ETC3250: Linear Regression

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Week 2 (a)

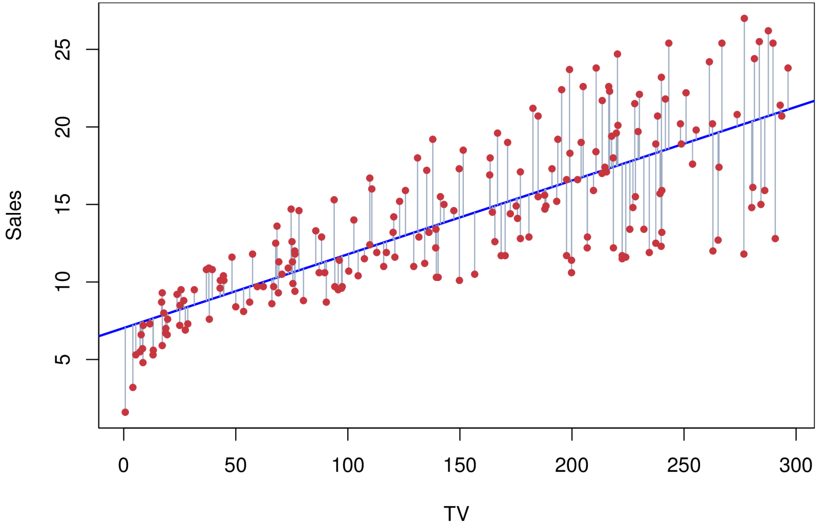
Multiple Regression

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \cdots + \beta_p X_{p,i} + e_i$$

- Each $X_{j,i}$ is numerical and is called a predictor.
- The coefficients $\beta_1, \ldots, \beta_p$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.
- Predictors may be transforms of other predictors. e.g., $X_2 = X_1^2$.

Multiple Regression

📊 The model describes a **line**, plane or hyperplane in the predictor space.



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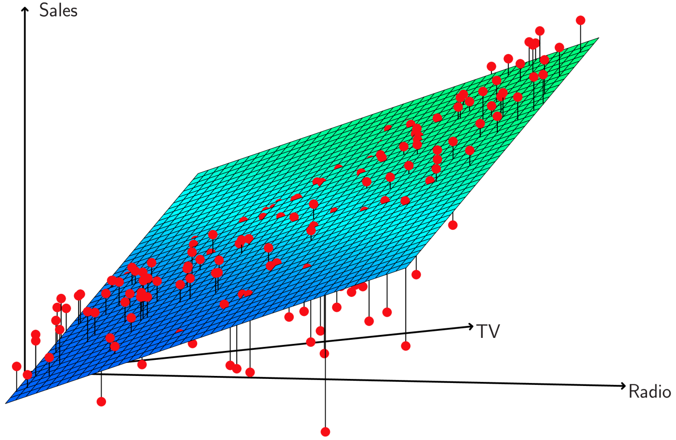
Multiple Regression

📊 The model describes a line, **plane** or hyperplane in the predictor space.



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Categorical Variables

Qualitative variables need to be converted to numeric.

$$x_i = \left\{ \begin{array}{ll} 1 & \text{if} \quad i^{\text{th}} \text{ obs is a koala} \\ 0 & \text{otherwise} \end{array} \right\}$$

which would result in the model

$$\hat{y}_i = \left\{ \begin{array}{ll} \beta_0 + \beta_1 & \text{if} \quad i^{\text{th}} \text{ obs is a koala} \\ \beta_0 & \text{otherwise} \end{array} \right\}$$

Categorical Variables

More than two categories

$$x_{i1} = \left\{ \begin{array}{ll} 1 & \text{if} \quad i^{\text{th}} \text{ obs is a koala} \\ 0 & \text{otherwise} \end{array} \right\}$$

$$x_{i2} = \left\{ \begin{array}{ll} 1 & \text{if} \quad i^{\text{th}} \text{ obs is a bilby} \\ 0 & \text{otherwise} \end{array} \right\}$$

which would result in the model using dummy variables.

$$\hat{y}_i = \left\{ \begin{array}{ll} \beta_0 + \beta_1 & \text{if} \quad i^{\text{th}} \text{ obs is a koala} \\ \beta_0 + \beta_2 & \text{if} \quad i^{\text{th}} \text{ obs is a bilby} \\ \beta_0 & \text{otherwise} \end{array} \right\}$$

Ordinary Least Squares (OLS)

Ordinary least squares is the simplest way to fit the model. Geometrically, this is the sum of the squared distances, parallel to the axis of the dependent variable, between each observed data point and the corresponding point on the regression surface – the smaller the sum of differences, the better the model fits the data.

Model Diagnostics

R^2 is the proportion of variation explained by the model, and measures the goodness of the fit, close to 1 the model explains most of the variability in Y, close to 0 it explains very little.

$$R^2 = 1 - \frac{RSS}{TSS}$$

where $RSS = \sum_{i=1}^{n}(y_i - \hat{y})^2$ (read: Residual Sum of Squares), and $TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2$ (read: Total Sum of Squares).

Model Diagnostics

Residual Standard Error (RSE) is an estimate of the standard deviation of ε. This is meaningful with the assumption that $\varepsilon \sim N(0, \sigma^2)$.

$$RSE = \sqrt{\frac{1}{n-p-1}RSS}$$

Model Diagnostics

F statistic tests whether any predictor explains response, by testing

$H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0$ vs H_A : at least one is not 0

Think about ...

- Is at least one of the predictors useful in predicting the response?
- Do all the predictors help to explain Y, or is only a subset of the predictors useful?
- How well does the model fit the data?
- Given a set of predictor values, what response value should we predict and how accurate is our prediction?

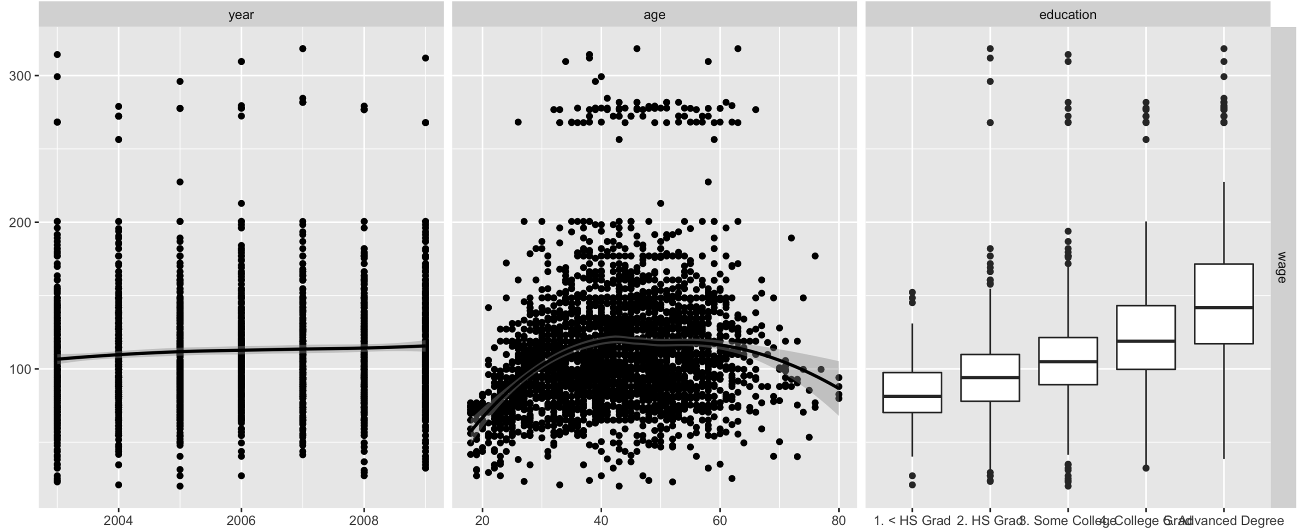
Example - Wages

Wage and other data for a group of 3000 male workers in the Mid-Atlantic region. Goal: Predict wage based on worker characteristics.

```
## Observations: 3,000
## Variables: 11
## $ year       <int> 2006, 2004, 2003, 2003, 2005, 2008, 2009, 2008, 2006, 2004…
## $ age        <int> 18, 24, 45, 43, 50, 54, 44, 30, 41, 52, 45, 34, 35, 39, 54…
## $ maritl     <fct> 1. Never Married, 1. Never Married, 2. Married, 2. Married…
## $ race       <fct> 1. White, 1. White, 1. White, 3. Asian, 1. White, 1. White…
## $ education  <fct> 1. < HS Grad, 4. College Grad, 3. Some College, 4. College…
## $ region     <fct> 2. Middle Atlantic, 2. Middle Atlantic, 2. Middle Atlantic…
## $ jobclass   <fct> 1. Industrial, 2. Information, 1. Industrial, 2. Informati…
## $ health     <fct> 1. <=Good, 2. >=Very Good, 1. <=Good, 2. >=Very Good, 1. <…
## $ health_ins <fct> 2. No, 2. No, 1. Yes, 1. Yes, 1. Yes, 1. Yes, 1. Yes, 1. Y…
## $ logwage    <dbl> 4.318063, 4.255273, 4.875061, 5.041393, 4.318063, 4.845098…
## $ wage       <dbl> 75.04315, 70.47602, 130.98218, 154.68529, 75.04315, 127.11…
```

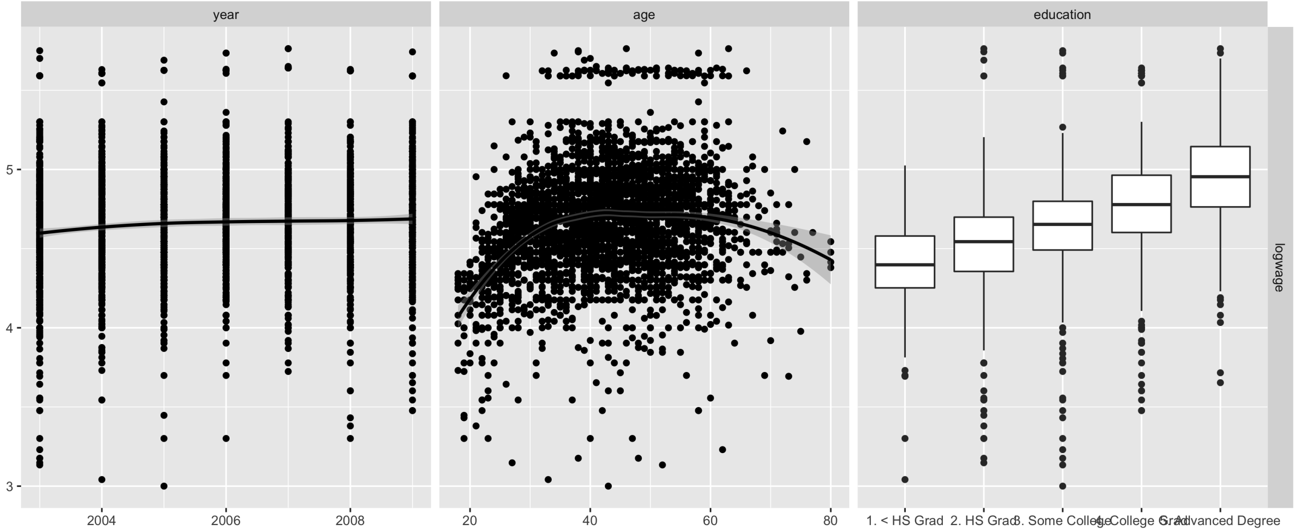
Take a look

What do the following pairwise comparisons of the variables `year`, `age` and, `education` against `wage` show us?



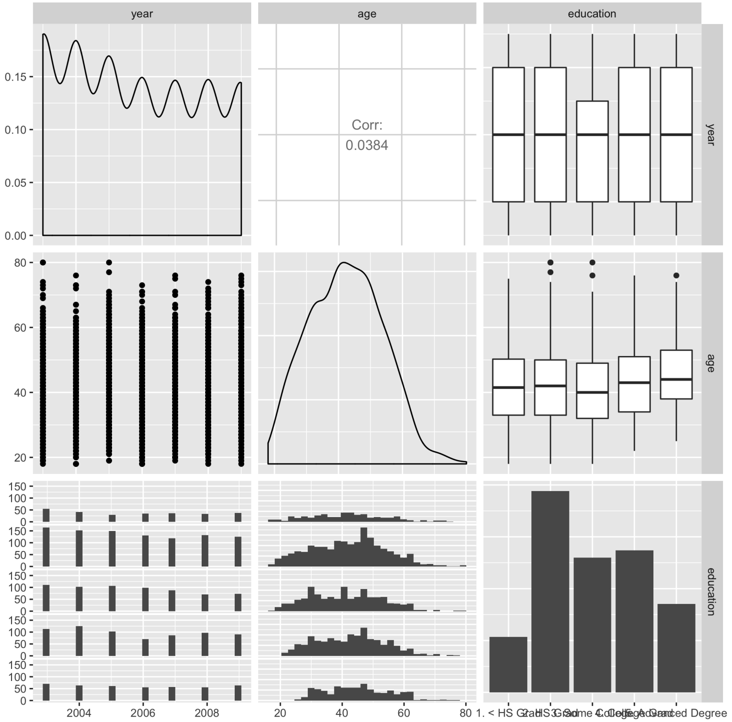
Take a look

If we examine `logwage` instead of `wage` as response variable - what changes?



Take a look

- Examine the predictors. Ideally values are spread out without any associations.



Model for wage data

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + e$$

where Y = `logwage`, X_1 = `year`, X_2 = `age`, and X_3 = `education`.

Fitting the model in R

```
lm(formula = logwage ~ year + age + education, data = Wage)

Coefficients:
                              Estimate Std. Error t value Pr(>|t|)
(Intercept)                 -1.745e+01  5.469e+00  -3.191  0.00143
year                         1.078e-02  2.727e-03   3.952 7.93e-05
age                          5.509e-03  4.813e-04  11.447  < 2e-16
education2. HS Grad          1.202e-01  2.086e-02   5.762 9.18e-09
education3. Some College     2.440e-01  2.195e-02  11.115  < 2e-16
education4. College Grad     3.680e-01  2.178e-02  16.894  < 2e-16
education5. Advanced Degree  5.411e-01  2.362e-02  22.909  < 2e-16

Residual standard error: 0.3023 on 2993 degrees of freedom
Multiple R-squared:  0.2631,    Adjusted R-squared:  0.2616
F-statistic: 178.1 on 6 and 2993 DF,  p-value: < 2.2e-16
```

Interpretation

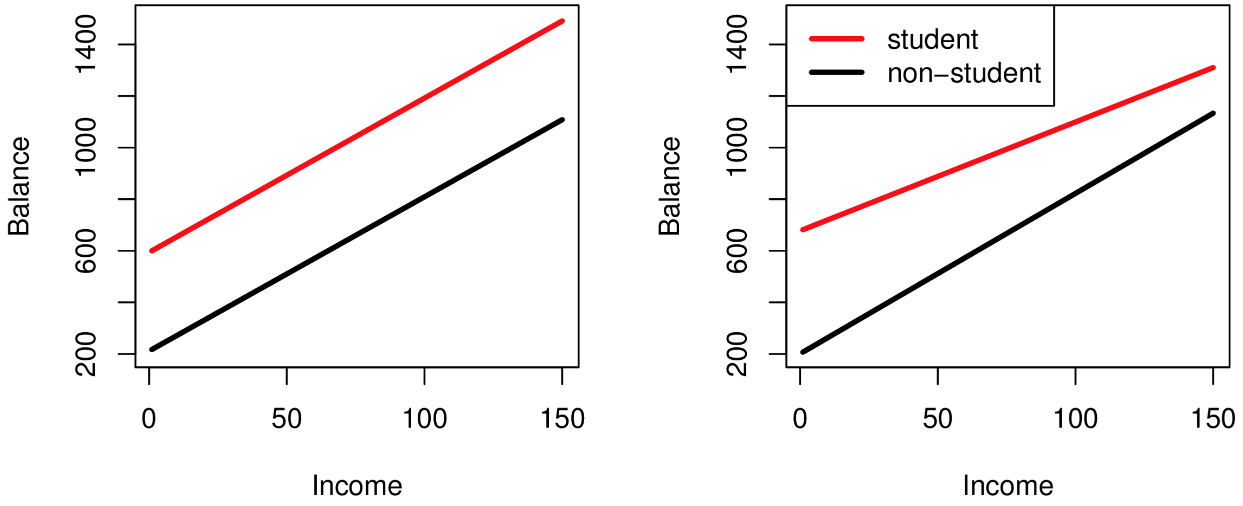
- The ideal scenario is when the predictors are uncorrelated.
  - Each coefficient can be interpreted and tested separately.
- Correlations amongst predictors cause problems.
  - The variance of all coefficients tends to increase, sometimes dramatically.
  - Interpretations become hazardous -- when X_j changes, everything else changes.
  - Predictions still work provided new X values are within the range of training X values.
- Claims of causality should be avoided for observational data.

Interactions

- An interaction occurs when the one variable changes the effect of a second variable. (e.g., spending on radio advertising increases the effectiveness of TV advertising).
- To model an interaction, include the product $X_1 X_2$ in the model in addition to X_1 and X_2.

- **Hierarchy principle**: If we include an interaction in a model, we should also include the main effects, even if the p-values associated with their coefficients are not significant. (This is because the interactions are almost impossible to interpret without the main effects.)

Interactions



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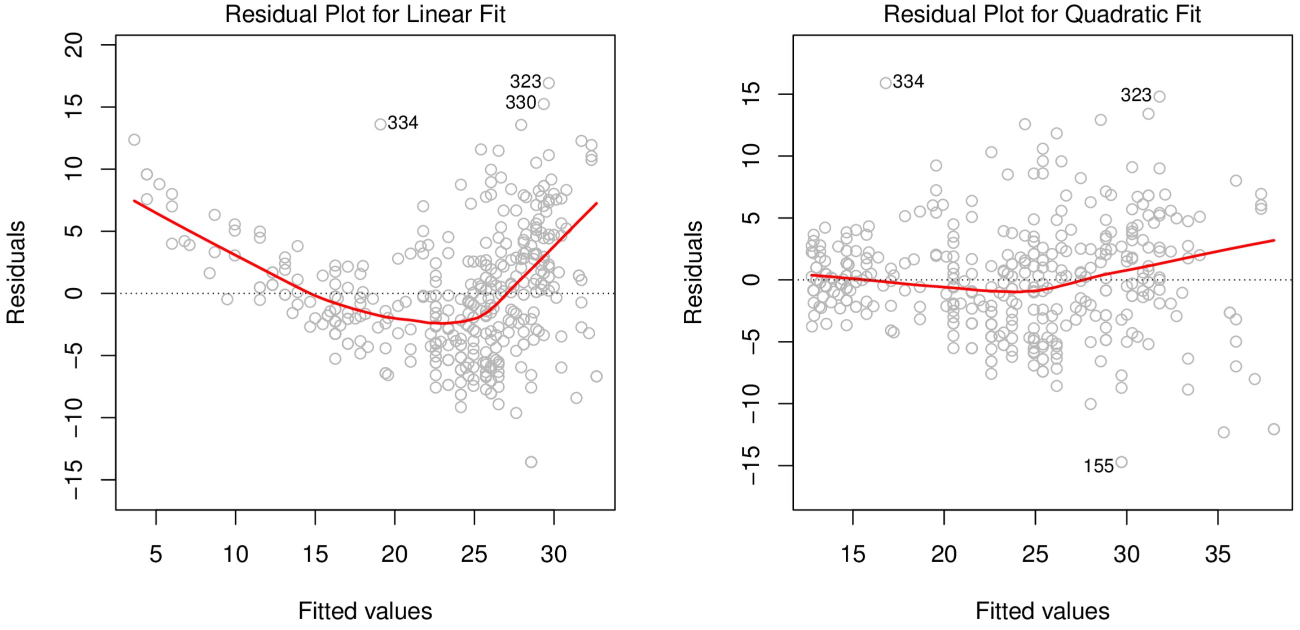
Residuals

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

Residuals



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Model

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \cdots + \beta_p X_{p,i} + e_i.$$

Let $Y = (Y_1, \ldots, Y_n)', e = (e_1, \ldots, e_n)', \beta = (\beta_0, \ldots, \beta_p)'$ and

$$X = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \ldots & X_{p,1} \\ 1 & X_{1,2} & X_{2,2} & \ldots & X_{p,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{1,n} & X_{2,n} & \ldots & X_{p,n} \end{bmatrix}.$$

Then

$$Y = X\beta + e$$

Least squares estimation

Minimize: $(Y - X\beta)'(Y - X\beta)$

Differentiate wrt β and equal to zero gives

$\hat{\beta} = (X'X)^{-1}X'Y$

(The "normal equation".)

$$\hat{\sigma}^2 = \frac{1}{n-p-1}(Y - X\hat{\beta})'(Y - X\hat{\beta})$$

Note: If you fall for the dummy variable trap, $(X'X)$ is a singular matrix.

Likelihood

If the errors are iid and normally distributed, then

$$Y \sim \mathcal{N}_n(X\beta, \sigma^2 I)$$

So the likelihood is

$$L = \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)\right)$$

which is maximized when $(Y - X\beta)'(Y - X\beta)$ is minimized.

So MLE $\equiv$ OLS.

Predictions

Optimal predictions

$$\hat{Y}^* = \text{E}(Y^*|X^*, Y, X) = X^*\hat{\beta} = X^*(X'X)^{-1}X'Y$$

where X^* is a row vector containing the values of the regressors for the predictions (in the same format as X).

Prediction variance

$$\text{Var}(Y^*|X^*, Y, X) = \sigma^2 \left[1 + X^*(X'X)^{-1}(X^*)'\right]$$

- This ignores any errors in X^*.
- 95% prediction intervals assuming normal errors: $\hat{Y}^* \pm 1.96\sqrt{\text{Var}(Y^*|Y, X, X^*)}$.

👩‍💻 Made by a human with a computer

Slides at https://iml.numbat.space.

Code and data at https://github.com/numbats/iml.

Created using R Markdown with flair by **xaringan**, and **kunoichi** (female ninja) style.