Foliations with small singular set in arbitrary characteristic

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Foliations, Complex Geometry, and Painlevé Equations - IMPA

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Work in progress with

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- João Pedro dos Santos (Université de Montpellier),
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Structure

- **Part I:** Introduction
- **Part II:** Some results

Part I: Introduction

Foliations

k = algebraically closed field of characteristic $p \geq 0$

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- $T_X/T_{\mathcal{F}}$ is torsion-free.

The singular set of $\mathcal{F}$ is given by:

$$\operatorname{sing}(\mathcal{F}) = \{q \in X \mid (T_X/T_{\mathcal{F}})_q \text{ is not a free } \mathcal{O}_{X,q}\text{-module}\}$$

Foliation with rational first integral

Definition

Let X be a smooth projective variety and $\mathcal{F}$ be a foliation on X. We say that $\mathcal{F}$ has a ***rational first integral****, or is* ***algebraically integrable****, if there is a rational map $f\colon X \dashrightarrow Y$, for some variety Y, such that $\mathcal{F}$ is defined by f, that is $\mathcal{F} = \ker(df)$*

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Examples:

- $\mathcal{F}$ is the foliation on $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ given by the first projection

$$\pi\colon \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}} \longrightarrow \mathbb{P}^1_{\mathrm{k}}$$

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- Let $F_0, \dots, F_q \in \mathrm{k}[x_0, \dots, x_n]$ be homogeneous of same degree, without common factors, and consider the foliation on $\mathbb{P}^n_{\mathrm{k}}$ given by the

$$\omega := i_R(dF_0 \wedge \cdots \wedge dF_q)$$

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  $\mathcal{F}$ is algebraically integrable and is given by the rational map:

$$\mathbb{P}^n_{\mathrm{k}} \dashrightarrow \mathbb{P}^q_{\mathrm{k}} \qquad \overline{x} \mapsto (F_0(\overline{x}) : \cdots : F_q(\overline{x}))$$

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Problem

Let $\mathcal{F}$ be a foliation on a variety X. Determine conditions that ensure that $\mathcal{F}$ admits a rational first integral.

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Example: Suppose $\mathrm{k} = \mathbb{C}$. The foliation

$$\mathcal{J}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

has no algebraic solutions[1]

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Theorem (Miyaoka)

[a b] *Let X be a smooth complex projective surface and suppose $K_{\mathcal{F}}$ is not pseudo-effective, that is, there is a ample divisor H such that*

$$K_{\mathcal{F}} \cdot H < 0$$

Then $\mathcal{F}$ has a rational first integral.

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When X is a rational surface and $\mathcal{F}$ is regular it follows that $K_{\mathcal{F}}$ **is not pseudo-effective.**

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Foliations on rationally connected varieties

Conjecture (Touzet)

[a] *Let $\mathcal{F}$ be a regular foliation on a rationally connected variety defined over $\mathbb{C}$. Then, $\mathcal{F}$ is algebraically integrable.*

[a] S. Druel - **Regular foliations on weak Fano manifolds**

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Some results:

Theorem (Druel)

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Theorem (J.P. Figueredo)

[a] *Let X be a rationally connected threefold with $-K_X$* ***nef****, and let $\mathcal{F}$ be a regular foliation on X. Then the foliation $\mathcal{F}$ is algebraically integrable.*

[a] J.P. Figueredo - **Codimension one regular foliations on rationally connected threefolds**

p-closed foliations

Problem

What happens if $\boldsymbol{char}(\mathrm{k}) = p > 0$*?*

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Definition

Let $\mathcal{F}$ be a foliation on a smooth variety over a field k*. We say that $\mathcal{F}$ **is p-closed** if $T_{\mathcal{F}}$ is stable under p-th powers; that is, if $\mathcal{F}$ is defined by a q-form ω, then*

$$i_v\omega = 0 \quad \textit{implies} \quad i_{v^p}\omega = 0.$$

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- A foliation $\mathcal{F}$ that admits a **rational first integral** is p**-closed**
- Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathrm{k}}$ given locally by the 1-form:

$$\omega = ydx + \alpha xdy.$$

Then $\mathcal{F}$ is p-closed if and only $\alpha \in \mathbb{F}_p$.

p-closed foliations

Theorem (p-closed = algebraically integrable)

[a] *A foliation is p-closed if and only if it is algebraically integrable*

[a] Peternell, Miyaoka - **Geometry of Higher Dimensional Algebraic Varieties**

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Theorem (Brunella-Nicolau)

[a] *Assume* $\operatorname{codim} \mathcal{F} = 1$. *Then* $\mathcal{F}$ ***algebraically integrable*** *if and only if it admits* ***infinitely many invariant*** *algebraic hypersurfaces.*

[a] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

Part III: Some results

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[2] Saito - **On a generalization of de Rham lemma**

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Proposition

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ given by a projective 1-form ω. Suppose that $\operatorname{sing}(\mathcal{F})$ *has* ***codimension at least*** 3. *Then* $d\omega = 0$.

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Idea:[2] Apply the de Rham-Saito's division lemma to the integrability condition, and compare the degrees.

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Corollary

*Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_k$. If $\mathrm{sing}(\mathcal{F})$ has **codimension at least** 3 then $p > 0$ and p divides $\deg(N_{\mathcal{F}})$.*

[2] Saito - **On a generalization of de Rham lemma**

Some restrictions

Theorem

[a] *Let $\mathcal{F}$ be a regular foliation on a smooth projective surface X. Then,*

$$N_{\mathcal{F}} \cdot N_{\mathcal{F}} = 0 \quad in \quad \mathrm{k}$$

[a] T. Fassarella, W. Mendson, F. Touzet, J. P. Santos - **Foliations with small singular set in arbitrary characteristic**, work in progress

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[a] Baum-Bott - **On the Zeroes of Meromorphic Vector-Fields**

[b] Brunella - **Birational geometry of foliations**

Regular foliations on surfaces

Remark

There are examples of regular foliations $\mathcal{F}$ in characteristic p with $N_{\mathcal{F}} \cdot N_{\mathcal{F}} \neq 0$.

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Let $\mathcal{F}$ be a foliation on a smooth projective surface X. If $\mathcal{F}$ is regular, then

$$N_{\mathcal{F}} \cdot K_{\mathcal{F}} = -c_2(T_X)$$

Idea: $\mathcal{F}$ is given by a nonzero section $v \in \mathrm{H}^0(X, T_X \otimes K_{\mathcal{F}})$, without zeros. In particular,

$$c_2(T_X \otimes K_{\mathcal{F}}) = 0.$$

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Example

In $\mathbb{P}^2_{\mathrm{k}}$ there are no regular foliations.

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2. $char(\mathrm{k}) = p > 0$, *p divides e and $\mathcal{F}$ is a Riccati foliation on $\mathbb{F}_e$ given by*

$$\omega = dz + \gamma$$

where z intends as coordinate of the fiber and $\gamma \in \mathrm{H}^0(\mathbb{P}^1_{\mathrm{k}}, \Omega^1_{\mathbb{P}^1_{\mathrm{k}}}(e))$.

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Remark: If $e = 1$, then $char(\mathrm{k}) \geq 0$, and $\mathcal{F}$ is the $\mathbb{P}^1_{\mathrm{k}}$-fibration of $\mathbb{F}_1$

Regular foliations on **del Pezzo** surfaces

Theorem

[a] *Let X be a smooth projective **del Pezzo** surface, over an algebraically closed field k of arbitrary characteristic. Let $\mathcal{F}$ be a regular foliation on X. Then X is either $\mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ or $\mathbb{F}_1$ and $\mathcal{F}$ leaves one of the standard $\mathbb{P}^1_{\mathrm{k}}$-fibrations invariant.*

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Let $\mathcal{F}$ be a regular foliation on X.

[3] Dolgachev - **Classical Algebraic Geometry: a modern view**

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Let $\mathcal{F}$ be a regular foliation on X.

- **step 1:**[3] The Mori cone $\overline{\mathrm{NE}}(X)$ can be written as

$$\overline{\mathrm{NE}}(X) = \mathbb{R}^+[C_1] + \cdots + \mathbb{R}^+[C_m]$$

where C_i are rational curves and $\mathbb{R}^+[C_i]$ are extremal rays

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- **step 5:** $K_{\mathcal{F}} \cdot C_i = -C_i^2 + tang(\mathcal{F}, C) \geq 1$, so $K_{\mathcal{F}}$ is nef

[3] Dolgachev - **Classical Algebraic Geometry: a modern view**

Idea

Let $\mathcal{F}$ be a regular foliation on X.

- **step 1:**[3] The Mori cone $\overline{\mathrm{NE}}(X)$ can be written as

$$\overline{\mathrm{NE}}(X) = \mathbb{R}^+[C_1] + \cdots + \mathbb{R}^+[C_m]$$

  where C_i are rational curves and $\mathbb{R}^+[C_i]$ are extremal rays

- **step 2:** $X = \mathbb{P}^1_{\mathrm{k}} \times \mathbb{P}^1_{\mathrm{k}}$ or $\mathbf{Bl}_{p_1,\ldots,p_l}(\mathbb{P}^2_{\mathrm{k}})$ with $1 \leq l \leq 8$ and C_i are smooth rational curves

- **step 3:** $N_{\mathcal{F}}$ is nef

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- **step 6:** $K_{\mathcal{F}}$ and $N_{\mathcal{F}}$ nef implies $K_{\mathcal{F}} \cdot N_{\mathcal{F}} \geq 0$, but this contradicts

$$K_{\mathcal{F}} \cdot N_{\mathcal{F}} = -c_2(X) < 0$$

[3] Dolgachev - **Classical Algebraic Geometry: a modern view**

Regular foliations on projective spaces

Theorem

[a] *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n_{\mathrm{k}}$ of codimension q, $0 < q < n$, and degree d. If $\mathcal{F}$ is regular, then $q = 1$, $char(\mathrm{k}) = 2$, $d = 0$ and n is odd. Furthermore, $\mathcal{F}$ is given by a global exact 1-form $\Omega = dF$, where F is a homogeneous polynomial of degree 2. In particular, $\mathcal{F}$ is 2-closed.*

[a] T. Fassarella, W. Mendson, F. Touzet, J. P. Santos - **Foliations with small singular set in arbitrary characteristic**, work in progress

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[a] T. Fassarella, W. Mendson, F. Touzet, J. P. Santos - **Foliations with small singular set in arbitrary characteristic**, work in progress

Idea: the subbundle $T_{\mathcal{F}} \subset T_{\mathbb{P}^n_{\mathrm{k}}}$ defines a nesting map $f\colon \mathbb{P}^n_{\mathrm{k}} \to \mathbb{G}(q, n)$, $x \mapsto T_x\mathcal{F}$.

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[a] *Let k be an algebraically closed field and $f\colon \mathbb{P}^n_{\mathrm{k}} \to \mathbb{G}(q, n)$ algebraic nesting map. Then n is odd, and $q = n - 1$.*

[a] C. De Concini, Z. Reichstein - **Nesting maps of Grassmannians**

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Corollary

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ of degree d and suppose that $\mathrm{sing}(\mathcal{F})$ *has* ***codimension at least*** 3. *Then $\mathcal{F}$ admits a reduced invariant hypersurface if and only if it admits infinitely many reduced invariant hypersurfaces.*

An interesting example

Example

[a] *Consider the projective* 1*-form*

$$\omega = d\Big(\sum_{i=0}^{n-1} x_i x_{i+1}^{2p-1}\Big) + x_n^p x_{n-1}^{p-1} dx_{n-1} - x_{n-1}^p x_n^{p-1} dx_n.$$

*ω defines a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ of degree $2p-2$ with **finite singular set**:* $\operatorname{sing}(\mathcal{F}) = \{(1:0:\cdots:0)\}$*. Also note that $\mathcal{F}$ has no algebraic invariant hypersurfaces.*

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Remark

[a] *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathrm{k}}$ and suppose that p does not divide* $\deg(N_{\mathcal{F}})$*. Then $\mathcal{F}$ admits a reduced invariant hypersurface.*

[a] W. Mendson, J.V. Pereira - **Codimension one foliations in positive characteristic**

Congratulations Frank =)

Obrigado!