violated at $x_0 = 0$

- 

Using original definition: the image of $f(x)$ consists of all the elements of y that an element of x is mapped to. If $f(x) = \{x \text{ if } x < 10, x + 10 \text{ if } x \geq 10\}$ and we set the inverse image open ball $B = (15, 25)$, then we have that $f^{-1}(15, 25) = \{x \in \mathbb{R} : f(x) \in (15, 25)\} = [10, 15)$. This is as $x \geq 10$ is not open in the domain for the open ball in the range $f^{-1}(15, 25)$

1.1.13 Concave Functions

f is concave if and only if $\forall x, y$ and all $t \in [0, 1]$ we have that $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$

1.1.14 Homogeneous of Degree k

A real value function f defined on a subset X of $\mathbb{R}^n$ is *homogeneous of degree k* if and only if $f(tx) = t^k f(x)$ for all $t \in \mathbb{R}_{++}$ and all $x \in X$

Properties: if you mutliply all inputs by a constant ($t > 0$) then you will generate an output equal to t^k times the function, for instance if $k = 1$ and you multiply all inputs by $t = 2$ then you will get twice the output

Consumer Theory

2.1 Preferences

In this section the concepts of preference relations will be explored. In addition to this, the consumer choice axioms will be outlined to provide a framework for proving and understanding utility representations

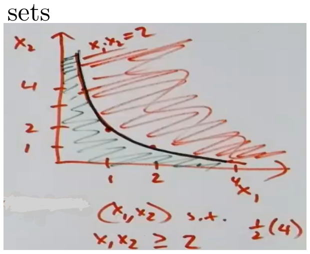
2.1.1 Preference Relation

Preferences are defined for each consumer over bundles of goods, with the notation $\succeq$ meaning *at least as good as*

Preferences Definition: preferences of a consumer on the set of bundles in $\mathbb{R}^n_+$ is a subset $\succeq$ of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. In mathematics, this subset is referred to as a *binary relation* on the set $\mathbb{R}^n_+$

- Note: $\mathbb{R}^n_+$ is the set of all bundles in the universe, while $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ is the set of all pairs of bundles in the universe. For instance, if we $A = \{a, b, c\}$ then $A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

Examples

- Example 1: $\succeq = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ represents the case where every possible bundle is at least as good as every other
- Example 2: $\succeq = \varnothing$ represents the case where no bundle is at least as good as any other (including itself
- Example 3: $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ x_i \geq y_i \text{ for all } i = 1, \ldots, n\}$ represents the case where only bundles where one has at least as much of each good as the other can be compared
- Example 4: $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ \prod_{i=1}^n x_i \geq \prod_{i=1}^n y_i\}$ represents the case where all bundles are compared and on is at least as good as another whenever the product of quantities is higher. See below for a graphical representation where $x_1 x_2 = 2$, red represents the better than sets and blue the worse than sets
    - 

2.1.2 Strict and Indifferent Preference Relations

To define instances where a consumer is indifferent between a set of bundles or strictly prefers a given bundle over another, we use indifferent and strict preference relations

Strict Preference Relation: when the binary relation $\succ$ on the consumption set X is defined as follows;

$x_1 \succ x_2$ if and only if $x_2 \not\succeq x_1$. This is read x_1 is strictly preferred to x_2

- Implication 1: from $\succeq = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ we have that $\succ = \varnothing$ and therefore there is no strict preference
- Implication 2: from $\succeq = \varnothing$ we have that $\succeq = \varnothing$ and therefore none are strict
- Implication 3: from $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ x_i \geq y_i \text{ for all } i = 1, \dots, n\}$ we get that

$$\succ = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ x_i \geq y_i \text{ for all } i = 1, \dots, n \text{ and } x_i > y_j \text{ for some } j = 1, \dots, n\}$$

  Therefore, a bundle is strictly preferred to y if and only if x has at least as much of each commodity as y and stricty more of some commodity

- Implication 4: from $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ \prod_{i=1}^n x_i \geq \prod_{i=1}^n y_i\}$ we get that

$$\succ = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ \prod_{i=1}^n x_i > \prod_{i=1}^n y_i\right\}$$

  Therefore, we have strict preference only when the product is higher

Indifference Reation: when the binary relation $\sim$ on the consumption set X is defined as follows; $x_1 \sim x_2$ if and only if $x_1 \succeq x_2$ a nd $x_2 \succeq x_1$. is is read as x_1 is indifference to x_2

- Implication 1: from $\succeq = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ we have that $\sim = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and therefore all bundles are indifferent
- Implication 2: from $\succeq = \varnothing$ we get $\sim = \varnothing$
- Implication 3: from $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ x_i \geq y_i \text{ for all } i = 1, \dots, n\}$ we get that

$$\sim = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ x = y\}$$

  Therefore, each bundle is indifferent only to itself

- Implication 4: from $\succeq = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ \prod_{i=1}^n x_i \geq \prod_{i=1}^n y_i\}$ we get that

$$\sim = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \ \prod_{i=1}^n x_i = \prod_{i=1}^n y_i\right\}$$

  Therefore, indifference when the product of the quantities is the same

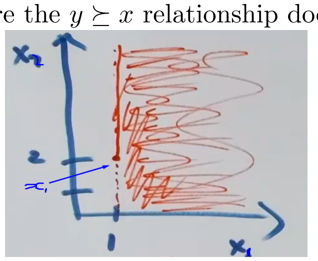
Combining Implications of Indifference and Strict Preference: from implications 2 we have that both indifference and strict preferences require that "one is atleast as good as the other". The combination of the strict preference implication i and indifference implication i yields example i from the preference relation section

2.1.3 Lexicographic Preferences

Describe preferences where an agent prefers any amount of one good (X) to any amount of another (Y), therefore if offered several bundles then the agent will choose the bundle that offers the most X no matter how much Y there is

Idea: first compare two bundles x and y according to the quantity of the the first good. If one bundle has more than the other of that good, then the one having more is strictly preferred. If neither has more of that good, go to the next good and compare in the same way. Repeat this until one bundle has more of good z than the other bundle, where z is the most recently compared good (i.e. the good with the lowest rank of those compared so far). If there is no bundle with more of good z, then the bundles are deemed to be indifferent to each other as they are the same bundle

Example for Two Goods: $\mathbb{R}^2_+ :\ x \succeq y$ if and only if (a) $x_1 > y_1$ or (b) $x_1 = y_1$ and $x_2 \geq y_2$

- Cases: for $(1, 2)$ vs $(2, 1)$ the latter has more of the first good so is strictly preferred, for $(1, 2)$ vs $(1, 1)$ both have same amounts of the first good but the former has more of the second good so it is preferred, for $(1, 2)$ and $(1, 2)$ both have the same amount of the first and second goods so they are indifferent. This is shown in the below figure where bundle $x = (x_1, x_2) = (2, 1)$ and what is marked in red are the $y \succeq x$ bundles that follow $\succeq$, note that the dotted red line is open to represent the boundary point where the $y \succeq x$ relationship does not hold
  - 

- Defining by Strict and Indifference Preferences: from $\succeq$ we have that:
  - A: $\sim = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ :\ x = y\}$ so each bundle is indifferent only to itself
  - B: $\succ = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ :\ x_1 > y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 > y_2)\}$ so a bundle is strictly preferred to another if and only if it has more of the first good or equal amounts of the first and more of the second.

2.1.4 Convex Preferences

Preferences $\succeq$ are convex if and only if for each $x \in \mathbb{R}^n_+$ the least as good as set $G(x)$ is a convex set

Alternative Definition: preferences are convex if and if for all $x, yin\mathbb{R}^+_n$ and all $t \in [0, 1]$: if $x \succeq y$ then $tx + (1 - t)y \succeq y$

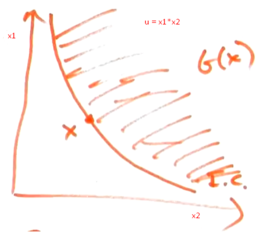
- Interpretation: averages are at least as good as extremes, i.e. you will have convex or straight shaped indifference curves

Strictly Convex Preferences: preferences $\succeq$ are strictly convex if and only if all distinct $x, y \in \mathbb{R}^n_+$ and all $t \in (0,1)$, if $x \succeq y$ then $tx + (1-t)y \succ y$

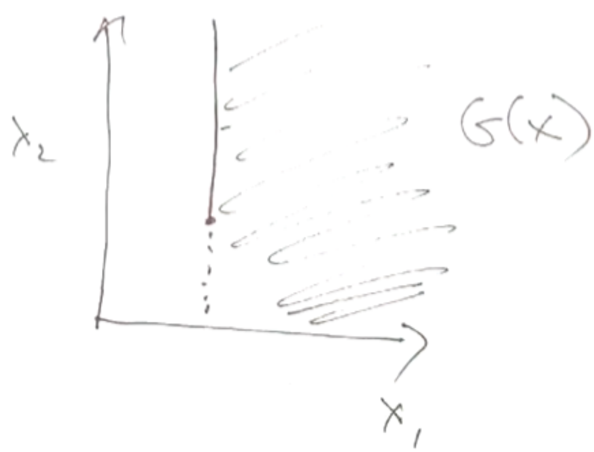
- Interpretation: avarages are better than extremenes, i.e. you will only have convex shaped indifference curves

Utility Representation: the utility representation of convex preferences is quasi-concavity

Example 1: the below figure has convex preferences

- 

Example 2: the below figure, though is not closed, has convex preferences

- 

2.1.5 Consumer Choice Axioms

All bundles in the subset can be compared (completeness), choices are consistent (transitivity), the consumer can choose all bundles in the subset (continuity), and more is always better (strict monotonicity). These four axioms must hold for preferences to be considered 'rational'

Axiom 1 - Completeness: for all x_1 and x_2 in X, either $x_1 \succeq x_2$ or $x_2 \succeq x_1$

Axion 2 - Transitivity: for any three elements x_1, x_2, x_3 in X, if $x_1 \succeq x_2$ and $x_2 \succeq x_3$ then $x_1 \succeq x_3$

- Note: although we require only that the consumer be capable of comparing two alternatives at a time, the assumption of transitivity requires that otther pairwise comparisons be linked together in a consistent way
- Transitivity Transfers to Indifference and Strict: suppose that preferences $\succeq$ are transitive, then:

- ◦ 1: indifference and strict preferences are transitive
  - ◦ 2: for all $x, y, z \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ if $x \succeq y$ and $y \succeq z$ (with one indifference), then $x \succeq z$
  - ◦ 3: for all $x, y, z \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ if $x \succeq y$ and $y \succeq z$ (with at least one strict), then $x \succ y$

Axiom 3 - Continuity: for all $x \in \mathbb{R}^n_+$, the "at least as good as set" $\succeq(x)$ and the "no better than" set $\preceq(x)$ are closed in $\mathbb{R}^n_+$

- Conditions from Definition preferences are continuious if $G(x) = \{y \in \mathbb{R}^n_+ : \ y \succeq x\}$ and $W(x) = \{y \in \mathbb{R}^n_+ : \ x \succeq y\}$ are both closed in $\mathbb{R}^n_+$
- Theorem 1: suppose that preferences are complete. Then, they are continuous if and only if for all $x \in \mathbb{R}^n_+$ the sets $SG(x) = \{y \in \mathbb{R}^n_+ : \ y \succ x\}$ and $SW(x) = \{y \in \mathbb{R}^n_+ : \ x \succ y\}$ are open in $\mathbb{R}^2_+$
  - ◦ Logic: $SG(x)$ follows by definition of continuity and completeness when taking the complement of $W(x)$, $SW(x)$ follows by definition of continuity and completeness when taking the complement of $G(x)$
  - ◦ Note: $SG(x)$ is referred to as the strictly greater than x set while $SW(x)$ is referred to as the strictly worse than x set

Axiom 4 - Strict Monotonicity: for all $x_0, x_1 \in \mathbb{R}^n_+$, if $x_0 \geq x_1$ then $x_0 \succeq x_1$ while if $x_0 > x_1$ then $x_0 \succ x_1$

- Alternative Definition: if x has at least as much of each good as y and strictly more of some good, then $x \succ y$
  - ◦ Note: by completeness of preferences, the alternative definition implies the first definition
- Exercise 1: prove that the alternative definition implies the first definition given the requirement that preferences are complete
  - ◦ Answer: the first part of the definition states that if $x \geq y$ then $x \succeq y$, which gives us case (a) where $x = y$ and case (b) where $x \neq y$. For (a) we have that by completeness $x \succeq y$, for (b) x has atleast more of some good which by completness implies that $x \succ y$. Thus the two cases, (a) and (b), given by the first part of the definition imply the alternative definition. The second part of the definition states that if $x > y$ then $x \succ y$, by completeness this implies that x *has atleast as much of good as y and strictly more of some good* which demonstrates equivalency to the alternative definition. Without completeness, these implications would not hold because it could mean that comparisons can only be made in the case where x has *strictly more of some good* and not *strictly more of all goods.*

2.1.6 Intermediate Value Version of Continuity

If the preferences are continuous and complete, then for any $x, y, z \in \mathbb{R}^n_+$, if $x \succ y$ and $y \succ z$, then there is some $\alpha \in (0, 1)$ such that the convex combination $\alpha x + (1 - \alpha)z$ is indifferent to y. This resembles the *intermediate value theorem*

Proof: this can be proved using completeness and continuity together with the least upper bound property (i.e. that every non-empty set of reals that is bounded above, has a least upper bound). We know that $x \succ y$ and $y \succ z$ and that the preferences are continuous. We need to show that there is some point on the line between x and z that is indifferent to y. Consider the set $A = \{\alpha \in [0,1] : \; y \succ \alpha x + (1-\alpha)z\}$. It stands that $\alpha = 0$ is an element of A since $0x + (1-0)z = z$ and $y \succ z$, therefore A is a set of reals. It also stands that $\alpha = 1$ is not an element of A since $1x + (1-1)z = x$ and $x \succ y$, therefore A is bounded above by 1 and must have a least upper bound. Call the least upper bound α^*, we can show that $\alpha^* x + (1-\alpha_*)z \sim y$ using the following steps:

- Step 1: show that $0 < \alpha^* < 1$ using completness and continuity around x and around z.

  For $0 < \alpha^*$ we have that by continuity theorem 1, since $y \succ z$, there is an open ball $B_\varepsilon(z)$ around z such that every z' in that ball that can be compared to y satisfies $y \succ z'$. By completeness, all the stuff in the ball can be compared to y, in particular the stuff that is also on the line between x and z. Note that if we have $\alpha < 0$ then there would be z's in the open ball around $B_\varepsilon(z)$ that do not satisfy and contradict $y \succ z'$. Therefore, it must be that $0 < \alpha^*$.

  For $\alpha_* < 1$ we have that by continuity theorem 1, since $x \succ y$, there is an open ball $B_\varepsilon(y)$ around y such that every y' in that ball that can be compared to x satisfies $x \succ y$. Likewise, by completeness, it must be that $\alpha^* < 1$.

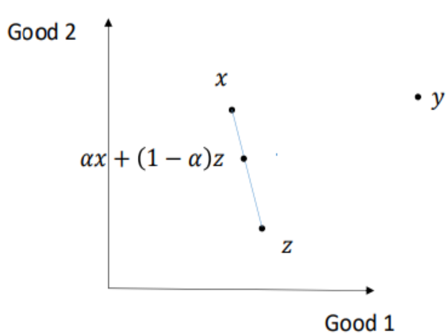
- Step 2: show that $\alpha^* x + (1-\alpha_*)z \sim y$ using completeness and continuity. By completeness either (1) $\alpha^* x + (1-\alpha^*)z \succ y$, (2) $y \succ \alpha^* x + (1-\alpha^*)z$, or (3) $\alpha^* x + (1-\alpha^*)z \sim y$.

  Option 1 cannot be the case as for $\alpha^* x + (1-\alpha^*)z \succ y$ to hold we cannot have α^* as our upper bound Option 2 cannot be the case since the least upper bound $1 \in \alpha^*$ the relationship $y \succ \alpha^* x + (1-\alpha^*)z$ does not hold and so α^* is not an upper bound since an open ball around the point of our least upper bound would include points where

  $$y \not\succ \alpha^* x + (1-\alpha^*)z$$

  Therefore, we are left with option (3), $\alpha^* x + (1-\alpha^*)z \sim y$.

Usefulness: this means that something on the line connecting x and z is indifferent to y



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2.1.7 Utility Representation

A utility representation of preferences $\succeq$ is a real valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ over the set of bundless such that for all bundles $x, y \in R_+$:

(a) if $x \succeq y$ then $u(x) \geq u(y)$

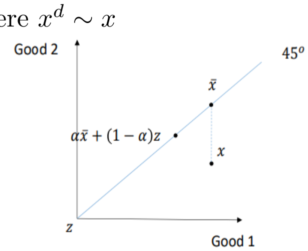
(b) if $u(x) \geq u(y)$ then $x \succeq y$

Alternative Definition: a real values function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing the relation $\succeq$ if for all $x_0, x_1 \in \mathbb{R}^n_+, u(x_0) \geq u(x_1) \iff x_0 \succeq x_1$

Strict Preferences: $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a utility representation of preferences $\succeq$ if and only if for all bundles $x, y \in \mathbb{R}^n_+$: (a) if $x \succ y$ then $u(x) > u(y)$, (b) if $u(x) > u(y)$ then $x \succ y$

Theorem 1 - Existence of a Utility Representation: if the binary relation $\succeq$ is complete, transitive, continuous, and strictly monotonic, then there exists a continuous real valued function, $u : \mathbb{R}^n_+ \to \mathbb{R}$, which represents $\succeq$

- Proof in Terms of 2 Goods: **step 1** - show every point x is indifferent to a point y on the 45° line which represents bundles with the same amount of each good, **step 2** - assign a utility $u(x)$ equal to the quantity of each good at the point on the diagonal that is indifferent to x, **step 3** - show that that if $x \succeq y$ then $u(x) \geq u(y)$, **step 4** - show that if $u(x) \geq u(y)$ then $x \succeq y$
  - Step 1: if x is on the 45° line, we are done as by completeness $x \sim x$. Consider x not on the 45° line, let $\widetilde{x}$ be the bundle on the 45° line where the amount of each good is the larger of x_1 and x_2 (e.g. if $x = (5, 2)$ then $\widetilde{x} = (5, 5)$). Since $\widetilde{x}$ has more of at least one good and at least as much of each of the others then $\widetilde{x} \succ x$ by strict monotonicity. Also, by strict monotinicity $\widetilde{x} \succ z$, where z is the origin $(0, 0)$. Hence, there is a point on the diagonal that is indifferent to x by the *intermediate value version of continuity* (i.e. $\exists \alpha, x = \alpha\widetilde{x} + (1 - \alpha)z$). Note that, by strict monotonicity, there is only one bundle on the diagonal that is indifferent to x referred to as x^d where $x^d \sim x$



    - *
  - Step 2: take the example where $x = (5, 2)$ and the point on the diagonal that is indifferent to x is $x^d = (2.5, 2.5)$ which yields $u(x) = 2.5$. Clearly, $u(x^d) = u(x)$ since $x^d = x$ in this case
  - Step 3: suppose $x \succeq y$ and that by definition $x^d \sim x, y \sim y^d$. Therefore by transitivity we have that $x^d \succeq y^d$ and by strict monotonicity we can infer that $u(x^d) \geq u(y^d)$.
    - Logic: since x^d and y^d are both on the diagonal whereby $x^d = (a, a)$ and $y^d = (b, b)$ for some quantities a and b. By strict monotonicity, combined with $x^d \succeq y^d$, we can infer that $a \geq b$.

This is as, if on the contrary $b > a$, then y^d would have more of each good than x^d. This would imply, by strict monotonicity, that $y^d \succ x^d$. Since this would contradict the finding that $x^d \succeq y^d$, then it must be that $a \geq b$. Since a represents the quantity of each good in x^d and b represents the quantity of each good in y^d, the utility function defined in step 2 will assign $u(x^d) = a$ and $u(y^d) = b$. Since $a \geq b$, we conclude that $u(x^d) \geq u(y^d)$.

From $u(x^d) \geq u(y^d)$ and $x^d \sim x, y \sim y^d$ we can infer that $u(x) \geq u(y)$. This is as by applying our utility function from step 2 we get $u(x) = u(x^d) \geq u(y^d) = u(y)$ and therefore $u(x) \geq u(y)$

- Step 4: homework from priscilla notes

Finite Sets: if the set of alternatives is finite or countably infinite, then we get a utility representation with only complete and transitive preferences

Huge Spaces: for the huge space $\mathbb{R}^n_+$, we can get a utility representation without monotonicity

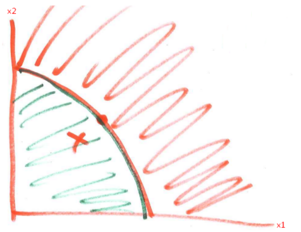
Properties: (1) preferences represented by a utility function will always be complete and transitive, (2) preferences represented by a continuous utility function will always be continuous

- Proof for (2): for $x \in \mathbb{R}^n_+$ we want to show $SG(x)$ is open. Suppose we have $x : \ (u(x), +\infty)$ which is open in $\mathbb{R}^n_+$. The inverse image of x we have that $u^{-1}(u(x), +\infty) = \{y \in \mathbb{R}^n_+ : u(y) \geq u(x)\} = SG(x)$ which by continuity is open. Therefore, since the inverse image is open, we have that the preferences represented by the continuous utility function are continuous.

2.1.8 Exercise 1

Answer if the below preferences are are complete, transitive, continuous, and strongly monotonic?

- Question 1: $x \succeq y$ if and only if $(x_1)^2 + x_2 \geq (y_1)^2 + y_2$
  - Complete: take x, y, we have that either $x_1^2 + x_2 \geq y_1^2 + y_2$ or $y_1^2 + y_2 > x_1^2 + x_2$. Therefore, it is complete.
  - Transitive: take x, y, z. Suppose $x \succeq y$ and $y \succeq z$. From the $\Rightarrow$ implication we have that $x_1^2 + x_2 \geq y_1^2 + y_2$ and $y_1^2 + y_2 \geq z_1^2 + z_2$, therefore $x_1^2 + x_2 \geq z_1^2 + z_2$. From the $\Leftarrow$ implication we have that $x \succeq z$
  - Continuous: since the preferences have a utility representation they must be continuous. Also, by drawing the indifference curves, it is clear that $G(x)$ and $W(x)$ is closed. Note that the complements $G(x) \to SG(x)$ and $W(x) \to SW(x)$ are open which implies that the preferences are continuous.

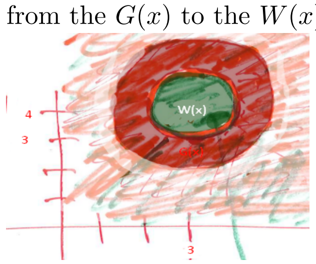


- ◦ Strongly Monotonic: suppose we have two bundles $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1^2 + x_2 = y_1^2 + y_2 \ \therefore \ x \sim y$. In this case, by adding 1 unit to either x_1 or x_2 we would have $x_1^{2*} + x_{2*} > y_1^2 + y_2 \ \therefore \ x^* \succ y$

- Question 2: $x \succeq y$ if and only if $(x_1 - 3)^2 + (x_2 - 4)^2 \geq (y_1 - 3)^2 + (y_2 - 4)^2$
  - ◦ Complete: take x, y, we have that either $(x_1 - 3)^2 + (x_2 - 4)^2 \geq (y_1 - 3)^2 + (y_2 - 4)^2$ or $(y_1 - 3)^2 + (y_2 - 4)^2 > (x_1 - 3)^2 + (x_2 - 4)^2$. Therefore, it is complete.
  - ◦ Transitive: take x, y, z. Suppose $x \succeq y$ and $y \succeq z$. From the $\Rightarrow$ implication we have that $(x_1 - 3)^2 + (x_2 - 4)^2 \geq (y_1 - 3)^2 + (y_2 - 4)^2$ and $(y_1 - 3)^2 + (y_2 - 4)^2 \geq (z_1 - 3)^2 + (z_2 - 4)^2$, therefore $(x_1 - 3)^2 + (x_2 - 4)^2 \geq (z_1 - 3)^2 + (z_2 - 4)^2$. From the $\Leftarrow$ implication we have that $x \succeq z$
  - ◦ Continuous: since the preferences have a utility representation they must be continuous
  - ◦ Strongly Monotonic: suppose $x = (3, 5)$ and $y = (2, 5)$, this yields:

$$(3 - 3)^2 + (5 - 4)^2 = 1 \quad \text{for } x$$
$$(2 - 3)^2 + (5 - 4)^2 = 2 \quad \text{for } y$$

    Therefore, though $x > y$ we have that $(x_1 - 3)^2 + (x_2 - 4)^2 < (y_1 - 3)^2 + (y_2 - 4)^2$. Therefore, the $\Leftarrow$ implication we have that $y \succ x$ and monotonicity is violated. The violation can also be seen in the below figure, where if you move from the south-west to the north-east you could move from the $G(x)$ to the $W(x)$ area of the donut.



2.1.9 Exercise 2

Larry has the utility function $u(x, y) = x^4 y^2$ over two goods whose quantities are denoted by $x \geq 0$ and $y \geq 0$.

Are Larry's Preferences complete, transitive, and continuous?

- Answer: if there is a utility representation of the preferences then the preferences must be complete and transitive (as is the case). Since the utility function is continuous, this implies that the preferences are continuous.

If the price of each good is 1 per unit and income is 30, solve for the optimal bundle using the three methods below?

- Indifference and Budget Line Tangency: Note that since the utility function is smooth (ie twice continuously differentiable), the indifference curves are convex, and monotonicity holds, it follows that interior solutions occur only where the indifference curve of the bundle is tangent to the budget line. This interior solution is captured by the following equations:

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y} \Rightarrow tfrac4x^3y^21 = \frac{2x^4y}{1} \tag{1}$$

$$p_x x + p_y y = I \Rightarrow x + y = 30 \tag{2}$$

  Note that equation (2) puts us on the budget line and equation (1) ensures that trading x for y (and vice versa) will not increase utility. Solving equation (1) yields $2y = x$. By substituting $2y = x$ into equation (2) we get $3y = 30 \ \therefore \ y = 10$. Since $y = 10$ we have from equation (2) that $x + 10 = 30$ and therefore $x = 20$. Our optimal bundle is $(x, y) = (20, 10)$

- Lagrangian: setting up the lagrangian we have $L = x^4y^2 + \lambda(x + y - 30)$. The partial derivatives of the lagrangian are therefore:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 4x^3y^2 = \lambda \tag{1}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2x^4y = \lambda \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x + y - 30 = 0 \tag{3}$$

  From equations (1) and (2) we have:

$$4x^3y^2 = \lambda = 2x^4y$$

$$\therefore \ y = 0.5x$$

  Plugging $y = 0.5x$ into equation (3) yields $0.5x + x = 30 \ \therefore \ x = 20$. Substituting x into the budget line yields $y + 20 = 30 \ \therefore \ y = 10$. Our optimal bundle is $(x, y) = (20, 10)$

- Cobb-Douglass Formula: the Cobb-Douglas utility function has the general form $u(x, y) = x^a y^b$. Since our equation mimics this, we have that $a = 4$ and $b = 2$. This is known to have a solution for all $a, b > 0$ which is given by:

$$x = tfracaI(a + b)p_x = \tfrac{4(30)}{(2+4)} = 20$$

$$y = tfracbI(a + b)p_y = \tfrac{2(30)}{(2+4)} = 10$$

Thus, our optimal bundle is $(x, y) = (20, 10)$

2.1.10 Exercise 3

Demand in a market is given by $D(p) = \frac{100}{p^2}$ for $p > 0$ and the cost of production is $C(Q) = 6\sqrt{Q} + Q$

Monopoly Market: suppose we have a monopoly seller. Invert the demand curve and find the monopoly output level Q and the corresponding price level p, or argue that one does not exist.

- Answer: inverting the demand function gives $P(Q) = \frac{10}{\sqrt{Q}}$. For a monopolist seller the optimal output is given by setting MR $=$ MC, so we must first find MR and MC:

$$\text{MC} = \frac{\partial C}{\partial Q} = 3Q^{-0.5} + 1$$
$$\text{MR} = \frac{\partial Q \times P}{\partial Q} = \frac{\partial 10 Q^{0.5}}{\partial Q} = 5Q^{-0.5}$$

Setting $MR = MC$ yields:

$$5Q^{-0.5} = 3Q^{-0.5} + 1$$
$$1 = Q^{-0.5}(5 - 3)$$
$$Q = 4$$

The optimal output $Q = 4$ results in a price of $P = \frac{10}{\sqrt{4}} = 5$. Note that this method is valid as the function is concave and allows us to find all interior solutions. As p is restricted to be above zero, and thus Q is restricted to be above zero, we are only dealing with interior points as possible solutions.

- Alternative Method: note that we can also insert $P(Q) = \frac{10}{\sqrt{Q}}$ into the profit function $\pi = P(Q)Q - C(Q)$, which yields:

$$\pi = \left(\frac{10}{\sqrt{Q}}\right)Q - (6\sqrt{Q} + Q) = 4\sqrt{Q} - Q$$

Taking first order conditions, with respect to Q, for profit maximization at an interior point gives us:

$$tfrac\partial\pi\partial Q = 0 \Rightarrow 2Q^{-0.5} - 1 = 0 \ \therefore \ Q = 4$$

Competitive Market: suppose we have sellers in a competitive market. Eithere find a competitive equilibrium or argue that one does not exist.

- Answer: Note that since we have a perfectly competitive market, we take each price as given so

$P(Q) = p$ and choose Q to maximize profit. Profit in this case is:

$$\pi = pQ - (6\sqrt{Q} + Q) = Q(p-1) - 6\sqrt{Q}$$

While this function is twice continuously differentiable, it is not concave since $\frac{\partial^2 \pi}{\partial Q^2} > 0$ making it a strictly convex function. Also note that marginal cost is a decreasing function of output since $\frac{\partial^2 \pi}{\partial Q^2} = \frac{\partial^2 C}{\partial Q^2} < 0$ (note that the two double derivative are equal due to $P = MC$ in perfect competition). Unlike the case with the monopolist seller, who is able to restrict quantity to increase prices, in a competitive market this is not. Therefore, at $P = MC$ profit will increase with each in output and sellers would never stop producing. As a result there is no competitive equilibrium. To break this down we have the following:

- Case (1): if $0 < p \leq 1$, then profit is diminishing with quantity. Therefore, each seller would like to supply as little as possible and so market supply is $S(p) = 0$. However, demand would be $D(p) = \frac{100}{p^2} \geq 100 > 0 = S(p)$. Overall, we would have a case where demand is high but production is nonexistent.
- Case (2): if $p > 1$, then each seller would like to move toward infinite output. This would cause supply to be greater than demand since $D(p) = \frac{100}{p^2} < 100 < S(p) = +\infty$.

Overall, in boths case (1) and (2) we do not have a competitive equilibrium.

- Note: if trying to solve this at $P = MC$ this will result in $Q = 49$ and $P = 1.428$. This yields negative profit where $\pi = Q(p-1) - 6\sqrt{Q} = 49(1.428 - 1) - 6\sqrt{49} = -21.028$. Profit at the competitive equilibrium cannot be equal to zero.

2.2 Quasi-Concavity and Utility Maximization

In this section we will be focusing on the consumer problem, which is to find an optimal bundle within the consumer's budget. We will first look at conditions that ensure an optimal bundle exists and add the assumption of quasi-concavity of a utility function. This will allow us to use math techniques to solve consumer problems.

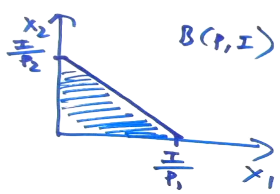
2.2.1 Consumer Problem

For the standard consumer problem, we will take the consumer to have a budget, $I \in \mathbb{R}_{++}$, and to be able to buy as much of each good as he can afford with his income at fixed positive per unit prices $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n_{++}$

The Consumer's Problem: to find a bundle $x \in B(p, I)$ such that $x \succeq y$ for all $y \in B(p, I)$, referred to as the optimal bundle

Affordable Bundles: the consumer's problem allows us to define affordable bundles as those that are in the consumer's budget set where $B(p, I) = \{x \in \mathbb{R}^n_+ : \ p \cdot x \le I\}$

- Note: $p \cdot x = \sum_{i=1}^{n} p_i x_i$

  $P_1 x_1 + P_2 x_2 \le I$
- 

2.2.2 Existence of an Optimal Bundle (Weierstrass Theorem)

Every continuous real-valued function f on a non-empty compact subset B of $\mathbb{R}^n$ obtains a maximum on B. In otherwords, there is an $x \in B$ such that $f(x) \ge f(y)$ for all $y \in B$

Implication: therefore, all we need to do is assume the preferences are represented by a continuous utility function and show that the budget set is compact and non-empty to solve the consumer problem

Requirements: the budget set is a closed, bounded, and non-empty subset of $\mathbb{R}$

- Example: the budget set figure above is closed, bounded, and non-empty and therefore when alying a continuious real value function to it we will find a maximum

2.2.3 Quasi-Concavity

A real valued function f on R^n_+ is quasi-concave if and only if for all $x, y \in \mathbb{R}^n_+$ and all $t \in [0, 1]$:
if $u(x) \ge u(y)$ then $u(tx + (1-t)y) \ge u(y)$

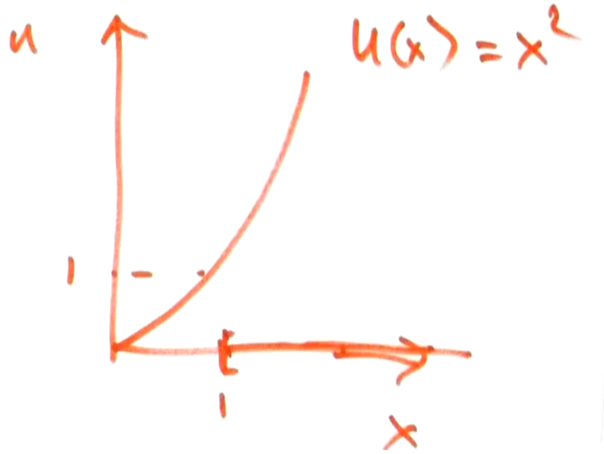
Relationship to Convex Preferences: quasi-concavity is the translation of convex preferences into a utility representation, if you have convex preferences then the utility function will be quasi-concave

Strictly Quasi-Concave: a real-valued function f on $\mathbb{R}^n_+$ is strictly quasi-concave if and only if for all distinct $x, y \in \mathbb{R}_n^+$ and all $t \in (0,1)$: if $u(x) \geq u(y)$ then $u(tx + (1-t)y) > u(y)$

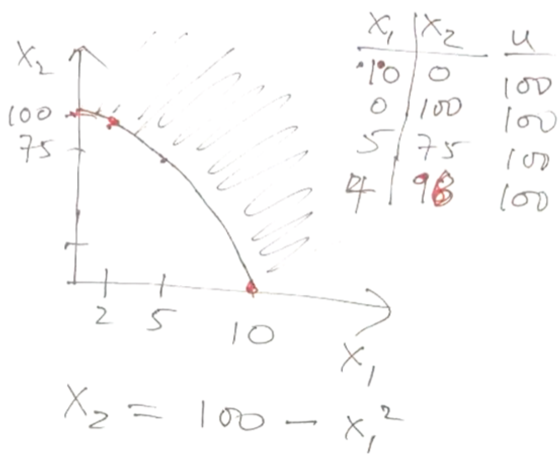
Theorem 1: every concave function is quasi-concave and every strictly concave function is strictly quasi-concave

- Logic: this result stems from the representation of convex preferences as indifference curves, note that if you draw indifference curves for a quasi-concave function then the indifference curve will be convex

Examples: (1) is $u(x) = x^2$ (strictly) quasi-concave? (2) is $u(x, y) = x^2 + y$ (strictly) quasi-concave?

- Answer 1: the function is strictly quasi-concave since y always has a lower utility than if the average of y and some higher value x. Also note that the set is convex and therefore the function must be quasi-concave. Note that though this is a convex function it is still quasi-concave.
  - 

- Answer 2: $u(x, y) = x^2 + y$ is not a convex set since the better than set is not convex (i.e. drawing a line between two bundles in the better than set would result in points being in the worse than set). Note that in this case averages are not better than extremes in this case, the line in the figure represents an indifference curve. Therefore, the function is not quasi-concave



You can also demonstrate that this set is not quasi-concave setting bundle 1 such that $2^2 + 14 = 18$ and bundle 2 such that $4^2 + 1 = 17$, in which case bundle 1 $\succ$ bundle 2. However, if we take $t = 0.5$ to form a new bundle, bundle 3, we have $3^2 + 7.5 = 16.5$. In this case, bundle 2 *succ* bundle 3 and therefore quasi-concavity is violated.

2.2.4 Lagrangian Approach to Bundle Optimality

If the utility function is continuously differentiable, that is, the partial derivatives are all continuous functions then we can use the Lagrangian Approach to find a solution to the consumer's problem

Lagrangian Approach: set up the Lagrangian where $L(x, \lambda) = u(x) + \lambda(p \cdot x - I)$ and look for solutions (x^*, λ^*), where $x_i^* > 0$ for each good i and $\lambda^* > 0$, to the system:
(1) $\frac{\partial L}{\partial x_i} = \frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i$ for each $i = 1, \ldots, n$ where you have n equations
(2) $\frac{\partial L}{\partial \lambda} = p \cdot x - I = 0$

Sufficiency of the Lagrangian Method: if $u(x)$ is continuously differentiable, stricty monotonic, and quasi-concave, then any solution to the Lagrangian approach is a solution to the consumer's problem

- Logic: system equation (1) implies *equalization of bang for buck* where for any two goods i and j we have

$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{p_i} = \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j}$$

  System equation (2) implies budget exhausted, i.e. that the optimal bundle is on the budget line
  - Bang for Buck across Goods: this is a multi-dimensional version of do it until the *marginal benefit equals marginal cost* (i.e. marginal gain equals marginal pain). This is your point of tangency between the budget set and the indifference curve

Uniqueness of Solution: if $u(x)$ is strictly quasi-concave, then there is at most one solution to the consumer's problem. Note that for cases of non-strictly quasi-concave preferences, we can have more than one solution (i.e. if you have perfect substitutes)

Example: for $u(x_1, x_2) = x_1 x_2$ where $p_1 = 10$, $p_2 = 5$, and $I = 50$ we set up the lagrangian as $L = x_1 \cdot x_2 - \lambda(10x_1 + 5x_2 - 50)$. The system of equations is given by:

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow x_2 - 10\lambda = 0 \ \therefore \ \lambda = \frac{x_2}{10} \tag{1}$$
$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow x_1 - 5\lambda = 0 \ \therefore \ \lambda = \frac{x_1}{5} \tag{2}$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 10x_1 + 5x_2 - 50 = 0 \tag{3}$$

Setting equations (1) and (2) equal to eachother yields:

$$\frac{x_2}{10} = \lambda = \frac{x_1}{5}$$
$$x_2 = 2x_1$$

Subtituting $x_2 = 2x_1$ into equation (3) gives us $10x_1 + 5(2x_1) = 50 \ \therefore \ x_1 = 2.5$. Since $x_1 = 2.5$ we have that $x_2 = 2x_1 = 5$. Note that $\lambda > 0$ since $\lambda = \frac{x_2}{10} = 0.5$, making the solution valid. Our optimal bundle is therefore $(x_1, x_2) = (2.5, 5)$

2.3 Demand and Consumer Welfare

Unlike previous sections, this section will be focusing on the total demand and consumer welfare instead of the demand of an individual consumer.

2.3.1 Demand

Demand of each good depends on potentially all prices and income. We write $x(p, I)$ to denote a bundle that maximizes utility subject to the budget constraint. This bundle $x(p, I)$ (or bundles) is called the demand at prices and income (p, I)

Demand Function: a demand function assigns a bundle/s $x(p, I)$ to each budget (p, I)

- Conditions for Uniqueness: if utility is strictly quasi-concave, therefore making the budget set convex, we get a unique demanded bundle for each budget that allows us to derive a demand function
  - Note: we will not have a unique demanded bundle if we have perfect substitutes (in this case utility won't be strictly quasi-concave) or other functions like $u(x) = x^2 + y^2$ where price is 1 for both. In this case, while demand cannot be a function, we can still analyze demand and properties of it
- Relationship with Utility Functions: if utility functions are differentiable and strictly convex, the demand functions will often be as well. In such cases, we can use partial derivatives to determine how prices and income affect demand for each good

Coobb-Douglas Example: suppose $u(x) = \prod_{j=1}^{n} x_j^{\alpha_j}$ where $\alpha_j > 0$ for all $j = 1, \ldots, n$, the demands are:

$$x_i(p, I) = \frac{\alpha_i I}{(\sum_{j=1}^{n} \alpha_j) p_i}$$

- Properties: is strictly quasi-concave, since it is the product of the quantities you will never have a quantity equal to zero so you will always consume on the interior, the function is differentiable so we can use lagrange techniques (which yields the above)

Perfect Substitutes Example: suppose $u(x) = \sum_{j=1}^{n} x_j$, the demands are:

$$x_i(p, I) = \begin{cases} 0 \text{ if } p_i \text{ is not the lowest price;} \\ \frac{I}{p_i} \text{ if } p_i \text{ is the unique lowest price;} \\ \text{non-unique otherwise} \end{cases}$$

- Properties: preferences are not strictly convex, you can have solutions if there are two or more goods with the same lowest price (i.e. goods are extremely price elastic), utility is quasi-concave

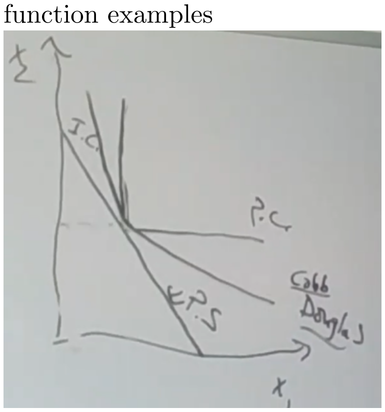
- Note: this is as the consumer will exhaust all his income one the cheapest good since all good are of equal value to him (i.e. they are perfect substitutes)

Perfect Complements Example: suppose $u(x) = \min\{x_1, \ldots, x_n\}$, the demands are:

$$x_i(p, I) = \frac{I}{\sum_{j=1}^{n} p_j}$$

- Properties: given that only the minimum amount of a good counts towards your utility then you will want to have an equal amount of each good - yielding the above demand function

Graphic Representation of Examples: the below figure shows the indifference curves for the above demand function examples



Quasi-linear Utility Functions: suppose $x(x) = v(x_1) + x_2$ where $v(x_1)$ does not depend on x_n. In this case, the demand for good 1 will not depend on income (provided income is sufficiently large and the solution is not a corner)

- Logic: good 1 is said to have *no-income effects* while good 2 is often taken to be *everything else* with a price of \$1 per unit
- Advantage: this makes consumer welfare easy to estimate as now it is just the area under the demand curve
- Linear Demand: a quadratic function for $v(x_1)$ would generate a linear demand for x_1 (i.e. $v(x_1) = ax_1 - bx_1^2$)
- Note: for a quasi-linear utility function to also be strictly quasi-concave, it is necessary and sufficient that v be quasi-concave

2.3.2 Indirect Utility Function

Given that utility is a function of bundles of goods, if a consumer is presumed to choose a bundle that maximizes utility, then we can plug the maximizing bundle $x(p, I)$ into the utility function to get what we call an indirect utility function $v(p, I)$

Mathematic Definition: $v(p, I) = \max\{U(x)|px \leq I\}$

- Note: in other words, the indirect utility function is a function/formula that solves the utility function given the utility functions constraints

Non-Unique Bundles: we do not need to have a unique bundle as a solution since all the best bundles will generate the same utility

Theorem 1: if $u(x)$ is a continuous and strongly monotonic real valued utility function on $\mathbb{R}^n_+$, then the indirect utility function $v(p, I)$ is:

- (1): continuous
- (2): homogeneous of degree zero
- (3): strictly increase in I and weakly increasing in p
- (4): the function $f(p, I) = -v(p, I)$ is quasi-concave in (p, I)

Roy's Identity: if $u(x)$ is differentiable and the partial derivative with respect to I is non-zero at a point (p', I') then it satisfies:

$$x_i(p', y') = -\frac{\frac{\partial v(p', I')}{\partial p_i}}{\frac{\partial v(p', I')}{\partial I}}$$

Relationship to the Expenditure Function: the expenditure function is the inverse of the indirect utility function

Example: the indirect utility function for Cobb-Douglas, $u = x_1 x_2$ where demand is $x_1 = \frac{I}{2p_1}$ and $x_2 = \frac{I}{2p_2}$, is $V(p, I) = \frac{I^2}{4p_1 p_2}$

- Logic: this comes from substituting the demand for x_1 and x_2 into the utility function, giving us an indirect utility function that yield maximum utility
- Properties: the indirect utility function is continuous, strictly increasing in I, strictly decreasing in p, and strictly monotonic

2.3.3 Dual Problem

A solution to the utility maximization problem also solves the expenditure minimization problem (and vice versa), the two problems are in essence a different way of stating the same thing

Utility Maximization Problem (u-max): find $x(p, I)$ that maximizes $u(x)$ subject to the budget (p, I)

Expenditure Minimization Problem (e-min): find $x(p, \overline{u})$ to minimize the expenditure $e(p, \overline{u}) = \sum_{i=1}^{n} p_i x_i$ incurred to obtain utility $\overline{u}$

- Translation: what is the cheapest way to get a specified level of utility given price and income (p, I)

- Solution: solving e-min can be done in the differentiable case using Lagrangian type methods. Note that under standard assumptions of quasi-concave utility, strong monotonicity and interior solution it satisfies bang for buck equalization (MRS equal price ratio) and utility is just equal to $\overline{u}$
- Theorem 1: if $u(x)$ is a continuous and strongly monotonic real valued utility function on $\mathbb{R}^n_+$ then the expenditure funtion $e(p,\overline{u})$ is:
  - (1): continuous
  - (2): homogeneous of degree one in p
  - (3): strictly increasing in $\overline{u}$ and weakly increasing in p
  - (4): concave in p

Conversion from u-max to e-min: if a bundle x solves u-max given budget (p, I), then x also solves e-min of obtaining $\overline{u} = u(x)$ given the same prices

- Note: $e(p, v(p, I)) = I$

Conversion from e-min to u-max: if a bundle x solves the e-min given $(p, \overline{u})$, then x also solves the u-max given budget $(p, \sum_{i=1}^{n} p_i x_i)$, where the income is the cost of the bundle x

- Note: $v(p, e(p, \overline{u})) = \overline{u}$

2.3.4 Welfare

Welfare refers to consumer surplus, typically the *area under the demand curve minus what the consumer pays*. The change in welfare therefore is the change in consumper surplus, typically the *change in the area under the demand curve minus what the consumer pays*

Assumptions: utility is quasi-linear

Basis: in general, welfare is based on the question of *how much are you willing to pay to change from situation A to situation B?*

Example 1: we want to focus on the change in the price of good 1 and its effect on consumer welfare given that $u(x_1, x_2) = \ln(x_1) + x_2$

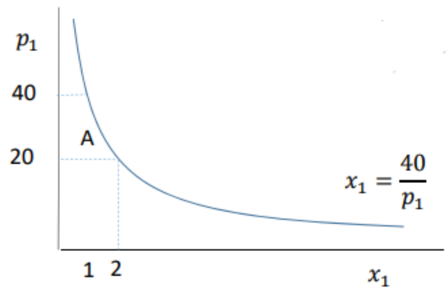
- Part 1: we first find demand for goods 1 and 2 as a function of $p_1, p_2, I > 0$. From this we get:

$$x_1 = \begin{cases} \frac{p_2}{p_1} \text{ if } I \geq p_2 \\ \frac{I}{p_1} \text{ if } I < p_2 \end{cases}$$

$$x_2 = \begin{cases} \frac{I}{p_2} - 1 \text{ if } I \geq p_2 \\ 0 \text{ if } I < p_2 \end{cases}$$

Notice that if income is high enough, your demand for the first good will not depend on income. Thus, we regard quasi-linear utility functions as representing a situation where the good in question (good 1) does not have income effects

- Part 2: we want to fix income I and p_2 as the demand curve is defined only for the price of the good in question, with all other things fixed we can focus on welfare changes from the price of good 1. Setting $I = 80$ and $p_2 = 40$ means that as $I \geq P_2$ we have that $x_1 = \frac{40}{p_1}$ and $x_2 = 1$.



In this case, the area A on the above figure represents the change in consumer surplus (CS) for a price drop from $p_1 = 40$ to $p_1' = 20$. This equals

$$\Delta CS = \int_{20}^{40} \frac{40}{p_1}\, dp_1 = 40[\ln(40) - \ln(20)] = 40 \ln(2) = 27.73$$

- Part 3: Note that at the high price of $p_1 = 40$, $p_2 = 40$, and $I = 80$, the optimal bundle is $(x_1, x_2) = (1, 1)$. When the price drops to $p_1' = 20$, the optimal bundle changes to $(x_1', x_2') = (2, 1)$. We must now examine *how much are you willing to pay at the new prices to have the price drop* by considering how much income we can take away at the new prices(p_1') to leave the consumer at the same level of utility as before the price change(p_1).

  Note that demand for the first good does not depend on income (no income effects), we can take money away from the consumer at the new prices without changing their demand for good 1. At the old prices utility was $u(1, 1) = \ln(1) + 1 = 1$. At the new prices, if we take away m from income, then demand for good 1 will remain at $x_1' = 2$ and demand for good 2 will drop to:

$$x_2' = \frac{80 - m}{40} - 1 = 1 - \frac{m}{40}$$

  Therefore, utility after we take away m from income will be $u' = \ln(2) + 1 - \frac{m}{40}$. We want to know when the utilities from the price drop scenario and the income drop scenario are equal, that is the m that solves $\ln(2) + 1 - \frac{m}{40} = 1$. This gives us the change in consumer surplus as:

$$m = 40 \ln(2) = \Delta CS$$

2.3.5 Compensating Variation (CV)

Example 1 of the welfare subsection is a demonstration of the *compensating variation* method of computing a welfare change. The compensating variation resulting from a change from (p, I) to (p', I') measures *how much the consumer is willing to pay* to move from (p, I) to (p', I')

Measurement of CV: we can measure CV using the expenditure function such that $CV = e(p', u) - I$, where u is the utility obtained at the initial solution (p, I)

Translation: CV measures how much income can be taken away from the person in the new situation, to make that person just as well off as in the initial situation

- Note: CV will be negative when the new situation makes the person worse off

Difference between CV and ΔCS: the use of CV is regarded as the correct measure of welfare change, since if income effects are not small then the change in consumer surplus will over or under state welfare changes. However in the case where income effects are negligible (i.e. when we have quasi-linear u) then the change in consumer surplus can be used instead.

- Note: the change in consumer surplus is based on the idea that how much you pay for the first few units does not affect your marginal willingness to pay for additional units how? . If this is not true, then the change in CS will be biased

Axiom 3 - Continuity: for all $x \in \mathbb{R}^n_+$, the "at least as good as set" $\succeq (x)$ and the "no better than" set $\preceq (x)$ are closed in $\mathbb{R}^n_+$

- Conditions from Definition preferences are continuious if $G(x) = \left\{y \in \mathbb{R}^n_+ : \ y \succeq x\right\}$ and $W(x) = \left\{y \in \mathbb{R}^n_+ : \ x \succeq y\right\}$ are both closed in $\mathbb{R}^n_+$
- Theorem 1: suppose that preferences are complete. Then, they are continuous if and only if for all $x \in \mathbb{R}^n_+$ the sets $SG(x) = \left\{y \in \mathbb{R}^n_+ : \ y \succ x\right\}$ and $SW(x) = \left\{y \in \mathbb{R}^n_+ : \ x \succ y\right\}$ are open in $\mathbb{R}^2_+$
  - Logic: $SG(x)$ follows by definition of continuity and completeness when taking the complement of $W(x)$, $SW(x)$ follows by definition of continuity and completeness when taking the complement of $G(x)$
  - Note: $SG(x)$ is referred to as the strictly greater than x set while $SW(x)$ is referred to as the strictly worse than x set