ON UNIMODULAR AND INVARIANT DOMAINS

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Abstract. We define a new class of local domains: unimodular domains and invariant domains. We formulate the Invariance Conjecture and we show that this conjecture is equivalent, in some sense, to the Jacobian Conjecture (over $\mathbb{C}$). Also, we investigate some cases of the Unimodular Conjecture (cf.[11, Essen-Lipton]).

1. Introduction

The following conjecture is well known.

Jacobian Conjecture. *Let R be a domain with $char(R) = 0$ and $F : R^n \longrightarrow R^n$ $(n > 1)$ a Keller map, i.e., a polynomial map with* $\det JF = 1$. *Then F is an isomophism.*

By "Lefschetz principle" (cf.[10, Lemma 1.1.14]) it is sufficient to consider the case $R = \mathbb{C}$. In [2] Bass, Connell and Wright showed that, to prove the above conjecture, it is enough to prove it for maps in the form $F = X + H$ where H is homogeneous and of degree 3 with JH nilpotent. A refinement, due to Essen-Bondt, ensures that it is in fact sufficient to consider maps in the form $F = X + H$ with $H = (H_1, ..., H_n)$ homogeneous, $deg(H) = 3$ and JH nilpotent and **symmetric**.

Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain and consider a linear map $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ i.e. $F = (F_1, \dots, F_n)$ with $F_i \in \mathcal{O}[X_1, \dots, X_n]$ homogeneous of degree 1. Suppose that the matrix $B := JF$ is invertible and let $A \in Mat_n(\mathcal{O})$ be such that $AB = BA = id_n$. The relation $BA = id_n$ implies that there exist $u_1, \dots, u_n \in \mathcal{O}$ such that $F_1(u_1, \dots, u_n) = 1$. In particular, by reduction $\mod \mathcal{M}$ we have $\overline{F} : k^n \longrightarrow k^n$ a non-zero map. The general case is an open problem, the so called (cf.[11])

Unimodular Conjecture. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $char(\mathcal{O}) = 0$ and $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ $(n > 1)$ a Keller map. Then the induced map $\overline{F} : k^n \longrightarrow k^n$ is a non-zero map.*

The interesting fact is that the Unimodular Conjecture is related to Jacobian Conjecture([11]):

Theorem (Essen-Lipton). *$\mathbb{Z}_p$ satisfies the Unimodular Conjecture for almost all primes p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

The objective of this paper is to give results in direction of the Unimodular Conjecture. Motivated by this conjecture we define the classes of *d-unimodular domains* and *invariant domains*. We propose a new conjecture, the Invariance Conjecture and we show it is equivalent, in some sense, to the Jacobian Conjecture (over $\mathbb{C}$). Furthermore, we give some contributions to the Unimodular Conjecture, in particular the following

Theorem. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $q := \#k < \infty$. Then $\mathcal{O}$ is a $(q-1)$-unimodular domain.*

Theorem. *There is a finite set of primes E such that for all prime $p \in \mathbb{Z} \setminus E$ we have*

$$\mathbb{Z}_p \text{ is an invariant domain} \Longleftrightarrow \mathbb{Z}_p \text{ is a unimodular domain.}$$

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2. Preliminaries

In this section we establish notations and present the results that will be used in the sequel.

Given a domain R we denote by $\mathcal{MP}_n(R)$ the collection of polynomial maps over R. If $F \in \mathcal{MP}_n(R)$ is a map, we say that F is **Keller** if $\det JF = 1$. We denote by $Aut_n(R)$ the group of polynomial maps which are isomophisms. If $F \in \mathcal{MP}_n(R)$ and $R \subset S$ for some domain S, we can look F as polynomial map over S. We denote this map by $F \otimes S$ (map obtained by scalar extension). The ring of p-adic integers is denoted by $\mathbb{Z}_p$.

We recall some facts:

Proposition 2.1. *Let $F \in \mathcal{MP}_n(R)$ and S a domain with $R \subset S$. Then*

$$F \in Aut_n(S) \Longleftrightarrow F \in Aut_n(R).$$

Proof. see [10, Lemma 1.1.8] □

Theorem 2.2. ***(Cynk-Rusek)*** *Fix an algebraically closed field k with $char(k) = 0$. Let $X \subset \mathbb{A}^n_k$ be an affine variety and $F : X \longrightarrow X$ a regular map. The following conditions are equivalent:*

(i) F is injective.

(ii) F is a bijection.

(iii) F is an automorphism.

Proof. see [5, Theorem 2.2],[10, Theorem 4.2.1] or [8, Theorem 1.6]. □

Immersion Lemma. *Let $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}}$. Then for infinitely of primes $p \in \mathbb{Z}$ there is an injection*

$$\phi_p : \mathbb{Z}[\alpha_1, ..., \alpha_n] \hookrightarrow \mathbb{Z}_p.$$

Proof. see [10, Theorem 10.3.1] □

Hensel Lemma. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring and $F_1(X_1, \dots, X_n), \dots, F_n(X_1, \dots, X_n) \in \mathcal{O}[X_1, \dots, X_n]$. Choose $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}^n$ such that*

$$F_1(\alpha_1, \dots, \alpha_n) \equiv \cdots \equiv F_n(\alpha_1, \dots, \alpha_n) \equiv 0 \mod \mathcal{M}^{2m+1}$$

where $m := ord_{\mathcal{M}}(\det JF(\alpha)) < \infty$. Then there is a unique $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{O}^n$ such that $F_1(\beta) = \cdots = F_n(\beta) = 0$ and $\beta_i \equiv \alpha_i \mod \mathcal{M}^{m+1}$ for all $i = 1, \dots, n$.

Proof. see [6, proposition 5.20]. □

By Hensel lemma we get the following

Proposition 2.3. *Let $f_1, \dots, f_n \in \mathcal{O}[X_1, \dots, X_n]$ be a Keller map where $(\mathcal{O}, \mathcal{M}, k)$ is a complete discrete valuation ring. If R is an $\mathcal{O}$-algebra denote by $X(R)$ the set of R-points of $Spec(\mathcal{O}[X_1, \dots, X_n]/\langle f_1, \dots, f_n\rangle)$. Then there is a bijection $X(\mathcal{O}) \cong X(k)$.*

Proof. As f is Keller we have $m = ord_{\mathcal{M}}(\det JF(\alpha)) = 0$ for all $\alpha \in \mathcal{O}^n$. The bijection is natural: given $P \in \mathcal{O}^n$ define $\varphi(P) \in X(k)$ the k-point obtained by reduction $\mod \mathcal{M}$. Hensel lemma implies that $\varphi : X(R) \longrightarrow X(k)$ is a bijection: injectivity by uniqueness and surjectivity by lifting. □

3. Unimodular Domains

Given $F \in \mathcal{MP}_n(\mathcal{O})$ over a local domain $(\mathcal{O}, \mathcal{M}, k)$ we will denote by $f \in \mathcal{MP}_n(k)$ the induced map over the residue field k.

Definition 3.1. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain. We say that $\mathcal{O}$ is a **unimodular domain** if the Unimodular Conjecture is true for $\mathcal{O}$. We say that a polynomial map $F \in \mathcal{MP}_n(\mathcal{O})$ is **unimodular** if it satisfies the condition in the Unimodular Conjecture.*

Proposition 3.2. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with k an infinite field. Then $\mathcal{O}$ is unimodular.*

Proof. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a Keller map. Let $f \in \mathcal{MP}_n(k)$ be the induced map over the residue field and suppose that $f(\alpha) = 0$ for all $\alpha \in k^n$. Since k is infinite we have $f \equiv 0$. So the coefficients that occur in F belong to the maximal ideal $\mathcal{M}$. In particular, $\det JF \in \mathcal{M}[X_1, .., X_n]$ a contradiction by Keller condition: $\det JF = 1$.

□

We recall the following proposition (cf.[11, Proposition 8]).

Proposition 3.3. *Suppose that the Jacobian Conjecture over $\mathbb{C}$ is true. Then every local domain $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = 0$ is unimodular.*

Proof. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a Keller map over a local domain $\mathcal{O}$ with $char(\mathcal{O}) = 0$. Since we assume the Jacobian Conjecture is true over $\mathbb{C}$ we have F an invertible map over $\mathcal{O}$ (cf.[10, Lemma 1.1.14]. So, there is a unique $G \in \mathcal{MP}_n(\mathcal{O})$ such that $F \circ G = X$. By reduction $\mod \mathcal{M}$ we see that the map $f \in \mathcal{MP}_n(k)$ is a bijection, in particular, non-zero map. □

We remark that the Unimodular Conjecture is false for local domains with $char(\mathcal{O}) = p > 0$ and residue field finite. For example: consider the local domain $(\mathbb{F}_p[[T]], T\mathbb{F}_p[[T]], \mathbb{F}_p)$ and take the polynomial map $F = (X_1 - X_1^p, \ldots, X_n - X_n^p) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$. Note that F is a Keller map but the induced map over the residue field is the zero map, since $\alpha^p = \alpha$ for all $\alpha \in \mathbb{F}_p$.

Remark 1. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain. The following table shows the complete set of relations between $char(\mathcal{O})$ and $char(k)$.*

$char(\mathcal{O})$	$char(k)$	$\#k$	*type*
$p = 0$	$q > 0$	∞	*unimodular*
$p = 0$	$q > 0$	$< \infty$	*unknown*
$p = 0$	$q = 0$	∞	*unimodular*
$p > 0$	$q = p$	$< \infty$	*non-unimodular*
$p > 0$	$q = p$	∞	*unimodular*

*Thus the interesting case is $(char(\mathcal{O}), char(k), \#k, type) = (0, p, < \infty, unknown)$ where $p > 0$. Indeed, Essen-Lipton theorem ensures that **unknown** = **unimodular** if and only if the Jacobian Conjecture over $\mathbb{C}$ is true.*

4. Invariance Conjecture and Invariant Domains

Invariance Conjecture. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $char(\mathcal{O}) = 0$ and $F \in \mathcal{MP}_n(\mathcal{O})$ a Keller unimodular map. Let $G \in Aut_n(\mathcal{O})$ be an affine Keller automorphism i.e. $G = AX + b$ where $A \in Sl_n(\mathcal{O})$. Then $F \circ G \circ F$ and $F - F(a)$ are unimodular maps for all $a \in \mathcal{O}^n$.*

Remark 2. *Note that in the above conjecture we ask the unimodular property to be invariant under translation and composition of a special type. Note also that, as in the unimodular case, if the residue field k is infinite then the Invariance Conjecture is true for any complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = p \geq 0$.*

Definition 4.1. *Let $(\mathcal{O},\, \mathcal{M}, k)$ be a local domain. Given a map $F \in \mathcal{MP}_n(\mathcal{O})$ we say that*

- *F is an **invariant map** if F is Keller, unimodular and satisfies the Invariance Conjecture condition.*
- *F is **strongly invariant** if it is invariant and for all Keller affine automorphisms $G_1, \dots, G_k \in Aut_n(\mathcal{O})$ the map $F_1 \circ F_2 \circ F_3 \circ \cdots \circ F_k$ is invariant where $F_j = G_j \circ F$.*

*The domain $\mathcal{O}$ is called an **invariant domain** if every polynomial map that is Keller and unimodular (in dimension $n > 1$) is invariant.*

Lemma 4.2. *If a map $F \in \mathcal{MP}_n(\mathcal{O})$ is strongly invariant then $F \circ G \circ F$ is strongly invariant for all Keller affine automorphism $G \in Aut_n(\mathcal{O})$.*

Proof. induction.

□

Proposition 4.3. *Let $(\mathcal{O},\, \mathcal{M}, k)$ be a local unimodular domain. Then $\mathcal{O}$ is invariant.*

Proof. By hypothesis a Keller map $F \in \mathcal{MP}_n(\mathcal{O})$ is unimodular. Since the Keller condition is invariant under composition and translation we have the result. □

The condition $char(\mathcal{O}) = 0$ is important.

Example 1. *Let $F_1, \dots, F_n \in \mathbb{F}_p[[T]][X_1, \dots, X_n]$ be defined by $F_j = 1 - X_j^p + X_j$ and consider the polynomial map $F = (F_1, \dots, F_n) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$. It is easy to check that* $\det JF = 1$ *and that F is unimodular. But*

$$F - F(1, \dots, 1) = (-X_1^p + X_1, \dots, -X_n^p + X_n).$$

So, in case $(\mathcal{O},\, \mathcal{M}, k) = (\mathbb{F}_p[[T]],\, T\mathbb{F}_p[[T]], \mathbb{F}_p)$ it follows that the property of invariance by translation is false.

Example 2. *Let $g(X) \in \mathbb{F}_p[X]$ be a polynomial wich maps $\{0, \dots, p-2\} \mapsto p-1$ and $p-1 \mapsto 0$. For example, take $p = 5$ and consider*

$$g(X) = -1 + X - X^2 + X^3 - X^4 \in \mathbb{F}_5[X].$$

It is easy to check that $g \circ g = 0$. Note that $g(0) \neq 0$. Define the polynomial map $F = (F_1, \dots, F_n) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$ with $F_j = X_j - X_j^p + g(X_j^p)$. We have F a Keller map with the induced map over the residue field non-zero. But by construction we have $F \circ F = 0$. Thus, in characteristic $p > 0$ the invariance by composition is false.

In the next theorem the argument is simitar to the argument given in [11, Theorem 4] with the observation that is sufficient to require the invariance property.

Theorem 4.4. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring with finite residue field. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a strongly invariant map. Then F is injective.*

Proof. Suppose false and let F be a strongly invariant map over $\mathcal{O}$ with $F(a_1) = \cdots = F(a_m) = c$ $(m > 1)$ for some $a_1, \dots, a_m \in \mathcal{O}^n$ with $a_i \neq a_j$, if $i \neq j$. We will show that there is a strongly invariant map G with $\#G^{-1}(c) > m$. By iteration we will get a Keller map $\widetilde{G} \in \mathcal{MP}_n(\mathcal{O})$ with $\#G^{-1}(c) > (\#k)^n$ a contradiction by proposition 2.3.

Since $F(a_1) = F(a_2)$ we have $\langle a_2 - a_1 \rangle = R$ ([10, Lemma 10.3.11]). On the other hand, since F is an invariant map it is ensured that there exists $b \in \mathcal{O}^n$ such that $F(b) - F(a_1)$ is unimodular, i.e., $\langle F(b) - F(a_1) \rangle = R$. In particular, $\langle a_2 - a_1 \rangle = \langle F(b) - F(a_1) \rangle = \langle F(b) - c \rangle = R$. So, we have $\{a_2, a_1\} \cong \{F(b), c\}$ (see [11, Transitivity, Proposition 1]). By [11, Theorem 2] we know that there is $H \in \mathcal{MP}_n(\mathcal{O})$, Keller affine automorphism such that $H(c) = a_1$ and $H(F(b)) = a_2$. Now define $G = F \circ H \circ F$. We have G strongly invariant map with $G(a_j) = F(H(c)) = F(a_1) = c$ for all j and $G(b) = F(H(F(b))) = F(a_2) = c$. Note that $b \neq a_j$ for all j. □

Theorem 4.5. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring with finite residue field. Suppose that $\mathcal{O}$ is an invariant domain. Then any unimodular Keller polynomial map $F \in \mathcal{MP}_n(\mathcal{O})$ is injective.*

5. Some results

Definition 5.1. *Pick $d \in \mathbb{Z}_{\geq 1}$ and let $(\mathcal{O}, \mathcal{M}, k)$ a local domain. We say that $\mathcal{O}$ is a d-unimodular map if any Keller map $F \in \mathcal{MP}_n(\mathcal{O})$ in dimension $n > 1$ with $deg(F) \leq d$ is unimodular.*

Note that any local domain $\mathcal{O}$ is 1-unimodular and $\mathcal{O}$ is a unimodular domain if and only if it is d-unimodular for all $d \in \mathbb{N}$. If $\mathcal{O}$ is d-unimodular then it is e-unimodular for all $e \leq d$. We will see later that $\mathbb{Z}_p$ is 3-unimodular for any prime $p > 3$. In case $char(\mathcal{O}) = p > 0$ and k finite we have that $\mathcal{O}$ isn't d-unimodular for infinitely many $d \in \mathbb{Z}$. Indeed, for each $m \in \mathbb{N}$ take $d = (\#k)^m$ and consider the map $F = (X_1 - X_1^d, \ldots, X_n - X_n^d) \in \mathcal{MP}_n(\mathcal{O})$.

Proposition 5.2. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ be a non constant polynomial map. Then for almost all primes $p \in \mathbb{Z}$ we have $F \otimes \mathbb{Z}_p$ unimodular map over $\mathbb{Z}_p$.*

Proof. Indeed, suppose $F_1(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n] \setminus \mathbb{Z}$. We can choose $d \in \mathbb{Z}^n$ such that $F_1(d) \neq 0$. Note that $F_1(d) \in \mathbb{Z}_p^*$ for all p such that $p \nmid F_1(d)$.

□

It is known that in order to prove the Jacobian Conjecture it is sufficient to consider polynomial maps of Druzkowski type, i.e., maps in the form $F = X + H$ with $H_j = (\sum_k a_{kj} X_k)^3$ and JH nilpotent (see [10, Theorem 6.3.2]). We call maps of the form $F = X + H$ with $H = \sum_k a_{kj} X_k^3$ quasi-Druzkowski maps.

Proposition 5.3. *For almost all primes p, the Unimodular Conjecture over $\mathbb{Z}_p$ is true for quasi-Druzkowski maps.*

Proof. Let F be a quasi-Druzkowski map with $H = (H_1, \ldots, H_n)$ where $H_j = \sum_k b_{kj} X_k^3$. We will show that there exist $u_1, \ldots, u_n \in \mathbb{Z}_p$, not all null , such that

$$u_1 H_1(X_1, \ldots, X_n) + \cdots + u_n H_n(X_1, \ldots, X_n) = 0.$$

Indeed, for this it is sufficient to find a non trivial solution for the homogeneous system:

$$u_1 b_{11} + u_2 b_{12} + \cdots + u_n b_{1n} = u_1 b_{21} + u_2 b_{22} + \cdots + u_n b_{2n} = \cdots = u_1 b_{n1} + u_2 b_{n2} + \cdots + u_n b_{nn} = 0.$$

Now since JH is nilpotent we have, in particular, $\det(b_{ij}) = 0$ and so there is a non-trivial solution $(u_1, \ldots, u_n) \in \mathbb{Q}_p^n$ for the system above. Without loss of generality we can suppose that $u_1 \in \mathbb{Z}_p^*$ and $u_j \in \mathbb{Z}_p$, if $j > 1$. Now consider $s := u_1 + u_2 p \cdots + u_n p \in \mathbb{Z}_p^*$. Note that, $(1, p, \ldots, p) \in \mathbb{Z}_p^n$ is such that

$$\langle F_1(1, p, \ldots, p), \ldots, F_n(1, p, \ldots, p) \rangle = \mathbb{Z}_p.$$

□

Remark 3. *The proposition above will be generalized later (see corollary 5.9).*

It was seen in the previous section that there are local domains $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = p > 0$ that are not unimodular domains. On the other hand we know that any local domain with infinite residue field is indeed a unimodular domain. In particular, if we consider the map $F = (X_1 - X_1^p, \ldots, X_n - X_n^p)$ over $(\overline{\mathbb{F}_p}[[T]], T\overline{\mathbb{F}_p}[[T]], \overline{\mathbb{F}_p})$ we have $\overline{F}(\alpha) \neq 0$ for some $\alpha \in \overline{\mathbb{F}_p}$ (= algebraically closure of $\mathbb{F}_p$). So, if we take $L =$ the field obtained by adjunction of α to $\mathbb{F}_p$ we see that our F is unimodular over the local domain $(L[[T]], TL[[T]], L)$. For the p-adic case there is an analogue:

Proposition 5.4. *Let $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map. Then there is a complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ that dominates $\mathbb{Z}_p$ such that $F \otimes \mathcal{O}$ is a unimodular map. Furthermore, $\mathcal{O}$ is a free $\mathbb{Z}_p$-module with*

$$rank_{\mathbb{Z}_p}(\mathcal{O}) = [k : \mathbb{F}_p].$$

Proof. In the proof we use a result of algebraic number theory ([9, Proposition 7.50]).

Consider the map $\overline{F} \in \mathcal{MP}_n(\mathbb{F}_p)$ induced over the residue field. By the previous remark we know that there exists $\alpha \in \overline{\mathbb{F}_p}^n$ such that $\overline{F}(\alpha) \neq 0$. By taking the field $k := \mathbb{F}_p(\alpha_1, \dots, \alpha_n)$ obtained by adjunction we can look at $\overline{F}$ as a polynomial map which is **non zero** over k. Now we recall the following theorem about unramified extensions of a local field L

Theorem. *Let L be a local field with residue field l. There exists a 1-1 correspondence between the following sets*

$$\{\textit{finite extensions unramified over } L\ \} \;\cong\; \{\textit{finite extensions of } l\}$$

given by $L' \mapsto l'$, where l' is the residue field associated to L'. Furthermore, in this correspondence we have $[L' : L] = [l' : l]$.

Applying the theorem above to $L = \mathbb{Q}_p$ with $l = \mathbb{F}_p$ we see that the extension $k|\mathbb{F}_p$ corresponds to a local field $K|\mathbb{Q}_p$ such that k is the residue field of K. Denote by $(\mathcal{O}, \mathcal{M}, k)$ the ring of integers of K. The ring $\mathcal{O}$ is the integral closure of $\mathbb{Z}_p$ in K and by a general result (cf.[1, proposition 5.17]) we know that $\mathcal{O}$ is a free $\mathbb{Z}_p$-module and $rank_{\mathbb{Z}_p}(\mathcal{O}) = [K : \mathbb{Q}_p] = [k : \mathbb{F}_p]$. So $F \otimes \mathcal{O} \in \mathcal{MP}_n(\mathcal{O})$ is a Keller map with non zero induced map over the residue field. □

The theorem below was gotten in an attempt of the author to show the following

Conjecture. *Denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Let $F : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ be a Keller map such that $F \otimes \mathcal{O}$ is injective. Then F is an isomorphim.*

Lemma 5.5. *Let $K|\mathbb{Q}_p$ be a finite Galois extension with $m := [K : \mathbb{Q}_p] > 1$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. Let $F \in \mathcal{MP}_n(\mathcal{O}_K)$ be a non-injective Keller unimodular map. Then there exists a non-injective Keller unimodular map $G \in \mathcal{MP}_{mn}(\mathbb{Z}_p)$.*

Proof. The same argument of [10, A Galois descent] works. The relevant fact is that $\mathcal{O}_K$ is a free $\mathbb{Z}_p$-module of $rank_{\mathbb{Z}_p}(\mathcal{O}_K) = m$. □

By theorem 4.4 we know that if $\mathbb{Z}_p$ is a unimodular domain then any Keller map over $\mathbb{Z}_p$ is injective. We can show a more general result

Theorem 5.6. *Assume that $\mathbb{Z}_p$ is an invariant domain for some prime p. Then for all Keller unimodular maps $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ and $K|\mathbb{Q}_p$ finite extension we have $F \otimes \mathcal{O}_K$ is an injective map.*

Proof. It is sufficient to show that $F \otimes \mathcal{O}$ is an injective map where $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. Indeed, if $\alpha \neq \beta \in \mathcal{O}$ are such that $F(\alpha) = F(\beta)$ consider the ring $R = \mathbb{Z}_p[\alpha, \beta]$ obtained by adjunction and let $K := Frac(R)$. We have $K|\mathbb{Q}_p$ a finite extension such that $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in K$. Note that $\alpha_i, \beta_i \in \mathcal{O}_K$ for all i. Without loss of generality we can suppose that $K|\mathbb{Q}_p$ is a Galois extension. So, $F \otimes \mathcal{O}_K$ is a Keller unimodular map over $\mathcal{O}_K$ that isn't injective. By the lemma above, we obtain $G \in \mathcal{MP}_N(\mathbb{Z}_p)$ a Keller unimodular map that isn't injective. Now since we assume that $\mathbb{Z}_p$ is an invariant domain and G is a strongly invariant map, by theorem 4.4, we know that G is an injective map. A contradiction. □

In the direction of the Unimodular Conjecture we have the interesting result:

Theorem 5.7. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $q := \#k < \infty$. Then $\mathcal{O}$ is a $(q-1)$-unimodular domain.*

Note that there are no restrictions about $char(\mathcal{O})$.

Proof. Let $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ be a Keller map with $deg(F) \leq q-1$. Denote by f_j the polynomial in $k[X_1, \ldots, X_n]$ obtained by reduction of $F_j \mod \mathcal{M}$. So we obtain a polynomial map $f = (f_1, \ldots, f_n) : k^n \longrightarrow k^n$. By Keller condition we have f_j is a non zero polynomial, for all indices j. We must prove that there exists $\alpha \in k^n$ such that $f(\alpha) \neq 0$.

Passing to algebraic closure consider the algebraic set $X \subset \overline{k}^n$ defined by equations $f_1 = \cdots = f_n = 0$. We affirm that $\dim X = 0$ Indeed, note that $X = f^{-1}(0)$, where f is the map over $\overline{k}$ defined by tuple $f_1, \ldots, f_n$. The Keller condition implies that $\delta := [\overline{k}(X_1, \ldots, X_n) : \overline{k}(f_1, \ldots, f_n)] < \infty$ (see [10, proposition 1.1.31]) and by [10, theorem 1.1.32] we know $\#f^{-1}(Q) \leq \delta$ for all $Q \in \overline{k}^n$. In particular, $\#X = \#f^{-1}(0) < \infty$. So $\dim X = 0$.

Now we make use of the following (cf.[3, 8, Bézout Inequality])

Bézout Inequality. *Let $X \subset \mathbb{A}^n_{\overline{k}}$ be an affine algebraic set given by the equations $f_1 = \cdots = f_r = 0$. Denote by $X(k)$ the set of k-points. If $\dim X = 0$ then*

$$\#X(k) \leq \#X \leq deg(f_1) \cdots deg(f_r).$$

Applying the theorem above we have

$$\#X(k) \leq deg(f_1) \cdots deg(f_n) \leq deg(F)^n < q^n$$

where we use the hypothesis: $q > deg(F)$. So $S = k^n \setminus X(k) \neq \varnothing$. Let $\alpha \in S$. By definition $f_j(\alpha) \neq 0$ for some index j. In particular, the map $f : k^n \longrightarrow k^n$ isn't zero. So F is a unimodular map. □

Corollary 5.8. *For all prime p, $\mathbb{F}_p[[T]]$ and $\mathbb{Z}_p$ are $(p-1)$-unimodular domains .*

Note that the bound $p-1$ is "maximal" for $\mathbb{F}_p[[T]]$.

Corollary 5.9. *$\mathbb{Z}_p$ is 3-unimodular domain for all prime $p > 3$. In particular, for almost all primes p the Unimodular Conjecture is true for maps of degree ≤ 3 over $\mathbb{Z}_p$.*

Proof. Since $p > 3$ we have $p - 1 \geq 3$ and so the result follows from theorem above. □

In dimension $n = 2$ and in characteristic 0 the theorem above can be refined. Indeed, we have the following

Theorem 5.10. *Let $f = (f_1, f_2) \in \mathcal{MP}_2(\mathcal{O})$ be a Keller map over a complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ with $q := \#k < \infty$ and $char(\mathcal{O}) = 0$. If $deg(f_1) < q^2$ then f is unimodular.*

Remark 4. *This is particular for $char(\mathcal{O}) = 0$. Indeed, consider the map*

$$f = (X_1 - X_1^p, X_2 - X_2^p) \in \mathcal{MP}_2(\mathbb{F}_p[[T]]).$$

f is a Keller map but is not unimodular. Furthermore $deg(f_1) = p < p^2$.

In order to prove the theorem above we use the following result(cf.[12]):

Theorem (Yitang Zhang). *Let $f = (f_1, f_2) \in \mathcal{MP}_2(K)$ be a Keller map over an algebraically closed field K with $char(K) = 0$. Then $[K(X, Y) : K(f_1, f_2)] \leq \mathbf{Min}\{deg(f_1), deg(f_2)\}$.*

Proof. (of theorem 5.10) Let $f = (f_1, f_2)$ be a Keller map over $\mathcal{O}$. By proposition 2.3 we have a bijection

$$S_1 := \{(u,v) \in \mathcal{O}^2 \mid f_1(u,v) = f_2(u,v) = 0\} \cong \{(a,b) \in k^2 \mid g_1(a,b) = g_2(a,b) = 0\} =: S_2$$

where g_1 and g_2 are the reductions of $f_1, f_2 \mod \mathcal{M}$.

By [10, theorem 1.1.32] we know that $\#S_1 \leq [K(X,Y) : K(f_1, f_2)]$ where $K = \overline{Frac(\mathcal{O})}$. By Zhang theorem and hypothesis we have $\#S_1 \leq \mathbf{Min}\{deg(f_1), deg(f_2)\} \leq q^2 - 1$. In particular there is $Q \in k^2 \setminus S_2$. So f is a unimodular map.

□

Theorem 5.11. *Let $p \in \mathbb{Z}$ be a prime. For each $d \in \mathbb{Z}_{\geq 1}$ we can find a finite extension $K|\mathbb{Q}_p$ such that the ring of integers $\mathcal{O}_K$ is a d-unimodular domain.*

Proof. Let $d \in \mathbb{N}$. If $d = 1$ take $K = \mathbb{Q}_p$. Suppose that $d > 1$. We know that for any Keller map $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ of degree d we have

$$\#X = deg(X) \leq deg(F)^n = d^n$$

where X is the algebraic set in $\mathbb{A}^n_{\overline{\mathbb{F}_p}}$ given by reduction of $F \mod p$. Let n be an integer such that $p^n > d$ and fix $\mathbb{F}_{p^n}$ the unique extension of $\mathbb{F}_p$ of degree n in $\overline{\mathbb{F}_p}$. We have seen in the proof of proposition 5.4 that there is a finite extension $K|\mathbb{Q}_p$ such that the residue field of $\mathcal{O}_K$ is $\mathbb{F}_{p^n}$. By construction, for all Keller map $G \in \mathcal{MP}_n(\mathcal{O}_K)$ with $deg(G) \leq d$ we have $\#\{g_1 = \cdots = g_n = 0\} \leq deg(G)^n \leq d^n < (p^n)^n$. So G is a unimodular map and $\mathcal{O}_K$ is a d-invariant domain. □

Proposition 5.12. *Suppose that for all $n \in \mathbb{N}$ and all $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(\mathbb{Z}_p)$ Keller map with $deg(F) < n$, F is unimodular. Then $\mathbb{Z}_p$ is a unimodular domain.*

Proof. Let $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map with $n \leq deg(F)$. Let $m \in \mathbb{Z}$ be an integer (to be determined) and consider the map

$$F^{[[m]]} = (F_1, \ldots, F_n, F_1, \ldots, F_n, \ldots, F_1, \ldots, F_n) \in \mathcal{MP}_{mn}(\mathbb{Z}_p)$$

which consists of m-repetitions of the tuple $F_1, \ldots, F_n$ where in each occurrence of such tuple we introduce n-distinct variables. By construction we have $F^{[[m]]}$ a Keller map and F is a unimodular map if and only if so is $F^{[[m]]}$. We can choose large m such that $deg(F) < mn$. Thus, we get the unimodularity of F. □

Let R be a domain and $f \in R[X_1, \ldots, X_n]$. Define $d(f) :=$ number of monomials in degree > 3 that occur in f. If $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(R)$ we define $d(F) := \sum_j d(F_j)$.

Proposition 5.13. *Let $p \in \mathbb{Z}_{>3}$ be a prime number and $f \in \mathcal{MP}_n(\mathbb{Z}_p)$ a Keller map. Suppose that*

$$d(f) \leq log(2)^{-1} log(nlog(p/3)/log(3)) \qquad (*)$$

where log is the natural logarithm. Then f is unimodular.

Proof. Let $f \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map. By the reduction theorem (cf. [2, (3.1) Proposition.] we can find invertible maps $G, H \in \mathcal{MP}_{n+m}(\mathbb{Z}_p)$ for some $m \in \mathbb{N}$ such that $g := G \circ f^{[m]} \circ H$ has degree ≤ 3 where $f^{[m]} = (f, X_{n+1}, \ldots, X_{n+m})$. Furthermore, we know that $G(0) = H(0) = 0$. Denote by $X_f(\mathbb{Z}_p)$ and $X_g(\mathbb{Z}_p)$ the set of $\mathbb{Z}_p$-points of f and g respectively. It is easy to check that $\#X_f(\mathbb{Z}_p) = \#X_g(\mathbb{Z}_p)$. Now, since $\mathbb{Z}_p$ is a 3-unimodular domain (corollary 5.9) we have $\#X_g(\mathbb{Z}_p) < 3^{n+m}$. By the proof of reduction theorem we get $m = 2^{d(f)}$. The inequality $(*)$ implies $3^{m+n} \leq p^n$ and so we have f a unimodular map. □

Theorem 5.14. *$\mathbb{Z}_p$ is an invariant domain for almost all prime p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

Proof. The implication $\Longleftarrow$ follows from 3.3. Suppose that the Invariance Conjecture is true over $\mathbb{Z}_p$ for almost all prime p. By a result of Connel-van den Dries (cf.[10, Proposition 1.1.19]) we know that it is sufficient to show the Jacobian Conjecture over $\mathbb{Z}$. So, suppose some Keller map $F \in \mathcal{MP}_n(\mathbb{Z})$ isn't invertible. Since F has coefficients in $\mathbb{Z}$ it follows that F is unimodular over $\mathbb{Z}_p$ for almost all primes p. Also, we know by hypothesis that $F \otimes \overline{\mathbb{Q}}$ isn't injective. By the immersion lemma, F isn't injective over $\mathbb{Z}_p$ for infinitely many primes p. Fix such a prime p such that $\mathbb{Z}_p$ is an invariant domain. So, we obtain $F \otimes \mathbb{Z}_p$ a Keller map non-injective over the invariant complete local domain. A contradiction by theorem 4.4. □

6. A REFINEMENT

Strong Immersion Lemma. *Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$ be algebraic numbers. Then there is a finite set E of rational primes such that for all prime $p \notin E$ we have an injective homomorphism*

$$\mathbb{Z}[\alpha_1, \ldots, \alpha_m] \hookrightarrow \mathcal{O}_{K,p}$$

where $\mathcal{O}_{K,p}$ is the ring of integers of some finite $K|\mathbb{Q}_p$.

Proof. The proof is similar to proof the immersion lemma (2). It is sufficient to prove the following

Fact. *Let $f(T) \in \mathbb{Z}[T] \setminus \mathbb{Z}$ be an irreducible polynomial. Then for almost all prime p there is a finite extension $K|\mathbb{Q}_p$ and $\alpha \in \mathcal{O}_{K,p}$ such that $f(\alpha) = 0$.*

Let d be the discriminant of the polynomial f and $E := \{p \mid p \text{ is prime with } p \mid d\}$. Let $p \in \mathbb{Z} \setminus E$ be a prime and take $\overline{f}(T) \in \mathbb{F}_p[T]$, via reduction $\mod p$. Let $\alpha \in \overline{\mathbb{F}_p}$ be a root of $\overline{f}(T)$ and take $\mathbb{F}_{p^k}$ the definition field of α. Then

$$\overline{f}(\alpha) = 0 \text{ and } \overline{f}'(\alpha) \neq 0 \text{ by condition } p \notin E.$$

Now we recall that there exists a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_{K,p}$ is a complete discrete valuation with residue field $\mathbb{F}_{p^k}$. Since $\mathcal{O}_{K,p}$ is a complete ring we can use the Hensel lemma to conclude that there is some $a \in \mathcal{O}_{K,p}$ such that $f(a) = 0$. □

Theorem 6.1. *$\mathbb{Z}_p$ is an invariant domain for infinitely many primes p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

Proof. The implication $\Longleftarrow$ is trivial. Suppose that $\mathbb{Z}_p$ is an invariant domain for infinitely many primes p but the Jacobian Conjecture is false. Let $F \in \mathcal{MP}_N(\mathbb{Z})$ be a counterexemple with $\det JF = 1$ (cf.[10, Proposition 1.1.19]). In particular, $F \otimes \overline{\mathbb{Q}}$ isn't injective. Let $\alpha \neq \beta \in \overline{\mathbb{Q}}^N$ be such that $F(\alpha) = F(\beta)$. By the strong immersion lemma we know that $R := \mathbb{Z}[\alpha, \beta] \hookrightarrow \mathcal{O}_{K,p}$ for almost all primes p. Fix a prime p such that $R \hookrightarrow \mathcal{O}_{K,p}$ and such that $\mathbb{Z}_p$ is an invariant domain. So, we obtain $F \otimes \mathcal{O}_{K,p}$ a Keller map, not injective, over the domain $\mathcal{O}_{K,p}$. By lemma 5.5 we know that there exists a Keller map G over $\mathbb{Z}_p$ that isn't injective. A contradiction by theorem 4.5. □

Theorem 6.2. *There is a finite set of primes E such that for all prime $p \in \mathbb{Z} \setminus E$ we have*

$$\mathbb{Z}_p \text{ is an invariant domain} \Longleftrightarrow \mathbb{Z}_p \text{ is a unimodular domain.}$$

Proof. The implication $\Longleftarrow$ is easy. Suppose $\Longrightarrow$ false. Then for infinitely many primes p we have $\mathbb{Z}_p$ is an invariant non-unimodular domain. Since $\mathbb{Z}_p$ is invariant for infinitely many primes we have that the Jacobian Conjecture is true by theorem 6.1. On the other hand since $\mathbb{Z}_p$ is not unimodular for infinitely many primes we know, by Essen-Lipton theorem, that the Jacobian Conjecture is false. Contradiction. □

7. Some problems

We have given a refinament to Essen-Lipton conjecture in sense that for "almost primes p" is replaced by "infinitely many primes p". It is interesting to investigate if "infinitely many primes p" can be replaced by "primes in a finite set" or better "**some** prime p".

We start with the following ad hoc definition

Definition 7.1. *Let R be an domain and $n \in \mathbb{N}$. We say that R is Keller-finite in dimension n if*

- *given $f_1, \cdots, f_n \in R[X_1, ..., X_n]$ polynomials with $\det J_f = 1$ the R-algebra $A := R[X_1, ..., X_n]/\langle f_1, ..., f_n \rangle$ is finitely generated as R-module.*

We say that R is Keller-finite if it is Keller-finite in dimension n for all $n \in \mathbb{N}$.

Proposition 7.2. *Let k be a algebraically closed field. Then k is Keller-finite.*

Proof. Let $R = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n \rangle$ and take $M \in Spec_m(R)$ a maximal ideal. By local noetherian ring theory we know that $\dim R_m \leq \dim_k T_P X$ for all $P \in X$ where $T_P X := Hom_k(\mathcal{M}_{X,P}/\mathcal{M}_{X,P}^2, k)$ is the tangent space. By the Jacobian criterion we have $\dim_k T_P X = n - rank(JF(P)) = n - n = 0$. So we conclude $\dim R_m = 0$. In particular, $\dim R = 0$ so that R is an artinian k-algebra. In particular, $\dim_k R < \infty$. □

Example 3. *$k = \mathbb{F}_p[[T]]$ is not Keller-finite.*

Indeed, take $f := TX^p - X$. Then, $k[X]/\langle f \rangle$ is not finitely generated as k-module since that X is not integral over $\mathbb{F}_p[[T]]$.

In this direction we have the following

Keller-finite Conjecture. *Every discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = 0$ and $\#k < \infty$ is Keller-finite.*

Theorem 7.3. *Suppose that Keller-finite conjecture is true. If $\mathbb{Z}_p$ is unimodular for* ***some*** *prime p then Jacobian Conjecture over $\mathbb{C}$ is true.*

Proof. Suppose that $\mathbb{Z}_p$ is unimodular for some prime p. Let $f = (f_1, ..., f_n) \in \mathcal{MP}_n(\overline{\mathbb{Q}_p})$ be an counterexemple for Jacobian Conjecture with $\det J_f = 1$ and coefficients in $\mathbb{Z}_p$ (cf.[10, Proposition 1.1.19]). In particular, f is not injective. So, there exists distincts $\alpha_1, \alpha_2 \in L^n$ such that $f(\alpha_1) = f(\alpha_2)$ for some finite extension $L|\mathbb{Q}_p$. By a translation we can assume that $f(\alpha_1) = 0$. Now, by hypothesis we have $\mathcal{O}_L$ a Keller-finite domain. So, the induced map

$$f : X := Spec(\mathcal{O}_L[X_1, ..., X_n]/\langle f_1, ..., f_n \rangle) \longrightarrow Spec(\mathcal{O}_L)$$

is a finite map and by valuative criterion[1] there exists a bijection $X(\mathcal{O}_L) \cong X(L)$ (cf.[7, Theorem 4.6]).

But, $\mathbb{Z}_p$ unimodular domain $\Longrightarrow f \otimes \mathcal{O}_L$ is a injective map (Theorem 5.6). In particular, $\#X(L) = \#X(\mathcal{O}_L) \leq 1 \Longrightarrow f \otimes L$ is injective. Contradiction. □

Here, we list some problems we wish to find a solution:

Problem 7.4. *Recall the reduction theorem: In order to show the Jacobian Conjecture it is sufficient to consider the maps of degree ≤ 3. So, we may ask*

- *Is there an analogue of the reduction theorem for the Unimodular Conjecture?*

[1] f is a proper map!

Problem 7.5. *Find a prime $p \in \mathbb{Z}$ such that $\mathbb{Z}_p$ is unimodular.*

It was seen that for each integer $d \in \mathbb{Z}$ there exists a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_K$ is d-unimodular. This motivates the following

Problem 7.6. *Given a prime $p \in \mathbb{Z}$ find (or show that this is impossible) a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_K$ is a unimodular domain.*

The following problem is equivalent to the Jacobian Conjecture

Problem 7.7. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ a Keller map. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Then, $F \otimes \mathcal{O}$ is injective.*

Indeed, the theorem of Connell-van den Dries (cf.[10, Proposition 1.1.19]) is more general: Let $P \in Spec(\mathcal{O})$ non-zero and consider $\mathcal{A} := \mathcal{O}_P$ localization over P. If the Jacobian Conjecture over $\mathbb{C}$ is false then there is a Keller map $F \in \mathcal{MP}_n(\mathbb{Z})$, counterexemple, such $F \otimes \mathcal{A}$ is injective. Details can be found in [4]

We can compare the problem above with the following

Theorem 7.8. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ be a Keller map. Suppose that $F \otimes \overline{\mathbb{Q}}$ is injective. Then F is an isomophism.*

The proof of this theorem uses Cynk-Rusek theorem.

Problem 7.9. *In theorem 6.2, what can be said about $\#E$?*

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