

# ETC3250: Categorical response

## Semester 1, 2020

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Econometrics and Business Statistics  
Monash University

Week 3 (a)

# Categorical responses

In classification, the output  $Y$  is a categorical variable. For example,

-  Loan approval:  $Y \in \{\text{successful, unsuccessful}\}$
-  Type of business culture:  
 $Y \in \{\text{clan, adhocracy, market, hierarchical}\}$
-  Historical document author:  $Y \in \{\text{Austen, Dickens, Imitator}\}$
-  Email:  $Y \in \{\text{spam, ham}\}$

Map the categories to a numeric variable, or possibly a binary matrix.

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An email comes into the server. Should it be moved into the inbox or the junk mail box, based on header text, sender, origin, time of day, ...?

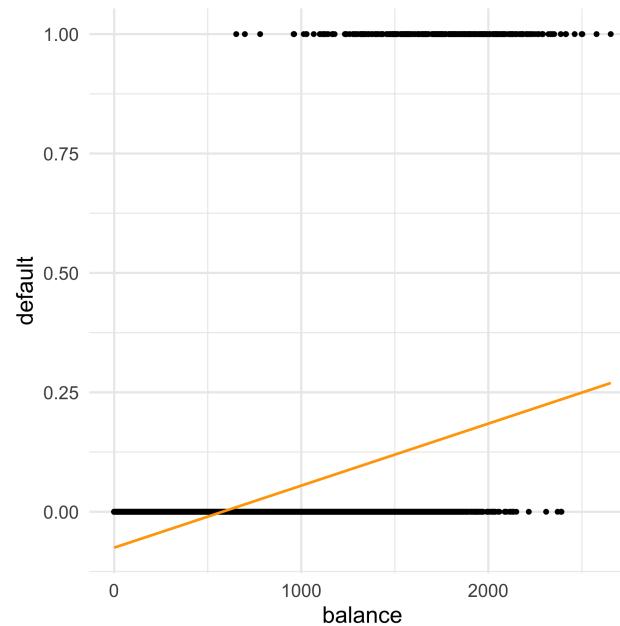
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# When linear regression is not appropriate

Consider the following data `simcredit` which looks at the default status based on credit balance.

**Question:** Why is a linear model not appropriate for this data?

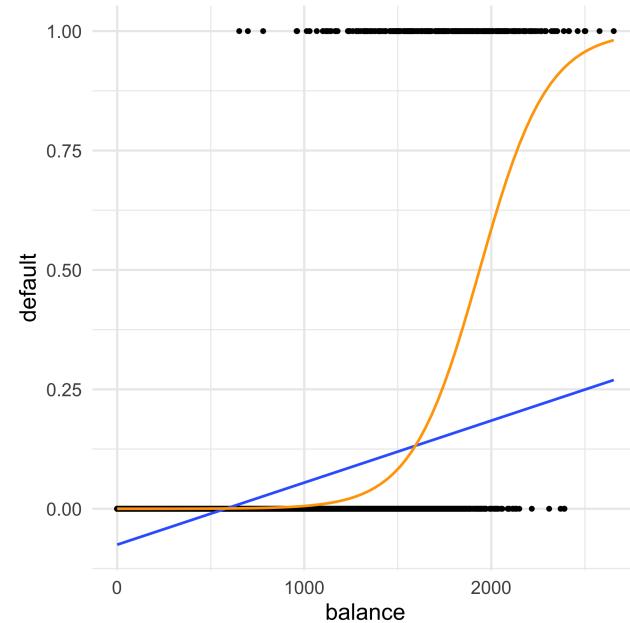
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# Modelling binary responses

- >To model binary data, we need to link our predictors to our response using a *link function*.
- There are many different types of link functions we could use, but we will focus today on the logistic link function.



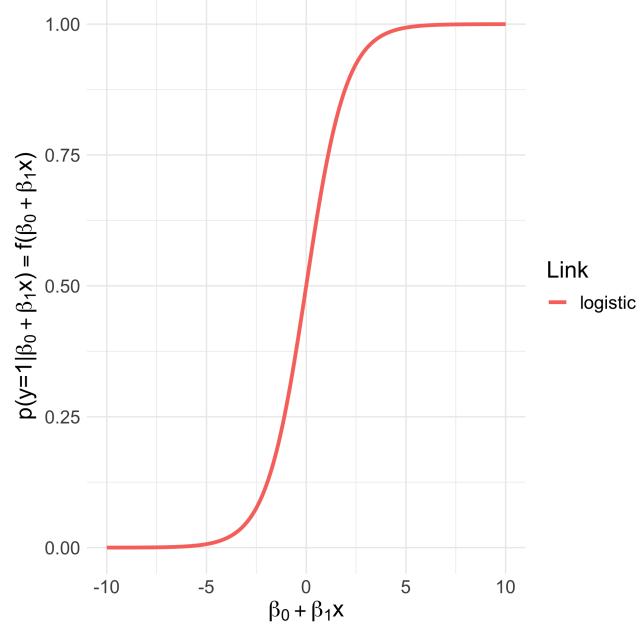
# The logistic function

Instead of predicting the outcome directly, we instead predict the probability of being class 1, given the linear combination of predictors, using the **logistic** link function.

$$p(y = 1 | \beta_0 + \beta_1 x) = f(\beta_0 + \beta_1 x)$$

where

$$f(\beta_0 + \beta_1 x) = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$



## Logistic function

Transform the function:

$$f(x) = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$\rightarrow f(x) = \frac{1}{1/e^{\beta_0 + \beta_1 x} + 1}$$

$$\rightarrow 1/f(x) = 1/e^{\beta_0 + \beta_1 x} + 1$$

$$\rightarrow 1/f(x) - 1 = 1/e^{\beta_0 + \beta_1 x}$$

$$\rightarrow \frac{1}{1/f(x)-1} = e^{\beta_0 + \beta_1 x}$$

$\rightarrow \dots$

$\rightarrow \dots$

$$\rightarrow \frac{f(x)}{1-f(x)} = e^{\beta_0 + \beta_1 x}$$

$$\rightarrow \log_e \frac{f(x)}{1-f(x)} = \beta_0 + \beta_1 x$$

The left-hand side  $\log_e \frac{f(x)}{1-f(x)}$  is called the **log-odds ratio** or **logit**.



## The logistic regression model

The fitted model, where  $P(Y = 0|X) = 1 - P(Y = 1|X)$ , is then written as:

$$\log_e \frac{P(Y=1|X)}{1-P(Y=1|X)} = \beta_0 + \beta_1 X$$

► Details

# Interpretation

## 📊 Linear regression

- ➊  $\beta_1$  gives the average change in  $Y$  associated with a one-unit increase in  $X$

## 📊 Logistic regression

- ➋ Increasing  $X$  by one unit changes the log odds by  $\beta_1$ , or equivalently it multiplies the odds by  $e^{\beta_1}$
- ⌚ However, because the model is not linear in  $X$ ,  $\beta_1$  does not correspond to the change in response associated with a one-unit increase in  $X$

## Maximum Likelihood Estimation

Given the logistic  $p(x_i) = \frac{1}{e^{-(\beta_0 + \beta_1 x_i)} + 1}$

We choose parameters  $\beta_0, \beta_1$  to maximize the likelihood of the data given the model. The likelihood function is

$$l_n(\beta_0, \beta_1) = \prod_{y_i=1,i}^n p(x_i) \prod_{y_i=0,i}^n (1 - p(x_i)).$$

It is more convenient to maximize the *log-likelihood*:

$$\max_{\beta_0, \beta_1} \log l_n(\beta_0, \beta_1) = \max_{\beta_0, \beta_1} - \sum_{i=1}^n \log (1 + e^{-(\beta_0 + \beta_1 x_i)})$$

## Making predictions

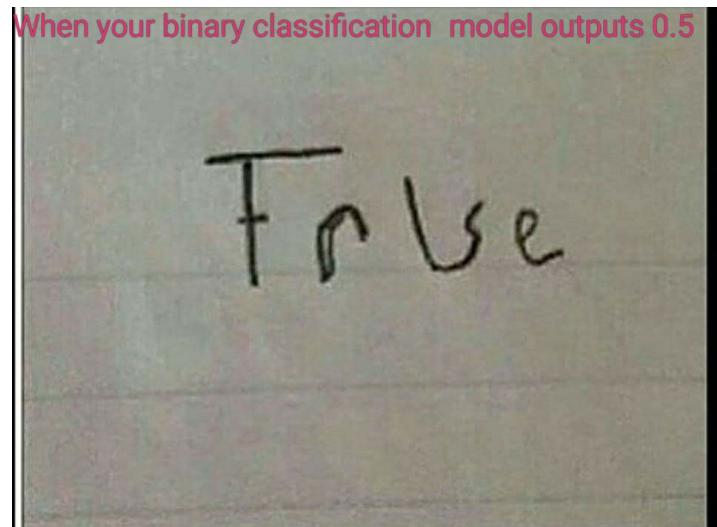
With estimates from the model fit,  $\hat{\beta}_0, \hat{\beta}_1$ , we can predict the probability of belonging to class 1 using:

$$p(y = 1 | \hat{\beta}_0 + \hat{\beta}_1 x) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x}}$$

This probability can be rounded to 0 or 1 for class prediction.

In R, we simply use the `predict()` function.

Of course, probabilities close to 0.5 are hard to classify!



Source: Statistical Statistics Memes

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## Fitting credit data in R

We use the `glm` function in R to fit a logistic regression model. The `glm` function can support many response types, so we specify `family="binomial"` to let R know that our response is *binary*.

```
library(broom)
fit <- glm(default~balance,
            data=simcredit, family="binomial")
tidy(fit)
```

```
## # A tibble: 2 x 5
##   term      estimate std.error statistic p.value
##   <chr>        <dbl>     <dbl>      <dbl>    <dbl>
## 1 (Intercept) -10.7      0.361     -29.5 3.62e-191
## 2 balance      0.00550   0.000220    25.0 1.98e-137
```

# Fitting credit data in R

We can use the `predict()` function to predict the probability of default, given credit balance. We then round these probabilities to predict default status.

```
probs <- predict(fit, simcredit ,type="response")  
head(probs, 4)
```

```
##           1            2            3            4  
## 0.0013056797 0.0021125949 0.0085947405 0.0004344368
```

```
head(round(probs), 4)
```

```
## 1 2 3 4  
## 0 0 0 0
```

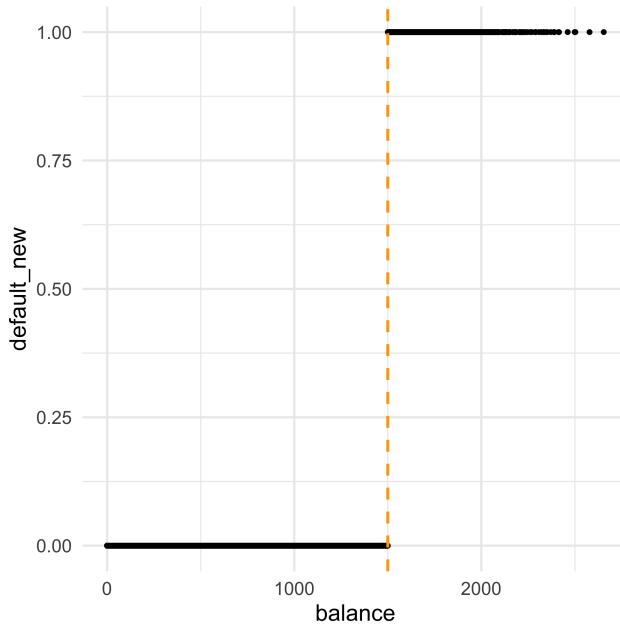


A warning for using GLMs!

# When logistic regression fails

Consider the case when the data is *perfectly* separated.

Here, we can see that all balances above \$1500 default.



## When logistic regression fails

If we fit a `glm` model to this data, the MLE fit will try and fit a step-wise function to this graph, pushing coefficients sizes towards infinity and produce large standard errors. R will warn us that the algorithm does not converge.

```
fit <- glm(default_new~balance,  
           data=simcredit, family="binomial")  
tidy(fit)
```

```
# A tibble: 2 x 5  
  term      estimate std.error statistic p.value  
  <chr>     <dbl>     <dbl>     <dbl>    <dbl>  
1 (Intercept) -41692.   57581.    -0.724   0.469  
2 balance      27.8      38.4      0.724   0.469
```

## Take home message - take note of R warnings

**when you get a warning in R and you ignore it  
but then it comes back to haunt you**



Source: R Memes for Statistical Fiends

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## Linear Discriminant Analysis

Logistic regression involves directly modeling  $P(Y = k|X = x)$  using the logistic function. Rounding the probabilities produces class predictions, in two class problems; selecting the class with the highest probability produces class predictions in multi-class problems.

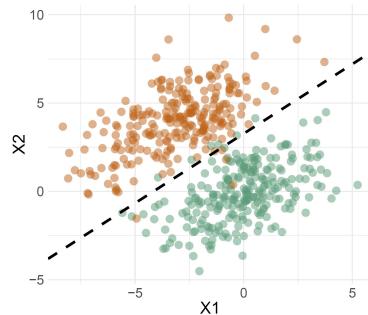
Another approach for building a classification model is linear discriminant analysis. This involves directly estimating the distribution of the predictors, separately for each class.

## Compare the pair

### Logistic Regression

Goal - directly estimate  $P(Y|X)$   
*(the dashed line)*

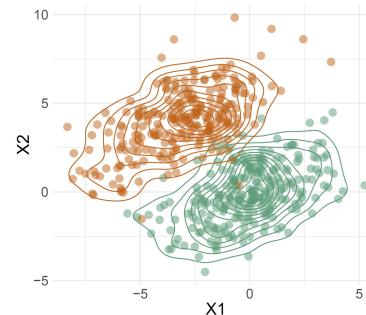
Assumptions - no assumptions on  
predictor space



### Linear Discriminant Analysis

Goal - estimate  $P(X|Y)$  (*the contours*) to then deduce  $P(Y|X)$

Assumptions - predictors are  
normally distributed



Assumptions are critical in LDA



Source: Statistical Statistics Memes

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## Bayes Theorem

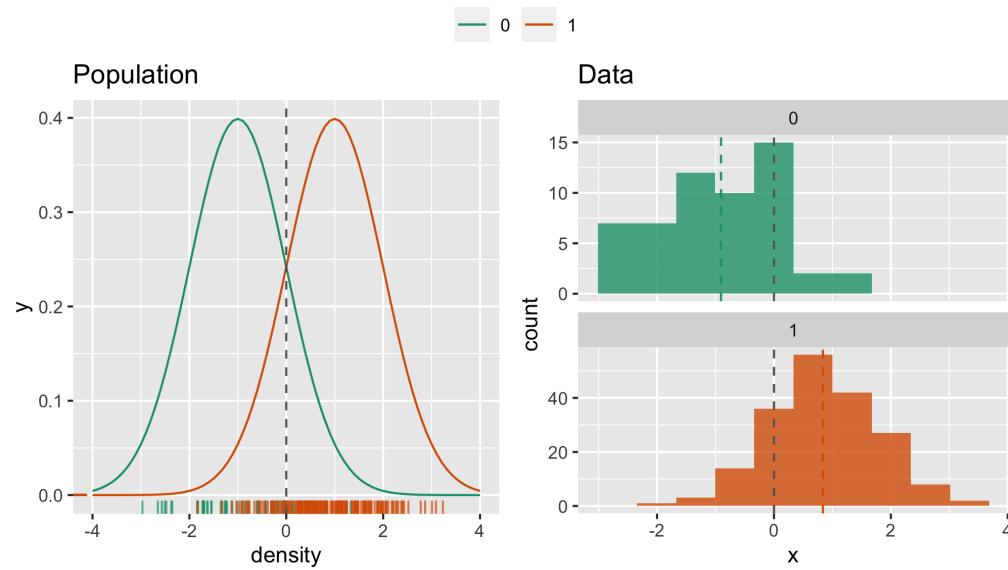
Let  $f_k(x)$  be the density function for predictor  $x$  for class  $k$ . If  $f$  is small, the probability that  $x$  belongs to class  $k$  is small, and conversely if  $f$  is large.

Bayes theorem (for  $K$  classes) states:

$$P(Y = k|X = x) = p_k(x) = \frac{\pi_k f_k(x)}{\sum_{i=1}^K \pi_i f_i(x)}$$

where  $\pi_k = P(Y = k)$  is the prior probability that the observation comes from class  $k$ .

## LDA with $p = 1$ predictors



## LDA with $p = 1$ predictors

We assume  $f_k(x)$  is univariate **Normal** (Gaussian):

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp \left( -\frac{1}{2\sigma_k^2}(x - \mu_k)^2 \right)$$

where  $\mu_k$  and  $\sigma_k^2$  are the mean and variance parameters for the  $k$ th class. Further assume that  $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_K^2$ ; then the conditional probabilities are

$$p_k(x) = \frac{\pi_k \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{1}{2\sigma^2}(x - \mu_k)^2 \right)}{\sum_{l=1}^K \pi_l \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{1}{2\sigma^2}(x - \mu_l)^2 \right)}$$

## LDA with $p = 1$ predictors

The Bayes classifier is assign new observation  $X = x_0$  to the class with the highest  $p_k(x_0)$ . A simplification of  $p_k(x_0)$  yields the discriminant functions:

$$\delta_k(x_0) = x_0 \frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$

and the rule Bayes classifier will assign  $x_0$  to the class with the largest value.

## LDA with $p = 1$ predictors

If  $K = 2$  and  $\pi_1 = \pi_2$ , we assign  $x_0$  to class if

$$\delta_1(x_0) > \delta_2(x_0)$$

$$x_0 \frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + \log(\pi) > x_0 \frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + \log(\pi)$$

which simplifies to  $x_0 > \frac{\mu_1 + \mu_2}{2}$ .

This is estimated on the data with  $x_0 > \frac{\bar{x}_1 + \bar{x}_2}{2}$ .

## Multivariate LDA

To indicate that a  $p$ -dimensional random variable  $X$  has a multivariate Gaussian distribution with  $E[X] = \mu$  and  $\text{Cov}(X) = \Sigma$ , we write  $X \sim N(\mu, \Sigma)$ .

The multivariate normal density function is:

$$f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)' \Sigma^{-1} (x - \mu)\right\}$$

with  $x, \mu$  are  $p$ -dimensional vectors,  $\Sigma$  is a  $p \times p$  variance-covariance matrix.

## Multivariate LDA

The discriminant functions are:

$$\delta_k(x) = x' \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k' \Sigma^{-1} \mu_k + \pi_k$$

and Bayes classifier is assign a new observation  $x_0$  to the class with the highest  $\delta_k(x_0)$ .

When  $K = 2$  and  $\pi_1 = \pi_2$  this reduces to

Assign observation  $x_0$  to class 1 if

$$x_0' \Sigma^{-1} (\mu_1 - \mu_2) > \frac{1}{2} (\mu_1 + \mu_2)' \Sigma^{-1} (\mu_1 - \mu_2)$$

## Multivariate LDA

**Discriminant space:** a benefit of LDA is that it provides a low-dimensional projection of the  $p$ -dimensional space, where the groups are the most separated. For  $K = 2$ , this is

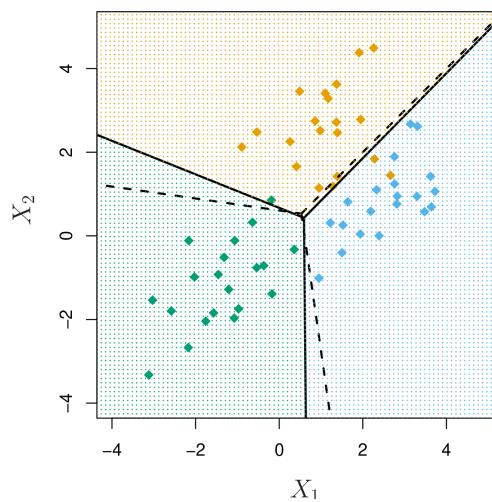
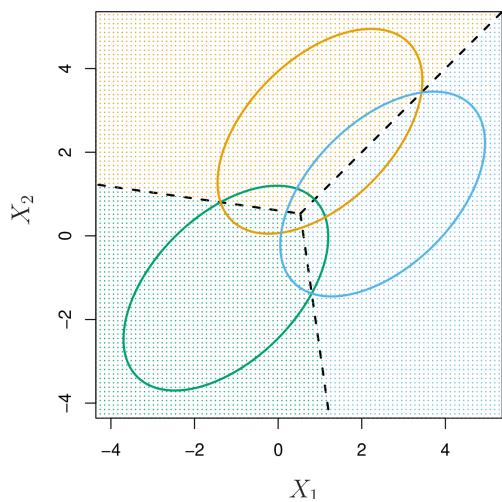
$$\Sigma^{-1}(\mu_1 - \mu_2)$$

For  $K > 2$ , the discriminant space is found by taking an eigen-decomposition of  $\Sigma^{-1}\Sigma_B$ , where

$$\Sigma_B = \frac{1}{K} \sum_{i=1}^K (\mu_i - \mu)(\mu_i - \mu)'$$

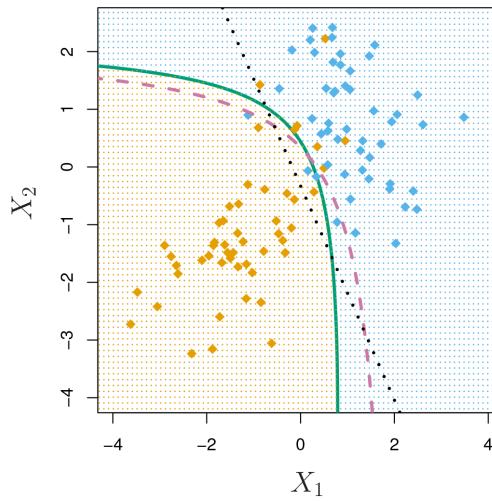
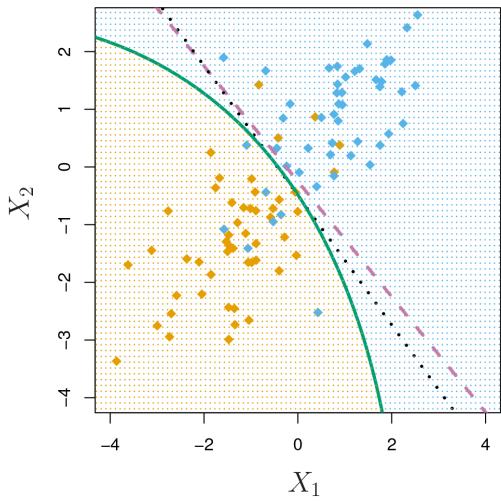
## Multivariate LDA

The dashed lines are the Bayes decision boundaries. Ellipses that contain 95% of the probability for each of the three classes are shown. Solid line corresponds to the class boundaries from the LDA model fit to the sample.



## Quadratic DA (QDA)

A quadratic boundary is obtained by relaxing the assumption of equal variance-covariance, and assume that  
 $\Sigma_k \neq \Sigma_l$ ,  $k \neq l, k, l = 1, \dots, K$





# Made by a human with a computer

Slides at <https://iml.numbat.space>.

Code and data at <https://github.com/numbats/iml>.

Created using R Markdown with flair by [xaringan](#), and  
[kunoichi](#) (female ninja) style.



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