AMaLGaM IDEAs in Noiseless Black-Box Optimization Benchmarking

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ABSTRACT

This paper describes the application of a Gaussian Estimation-of-Distribution (EDA) for real-valued optimization to the noiseless part of a benchmark introduced in 2009 called BBOB (Black-Box Optimization Benchmarking). Specifically, the EDA considered here is the recently introduced parameter-free version of the Adapted Maximum-Likelihood Gaussian Model Iterated Density-Estimation Evolutionary Algorithm (AMaLGaM-ID𝔼A). Also the version with incremental model building (iAMaLGaM-ID𝔼A) is considered.

Categories and Subject Descriptors

G.1.6 [**Numerical Analysis**]: OptimizationGlobal Optimization, Unconstrained Optimization; F.2.1 [**Analysis of Algorithms and Problem Complexity**]: Numerical Algorithms and Problems

General Terms

Algorithms

Keywords

Benchmarking, Black-box optimization, Evolutionary computation

1. METHOD

Estimation-of-distribution algorithms (EDAs) [7, 8] are an important strand of research on black-box optimization (BBO). EDAs attempt to automatically exploit features of a problem's structure by probabilistically modeling the search space based on previously evaluated solutions and generating new solutions by sampling the probabilistic model.

The general EDA procedure is as follows. A population $\mathcal{P}$ of n solutions is maintained. Through selection, a vector $\mathcal{S}$ is selected from $\mathcal{P}$. A probability distribution over the solution space is then estimated using $\mathcal{S}$ as a data set. New solutions are generated by sampling the estimated probability distribution. Finally, the newly generated samples are incorporated into the population and the process repeats until a termination criterion has been satisfied.

The EDA considered here is the Adapted Maximum-Likelihood Gaussian Model Iterated Density-Estimation Evolutionary Algorithm (AMaLGaM-ID𝔼A, or AMaLGaM for short). In AMaLGaM, the probability distribution used is the normal, also known as the Gaussian, distribution. This EDA uses maximum–likelihood estimates for the mean and the covariance matrix, estimated from the selected solutions. It has a mechanism that scales up the covariance matrix when required to prevent premature convergence on slopes. It furthermore has a mechanism that anticipates the mean shift in the next generation to speed up descent (in case of minimization) along slopes. For a more extensive description, we refer the interested reader to the literature [1].

In addition to the above base procedure, recently a parameter-free version of AMaLGaM was introduced [3]. After experimental analysis, settings were proposed for all parameters. Guidelines were developed for the minimally required population size that allows unimodal problems to be solved. On multimodal problems a restart mechanism is required to increase the probability of success. The specific restart scheme considered increases the number of solutions upon each restart by alternating between two approaches: a single run with a larger population and more parallel runs. To maximize the joint global effect of the parallel runs, their locality is increased by starting them in separate regions that are obtained from clustering the search space first. When increasing the number of parallel runs, the subpopulation size is also increased slightly so as to increase the robustness of the more localized searches.

Distribution estimation in AMaLGaM is done anew from scratch each generation. Subsequent iterations however have much in common and therefore the required population size can be reduced by incremental learning, i.e. combining the distribution estimated from $\mathcal{S}$ with the distribution used in the previous generation. In iAMaLGaM a memory-decay approach is taken to this end. On unimodal problems the required population size was found to indeed be significantly reduced while at the same time requiring less function evaluations to reach the same solution quality. Results on multimodal landscapes indicated however that if memory resources are not very important, a larger base–population size helps in optimizing multimodal problems, thus favoring the non-incremental approach. For this reason we tested both AMaLGaM and iAMaLGaM on the BBOB benchmark.

Next to the full covariance matrix, two other versions of AMaLGaM exist that reduce the number of distribution parameters to be estimated. One version uses Bayesian factorizations to select only the most important covariances while another version allows only variances. If only a few dependencies between problem variables exist, these methods outperform the use of the full covariance matrix in asymptotic complexity for the scalability in terms of required function evaluations and required time. These restrictions however also render the EDA non-rotationally invariant and therefore less generally applicable. For this reason and for the sake of space, we do not submit these variants to the BBOB benchmark here. A closer look at the differences with the full covariance matrix can be found in [3]; BBOB benchmarks for additional variants are given in [2].

For technical completeness, pseudo-code is presented below. A note on the pseudo-code: in iAMaLGaM, for $\hat{\mathbf{\Sigma}}(0)$ a matrix with the ML variances on the diagonal and zeros off the diagonal is used. Also, $\hat{\boldsymbol{\mu}}^{\text{Shift}}(t)$ is non–existent for $t = 0$ and for $t = 1$ it is $\hat{\boldsymbol{\mu}}(1) - \hat{\boldsymbol{\mu}}(0)$. SDR stands for standard-deviation ratio, NIS stands for no-improvement stretch.

(i)AMaLGaM-Free
$s \leftarrow 0; n^{\text{Base}} \leftarrow 17 + 3D^{1.5}$ (iAMaLGaM: $n^{\text{Base}} \leftarrow 10D^{0.5}$)
do
  if $(s$ **mod** $2) = 0$ **then** $n \leftarrow (1 + s/2)n^{\text{Base}}; p \leftarrow 2^{s/2}$
  else $n \leftarrow 2^{1+s/2}n^{\text{Base}}; p \leftarrow 1$
  Run (i)AMaLGaM with population size n and p parallel runs, starting from the clustering of np randomly generated solutions into p clusters and using $\eta^{\text{DEC}} \leftarrow 0.9; \eta^{\text{INC}} \leftarrow 1/\eta^{\text{DEC}}; \theta^{\text{SDR}} \leftarrow 1; \tau \leftarrow 0.35; \alpha^{\text{AMS}} \leftarrow \frac{1}{2}\tau(n/(n-1)); \delta^{\text{AMS}} \leftarrow 2; \text{NIS}^{\text{MAX}} \leftarrow 25 + D$
  $s \leftarrow s + 1$
while optimum not found and max. eval. not reached

(i)AMaLGaM
$\eta^{\mathbf{\Sigma}} \leftarrow 1; \eta^{\text{Shift}} \leftarrow 1$
(iAMaLGaM: $\eta^{\mathbf{\Sigma}} \leftarrow 1 - e^{-1.1\lfloor\tau n\rfloor^{1.2}/D^{1.6}}; \eta^{\text{Shift}} \leftarrow 1 - e^{-1.2\lfloor\tau n\rfloor^{0.31}/D^{0.50}}$)
$c^{\text{Multiplier}} \leftarrow 1; n^{\text{AMS}} \leftarrow \alpha^{\text{AMS}}(n-1); \text{NIS} \leftarrow 0; t \leftarrow 0$
do
  $\boldsymbol{\mathcal{S}} \leftarrow$ the best $\lfloor\tau n\rfloor$ solutions in $\boldsymbol{\mathcal{P}}$ (truncation selection)
  $\hat{\boldsymbol{\mu}}(t) \leftarrow \frac{1}{|\boldsymbol{\mathcal{S}}|}\sum_{i=0}^{|\boldsymbol{\mathcal{S}}|-1} \boldsymbol{\mathcal{S}}_i$
  $\hat{\mathbf{\Sigma}}(t) \leftarrow (1-\eta^{\mathbf{\Sigma}})\hat{\mathbf{\Sigma}}(t-1) + \eta^{\mathbf{\Sigma}} \frac{1}{|\boldsymbol{\mathcal{S}}|}\sum_{i=0}^{|\boldsymbol{\mathcal{S}}|-1} (\boldsymbol{\mathcal{S}}_i - \hat{\boldsymbol{\mu}}(t))(\boldsymbol{\mathcal{S}}_i - \hat{\boldsymbol{\mu}}(t))^T$
  $\hat{\boldsymbol{\mu}}^{\text{Shift}}(t) \leftarrow (1-\eta^{\text{Shift}})\hat{\boldsymbol{\mu}}^{\text{Shift}}(t-1) + \eta^{\text{Shift}}(\hat{\boldsymbol{\mu}}(t) - \hat{\boldsymbol{\mu}}(t-1))$
  $\hat{\boldsymbol{\mu}} \leftarrow \hat{\boldsymbol{\mu}}(t); \hat{\mathbf{\Sigma}} \leftarrow c^{\text{Multiplier}}\hat{\mathbf{\Sigma}}(t); \boldsymbol{L}\boldsymbol{L}^* \leftarrow$ Cholesky decomp. of $\hat{\mathbf{\Sigma}}$
  $\boldsymbol{\mathcal{P}}_0 \leftarrow$ the best solution in $\boldsymbol{\mathcal{S}}$
  $\boldsymbol{\mathcal{P}}_{1\ldots n-1} \leftarrow n-1$ samples from $\mathcal{N}(\hat{\boldsymbol{\mu}}, \hat{\mathbf{\Sigma}}) = \hat{\boldsymbol{\mu}} + \boldsymbol{L}\mathcal{N}(\mathbf{0}, \boldsymbol{I})$
  for n^{AMS} random solutions $\boldsymbol{\mathcal{P}}_j$ $(1 \le j \le n-1)$
  do $\boldsymbol{\mathcal{P}}_j \leftarrow \boldsymbol{\mathcal{P}}_j + \delta^{\text{AMS}} c^{\text{Multiplier}} \hat{\boldsymbol{\mu}}^{\text{Shift}}(t)$
  if any $\boldsymbol{\mathcal{P}}_i$ better than $\boldsymbol{\mathcal{P}}_0$ $(1 \le i \le n-1)$
  then
    $\text{NIS} \leftarrow 0$
    if $c^{\text{Multiplier}} < 1$ **then** $c^{\text{Multiplier}} \leftarrow 1$
    $\boldsymbol{x}^{\text{avg-imp}} \leftarrow$ average of all $\boldsymbol{\mathcal{P}}_i$ better than $\boldsymbol{\mathcal{P}}_0$ $(1 \le i \le n-1)$
    $\text{SDR} \leftarrow \max_{0 \le i \le D-1}\{|(\boldsymbol{L}^{-1}(\boldsymbol{x}^{\text{avg-imp}} - \hat{\boldsymbol{\mu}}))_i|\}$
    if $\text{SDR} > \theta^{\text{SDR}}$ **then** $c^{\text{Multiplier}} \leftarrow \eta^{\text{INC}} c^{\text{Multiplier}}$
  else
    if $c^{\text{Multiplier}} \le 1$ **then** $\text{NIS} \leftarrow \text{NIS} + 1$
    if $(c^{\text{Multiplier}} > 1)$ **or** $(\text{NIS} \ge \text{NIS}^{\text{MAX}})$
    then $c^{\text{Multiplier}} \leftarrow \eta^{\text{DEC}} c^{\text{Multiplier}}$
    if $(c^{\text{Multiplier}} < 1)$ **and** $(\text{NIS} < \text{NIS}^{\text{MAX}})$ **then** $c^{\text{Multiplier}} \leftarrow 1$
  $t \leftarrow t + 1$
while opt. not found, max. eval. not reached and $c^{\text{Multiplier}} \ge 10^{-10}$

2. PARAMETERS AND OTHER SETTINGS

For initialization, a uniform sampling in $[-5, 5]^D$ was used, where D denotes the dimension of the search space. The experiments according to [5] on the benchmark functions given in [4, 6] have been conducted using the provided C-code. The AMaLGaM implementation used is also in C. A maximum of $10^6 D$ function evaluations is allowed. No changes were made to parameter-free AMaLGaM as described in [3] and as outlined above. Therefore no parameter tuning was required and the crafting effort CrE [5] is zero.

3. CPU TIMING EXPERIMENT

For the timing experiment the full covariance matrix variant for both AMaLGaM and iAMaLGaM were run with a maximum of $10^6 D$ function evaluations and restarted until 30 seconds had passed (according to Figure 2 in [5]). The experiments have been conducted on an Intel Q6600 Core2Quad 2.4 GHz processor under Fedora Linux release 10 (Cambridge). In 2, 3, 5, 10, 20 and 40 dimensions, the time in 10^{-7} seconds per function evaluation was as follows:

	2	**3**	**5**	**10**	**20**	**40**
AMaLGaM	1.9	2.2	3.0	5.0	10	24
iAMaLGaM	1.9	2.3	3.0	5.3	11	29

4. RESULTS AND CONCLUSION

Results from experiments according to [5] on the benchmark functions given in [4, 6] are presented in Figures 1 and 2 and in Table 1 for AMaLGaM and in Figures 3 and 4 and in Table 2 for iAMaLGaM.

Problems with weak structure appear to be the hardest for (i)AMaLGaM. Even within $10^6 D$ evaluations the optimum cannot be found within a desirable precision, especially for larger D. The difference between AMaLGaM and iAMaLGaM is not large which supports the design of the population-size reducing incremental-learning approach used. Consistent with earlier findings, the incremental approach is better on unimodal functions, whereas the non-incremental approach is (slightly) better on multimodal functions, most likely due to the larger base population-size.

5. REFERENCES

[1] P. A. N. Bosman, J. Grahl, and D. Thierens. Enhancing the performance of maximum–likelihood Gaussian EDAs using anticipated mean shift. In G. Rudolph et al., editors, *Parallel Problem Solving from Nature — PPSN X*, pages 133–134, Berlin, 2008. Springer–Verlag.

[2] P. A. N. Bosman, J. Grahl, and D. Thierens. A parameter-free Gaussian EDA called AMaLGaM-IDEA: algorithms and benchmarks. CWI technical report (*To Appear*), 2009.

[3] P.A.N. Bosman. On emprical memory design, faster selection of Bayesian factorizations and parameter-free Gaussian EDAs. In G. Raidl et al., editors, *Proc. of the Genetic and Evolutionary Computation Conference — GECCO–2009*, New York, New York, 2009. ACM Press. (*To Appear*).

[4] S. Finck, N. Hansen, R. Ros, and A. Auger. Real-parameter black-box optimization benchmarking 2009: Presentation of the noiseless functions. Technical Report 2009/20, Research Center PPE, 2009.

[5] N. Hansen, A. Auger, S. Finck, and R. Ros. Real-parameter black-box optimization benchmarking 2009: Experimental setup. Technical Report RR-6828, INRIA, 2009.

[6] N. Hansen, S. Finck, R. Ros, and A. Auger. Real-parameter black-box optimization benchmarking 2009: Noiseless functions definitions. Technical Report RR-6829, INRIA, 2009.

[7] J. A. Lozano, P. Larrañaga, I. Inza, and E. Bengoetxea. *Towards a New Evolutionary Computation. Advances in Estimation of Distribution Algorithms.* Springer–Verlag, Berlin, 2006.

[8] M. Pelikan, K. Sastry, and E. Cantú-Paz. *Scalable Optimization via Probabilistic Modeling: From Algorithms to Applications.* Springer–Verlag, Berlin, 2006.

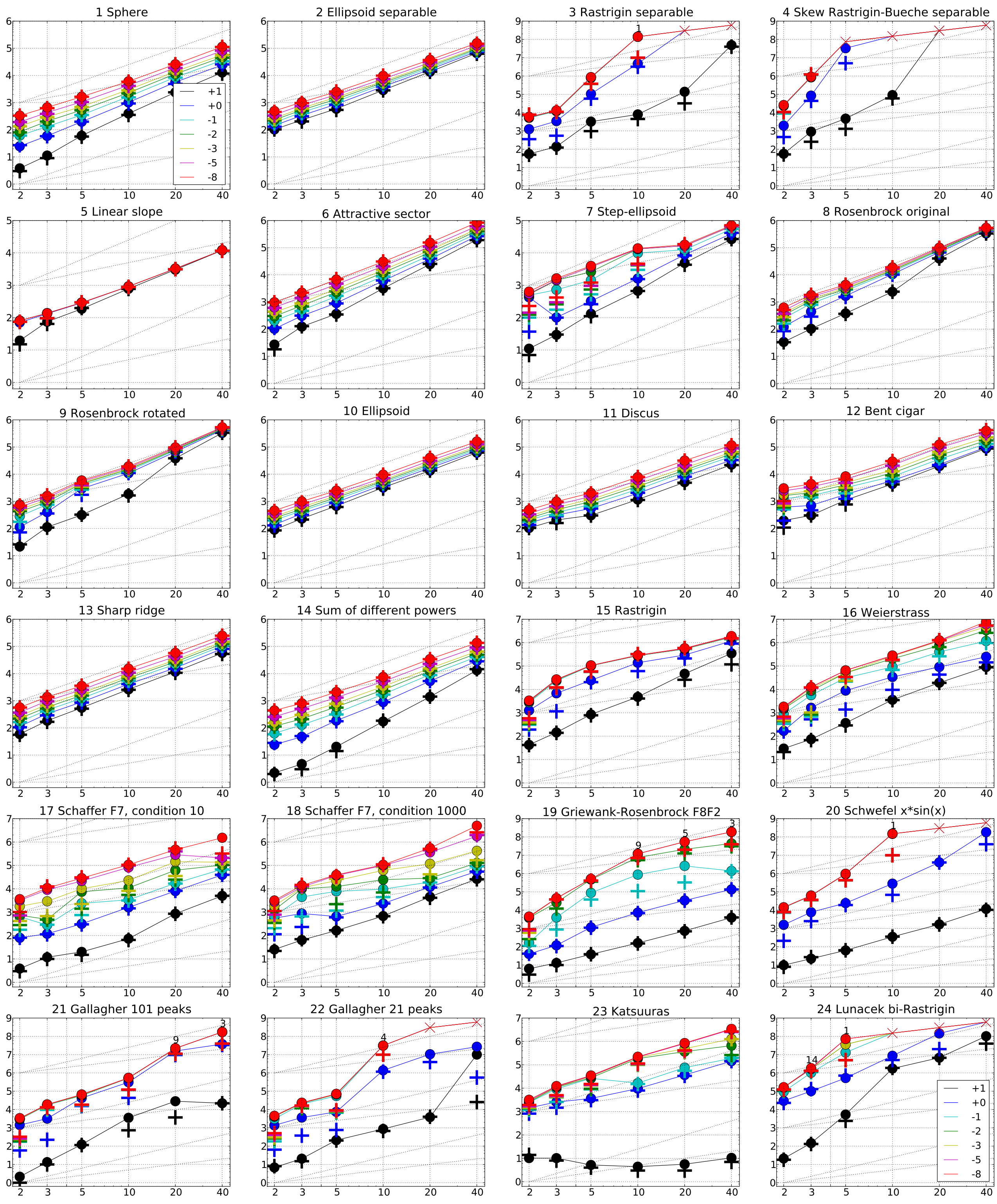

Figure 1: AMaLGaM: Expected Running Time (ERT, ●) to reach $f_{\mathrm{opt}} + \Delta f$ and median number of function evaluations of successful trials (+), shown for $\Delta f = 10, 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-5}, 10^{-8}$ (the exponent is given in the legend of f_1 and f_{24}) versus dimension in log-log presentation. The $\mathrm{ERT}(\Delta f)$ equals to $\#\mathrm{FEs}(\Delta f)$ divided by the number of successful trials, where a trial is successful if $f_{\mathrm{opt}} + \Delta f$ was surpassed during the trial. The $\#\mathrm{FEs}(\Delta f)$ are the total number of function evaluations while $f_{\mathrm{opt}} + \Delta f$ was not surpassed during the trial from all respective trials (successful and unsuccessful), and f_{opt} denotes the optimal function value. Crosses ($\times$) indicate the total number of function evaluations $\#\mathrm{FEs}(-\infty)$. Numbers above ERT-symbols indicate the number of successful trials. Annotated numbers on the ordinate are decimal logarithms. Additional grid lines show linear and quadratic scaling.

Δf	**f_1 in 5-D**, N=15, mFE=2108					**f_1 in 20-D**, N=15, mFE=32945				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	6.1e1	5.2e1	6.9e1	6.1e1	15	2.4e3	2.3e3	2.5e3	2.4e3
1	15	2.0e2	1.8e2	2.1e2	2.0e2	15	5.7e3	5.2e3	6.2e3	5.7e3
1e−1	15	3.5e2	3.3e2	3.7e2	3.5e2	15	8.5e3	7.9e3	9.2e3	8.5e3
1e−3	15	7.1e2	6.7e2	7.5e2	7.1e2	15	1.4e4	1.3e4	1.5e4	1.4e4
1e−5	15	1.1e3	1.0e3	1.1e3	1.1e3	15	1.9e4	1.8e4	2.0e4	1.9e4
1e−8	15	1.6e3	1.6e3	1.7e3	1.6e3	15	2.6e4	2.5e4	2.7e4	2.6e4

Δf	**f_2 in 5-D**, N=15, mFE=2941					**f_2 in 20-D**, N=15, mFE=46861				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.9e2	5.5e2	6.4e2	5.9e2	15	1.4e4	1.3e4	1.4e4	1.4e4
1	15	8.7e2	8.0e2	9.5e2	8.7e2	15	1.7e4	1.6e4	1.7e4	1.7e4
1e−1	15	1.1e3	1.0e3	1.2e3	1.1e3	15	1.9e4	1.8e4	2.0e4	1.9e4
1e−3	15	1.6e3	1.5e3	1.7e3	1.6e3	15	2.4e4	2.3e4	2.5e4	2.4e4
1e−5	15	1.9e3	1.8e3	2.0e3	1.9e3	15	3.0e4	2.9e4	3.1e4	3.0e4
1e−8	15	2.5e3	2.4e3	2.6e3	2.5e3	15	3.8e4	3.6e4	3.9e4	3.8e4

Δf	**f_3 in 5-D**, N=15, mFE=2.66e6					**f_3 in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.2e3	9.4e2	5.5e3	3.2e3	15	1.4e5	9.4e4	1.8e5	1.4e5
1	15	1.1e5	7.7e4	1.4e5	1.1e5	0	*40e-1*	*30e-1*	*50e-1*	5.6e5
1e−1	15	7.9e5	5.2e5	1.1e6	7.9e5	.	.	.	.	.
1e−3	15	8.5e5	5.6e5	1.1e6	8.5e5	.	.	.	.	.
1e−5	15	8.6e5	5.6e5	1.2e6	8.6e5	.	.	.	.	.
1e−8	15	8.7e5	5.8e5	1.2e6	8.7e5	.	.	.	.	.

Δf	**f_4 in 5-D**, N=15, mFE=5.01e6					**f_4 in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	4.7e3	2.3e3	7.1e3	4.7e3	0	*14e+0*	*13e+0*	*15e+0*	4.5e6
1	2	3.3e7	1.8e7	>7e7	5.0e6	.	.	.	.	.
1e−1	0	*20e-1*	*99e-2*	*30e-1*	8.9e5	.	.	.	.	.
1e−3	.	.	.	.	.	.	.	.	.	.
1e−5	.	.	.	.	.	.	.	.	.	.
1e−8	.	.	.	.	.	.	.	.	.	.

Δf	**f_5 in 5-D**, N=15, mFE=491					**f_5 in 20-D**, N=15, mFE=4545				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.9e2	1.8e2	2.0e2	1.9e2	15	3.0e3	2.8e3	3.2e3	3.0e3
1	15	2.8e2	2.6e2	3.1e2	2.8e2	15	3.2e3	3.0e3	3.4e3	3.2e3
1e−1	15	2.9e2	2.7e2	3.1e2	2.9e2	15	3.3e3	3.1e3	3.5e3	3.3e3
1e−3	15	2.9e2	2.7e2	3.1e2	2.9e2	15	3.3e3	3.0e3	3.5e3	3.3e3
1e−5	15	2.9e2	2.7e2	3.1e2	2.9e2	15	3.3e3	3.1e3	3.5e3	3.3e3
1e−8	15	2.9e2	2.7e2	3.1e2	2.9e2	15	3.3e3	3.0e3	3.5e3	3.3e3

Δf	**f_6 in 5-D**, N=15, mFE=7498					**f_6 in 20-D**, N=15, mFE=168697				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.6e2	3.2e2	4.0e2	3.6e2	15	2.5e4	2.4e4	2.6e4	2.5e4
1	15	9.1e2	8.3e2	1.0e3	9.1e2	15	3.8e4	3.7e4	4.0e4	3.8e4
1e−1	15	1.6e3	1.5e3	1.8e3	1.6e3	15	5.2e4	5.0e4	5.4e4	5.2e4
1e−3	15	3.0e3	2.8e3	3.3e3	3.0e3	15	8.1e4	7.9e4	8.3e4	8.1e4
1e−5	15	4.4e3	4.2e3	4.7e3	4.4e3	15	1.1e5	1.1e5	1.1e5	1.1e5
1e−8	15	6.7e3	6.4e3	6.9e3	6.7e3	15	1.5e5	1.5e5	1.6e5	1.5e5

Δf	**f_7 in 5-D**, N=15, mFE=14818					**f_7 in 20-D**, N=15, mFE=21017				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.3e2	1.0e2	1.6e2	1.3e2	15	4.9e3	4.4e3	5.4e3	4.9e3
1	15	3.2e2	2.8e2	3.8e2	3.2e2	15	8.8e3	8.2e3	9.4e3	8.8e3
1e−1	15	1.4e3	5.3e2	2.3e3	1.4e3	15	1.3e4	1.2e4	1.3e4	1.3e4
1e−3	15	3.7e3	1.9e3	5.5e3	3.7e3	15	1.7e4	1.6e4	1.7e4	1.7e4
1e−5	15	3.7e3	1.9e3	5.5e3	3.7e3	15	1.7e4	1.6e4	1.7e4	1.7e4
1e−8	15	4.0e3	2.2e3	5.7e3	4.0e3	15	1.8e4	1.7e4	1.9e4	1.8e4

Δf	**f_8 in 5-D**, N=15, mFE=5440					**f_8 in 20-D**, N=15, mFE=116157				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.8e2	3.4e2	4.1e2	3.8e2	15	4.0e4	3.9e4	4.1e4	4.0e4
1	15	1.7e3	1.5e3	1.8e3	1.7e3	15	6.8e4	6.6e4	7.0e4	6.8e4
1e−1	15	2.6e3	2.4e3	2.8e3	2.6e3	15	7.7e4	7.5e4	7.8e4	7.7e4
1e−3	15	3.3e3	3.1e3	3.6e3	3.3e3	15	8.6e4	8.5e4	8.8e4	8.6e4
1e−5	15	3.8e3	3.5e3	4.0e3	3.8e3	15	9.2e4	9.0e4	9.4e4	9.2e4
1e−8	15	4.3e3	4.0e3	4.5e3	4.3e3	15	1.0e5	9.7e4	1.0e5	1.0e5

Δf	**f_9 in 5-D**, N=15, mFE=25268					**f_9 in 20-D**, N=15, mFE=112465				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.2e2	2.9e2	3.4e2	3.2e2	15	3.8e4	3.8e4	3.9e4	3.8e4
1	15	2.9e3	1.6e3	4.3e3	2.9e3	15	6.7e4	6.6e4	6.8e4	6.7e4
1e−1	15	3.9e3	2.5e3	5.4e3	3.9e3	15	7.5e4	7.4e4	7.6e4	7.5e4
1e−3	15	4.8e3	3.4e3	6.4e3	4.8e3	15	8.4e4	8.2e4	8.5e4	8.4e4
1e−5	15	5.3e3	3.8e3	6.9e3	5.3e3	15	8.9e4	8.8e4	9.1e4	8.9e4
1e−8	15	5.9e3	4.4e3	7.5e3	5.9e3	15	9.7e4	9.5e4	9.9e4	9.7e4

Δf	**f_{10} in 5-D**, N=15, mFE=3382					**f_{10} in 20-D**, N=15, mFE=46293				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	6.8e2	6.1e2	7.6e2	6.8e2	15	1.3e4	1.3e4	1.4e4	1.3e4
1	15	9.2e2	8.3e2	1.0e3	9.2e2	15	1.7e4	1.6e4	1.8e4	1.7e4
1e−1	15	1.2e3	1.1e3	1.3e3	1.2e3	15	2.0e4	1.9e4	2.1e4	2.0e4
1e−3	15	1.6e3	1.4e3	1.7e3	1.6e3	15	2.6e4	2.5e4	2.7e4	2.6e4
1e−5	15	1.9e3	1.8e3	2.0e3	1.9e3	15	3.2e4	3.0e4	3.3e4	3.2e4
1e−8	15	2.5e3	2.3e3	2.6e3	2.5e3	15	4.0e4	3.9e4	4.2e4	4.0e4

Δf	**f_{11} in 5-D**, N=15, mFE=2549					**f_{11} in 20-D**, N=15, mFE=40045				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.0e2	2.8e2	3.3e2	3.0e2	15	5.1e3	4.7e3	5.4e3	5.1e3
1	15	5.5e2	4.9e2	6.1e2	5.5e2	15	8.3e3	7.8e3	8.8e3	8.3e3
1e−1	15	7.6e2	6.9e2	8.4e2	7.6e2	15	1.2e4	1.1e4	1.2e4	1.2e4
1e−3	15	1.2e3	1.1e3	1.3e3	1.2e3	15	1.7e4	1.6e4	1.8e4	1.7e4
1e−5	15	1.5e3	1.4e3	1.6e3	1.5e3	15	2.3e4	2.1e4	2.4e4	2.3e4
1e−8	15	2.0e3	1.9e3	2.2e3	2.0e3	15	3.2e4	3.0e4	3.3e4	3.2e4

Δf	**f_{12} in 5-D**, N=15, mFE=19209					**f_{12} in 20-D**, N=15, mFE=144557				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.1e3	8.3e2	1.4e3	1.1e3	15	2.0e4	1.9e4	2.1e4	2.0e4
1	15	1.8e3	1.2e3	2.5e3	1.8e3	15	2.3e4	2.1e4	2.4e4	2.3e4
1e−1	15	3.2e3	2.3e3	4.1e3	3.2e3	15	3.6e4	3.3e4	3.9e4	3.6e4
1e−3	15	4.9e3	3.8e3	6.0e3	4.9e3	15	6.4e4	6.0e4	6.7e4	6.4e4
1e−5	15	6.6e3	5.3e3	8.0e3	6.6e3	15	9.5e4	9.1e4	9.8e4	9.5e4
1e−8	15	8.3e3	6.8e3	9.9e3	8.3e3	15	1.2e5	1.2e5	1.3e5	1.2e5

Δf	**f_{13} in 5-D**, N=15, mFE=4019					**f_{13} in 20-D**, N=15, mFE=70149				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.5e2	4.9e2	6.1e2	5.5e2	15	1.2e4	1.1e4	1.3e4	1.2e4
1	15	8.9e2	8.2e2	9.7e2	8.9e2	15	1.6e4	1.5e4	1.7e4	1.6e4
1e−1	15	1.2e3	1.1e3	1.3e3	1.2e3	15	2.2e4	2.1e4	2.3e4	2.2e4
1e−3	15	1.9e3	1.8e3	2.0e3	1.9e3	15	3.1e4	3.0e4	3.2e4	3.1e4
1e−5	15	2.5e3	2.4e3	2.7e3	2.5e3	15	4.2e4	4.2e4	4.3e4	4.2e4
1e−8	15	3.5e3	3.4e3	3.6e3	3.5e3	15	5.8e4	5.7e4	5.9e4	5.8e4

Δf	**f_{14} in 5-D**, N=15, mFE=2892					**f_{14} in 20-D**, N=15, mFE=42885				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.0e1	1.5e1	2.6e1	2.0e1	15	1.4e3	1.3e3	1.5e3	1.4e3
1	15	1.9e2	1.7e2	2.1e2	1.9e2	15	5.3e3	4.8e3	5.8e3	5.3e3
1e−1	15	3.5e2	3.3e2	3.8e2	3.5e2	15	9.1e3	8.4e3	9.9e3	9.1e3
1e−3	15	8.1e2	7.5e2	8.7e2	8.1e2	15	1.6e4	1.5e4	1.6e4	1.6e4
1e−5	15	1.3e3	1.2e3	1.4e3	1.3e3	15	2.3e4	2.2e4	2.5e4	2.3e4
1e−8	15	2.1e3	1.9e3	2.2e3	2.1e3	15	3.4e4	3.3e4	3.6e4	3.4e4

Δf	**f_{15} in 5-D**, N=15, mFE=622081					**f_{15} in 20-D**, N=15, mFE=1.12e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	8.5e2	7.7e2	9.4e2	8.5e2	15	4.7e4	2.7e4	6.6e4	4.7e4
1	15	2.5e4	2.1e4	2.9e4	2.5e4	15	3.0e5	2.5e5	3.4e5	3.0e5
1e−1	15	1.0e5	6.1e4	1.5e5	1.0e5	15	5.3e5	4.3e5	6.3e5	5.3e5
1e−3	15	1.0e5	6.1e4	1.5e5	1.0e5	15	5.4e5	4.5e5	6.4e5	5.4e5
1e−5	15	1.0e5	6.2e4	1.5e5	1.0e5	15	5.6e5	4.6e5	6.6e5	5.6e5
1e−8	15	1.1e5	6.4e4	1.5e5	1.1e5	15	5.8e5	4.8e5	6.8e5	5.8e5

Δf	**f_{16} in 5-D**, N=15, mFE=255129					**f_{16} in 20-D**, N=15, mFE=2.48e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.7e2	2.8e2	4.6e2	3.7e2	15	1.9e4	1.8e4	2.1e4	1.9e4
1	15	8.9e3	3.3e3	1.5e4	8.9e3	15	9.1e4	6.3e4	1.2e5	9.1e4
1e−1	15	3.1e4	1.9e4	4.3e4	3.1e4	15	4.0e5	3.0e5	5.1e5	4.0e5
1e−3	15	5.1e4	3.1e4	7.2e4	5.1e4	15	1.2e6	9.1e5	1.4e6	1.2e6
1e−5	15	6.4e4	4.3e4	8.7e4	6.4e4	15	1.2e6	9.7e5	1.5e6	1.2e6
1e−8	15	6.5e4	4.4e4	8.8e4	6.5e4	15	1.3e6	1.0e6	1.5e6	1.3e6

Δf	**f_{17} in 5-D**, N=15, mFE=48296					**f_{17} in 20-D**, N=15, mFE=611698				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.0e1	1.2e1	2.9e1	2.0e1	15	8.6e2	7.1e2	1.0e3	8.6e2
1	15	3.2e2	2.8e2	3.6e2	3.2e2	15	8.0e3	7.4e3	8.5e3	8.0e3
1e−1	15	2.4e3	7.3e2	4.1e3	2.4e3	15	1.7e4	1.6e4	1.8e4	1.7e4
1e−3	15	1.0e4	5.5e3	1.5e4	1.0e4	15	1.4e5	8.7e4	2.0e5	1.4e5
1e−5	15	2.1e4	1.7e4	2.6e4	2.1e4	15	2.8e5	2.1e5	3.6e5	2.8e5
1e−8	15	2.9e4	2.5e4	3.2e4	2.9e4	15	4.7e5	4.1e5	5.2e5	4.7e5

Δf	**f_{18} in 5-D**, N=15, mFE=91310					**f_{18} in 20-D**, N=15, mFE=1.45e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.6e2	1.4e2	1.9e2	1.6e2	15	4.5e3	4.0e3	5.1e3	4.5e3
1	15	6.5e2	5.7e2	7.4e2	6.5e2	15	1.2e4	1.1e4	1.3e4	1.2e4
1e−1	15	7.8e3	4.1e3	1.2e4	7.8e3	15	2.0e4	1.9e4	2.1e4	2.0e4
1e−3	15	2.2e4	1.5e4	3.0e4	2.2e4	15	1.2e5	6.5e4	1.7e5	1.2e5
1e−5	15	3.7e4	3.0e4	4.4e4	3.7e4	15	3.6e5	3.1e5	4.2e5	3.6e5
1e−8	15	3.9e4	3.2e4	4.7e4	3.9e4	15	5.5e5	4.7e5	6.4e5	5.5e5

Δf	**f_{19} in 5-D**, N=15, mFE=1.35e6					**f_{19} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.8e1	3.4e1	4.3e1	3.8e1	15	7.4e2	6.9e2	7.8e2	7.4e2
1	15	1.1e3	8.9e2	1.4e3	1.1e3	15	3.4e4	3.2e4	3.5e4	3.4e4
1e−1	15	8.8e4	5.7e4	1.2e5	8.8e4	14	2.6e6	2.7e5	5.3e6	1.6e6
1e−3	15	5.1e5	4.0e5	6.2e5	5.1e5	5	5.5e7	3.8e7	9.3e7	2.0e7
1e−5	15	5.1e5	4.0e5	6.3e5	5.1e5	5	5.5e7	3.8e7	9.3e7	2.0e7
1e−8	15	5.1e5	4.0e5	6.3e5	5.1e5	5	5.5e7	3.7e7	9.3e7	2.0e7

Δf	**f_{20} in 5-D**, N=15, mFE=3.04e6					**f_{20} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	6.3e1	5.5e1	7.1e1	6.3e1	15	1.7e3	1.6e3	1.8e3	1.7e3
1	15	2.4e4	1.6e4	3.3e4	2.4e4	15	4.1e6	3.5e6	4.7e6	4.1e6
1e−1	15	9.2e5	6.3e5	1.2e6	9.2e5	0	*68e-2*	*47e-2*	*77e-2*	1.8e7
1e−3	15	9.5e5	6.4e5	1.3e6	9.5e5	.	.	.	.	.
1e−5	15	9.6e5	6.4e5	1.3e6	9.6e5	.	.	.	.	.
1e−8	15	9.7e5	6.6e5	1.3e6	9.7e5	.	.	.	.	.

Δf	**f_{21} in 5-D**, N=15, mFE=484694					**f_{21} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.2e2	9.4e1	1.6e2	1.2e2	15	2.8e4	3.9e3	5.3e4	2.8e4
1	15	4.3e4	1.0e4	7.7e4	4.3e4	10	1.6e7	1.0e7	2.3e7	8.6e6
1e−1	15	5.8e4	2.2e4	9.5e4	5.8e4	9	2.2e7	1.6e7	3.2e7	1.2e7
1e−3	15	6.2e4	2.5e4	1.1e5	6.2e4	9	2.2e7	1.6e7	3.2e7	1.2e7
1e−5	15	6.5e4	2.3e4	1.1e5	6.5e4	9	2.2e7	1.6e7	3.2e7	1.2e7
1e−8	15	6.7e4	2.8e4	1.1e5	6.7e4	9	2.2e7	1.6e7	3.2e7	1.2e7

Δf	**f_{22} in 5-D**, N=15, mFE=291492					**f_{22} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.1e2	1.7e2	2.6e2	2.1e2	15	3.8e3	3.3e3	4.3e3	3.8e3
1	15	7.4e3	3.5e3	1.2e4	7.4e3	11	1.1e7	6.9e6	1.5e7	8.6e6
1e−1	15	5.5e4	2.9e4	8.3e4	5.5e4	0	*69e-2*	*69e-2*	*20e-1*	4.0e4
1e−3	15	7.0e4	3.7e4	1.0e5	7.0e4	.	.	.	.	.
1e−5	15	7.1e4	3.8e4	1.0e5	7.1e4	.	.	.	.	.
1e−8	15	7.2e4	3.7e4	1.1e5	7.2e4	.	.	.	.	.

Δf	**f_{23} in 5-D**, N=15, mFE=98806					**f_{23} in 20-D**, N=15, mFE=5.83e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.2e0	3.8e0	6.6e0	5.2e0	15	5.6e0	3.8e0	7.5e0	5.6e0
1	15	3.6e3	3.2e3	4.1e3	3.6e3	15	3.7e4	3.3e4	4.0e4	3.7e4
1e−1	15	2.6e4	1.7e4	3.6e4	2.6e4	15	7.6e4	5.1e4	1.0e5	7.6e4
1e−3	15	3.2e4	2.2e4	4.2e4	3.2e4	15	4.9e5	3.7e5	6.2e5	4.9e5
1e−5	15	3.3e4	2.3e4	4.4e4	3.3e4	15	8.1e5	4.1e5	1.2e6	8.1e5
1e−8	15	3.5e4	2.4e4	4.6e4	3.5e4	15	8.5e5	4.4e5	1.3e6	8.5e5

Δf	**f_{24} in 5-D**, N=15, mFE=5.01e6					**f_{24} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.3e3	3.0e3	7.9e3	5.3e3	15	6.8e6	5.2e6	8.4e6	6.8e6
1	15	5.3e5	4.5e5	6.1e5	5.3e5	2	1.4e8	7.4e7	>3e8	2.0e7
1e−1	5	1.3e7	9.0e6	2.3e7	4.9e6	0	*23e-1*	*90e-2*	*37e-1*	7.9e6
1e−3	2	3.6e7	1.9e7	>7e7	5.0e6	.	.	.	.	.
1e−5	1	7.2e7	3.5e7	>7e7	5.0e6	.	.	.	.	.
1e−8	1	7.2e7	3.5e7	>7e7	5.0e6	.	.	.	.	.

Table 1: AMaLGaM: Shown are, for a given target difference to the optimal function value Δf: the number of successful trials (#); the expected running time to surpass $f_{\mathrm{opt}} + \Delta f$ (ERT, see Figure 1); the 10%-tile and 90%-tile of the bootstrap distribution of ERT; the average number of function evaluations in successful trials or, if none was successful, as last entry the median number of function evaluations to reach the best function value ($\mathbf{RT_{succ}}$). If $f_{\mathrm{opt}} + \Delta f$ was never reached, figures in *italics* denote the best achieved Δf-value of the median trial and the 10% and 90%-tile trial. Furthermore, N denotes the number of trials, and mFE denotes the maximum of number of function evaluations executed in one trial. See Figure 1 for the names of functions.

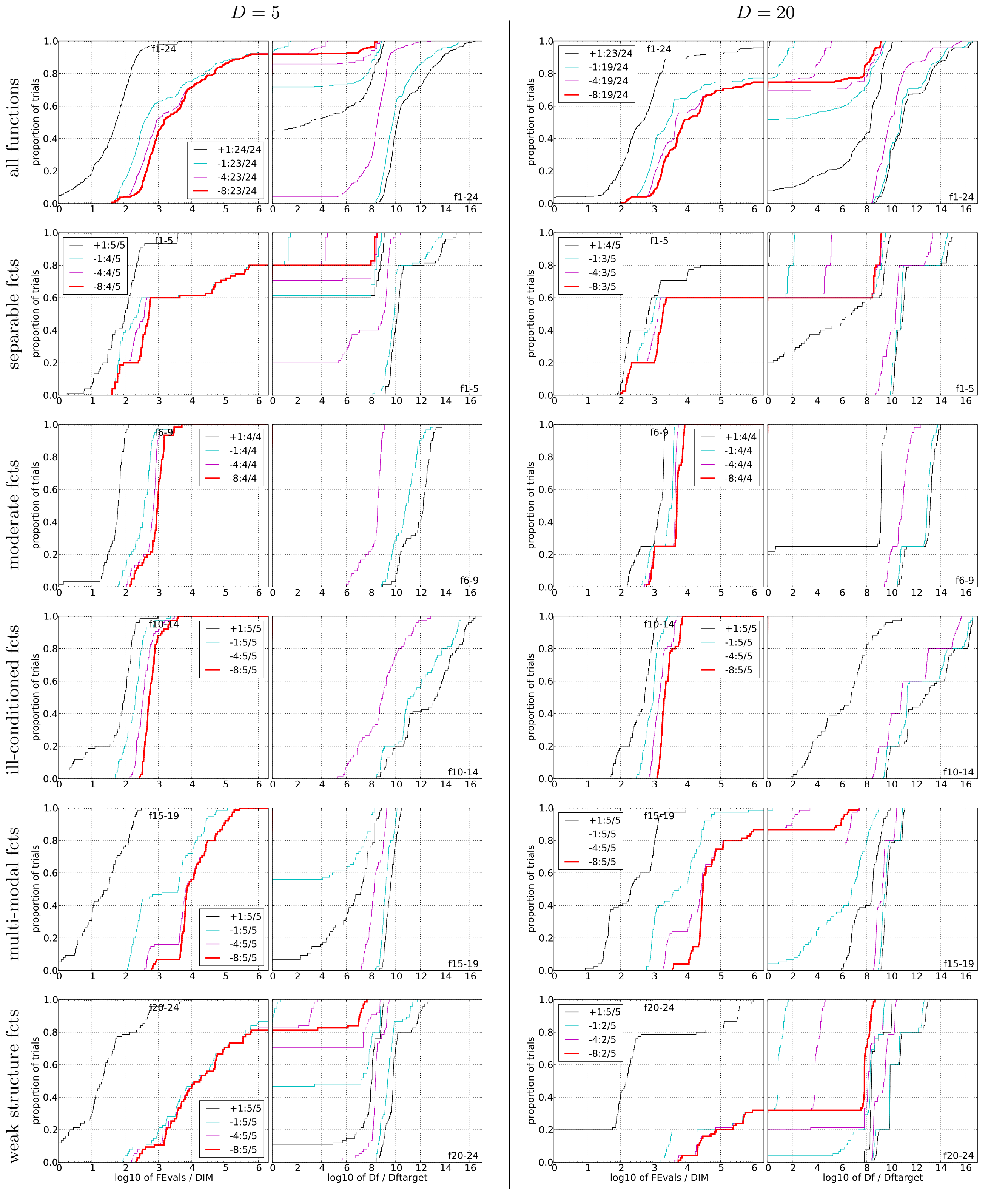

Figure 2: AMaLGaM: Empirical cumulative distribution functions (ECDFs), plotting the fraction of trials versus running time (left) or Δf. Left subplots: ECDF of the running time (number of function evaluations), divided by search space dimension D, to fall below $f_{\mathrm{opt}} + \Delta f$ with $\Delta f = 10^k$, where k is the first value in the legend. Right subplots: ECDF of the best achieved Δf divided by 10^k (upper left lines in continuation of the left subplot), and best achieved Δf divided by 10^{-8} for running times of $D, 10\,D, 100\,D\ldots$ function evaluations (from right to left cycling black-cyan-magenta). Top row: all results from all functions; second row: separable functions; third row: misc. moderate functions; fourth row: ill-conditioned functions; fifth row: multi-modal functions with adequate structure; last row: multi-modal functions with weak structure. The legends indicate the number of functions that were solved in at least one trial. FEvals denotes number of function evaluations, D and DIM denote search space dimension, and Δf and Df denote the difference to the optimal function value.

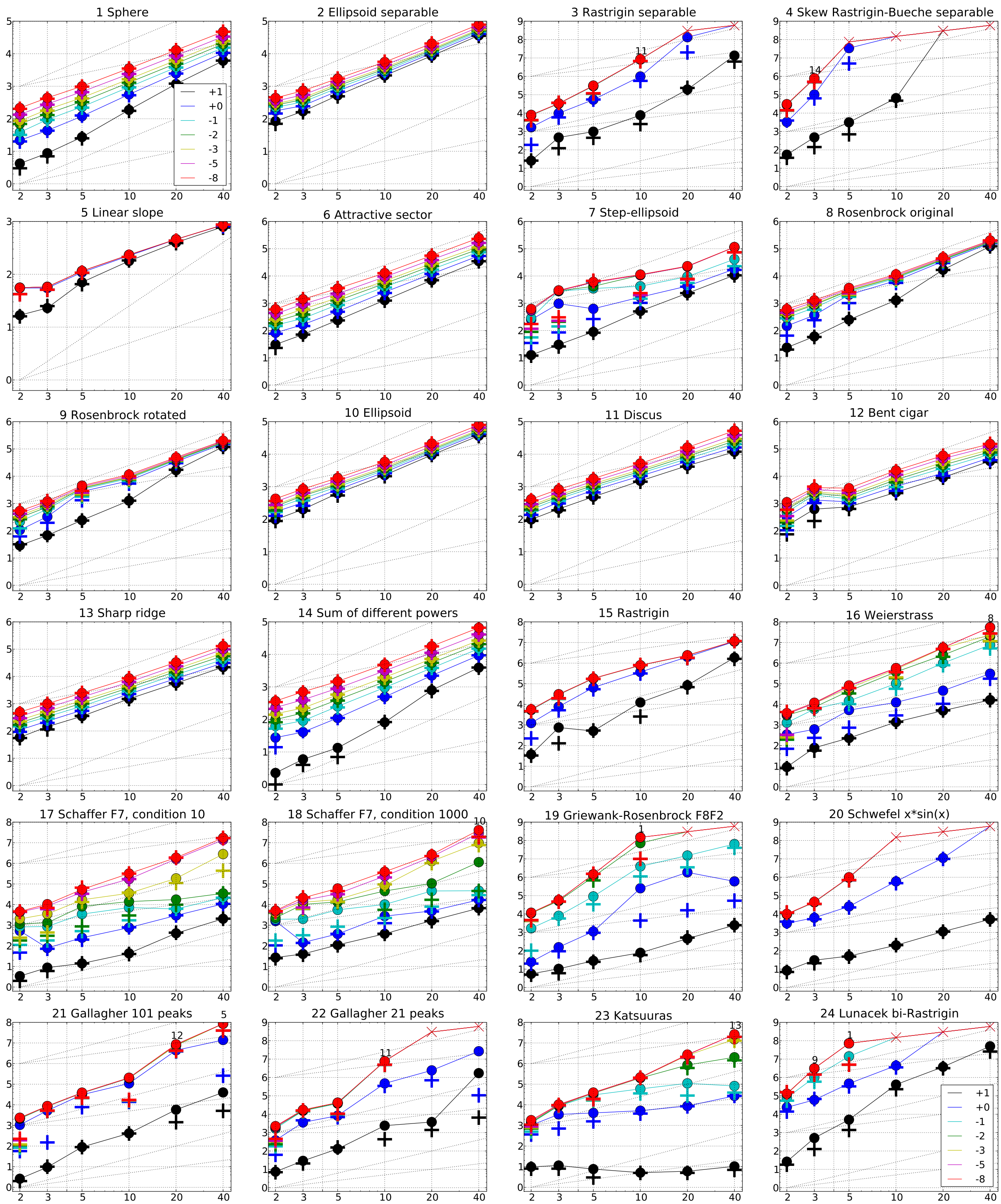

Figure 3: iAMaLGaM: Expected Running Time (ERT, ●) to reach $f_{\mathrm{opt}} + \Delta f$ and median number of function evaluations of successful trials (+), shown for $\Delta f = 10, 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-5}, 10^{-8}$ (the exponent is given in the legend of f_1 and f_{24}) versus dimension in log-log presentation. The $\mathrm{ERT}(\Delta f)$ equals to $\#\mathrm{FEs}(\Delta f)$ divided by the number of successful trials, where a trial is successful if $f_{\mathrm{opt}} + \Delta f$ was surpassed during the trial. The $\#\mathrm{FEs}(\Delta f)$ are the total number of function evaluations while $f_{\mathrm{opt}} + \Delta f$ was not surpassed during the trial from all respective trials (successful and unsuccessful), and f_{opt} denotes the optimal function value. Crosses ($\times$) indicate the total number of function evaluations $\#\mathrm{FEs}(-\infty)$. Numbers above ERT-symbols indicate the number of successful trials. Annotated numbers on the ordinate are decimal logarithms. Additional grid lines show linear and quadratic scaling.

Δf	**f_1 in 5-D**, N=15, mFE=1198					**f_1 in 20-D**, N=15, mFE=13503				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.7e1	2.2e1	3.3e1	2.7e1	15	1.2e3	1.1e3	1.2e3	1.2e3
1	15	1.2e2	1.1e2	1.3e2	1.2e2	15	2.5e3	2.4e3	2.5e3	2.5e3
1e−1	15	2.3e2	2.2e2	2.5e2	2.3e2	15	3.8e3	3.8e3	3.8e3	3.8e3
1e−3	15	4.4e2	4.2e2	4.7e2	4.4e2	15	6.3e3	6.2e3	6.3e3	6.3e3
1e−5	15	6.8e2	6.5e2	7.0e2	6.8e2	15	8.8e3	8.8e3	8.9e3	8.8e3
1e−8	15	1.0e3	9.7e2	1.0e3	1.0e3	15	1.3e4	1.3e4	1.3e4	1.3e4

Δf	**f_2 in 5-D**, N=15, mFE=2206					**f_2 in 20-D**, N=15, mFE=22791				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.2e2	4.7e2	5.8e2	5.2e2	15	8.6e3	8.3e3	8.8e3	8.6e3
1	15	7.1e2	6.5e2	7.7e2	7.1e2	15	1.0e4	1.0e4	1.1e4	1.0e4
1e−1	15	8.8e2	8.1e2	9.5e2	8.8e2	15	1.2e4	1.1e4	1.2e4	1.2e4
1e−3	15	1.2e3	1.1e3	1.2e3	1.2e3	15	1.4e4	1.4e4	1.4e4	1.4e4
1e−5	15	1.4e3	1.3e3	1.5e3	1.4e3	15	1.7e4	1.6e4	1.7e4	1.7e4
1e−8	15	1.7e3	1.6e3	1.8e3	1.7e3	15	2.1e4	2.0e4	2.1e4	2.1e4

Δf	**f_3 in 5-D**, N=15, mFE=2.13e6					**f_3 in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	9.8e2	4.4e2	1.5e3	9.8e2	15	1.9e5	1.5e5	2.3e5	1.9e5
1	15	5.4e4	3.6e4	7.3e4	5.4e4	2	1.3e8	7.1e7	>3e8	2.0e7
1e−1	15	2.9e5	1.5e5	4.5e5	2.9e5	0	*20e−1*	*99e−2*	*40e−1*	6.3e6
1e−3	15	3.0e5	1.5e5	4.6e5	3.0e5	.	.	.	.	.
1e−5	15	3.1e5	1.6e5	4.7e5	3.1e5	.	.	.	.	.
1e−8	15	3.1e5	1.6e5	4.9e5	3.1e5	.	.	.	.	.

Δf	**f_4 in 5-D**, N=15, mFE=5.01e6					**f_4 in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.1e3	1.9e3	4.4e3	3.1e3	0	*13e+0*	*12e+0*	*15e+0*	7.9e6
1	2	3.4e7	1.7e7	>7e7	2.8e6	.	.	.	.	.
1e−1	0	*20e−1*	*99e−2*	*20e−1*	7.1e5	.	.	.	.	.
1e−3	.	.	.	.	.	.	.	.	.	.
1e−5	.	.	.	.	.	.	.	.	.	.
1e−8	.	.	.	.	.	.	.	.	.	.

Δf	**f_5 in 5-D**, N=15, mFE=232					**f_5 in 20-D**, N=15, mFE=689				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	7.1e1	6.2e1	7.9e1	7.1e1	15	4.0e2	3.7e2	4.3e2	4.0e2
1	15	1.1e2	9.4e1	1.2e2	1.1e2	15	4.6e2	4.3e2	4.8e2	4.6e2
1e−1	15	1.2e2	1.0e2	1.3e2	1.2e2	15	4.6e2	4.4e2	4.8e2	4.6e2
1e−3	15	1.2e2	1.0e2	1.3e2	1.2e2	15	4.6e2	4.4e2	4.8e2	4.6e2
1e−5	15	1.2e2	1.0e2	1.3e2	1.2e2	15	4.6e2	4.4e2	4.8e2	4.6e2
1e−8	15	1.2e2	1.0e2	1.3e2	1.2e2	15	4.6e2	4.4e2	4.8e2	4.6e2

Δf	**f_6 in 5-D**, N=15, mFE=4600					**f_6 in 20-D**, N=15, mFE=68285				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.4e2	2.1e2	2.7e2	2.4e2	15	7.0e3	6.8e3	7.2e3	7.0e3
1	15	5.0e2	4.5e2	5.5e2	5.0e2	15	1.2e4	1.2e4	1.2e4	1.2e4
1e−1	15	8.9e2	7.9e2	9.9e2	8.9e2	15	1.8e4	1.7e4	1.8e4	1.8e4
1e−3	15	1.6e3	1.5e3	1.7e3	1.6e3	15	2.9e4	2.7e4	3.0e4	2.9e4
1e−5	15	2.3e3	2.1e3	2.4e3	2.3e3	15	4.0e4	3.8e4	4.2e4	4.0e4
1e−8	15	3.5e3	3.3e3	3.6e3	3.5e3	15	5.6e4	5.4e4	5.8e4	5.6e4

Δf	**f_7 in 5-D**, N=15, mFE=20656					**f_7 in 20-D**, N=15, mFE=71517				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	8.9e1	7.6e1	1.0e2	8.9e1	15	2.3e3	2.3e3	2.4e3	2.3e3
1	15	6.3e2	2.6e2	1.0e3	6.3e2	15	4.3e3	4.0e3	4.6e3	4.3e3
1e−1	15	3.5e3	2.7e3	4.3e3	3.5e3	15	9.5e3	6.8e3	1.2e4	9.5e3
1e−3	15	5.8e3	4.5e3	7.5e3	5.8e3	15	2.2e4	1.5e4	2.9e4	2.2e4
1e−5	15	5.8e3	4.5e3	7.5e3	5.8e3	15	2.2e4	1.5e4	2.9e4	2.2e4
1e−8	15	6.1e3	4.7e3	7.8e3	6.1e3	15	2.3e4	1.6e4	3.0e4	2.3e4

Δf	**f_8 in 5-D**, N=15, mFE=11429					**f_8 in 20-D**, N=15, mFE=116850				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.5e2	2.3e2	2.7e2	2.5e2	15	1.7e4	1.6e4	1.7e4	1.7e4
1	15	2.1e3	1.1e3	3.0e3	2.1e3	15	3.4e4	2.9e4	4.0e4	3.4e4
1e−1	15	2.6e3	1.6e3	3.6e3	2.6e3	15	3.8e4	3.3e4	4.4e4	3.8e4
1e−3	15	3.0e3	2.1e3	4.0e3	3.0e3	15	4.2e4	3.7e4	4.7e4	4.2e4
1e−5	15	3.3e3	2.4e3	4.4e3	3.3e3	15	4.5e4	3.9e4	5.0e4	4.5e4
1e−8	15	3.7e3	2.7e3	4.7e3	3.7e3	15	4.9e4	4.3e4	5.4e4	4.9e4

Δf	**f_9 in 5-D**, N=15, mFE=13398					**f_9 in 20-D**, N=15, mFE=121138				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.4e2	2.3e2	2.6e2	2.4e2	15	1.7e4	1.6e4	1.8e4	1.7e4
1	15	2.8e3	1.7e3	3.9e3	2.8e3	15	3.4e4	2.9e4	4.0e4	3.4e4
1e−1	15	3.3e3	2.1e3	4.5e3	3.3e3	15	3.8e4	3.3e4	4.4e4	3.8e4
1e−3	15	3.8e3	2.6e3	5.1e3	3.8e3	15	4.2e4	3.7e4	4.8e4	4.2e4
1e−5	15	4.1e3	2.9e3	5.4e3	4.1e3	15	4.5e4	3.9e4	5.1e4	4.5e4
1e−8	15	4.5e3	3.3e3	5.8e3	4.5e3	15	4.9e4	4.3e4	5.5e4	4.9e4

Δf	**f_{10} in 5-D**, N=15, mFE=2458					**f_{10} in 20-D**, N=15, mFE=23694				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	6.3e2	5.6e2	7.1e2	6.3e2	15	9.5e3	9.1e3	9.8e3	9.5e3
1	15	8.1e2	7.2e2	9.1e2	8.1e2	15	1.1e4	1.1e4	1.1e4	1.1e4
1e−1	15	1.0e3	8.9e2	1.1e3	1.0e3	15	1.2e4	1.2e4	1.3e4	1.2e4
1e−3	15	1.3e3	1.2e3	1.4e3	1.3e3	15	1.5e4	1.5e4	1.5e4	1.5e4
1e−5	15	1.5e3	1.4e3	1.6e3	1.5e3	15	1.8e4	1.7e4	1.8e4	1.8e4
1e−8	15	1.9e3	1.8e3	2.0e3	1.9e3	15	2.1e4	2.1e4	2.2e4	2.1e4

Δf	**f_{11} in 5-D**, N=15, mFE=2248					**f_{11} in 20-D**, N=15, mFE=18491				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	4.9e2	4.3e2	5.5e2	4.9e2	15	4.4e3	4.2e3	4.7e3	4.4e3
1	15	7.8e2	6.9e2	8.6e2	7.8e2	15	6.0e3	5.7e3	6.3e3	6.0e3
1e−1	15	9.4e2	8.6e2	1.0e3	9.4e2	15	7.2e3	6.9e3	7.5e3	7.2e3
1e−3	15	1.2e3	1.1e3	1.3e3	1.2e3	15	9.8e3	9.5e3	1.0e4	9.8e3
1e−5	15	1.5e3	1.4e3	1.6e3	1.5e3	15	1.2e4	1.2e4	1.3e4	1.2e4
1e−8	15	1.8e3	1.7e3	1.9e3	1.8e3	15	1.6e4	1.6e4	1.6e4	1.6e4

Δf	**f_{12} in 5-D**, N=15, mFE=7372					**f_{12} in 20-D**, N=15, mFE=67898				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	7.5e2	6.6e2	8.4e2	7.5e2	15	9.0e3	8.9e3	9.2e3	9.0e3
1	15	1.1e3	1.0e3	1.3e3	1.1e3	15	1.3e4	1.0e4	1.5e4	1.3e4
1e−1	15	1.5e3	1.3e3	1.6e3	1.5e3	15	2.0e4	1.7e4	2.2e4	2.0e4
1e−3	15	2.2e3	1.9e3	2.4e3	2.2e3	15	3.2e4	3.0e4	3.5e4	3.2e4
1e−5	15	2.8e3	2.4e3	3.1e3	2.8e3	15	4.4e4	4.2e4	4.7e4	4.4e4
1e−8	15	3.6e3	3.2e3	4.1e3	3.6e3	15	5.7e4	5.5e4	5.9e4	5.7e4

Δf	**f_{13} in 5-D**, N=15, mFE=3025					**f_{13} in 20-D**, N=15, mFE=40034				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	3.5e2	3.3e2	3.6e2	3.5e2	15	5.6e3	5.6e3	5.7e3	5.6e3
1	15	5.8e2	5.5e2	6.0e2	5.8e2	15	8.4e3	8.2e3	8.5e3	8.4e3
1e−1	15	8.1e2	7.8e2	8.5e2	8.1e2	15	1.2e4	1.1e4	1.2e4	1.2e4
1e−3	15	1.3e3	1.2e3	1.4e3	1.3e3	15	1.9e4	1.8e4	2.0e4	1.9e4
1e−5	15	1.8e3	1.7e3	1.8e3	1.8e3	15	2.4e4	2.4e4	2.5e4	2.4e4
1e−8	15	2.5e3	2.4e3	2.6e3	2.5e3	15	3.3e4	3.2e4	3.4e4	3.3e4

Δf	**f_{14} in 5-D**, N=15, mFE=1975					**f_{14} in 20-D**, N=15, mFE=20254				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.3e1	9.9e0	1.7e1	1.3e1	15	7.9e2	7.5e2	8.4e2	7.9e2
1	15	1.1e2	9.7e1	1.2e2	1.1e2	15	2.2e3	2.2e3	2.3e3	2.2e3
1e−1	15	2.5e2	2.3e2	2.7e2	2.5e2	15	3.8e3	3.8e3	3.9e3	3.8e3
1e−3	15	5.7e2	5.4e2	6.1e2	5.7e2	15	7.3e3	7.2e3	7.4e3	7.3e3
1e−5	15	9.1e2	8.6e2	9.6e2	9.1e2	15	1.1e4	1.1e4	1.2e4	1.1e4
1e−8	15	1.5e3	1.4e3	1.6e3	1.5e3	15	1.8e4	1.7e4	1.8e4	1.8e4

Δf	**f_{15} in 5-D**, N=15, mFE=485369					**f_{15} in 20-D**, N=15, mFE=4.85e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.1e2	4.6e2	5.6e2	5.1e2	15	8.4e4	6.3e4	1.1e5	8.4e4
1	15	6.6e4	4.6e4	8.6e4	6.6e4	15	2.0e6	1.7e6	2.4e6	2.0e6
1e−1	15	1.8e5	1.4e5	2.2e5	1.8e5	15	2.3e6	2.0e6	2.7e6	2.3e6
1e−3	15	1.8e5	1.4e5	2.3e5	1.8e5	15	2.4e6	2.0e6	2.7e6	2.4e6
1e−5	15	1.8e5	1.4e5	2.3e5	1.8e5	15	2.4e6	2.0e6	2.7e6	2.4e6
1e−8	15	1.8e5	1.4e5	2.3e5	1.8e5	15	2.4e6	2.1e6	2.8e6	2.4e6

Δf	**f_{16} in 5-D**, N=15, mFE=223478					**f_{16} in 20-D**, N=15, mFE=1.58e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.3e2	1.7e2	3.0e2	2.3e2	15	5.0e3	4.7e3	5.2e3	5.0e3
1	15	5.3e3	2.5e3	8.2e3	5.3e3	15	4.6e4	2.9e4	6.4e4	4.6e4
1e−1	15	1.5e4	1.1e4	1.9e4	1.5e4	15	1.0e6	7.6e5	1.3e6	1.0e6
1e−3	15	6.1e4	4.7e4	7.6e4	6.1e4	15	5.4e6	4.1e6	6.7e6	5.4e6
1e−5	15	7.5e4	5.7e4	9.6e4	7.5e4	15	5.6e6	4.3e6	7.0e6	5.6e6
1e−8	15	8.3e4	6.6e4	1.0e5	8.3e4	15	5.7e6	4.5e6	7.1e6	5.7e6

Δf	**f_{17} in 5-D**, N=15, mFE=91729					**f_{17} in 20-D**, N=15, mFE=2.37e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.4e1	1.1e1	1.7e1	1.4e1	15	4.1e2	3.3e2	4.8e2	4.1e2
1	15	2.5e2	2.1e2	2.9e2	2.5e2	15	3.0e3	2.9e3	3.1e3	3.0e3
1e−1	15	3.5e3	1.0e3	5.9e3	3.5e3	15	6.1e3	6.0e3	6.3e3	6.1e3
1e−3	15	1.7e4	1.1e4	2.3e4	1.7e4	15	1.9e5	1.0e5	2.8e5	1.9e5
1e−5	15	4.0e4	3.0e4	5.0e4	4.0e4	15	1.6e6	1.4e6	1.8e6	1.6e6
1e−8	15	5.4e4	4.4e4	6.5e4	5.4e4	15	1.9e6	1.8e6	2.0e6	1.9e6

Δf	**f_{18} in 5-D**, N=15, mFE=140996					**f_{18} in 20-D**, N=15, mFE=4.90e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.1e2	9.6e1	1.2e2	1.1e2	15	1.7e3	1.6e3	1.8e3	1.7e3
1	15	3.8e2	3.6e2	4.0e2	3.8e2	15	4.9e3	4.7e3	5.2e3	4.9e3
1e−1	15	5.6e3	3.5e3	7.7e3	5.6e3	15	4.6e4	2.4e4	7.0e4	4.6e4
1e−3	15	1.8e4	1.3e4	2.2e4	1.8e4	15	1.3e6	1.0e6	1.5e6	1.3e6
1e−5	15	3.4e4	2.6e4	4.2e4	3.4e4	15	2.1e6	1.9e6	2.2e6	2.1e6
1e−8	15	6.1e4	4.9e4	7.2e4	6.1e4	15	2.7e6	2.3e6	3.1e6	2.7e6

Δf	**f_{19} in 5-D**, N=15, mFE=3.20e6					**f_{19} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	2.8e1	2.2e1	3.4e1	2.8e1	15	4.6e2	4.1e2	5.0e2	4.6e2
1	15	1.1e3	8.9e2	1.4e3	1.1e3	14	1.8e6	2.0e4	3.7e6	1.8e6
1e−1	15	9.0e4	5.2e4	1.3e5	9.0e4	9	1.5e7	9.8e6	2.3e7	1.0e7
1e−3	15	1.4e6	1.1e6	1.8e6	1.4e6	0	*72e−3*	*40e−3*	*63e−2*	4.5e6
1e−5	15	1.4e6	1.1e6	1.8e6	1.4e6	.	.	.	.	.
1e−8	15	1.4e6	1.1e6	1.8e6	1.4e6	.	.	.	.	.

Δf	**f_{20} in 5-D**, N=15, mFE=1.89e6					**f_{20} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.1e1	4.6e1	5.7e1	5.1e1	15	1.1e3	1.1e3	1.2e3	1.1e3
1	15	2.6e4	1.7e4	3.4e4	2.6e4	13	1.1e7	8.6e6	1.4e7	9.4e6
1e−1	15	9.4e5	7.6e5	1.1e6	9.4e5	0	*88e−2*	*61e−2*	*11e−1*	1.0e7
1e−3	15	9.7e5	7.8e5	1.2e6	9.7e5	.	.	.	.	.
1e−5	15	9.8e5	7.9e5	1.2e6	9.8e5	.	.	.	.	.
1e−8	15	1.0e6	8.0e5	1.2e6	1.0e6	.	.	.	.	.

Δf	**f_{21} in 5-D**, N=15, mFE=216292					**f_{21} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	9.0e1	7.2e1	1.1e2	9.0e1	15	5.8e3	1.6e3	1.0e4	5.8e3
1	15	3.1e4	1.5e4	4.9e4	3.1e4	14	4.4e6	2.6e6	6.5e6	3.9e6
1e−1	15	3.7e4	2.1e4	5.5e4	3.7e4	13	7.6e6	4.6e6	1.1e7	5.4e6
1e−3	15	3.8e4	2.2e4	5.7e4	3.8e4	13	7.7e6	4.6e6	1.1e7	5.4e6
1e−5	15	3.8e4	2.1e4	5.6e4	3.8e4	12	8.5e6	5.0e6	1.3e7	5.2e6
1e−8	15	3.9e4	2.2e4	5.8e4	3.9e4	12	8.6e6	4.9e6	1.3e7	5.3e6

Δf	**f_{22} in 5-D**, N=15, mFE=333364					**f_{22} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	1.3e2	1.0e2	1.5e2	1.3e2	15	3.8e3	1.6e3	6.0e3	3.8e3
1	15	8.3e3	5.4e3	1.1e4	8.3e3	15	2.4e6	1.5e6	3.4e6	2.4e6
1e−1	15	3.8e4	1.5e4	6.4e4	3.8e4	0	*69e−2*	*69e−2*	*69e−2*	7.9e5
1e−3	15	4.2e4	1.7e4	6.9e4	4.2e4	.	.	.	.	.
1e−5	15	4.2e4	1.8e4	7.1e4	4.2e4	.	.	.	.	.
1e−8	15	4.3e4	1.8e4	6.8e4	4.3e4	.	.	.	.	.

Δf	**f_{23} in 5-D**, N=15, mFE=158684					**f_{23} in 20-D**, N=15, mFE=6.43e6				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	7.7e0	4.7e0	1.1e1	7.7e0	15	6.0e0	4.3e0	7.7e0	6.0e0
1	15	4.0e3	2.3e3	5.7e3	4.0e3	15	8.8e3	8.2e3	9.4e3	8.8e3
1e−1	15	3.0e4	1.9e4	4.2e4	3.0e4	15	1.1e5	6.9e4	1.5e5	1.1e5
1e−3	15	3.7e4	2.6e4	4.9e4	3.7e4	15	2.5e6	2.0e6	3.1e6	2.5e6
1e−5	15	3.8e4	2.6e4	5.1e4	3.8e4	15	2.7e6	2.2e6	3.2e6	2.7e6
1e−8	15	4.0e4	2.8e4	5.2e4	4.0e4	15	2.7e6	2.2e6	3.3e6	2.7e6

Δf	**f_{24} in 5-D**, N=15, mFE=5.01e6					**f_{24} in 20-D**, N=15, mFE=2.00e7				
	#	ERT	10%	90%	RT_{succ}	#	ERT	10%	90%	RT_{succ}
10	15	5.1e3	3.5e3	6.8e3	5.1e3	15	3.8e6	3.2e6	4.4e6	3.8e6
1	15	4.7e5	3.7e5	5.6e5	4.7e5	0	*21e−1*	*13e−1*	*29e−1*	1.8e7
1e−1	5	1.4e7	9.2e6	2.5e7	4.2e6	.	.	.	.	.
1e−3	1	7.2e7	3.4e7	>7e7	5.0e6	.	.	.	.	.
1e−5	1	7.2e7	3.4e7	>7e7	5.0e6	.	.	.	.	.
1e−8	1	7.2e7	3.4e7	>7e7	5.0e6	.	.	.	.	.

Table 2: iAMaLGaM: Shown are, for a given target difference to the optimal function value Δf: the number of successful trials (#); the expected running time to surpass $f_{\rm opt} + \Delta f$ (ERT, see Figure 1); the 10%-tile and 90%-tile of the bootstrap distribution of ERT; the average number of function evaluations in successful trials or, if none was successful, as last entry the median number of function evaluations to reach the best function value ($\mathbf{RT_{succ}}$). If $f_{\rm opt} + \Delta f$ was never reached, figures in *italics* denote the best achieved Δf-value of the median trial and the 10% and 90%-tile trial. Furthermore, N denotes the number of trials, and mFE denotes the maximum of number of function evaluations executed in one trial. See Figure 1 for the names of functions.

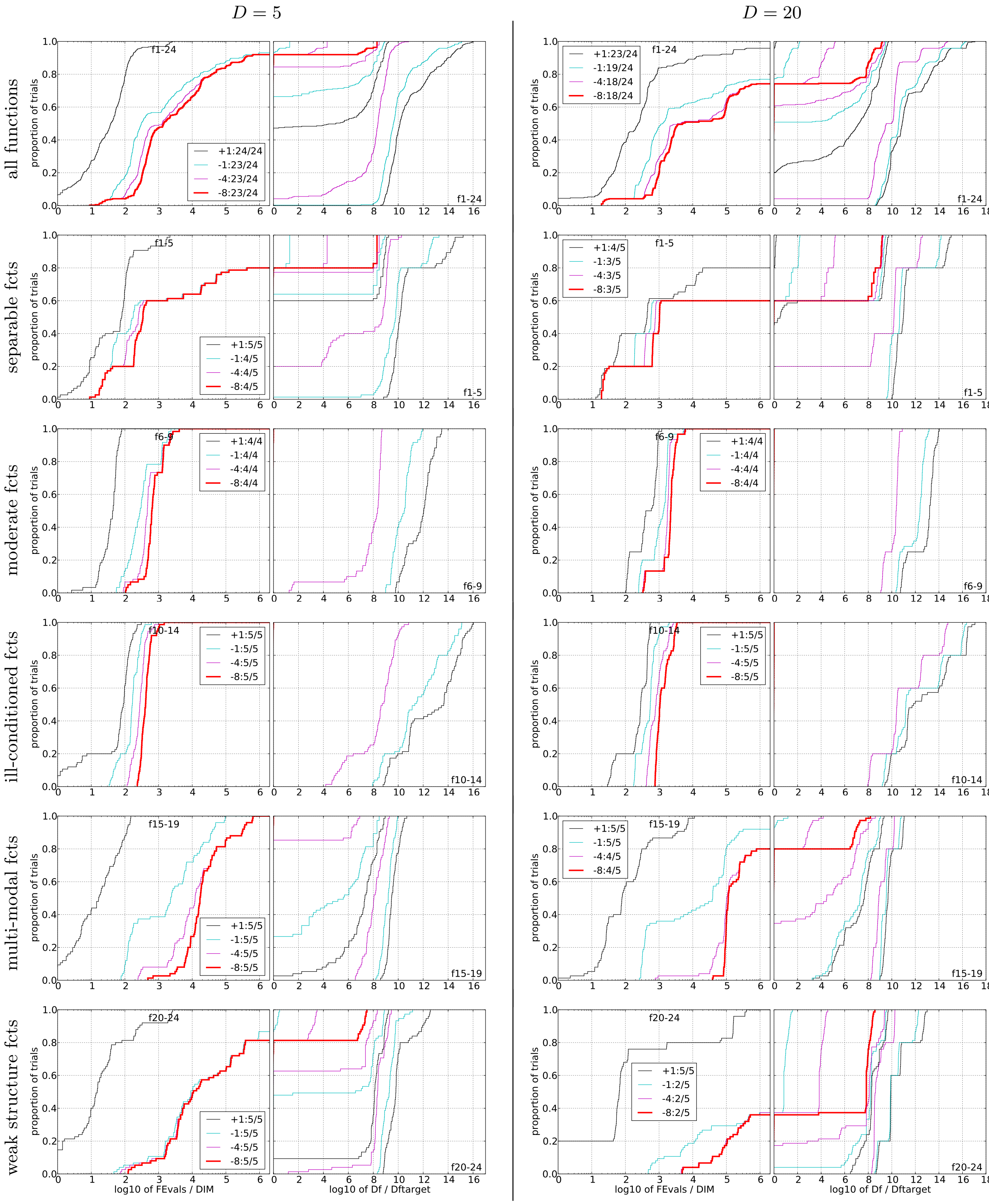

Figure 4: iAMaLGaM: Empirical cumulative distribution functions (ECDFs), plotting the fraction of trials versus running time (left) or Δf. Left subplots: ECDF of the running time (number of function evaluations), divided by search space dimension D, to fall below $f_{\text{opt}} + \Delta f$ with $\Delta f = 10^k$, where k is the first value in the legend. Right subplots: ECDF of the best achieved Δf divided by 10^k (upper left lines in continuation of the left subplot), and best achieved Δf divided by 10^{-8} for running times of $D, 10\,D, 100\,D\ldots$ function evaluations (from right to left cycling black-cyan-magenta). Top row: all results from all functions; second row: separable functions; third row: misc. moderate functions; fourth row: ill-conditioned functions; fifth row: multi-modal functions with adequate structure; last row: multi-modal functions with weak structure. The legends indicate the number of functions that were solved in at least one trial. FEvals denotes number of function evaluations, D and DIM denote search space dimension, and Δf and Df denote the difference to the optimal function value.