ON UNIMODULAR AND INVARIANT DOMAINS

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ABSTRACT. We define a new class of local domains: unimodular domains and invariant domains. We formulate the Invariance Conjecture and we show that this conjecture is equivalent, in some sense, to the Jacobian Conjecture (over $\mathbb{C}$). Also, we make a contribution to the Unimodular Conjecture (cf.[10, Essen-Lipton]).

1. INTRODUCTION

The following conjecture is well known.

Jacobian Conjecture. *Let R be a domain with $char(R) = 0$ and $F : R^n \longrightarrow R^n$ $(n > 1)$ a Keller map, i.e., a polynomial map with $\det JF = 1$. Then F is an isomophism.*

By "Lefschetz principle" (cf.[9, Lemma 1.1.14]) it is sufficient to consider the case $R = \mathbb{C}$. In [2] Bass, Connell and Wright showed that, to prove the above conjecture, it is enough to prove it for maps in the form $F = X + H$ where H is homogeneous and of degree 3 with JH nilpotent. A refinement, due to Essen-Bondt, ensures that it is in fact sufficient to consider maps in the form $F = X + H$ with $H = (H_1, ..., H_n)$ homogeneous, $deg(H) = 3$ and JH nilpotent and **symmetric**.

Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain and consider a linear map $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ i.e. $F = (F_1, \ldots, F_n)$ with $F_i \in \mathcal{O}[X_1, \ldots, X_n]$ homogeneous of degree 1. Suppose that the matrix $B := JF$ is invertible and let $A \in Mat_n(\mathcal{O})$ be such that $AB = BA = id_n$. The relation $BA = id_n$ implies that there exist $u_1, \ldots, u_n \in \mathcal{O}$ such that $F_1(u_1, \ldots, u_n) = 1$. In particular, by reduction mod $\mathcal{M}$ we have $\overline{F} : k^n \longrightarrow k^n$ a non-zero map. The general case is an open problem, the so called (cf.[10])

Unimodular Conjecture. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $char(\mathcal{O}) = 0$ and $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ $(n > 1)$ a Keller map. Then the induced map $\overline{F} : k^n \longrightarrow k^n$ is a non-zero map.*

The interesting fact is that the Unimodular Conjecture is related to Jacobian Conjecture([10]):

Theorem (Essen-Lipton). *$\mathbb{Z}_p$ satisfies the Unimodular Conjecture for almost all primes p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

The objective of this paper is to give results in direction of the Unimodular Conjecture. Motivated by this conjecture we define the classes of *d-unimodular domains* and *invariant domains*. We propose a new conjecture, the Invariance Conjecture and we show it is equivalent, in some sense, to the Jacobian Conjecture (over $\mathbb{C}$). Furthermore, we give some contributions to the Unimodular Conjecture, in particular the following

Theorem. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $q := \#k < \infty$. Then $\mathcal{O}$ is $(q-1)$-unimodular.*

Theorem. *Let $p \in \mathbb{Z}$ be a prime. For each $d \in \mathbb{Z}_{\geq 1}$ there exists a finite extension $K|\mathbb{Q}_p$ such that the ring of integers $\mathcal{O}_K$ is a d-unimodular domain.*

2. Preliminaries

In this section we establish notations and present the results that will be used in the sequel.

Given a domain R we denote by $\mathcal{MP}_n(R)$ the collection of polynomial maps over R. If $F \in \mathcal{MP}_n(R)$ is a map, we say that F is **Keller** if $\det JF = 1$. We denote by $Aut_n(R)$ the group of polynomial maps which are isomophisms. If $F \in \mathcal{MP}_n(R)$ and $R \subset S$ for some domain S, we can look F as polynomial map over S. We denote this map by $F \otimes S$ (map obtained by scalar extension). The ring of p-adic integers is denoted by $\mathbb{Z}_p$.

We recall some facts:

Proposition 2.1. *Let $F \in \mathcal{MP}_n(R)$ and S a domain with $R \subset S$. Then*

$$F \in Aut_n(S) \Longleftrightarrow F \in Aut_n(R).$$

Proof. see [9, Lemma 1.1.8]

$\square$

Theorem 2.2. *(Cynk-Rusek) Fix an algebraically closed field k with $char(k) = 0$. Let $X \subset \mathbb{A}_k^n$ be an affine variety and $F : X \longrightarrow X$ a regular map. The following conditions are equivalent:*

(i) F is injective.

(ii) F is a bijection.

(iii) F is an automorphism.

Proof. see [5, Theorem 2.2],[9, Theorem 4.2.1] or [7, Theorem 1.6]. $\square$

Immersion Lemma. *Let $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}}$. Then for infinitely of primes $p \in \mathbb{Z}$ there is an injection*

$$\phi_p : \mathbb{Z}[\alpha_1, ..., \alpha_n] \hookrightarrow \mathbb{Z}_p.$$

Proof. see [9, Theorem 10.3.1] $\square$

Hensel Lemma. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring and $F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n) \in \mathcal{O}[X_1, \ldots, X_n]$. Choose $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}^n$ such that*

$$F_1(\alpha_1, \ldots, \alpha_n) \equiv \cdots \equiv F_n(\alpha_1, \ldots, \alpha_n) \equiv 0 \mod \mathcal{M}^{2m+1}$$

where $m := ord_{\mathcal{M}}(\det JF(\alpha)) < \infty$. Then there is a unique $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{O}^n$ such that $F_1(\beta) = \cdots = F_n(\beta) = 0$ and $\beta_i \equiv \alpha_i \mod \mathcal{M}^{m+1}$ for all $i = 1, \ldots, n$.

Proof. see [6, proposition 5.20]. $\square$

By Hensel lemma we get the following

Proposition 2.3. *Let $f_1, \ldots, f_n \in \mathcal{O}[X_1, \ldots, X_n]$ be a Keller map where $(\mathcal{O}, \mathcal{M}, k)$ is a complete discrete valuation ring. If R is an $\mathcal{O}$-algebra denote by $X(R)$ the set of R-points of $Spec(\mathcal{O}[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n \rangle)$. Then there is a bijection $X(\mathcal{O}) \cong X(k)$.*

Proof. As f is Keller we have $m = ord_{\mathcal{M}}(\det JF(\alpha)) = 0$ for all $\alpha \in \mathcal{O}^n$. The bijection is natural: given $P \in \mathcal{O}^n$ define $\varphi(P) \in X(k)$ the k-point obtained by reduction $\mod \mathcal{M}$. Hensel lemma implies that $\varphi : X(R) \longrightarrow X(k)$ is a bijection: injectivity by uniqueness and surjectivity by lifting. $\square$

3. Unimodular Domains

Given $F \in \mathcal{MP}_n(\mathcal{O})$ over a local domain $(\mathcal{O}, \mathcal{M}, k)$ we will denote by $f \in \mathcal{MP}_n(k)$ the induced map over the residue field k.

Definition 3.1. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain. We say that $\mathcal{O}$ is a **unimodular domain** if the Unimodular Conjecture is true for $\mathcal{O}$. We say that a polynomial map $F \in \mathcal{MP}_n(\mathcal{O})$ is **unimodular** if it satisfies the condition in the Unimodular Conjecture.*

Proposition 3.2. [1] *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with k an infinite field. Then $\mathcal{O}$ is unimodular.*

Proof. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a Keller map. Let $f \in \mathcal{MP}_n(k)$ be the induced map over the residue field and suppose that $f(\alpha) = 0$ for all $\alpha \in k^n$. Since k is infinite we have $f \equiv 0$. So the coefficients that occur in F belong to the maximal ideal $\mathcal{M}$. In particular, $\det JF \in \mathcal{M}[X_1, .., X_n]$ a contradiction by Keller condition: $\det JF = 1$.

$\square$

We recall the following proposition (cf.[10, Proposition 8]).

Proposition 3.3. *Suppose that the Jacobian Conjecture over $\mathbb{C}$ is true. Then every local domain $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = 0$ is unimodular.*

Proof. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a Keller map over a local domain $\mathcal{O}$ with $char(\mathcal{O}) = 0$. Since we assume the Jacobian Conjecture is true over $\mathbb{C}$ we have F an invertible map over $\mathcal{O}$ (cf.[9, Lemma 1.1.14]. So, there is a unique $G \in \mathcal{MP}_n(\mathcal{O})$ such that $F \circ G = X$. By reduction $\mod \mathcal{M}$ we see that the map $f \in \mathcal{MP}_n(k)$ is a bijection, in particular, non-zero map. $\square$

We remark that the Unimodular Conjecture is false for local domains with $char(\mathcal{O}) = p > 0$ and residue field finite. For example: consider the local domain $(\mathbb{F}_p[[T]], T\mathbb{F}_p[[T]], \mathbb{F}_p)$ and take the polynomial map $F = (X_1 - X_1^p, \ldots, X_n - X_n^p) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$. Note that F is a Keller map but the induced map over the residue field is the zero map, since $\alpha^p = \alpha$ for all $\alpha \in \mathbb{F}_p$.

Remark 1. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain. The following table shows the complete set of relations between $char(\mathcal{O})$ and $char(k)$.*

$char(\mathcal{O})$	$char(k)$	$\#k$	type
$p = 0$	$q > 0$	∞	unimodular
$p = 0$	$q > 0$	$< \infty$	unknown
$p = 0$	$q = 0$	∞	unimodular
$p > 0$	$q = p$	$< \infty$	non-unimodular
$p > 0$	$q = p$	∞	unimodular

*Thus the interesting case is $(char(\mathcal{O}), char(k), \#k, type) = (0, p, < \infty, unknown)$ where $p > 0$. Indeed, Essen-Lipton theorem ensures that **unknown** = **unimodular** if and only if the Jacobian Conjecture over $\mathbb{C}$ is true.*

4. Invariance Conjecture and Invariant Domains

Invariance Conjecture. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $char(\mathcal{O}) = 0$ and $F \in \mathcal{MP}_n(\mathcal{O})$ a Keller unimodular map. Let $G \in Aut_n(\mathcal{O})$ be an affine Keller automorphism i.e. $G = AX + b$ where $A \in Sl_n(\mathcal{O})$. Then $F \circ G \circ F$ and $F - F(a)$ are unimodular maps for all $a \in \mathcal{O}^n$.*

Remark 2. *Note that in the above conjecture we ask the unimodular property to be invariant under translation and composition of a special type. Note also that, as in the unimodular case, if the residue field k is infinite then the Invariance Conjecture is true for any complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = p \geq 0$.*

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[1]This is more general than [10, Proposition 9].

Definition 4.1. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain. Given a map $F \in \mathcal{MP}_n(\mathcal{O})$ we say that*

- *F is an **invariant map** if F is Keller, unimodular and satisfies the Invariance Conjecture condition.*

- *F is **strongly invariant** if it is invariant and for all Keller affine automorphisms $G_1, \ldots, G_k \in Aut_n(\mathcal{O})$ the map $F_1 \circ F_2 \circ F_3 \circ \cdots \circ F_k$ is invariant where $F_j = G_j \circ F$.*

*The domain $\mathcal{O}$ is called an **invariant domain** if every polynomial map that is Keller and unimodular (in dimension $n > 1$) is invariant.*

Lemma 4.2. *If a map $F \in \mathcal{MP}_n(\mathcal{O})$ is strongly invariant then $F \circ G \circ F$ is strongly invariant for all Keller affine automorphism $G \in Aut_n(\mathcal{O})$.*

Proof. Indeed, by induction it is sufficient to consider the case $k = 2$. For this, let $G_1, G_2 \in Aut_n(\mathcal{O})$ be a Keller affine automorphisms and remark that $G_1 \circ (F \circ G \circ F) \circ G_2 \circ (F \circ G \circ F) = (G_1 \circ F) \circ (G \circ F) \circ (G_2 \circ F) \circ (G \circ F)$. So it is invariant by hypothesis on F. $\qquad\square$

Proposition 4.3. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local unimodular domain. Then $\mathcal{O}$ is invariant.*

Proof. By hypothesis a Keller map $F \in \mathcal{MP}_n(\mathcal{O})$ is unimodular. Since the Keller condition is invariant under composition and translation we have the result. $\qquad\square$

The condition $char(\mathcal{O}) = 0$ is important.

Example 1. *Let $F_1, \ldots, F_n \in \mathbb{F}_p[[T]][X_1, \ldots, X_n]$ be defined by $F_j = 1 - X_j^p + X_j$ and consider the polynomial map $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$. It is easy to check that $\det JF = 1$ and that F is unimodular. But*

$$F - F(1, \ldots, 1) = (-X_1^p + X_1, \ldots, -X_n^p + X_n).$$

So, in case $(\mathcal{O}, \mathcal{M}, k) = (\mathbb{F}_p[[T]], T\mathbb{F}_p[[T]], \mathbb{F}_p)$ it follows that the property of invariance by translation is false.

Example 2. *Let $g(X) \in \mathbb{F}_p[X]$ be a polynomial wich maps $\{0, \ldots, p-2\} \mapsto p-1$ and $p-1 \mapsto 0$. For example, take $p = 5$ and consider*

$$g(X) = -1 + X - X^2 + X^3 - X^4 \in \mathbb{F}_5[X].$$

It is easy to check that $g \circ g = 0$. Note that $g(0) \neq 0$. Define the polynomial map $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(\mathbb{F}_p[[T]])$ with $F_j = X_j - X_j^p + g(X_j^p)$. We have F a Keller map with the induced map over the residue field non-zero. But by construction we have $F \circ F = 0$. Thus, in characteristic $p > 0$ the invariance by composition is false.

In the next theorem the argument is simitar to the argument given in [10, Theorem 4] with the observation that is sufficient to require the invariance property.

Theorem 4.4. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring with finite residue field. Let $F \in \mathcal{MP}_n(\mathcal{O})$ be a strongly invariant map. Then F is injective.*

Proof. Suppose false and let F be a strongly invariant map over $\mathcal{O}$ with $F(a_1) = \cdots = F(a_m) = c$ $(m > 1)$ for some $a_1, \ldots, a_m \in \mathcal{O}^n$ with $a_i \neq a_j$, if $i \neq j$. We will show that there is a strongly invariant map G with $\#G^{-1}(c) > m$. By iteration we will get a Keller map $\widetilde{G} \in \mathcal{MP}_n(\mathcal{O})$ with $\#G^{-1}(c) > (\#k)^n$ a contradiction by proposition 2.3.

Since $F(a_1) = F(a_2)$ we have $\langle a_2 - a_1 \rangle = R$ ([9, Lemma 10.3.11]). On the other hand, since F is an invariant map it is ensured that there exists $b \in \mathcal{O}^n$ such that $F(b) - F(a_1)$ is unimodular, i.e., $\langle F(b) - F(a_1) \rangle = R$. In

particular, $\langle a_2 - a_1 \rangle = \langle F(b) - F(a_1) \rangle = \langle F(b) - c \rangle = R$. So, we have $\{a_2, a_1\} \cong \{F(b), c\}$ (see [10, Transitivity, Proposition 1]). By [10, Theorem 2] we know that there is $H \in \mathcal{MP}_n(\mathcal{O})$, Keller affine automorphism such that $H(c) = a_1$ and $H(F(b)) = a_2$. Now define $G = F \circ H \circ F$. We have G strongly invariant map with $G(a_j) = F(H(c)) = F(a_1) = c$ for all j and $G(b) = F(H(F(b))) = F(a_2) = c$. Note that $b \neq a_j$ for all j. $\qquad \square$

Theorem 4.5. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a complete discrete valuation ring with finite residue field. Suppose that $\mathcal{O}$ is an invariant domain. Then any unimodular Keller polynomial map $F \in \mathcal{MP}_n(\mathcal{O})$ is injective.*

5. SOME RESULTS

Definition 5.1. *Pick $d \in \mathbb{Z}_{\geq 1}$ and let $(\mathcal{O}, \mathcal{M}, k)$ a local domain. We say that $\mathcal{O}$ is a d-unimodular map if any Keller map $F \in \mathcal{MP}_n(\mathcal{O})$ in dimension $n > 1$ with $\deg(F) \leq d$ is unimodular.*

Note that any local domain $\mathcal{O}$ is 1-unimodular and $\mathcal{O}$ is a unimodular domain if and only if it is d-unimodular for all $d \in \mathbb{N}$. If $\mathcal{O}$ is d-unimodular then it is e-unimodular for all $e \leq d$. We will see later that $\mathbb{Z}_p$ is 3-unimodular for any prime $p > 3$. In case $char(\mathcal{O}) = p > 0$ and k finite we have that $\mathcal{O}$ isn't d-unimodular for infinitely many $d \in \mathbb{Z}$. Indeed, for each $m \in \mathbb{N}$ take $d = (\#k)^m$ and consider the map $F = (X_1 - X_1^d, \ldots, X_n - X_n^d) \in \mathcal{MP}_n(\mathcal{O})$.

Proposition 5.2. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ be a non constant polynomial map. Then for almost all primes $p \in \mathbb{Z}$ we have $F \otimes \mathbb{Z}_p$ unimodular map over $\mathbb{Z}_p$.*

Proof. Indeed, suppose $F_1(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n] \setminus \mathbb{Z}$. We can choose $d \in \mathbb{Z}^n$ such that $F_1(d) \neq 0$. Note that $F_1(d) \in \mathbb{Z}_p^*$ for all p such that $p \nmid F_1(d)$. $\qquad \square$

It is known that in order to prove the Jacobian Conjecture it is sufficient to consider polynomial maps of Druzkowski type, i.e., maps in the form $F = X + H$ with $H_j = (\sum_k a_{kj} X_k)^3$ and JH nilpotent (see [9, Theorem 6.3.2]). We call maps of the form $F = X + H$ with $H = \sum_k a_{kj} X_k^3$ quasi-Druzkowski maps.

Proposition 5.3. *For almost all primes p, the Unimodular Conjecture over $\mathbb{Z}_p$ is true for quasi-Druzkowski maps.*

Proof. Let F be a quasi-Druzkowski map with $H = (H_1, \ldots, H_n)$ where $H_j = \sum_k b_{kj} X_k^3$. We will show that there exist $u_1, \ldots, u_n \in \mathbb{Z}_p$, not all null , such that

$$u_1 H_1(X_1, \ldots, X_n) + \cdots + u_n H_n(X_1, \ldots, X_n) = 0.$$

Indeed, for this it is sufficient to find a non trivial solution for the homogeneous system:

$$u_1 b_{11} + u_2 b_{12} + \cdots + u_n b_{1n} = u_1 b_{21} + u_2 b_{22} + \cdots + u_n b_{2n} = \cdots = u_1 b_{n1} + u_2 b_{n2} + \cdots + u_n b_{nn} = 0.$$

Now since JH is nilpotent we have, in particular, $\det(b_{ij}) = 0$ and so there is a non-trivial solution $(u_1, \ldots, u_n) \in \mathbb{Q}_p^n$ for the system above. Without loss of generality we can suppose that $u_1 \in \mathbb{Z}_p^*$ and $u_j \in \mathbb{Z}_p$, if $j > 1$. Now consider $s := u_1 + u_2 p \cdots + u_n p \in \mathbb{Z}_p^*$. Note that, $(1, p, \ldots, p) \in \mathbb{Z}_p^n$ is such that

$$\langle F_1(1, p, \ldots, p), \ldots, F_n(1, p, \ldots, p) \rangle = \mathbb{Z}_p.$$

$\qquad \square$

Remark 3. *The proposition above will be generalized later (see corollary 5.9).*

It was seen in the previous section that there are local domains $(\mathcal{O}, \mathcal{M}, k)$ with $char(\mathcal{O}) = p > 0$ that are not unimodular domains. On the other hand we know that any local domain with infinite residue field is indeed a unimodular domain. In particular, if we consider the map $F = (X_1 - X_1^p, \ldots, X_n - X_n^p)$ over $(\overline{\mathbb{F}_p}[[T]], T\overline{\mathbb{F}_p}[[T]], \overline{\mathbb{F}_p})$ we have $\overline{F}(\alpha) \neq 0$ for some $\alpha \in \overline{\mathbb{F}_p}$ (= algebraically closure of $\mathbb{F}_p$). So, if we take $L =$ the field obtained by adjunction of α to $\mathbb{F}_p$ we see that our F is unimodular over the local domain $(L[[T]], TL[[T]], L)$. For the p-adic case there is an analogue:

Theorem 5.4. *Let $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map. Then there is a complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ that dominates $\mathbb{Z}_p$ such that $F \otimes \mathcal{O}$ is a unimodular map. Furthermore, $\mathcal{O}$ is a free $\mathbb{Z}_p$-module with*

$$rank_{\mathbb{Z}_p}(\mathcal{O}) = [k : \mathbb{F}_p].$$

Proof. In the proof we use a result of algebraic number theory ([8, Proposition 7.50]).

Consider the map $\overline{F} \in \mathcal{MP}_n(\mathbb{F}_p)$ induced over the residue field. By the previous remark we know that there exists $\alpha \in \overline{\mathbb{F}_p}^n$ such that $\overline{F}(\alpha) \neq 0$. By taking the field $k := \mathbb{F}_p(\alpha_1, \ldots, \alpha_n)$ obtained by adjunction we can look at $\overline{F}$ as a polynomial map which is **non zero** over k. Now we recall the following theorem about unramified extensions of a local field L

Theorem. [2] *Let L be a local field with residue field l. There exists a 1-1 correspondence between the following sets*

$$\{\text{finite extensions unramified over } L \} \cong \{\text{finite extensions of } l\}$$

given by $L' \mapsto l'$, where l' is the residue field associated to L'. Furthermore, in this correspondence we have $[L' : L] = [l' : l]$.

Applying the theorem above to $L = \mathbb{Q}_p$ with $l = \mathbb{F}_p$ we see that the extension $k|\mathbb{F}_p$ corresponds to a local field $K|\mathbb{Q}_p$ such that k is the residue field of K. Denote by $(\mathcal{O}, \mathcal{M}, k)$ the ring of integers of K. The ring $\mathcal{O}$ is the integral closure of $\mathbb{Z}_p$ in K and by a general result (cf.[1, proposition 5.17]) we know that $\mathcal{O}$ is a free $\mathbb{Z}_p$-module and $rank_{\mathbb{Z}_p}(\mathcal{O}) = [K : \mathbb{Q}_p] = [k : \mathbb{F}_p]$. So $F \otimes \mathcal{O} \in \mathcal{MP}_n(\mathcal{O})$ is a Keller map with non zero induced map over the residue field. □

The theorem below was gotten in an attempt of the author to show the following

Conjecture. [3] *Denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Let $F : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ be a Keller map such that $F \otimes \mathcal{O}$ is injective. Then F is an isomorphim.*

Lemma 5.5. *Let $K|\mathbb{Q}_p$ be a finite Galois extension with $m := [K : \mathbb{Q}_p] > 1$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. Let $F \in \mathcal{MP}_n(\mathcal{O}_K)$ be a non-injective Keller unimodular map. Then there exists a non-injective Keller unimodular map $G \in \mathcal{MP}_{mn}(\mathbb{Z}_p)$.*

Proof. The same argument of [9, A Galois descent] works. The relevant fact is that $\mathcal{O}_K$ is a free $\mathbb{Z}_p$-module of $rank_{\mathbb{Z}_p}(\mathcal{O}_K) = m$. □

By theorem 4.4 we know that if $\mathbb{Z}_p$ is a unimodular domain then any Keller map over $\mathbb{Z}_p$ is injective. We can show a more general result

Theorem 5.6. *Assume that $\mathbb{Z}_p$ is an invariant domain for some prime p. Then for all Keller unimodular maps $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ and $K|\mathbb{Q}_p$ finite extension we have $F \otimes \mathcal{O}_K$ is an injective map.*

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[2]local field = complete field with respect to a discrete valuation with residue field finite.

[3]see mathoverflow "Polynomial maps over $\mathbb{Z}$" for an incomplete argument due to Jason Starr

Proof. It is sufficient to show that $F \otimes \mathcal{O}$ is an injective map where $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. Indeed, if $\alpha \neq \beta \in \mathcal{O}$ are such that $F(\alpha) = F(\beta)$ consider the ring $R = \mathbb{Z}_p[\alpha, \beta]$ obtained by adjunction and let $K := Frac(R)$. We have $K|\mathbb{Q}_p$ a finite extension such that $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in K$. Note that $\alpha_i, \beta_i \in \mathcal{O}_K$ for all i. Without loss of generality we can suppose that $K|\mathbb{Q}_p$ is a Galois extension. So, $F \otimes \mathcal{O}_K$ is a Keller unimodular map over $\mathcal{O}_K$ that isn't injective. By the lemma above, we obtain $G \in \mathcal{MP}_N(\mathbb{Z}_p)$ a Keller unimodular map that isn't injective. Now since we assume that $\mathbb{Z}_p$ is an invariant domain and G is a strongly invariant map, by theorem 4.4, we know that G is an injective map. A contradiction. $\square$

In the direction of the Unimodular Conjecture we have the interesting result:

Theorem 5.7. *Let $(\mathcal{O}, \mathcal{M}, k)$ be a local domain with $q := \#k < \infty$. Then $\mathcal{O}$ is a $(q-1)$-unimodular domain.*

Note that there are no restrictions about $char(\mathcal{O})$.

Proof. Let $F : \mathcal{O}^n \longrightarrow \mathcal{O}^n$ be a Keller map with $deg(F) \leq q - 1$. Denote by f_j the polynomial in $k[X_1, \ldots, X_n]$ obtained by reduction of F_j mod $\mathcal{M}$. So we obtain a polynomial map $f = (f_1, \ldots, f_n) : k^n \longrightarrow k^n$. By Keller condition we have f_j is a non zero polynomial, for all indices j. We must prove that there exists $\alpha \in k^n$ such that $f(\alpha) \neq 0$.

Passing to algebraic closure consider the algebraic set $X \subset \overline{k}^n$ defined by equations $f_1 = \cdots = f_n = 0$. We affirm that $\dim X = 0$ [4]. Indeed, note that $X = f^{-1}(0)$, where f is the map over $\overline{k}$ defined by tuple $f_1, \ldots, f_n$. The Keller condition implies that $\delta := [\overline{k}(X_1, \ldots, X_n) : \overline{k}(f_1, \ldots, f_n)] < \infty$ (see [9, proposition 1.1.31]) and by [9, theorem 1.1.32] we know $\#f^{-1}(Q) \leq \delta$ for all $Q \in \overline{k}^n$. In particular, $\#X = \#f^{-1}(0) < \infty$. So $\dim X = 0$.

Now we make use of the following (cf.[3, 7, Bézout Inequality])

Bézout Inequality. *Let $X \subset \mathbb{A}_{\overline{k}}^n$ be an affine algebraic set given by the equations $f_1 = \cdots = f_r = 0$. Denote by $X(k)$ the set of k-points. If $\dim X = 0$ then*

$$\#X(k) \leq \#X \leq deg(f_1) \cdots deg(f_r).$$

Applying the theorem above we have

$$\#X(k) \leq deg(f_1) \cdots deg(f_n) \leq deg(F)^n < q^n$$

where we use the hypothesis: $q > deg(F)$. So $S = k^n \setminus X(k) \neq \varnothing$. Let $\alpha \in S$. By definition $f_j(\alpha) \neq 0$ for some index j. In particular, the map $f : k^n \longrightarrow k^n$ isn't zero. So F is a unimodular map. $\square$

Corollary 5.8. *For all prime p, $\mathbb{F}_p[[T]]$ and $\mathbb{Z}_p$ are $(p-1)$-unimodular domains .*

Note that the bound $p - 1$ is "maximal" for $\mathbb{F}_p[[T]]$.

Corollary 5.9. *$\mathbb{Z}_p$ is 3-unimodular domain for all prime $p > 3$. In particular, for almost all primes p the Unimodular Conjecture is true for maps of degree ≤ 3 over $\mathbb{Z}_p$.*

Proof. Since $p > 3$ we have $p - 1 \geq 3$ and so the result follows from theorem above. $\square$

In dimension $n = 2$ and in characteristic 0 the theorem above can be refined. Indeed, we have the following

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[4]Second proof: consider the $\overline{k}$-algebra $R = \overline{k}[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n \rangle$ and take $M \in Spec_m(R)$ a maximal ideal. By local noetherian ring theory we know that $\dim R_m \leq dim_{\overline{k}} T_P X$ for all $P \in X$ where $T_P X := Hom_{\overline{k}}(\mathcal{M}_{X,P}/\mathcal{M}_{X,P}^2, \overline{k})$ is the tangent space. By the Jacobian criterion we have $dim_{\overline{k}} T_P X = n - rank(JF(P)) = n - n = 0$. So we conclude $\dim R_m = 0$. In particular, $\dim R = 0$ so that R is an artinian $\overline{k}$-algebra. In particular, $Spec_m(R)$ is a finite set. By correspondence, we conclude that X is a finite set. So $\dim X = 0$.

Theorem 5.10. *Let $f = (f_1, f_2) \in \mathcal{MP}_2(\mathcal{O})$ be a Keller map over a complete discrete valuation ring $(\mathcal{O}, \mathcal{M}, k)$ with $q := \#k < \infty$ and $char(\mathcal{O}) = 0$. If $deg(f_1) < q^2$ then f is unimodular.*

Remark 4. *This is particular for $char(\mathcal{O}) = 0$. Indeed, consider the map*

$$f = (X_1 - X_1^p, X_2 - X_2^p) \in \mathcal{MP}_2(\mathbb{F}_p[[T]]).$$

f is a Keller map but is not unimodular. Furthermore $deg(f_1) = p < p^2$.

In order to prove the theorem above we use the following result:

Theorem (Yitang Zhang). [5] *Let $f = (f_1, f_2) \in \mathcal{MP}_2(K)$ be a Keller map over an algebraically closed field K with $char(K) = 0$. Then $[K(X, Y) : K(f_1, f_2)] \leq \mathbf{Min}\{deg(f_1), deg(f_2)\}$.*

Proof. (of theorem 5.10) Let $f = (f_1, f_2)$ be a Keller map over $\mathcal{O}$. By proposition 2.3 we have a bijection

$$S_1 := \{(u, v) \in \mathcal{O}^2 \mid f_1(u, v) = f_2(u, v) = 0\} \cong \{(a, b) \in k^2 \mid g_1(a, b) = g_2(a, b) = 0\} =: S_2$$

where g_1 and g_2 are the reductions of $f_1, f_2 \mod \mathcal{M}$.

By [9, theorem 1.1.32] we know that $\#S_1 \leq [K(X, Y) : K(f_1, f_2)]$ where $K = \overline{Frac(\mathcal{O})}$. By Zhang theorem and hypothesis we have $\#S_1 \leq \mathbf{Min}\{deg(f_1), deg(f_2)\} \leq q^2 - 1$. In particular there is $Q \in k^2 \setminus S_2$. So f is a unimodular map.

$\square$

Theorem 5.11. [6] *Let $p \in \mathbb{Z}$ be a prime. For each $d \in \mathbb{Z}_{\geq 1}$ we can find a finite extension $K|\mathbb{Q}_p$ such that the ring of integers $\mathcal{O}_K$ is a d-unimodular domain.*

Proof. Let $d \in \mathbb{N}$. If $d = 1$ take $K = \mathbb{Q}_p$. Suppose that $d > 1$. We know that for any Keller map $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ of degree d we have

$$\#X = deg(X) \leq deg(F)^n = d^n$$

where X is the algebraic set in $\mathbb{A}^n_{\mathbb{F}_p}$ given by reduction of $F \mod p$. Let n be an integer such that $p^n > d$ and fix $\mathbb{F}_{p^n}$ the unique extension of $\mathbb{F}_p$ of degree n in $\overline{\mathbb{F}_p}$. We have seen in the proof of theorem 5.4 that there is a finite extension $K|\mathbb{Q}_p$ such that the residue field of $\mathcal{O}_K$ is $\mathbb{F}_{p^n}$. By construction, for all Keller map $G \in \mathcal{MP}_n(\mathcal{O}_K)$ with $deg(G) \leq d$ we have $\#\{g_1 = \cdots = g_n = 0\} \leq deg(G)^n \leq d^n < (p^n)^n$. So G is a unimodular map and $\mathcal{O}_K$ is a d-invariant domain. $\square$

Proposition 5.12. *Suppose that for all $n \in \mathbb{N}$ and all $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(\mathbb{Z}_p)$ Keller map with $deg(F) < n$, F is unimodular. Then $\mathbb{Z}_p$ is a unimodular domain.*

Proof. Let $F \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map with $n \leq deg(F)$. Let $m \in \mathbb{Z}$ be an integer (to be determined) and consider the map

$$F^{[[m]]} = (F_1, \ldots, F_n, F_1, \ldots, F_n, \ldots, F_1, \ldots, F_n) \in \mathcal{MP}_{mn}(\mathbb{Z}_p)$$

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[5]see: Zhang, Y.: " The Jacobian conjecture and the degree of field extension-Thesis, (1991)".
[6]This proposition motivates the following question:

- Let p be a prime and consider the p-adic ring. Is there a finite extension $K/\mathbb{Q}_p$ such that $\mathcal{O}_K$, the ring of integers, is a unimodular domain?

which consists of m-repetitions of the tuple $F_1, \ldots, F_n$ where in each occurrence of such tuple we introduce n-distinct variables. By construction we have $F^{[[m]]}$ a Keller map and F is a unimodular map if and only if so is $F^{[[m]]}$. We can choose large m such that $deg(F) < mn$. Thus, we get the unimodularity of F. $\quad\square$

Let R be a domain and $f \in R[X_1, \ldots, X_n]$. Define $d(f) :=$ number of monomials in degree > 3 that occur in f. If $F = (F_1, \ldots, F_n) \in \mathcal{MP}_n(R)$ we define $d(F) := \sum_j d(F_j)$.

Proposition 5.13. *Let $p \in \mathbb{Z}_{>3}$ be a prime number and $f \in \mathcal{MP}_n(\mathbb{Z}_p)$ a Keller map. Suppose that*

$$d(f) \leq log(2)^{-1} log(n log(p/3)/log(3)) \qquad (*)$$

where log is the natural logarithm. Then f is unimodular.

Proof. Let $f \in \mathcal{MP}_n(\mathbb{Z}_p)$ be a Keller map. By the reduction theorem (cf. [2, (3.1) Proposition.] we can find invertible maps $G, H \in \mathcal{MP}_{n+m}(\mathbb{Z}_p)$ for some $m \in \mathbb{N}$ such that $g := G \circ f^{[m]} \circ H$ has degree ≤ 3 where $f^{[m]} = (f, X_{n+1}, \ldots, X_{n+m})$. Furthermore, we know that $G(0) = H(0) = 0$. Denote by $X_f(\mathbb{Z}_p)$ and $X_g(\mathbb{Z}_p)$ the set of $\mathbb{Z}_p$-points of f and g respectively. It is easy to check that $\#X_f(\mathbb{Z}_p) = \#X_g(\mathbb{Z}_p)$. Now, since $\mathbb{Z}_p$ is a 3-unimodular domain (corollary 5.9) we have $\#X_g(\mathbb{Z}_p) < 3^{n+m}$. By the proof of reduction theorem we get $m = 2^{d(f)}$. The inequality $(*)$ implies $3^{m+n} \leq p^n$ and so we have f a unimodular map. $\quad\square$

Theorem 5.14. *$\mathbb{Z}_p$ is an invariant domain for almost all prime p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

Proof. The implication $\Longleftarrow$ follows from 3.3. Suppose that the Invariance Conjecture is true over $\mathbb{Z}_p$ for almost all prime p. By a result of Connel-van den Dries (cf.[9, Proposition 1.1.19]) we know that it is sufficient to show the Jacobian Conjecture over $\mathbb{Z}$. So, suppose some Keller map $F \in \mathcal{MP}_n(\mathbb{Z})$ isn't invertible. Since F has coefficients in $\mathbb{Z}$ it follows that F is unimodular over $\mathbb{Z}_p$ for almost all primes p. Also, we know by hypothesis that $F \otimes \overline{\mathbb{Q}}$ isn't injective. By the immersion lemma, F isn't injective over $\mathbb{Z}_p$ for infinitely many primes p. Fix such a prime p such that $\mathbb{Z}_p$ is an invariant domain. So, we obtain $F \otimes \mathbb{Z}_p$ a Keller map non-injective over the invariant complete local domain. A contradiction by theorem 4.4. $\quad\square$

6. A REFINEMENT

Strong Immersion Lemma. *Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$ be algebraic numbers. Then there is a finite set E of rational primes such that for all prime $p \notin E$ we have an injective homomorphism*

$$\mathbb{Z}[\alpha_1, \ldots, \alpha_m] \hookrightarrow \mathcal{O}_{K,p}$$

where $\mathcal{O}_{K,p}$ is the ring of integers of some finite $K|\mathbb{Q}_p$.

Proof. The proof is similar to proof the immersion lemma (2). It is sufficient to prove the following

Fact. *Let $f(T) \in \mathbb{Z}[T] \setminus \mathbb{Z}$ be an irreducible polynomial. Then for almost all prime p there is a finite extension $K|\mathbb{Q}_p$ and $\alpha \in \mathcal{O}_{K,p}$ such that $f(\alpha) = 0$.*

Let d be the discriminant of the polynomial f and $E := \{p \mid p \text{ is prime with } p \mid d\}$. Let $p \in \mathbb{Z} \setminus E$ be a prime and take $\overline{f}(T) \in \mathbb{F}_p[T]$, via reduction $\mod p$. Let $\alpha \in \overline{\mathbb{F}_p}$ be a root of $\overline{f}(T)$ and take $\mathbb{F}_{p^k}$ the definition field of α. Then

$$\overline{f}(\alpha) = 0 \text{ and } \overline{f}'(\alpha) \neq 0 \text{ by condition } p \notin E.$$

Now we recall that there exists a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_{K,p}$ is a complete discrete valuation with residue field $\mathbb{F}_{p^k}$. Since $\mathcal{O}_{K,p}$ is a complete ring we can use the Hensel lemma to conclude that there is some $a \in \mathcal{O}_{K,p}$ such that $f(a) = 0$. $\quad\square$

Theorem 6.1. $\mathbb{Z}_p$ *is an invariant domain for infinitely many primes p if and only if the Jacobian Conjecture (over $\mathbb{C}$) is true.*

Proof. The implication $\Longleftarrow$ is trivial. Suppose that $\mathbb{Z}_p$ is an invariant domain for infinitely many primes p but the Jacobian Conjecture is false. Let $F \in \mathcal{MP}_N(\mathbb{Z})$ be a counterexemple with $\det JF = 1$ (cf.[9, Proposition 1.1.19]). In particular, $F \otimes \overline{\mathbb{Q}}$ isn't injective. Let $\alpha \neq \beta \in \overline{\mathbb{Q}}^N$ be such that $F(\alpha) = F(\beta)$. By the strong immersion lemma we know that $R := \mathbb{Z}[\alpha, \beta] \hookrightarrow \mathcal{O}_{K,p}$ for almost all primes p. Fix a prime p such that $R \hookrightarrow \mathcal{O}_{K,p}$ and such that $\mathbb{Z}_p$ is an invariant domain. So, we obtain $F \otimes \mathcal{O}_{K,p}$ a Keller map, not injective, over the domain $\mathcal{O}_{K,p}$. By lemma 5.5 we know that there exists a Keller map G over $\mathbb{Z}_p$ that isn't injective. A contradiction by theorem 4.5. $\qquad\square$

Theorem 6.2. *There is a finite set of primes E such that for all prime $p \in \mathbb{Z} \setminus E$ we have*

$$\mathbb{Z}_p \text{ is an invariant domain} \Longleftrightarrow \mathbb{Z}_p \text{ is a unimodular domain.}$$

Proof. The implication $\Longleftarrow$ is easy. Suppose $\Longrightarrow$ false. Then for infinitely many primes p we have $\mathbb{Z}_p$ is an invariant non-unimodular domain. Since $\mathbb{Z}_p$ is invariant for infinitely many primes we have that the Jacobian Conjecture is true by theorem 6.1. On the other hand since $\mathbb{Z}_p$ is not unimodular for infinitely many primes we know, by Essen-Lipton theorem, that the Jacobian Conjecture is false. Contradiction. $\qquad\square$

7. Some problems

Here, we list some problems we wish to find a solution:

Problem 7.1. *Recall the reduction theorem: In order to show the Jacobian Conjecture it is sufficient to consider the maps of degree ≤ 3. So, we may ask*

- *Is there an analogue of the reduction theorem for the Unimodular Conjecture?*

Problem 7.2. *Find a prime $p \in \mathbb{Z}$ such that $\mathbb{Z}_p$ is unimodular.*

It was seen that for each integer $d \in \mathbb{Z}$ there exists a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_K$ is d-unimodular. This motivates the following

Problem 7.3. *Given a prime $p \in \mathbb{Z}$ find (or show that this is impossible) a finite extension $K|\mathbb{Q}_p$ such that $\mathcal{O}_K$ is a unimodular domain.*

The following problem is equivalent to the Jacobian Conjecture [7]

Problem 7.4. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ a Keller map. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Then, $F \otimes \mathcal{O}$ is injective.*

We can compare the problem above with the following

Theorem 7.5. *Let $F \in \mathcal{MP}_n(\mathbb{Z})$ be a Keller map. Suppose that $F \otimes \overline{\mathbb{Q}}$ is injective. Then F is an isomophism.*

The proof of this theorem uses Cynk-Rusek theorem.

Problem 7.6. *In theorem 6.2, what can be said about $\#E$?*

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[7]indeed, the theorem of Connell-van den Dries (cf.[9, Proposition 1.1.19]) is more general: Let $P \in Spec(\mathcal{O})$ non-zero and consider $\mathcal{A} := \mathcal{O}_P$ localization over P. If the Jacobian Conjecture over $\mathbb{C}$ is false then there is a Keller map $F \in \mathcal{MP}_n(\mathbb{Z})$, counterexemple, such $F \otimes \mathcal{A}$ is injective. Details can be found in [4]

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