Non-algebraicity of foliations via reduction mod 2

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Structure

- **Part I:** Introduction
- **Part II:** Foliations in characteristic p
- **Part III:** A criterion via reduction **mod** 2

Part I: Introduction

Foliations

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foliation = foliation on $\mathbb{P}^2_K$

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Let $d \in \mathbb{Z}_{>0}$

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Explicitly:

- By Euler exact sequence the 1-form ω can be seen as a projective 1-form:

$$\omega = Adx + Bdy + Cdz$$

on $\mathbb{A}^3_K$ such that $A, B, C \in K[x, y, z]$ are homogeneous of degree $d+1$ and $Ax + By + Cz = 0$ with

$$\mathrm{sing}(\omega) = \mathcal{Z}(A, B, C) = \{p \in \mathbb{P}^2_K \mid A(p) = B(p) = C(p) = 0\}$$

a finite set.

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Definition

*The C is $\mathcal{F}$-**invariant**, or is a **algebraic solution of** $\mathcal{F}$, if there exists a homogeneous 2-form σ in $\mathbb{A}^3_K$ such that*

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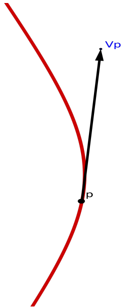

Example: Foliations with algebraic curves

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- **Logarithmic foliations**: let $d_1, d_2, \ldots, d_r \in \mathbb{Z}_{>0}$ and $F_1, \ldots, F_r \in K[x, y, z]$ homogeneous of degree $d_i = \deg(F_i)$. Suppose that $F_1, \ldots, F_r$ are irreducible and coprimes. Let $\alpha_1, \ldots, \alpha_r \in K^*$ be constants such that $\sum_{i=1}^{r} \alpha_i d_i = 0$ and consider the 1-form

$$\Omega = F_1 F_2 \cdots F_{r-1} F_r \sum_{i=1}^{r} \alpha_i \frac{dF_i}{F_i}.$$

  The 1-form Ω define, $\mathcal{F}_\Omega$, a foliation of degree $d = \sum_i d_i - 2$ on $\mathbb{P}^2_K$. The foliation $\mathcal{F}_\Omega$ is called **logarithmic foliation** of type $(d_1, \ldots, d_r)$. The curves $C_i = \{F_i = 0\}$ are $\mathcal{F}_\Omega$-invariant curves.

Jouanolou: foliations without algebraic solutions

Let $d \in \mathbb{Z}_{>1}$ and consider the foliation on $\mathbb{P}^2_K$ given by the 1-form:

$$\mathcal{J}_d \colon \Omega_d = (x^d z - y^{d+1})dx + (xy^d - z^{d+1})dy + (z^d y - x^{d+1})dz$$

$$v_d = z^d \partial_x + x^d \partial_y + y^d \partial_z$$

[1] Zoladek - **New examples of holomorphic foliations without algebraic leaves**

[2] J.V. Pereira, P. F. Sanchez - **Automorphisms and non-integrability**

[3] Claudia R. Alcántara - **Foliations on $\mathbb{CP}^2$ of degree d with a singular point with Milnor number $d^2 + d + 1$**

[4] S. C. Coutinho, Filipe Ramos Ferreira - **Foliations with one singularity and finite isotropy group**

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Theorem (Jouanolou)

[a] *Suppose $K = \mathbb{C}$. The foliation $\mathcal{J}_d$ has no algebraic solutions*

[a] Jouanolou - **Equations de Pfaff algébriques**

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The result implies that in $\mathbb{P}^2_{\mathbb{C}}$, **almost all** foliations have no algebraic invariant curves. However, for a given foliation, it is not easy to determine whether it admits an algebraic solution or not.[1][2][3][4][5]

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Part II: Foliations in characteristic $p > 0$

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and suppose that $p \nmid (d+2)$. Write $d\omega = (d+2)(Ldy \wedge dz - Mdx \wedge dz + Ndx \wedge dy)$ and let v_ω the vector field of degree d associated with $\mathcal{F}$ given by:

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The p-**divisor** is defined by:

$$\Delta_{\mathcal{F}} = \{i_{v_\omega^p}\omega = 0\} \in \mathrm{Div}(\mathbb{P}^2_K).$$

Note that $\Delta_{\mathcal{F}}$ has **degree** $p(d-1) + d + 2$.

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Definition

$\mathcal{F}$ is p***-closed*** *if* $\Delta_{\mathcal{F}} = 0$.

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ω defines a foliation of degree 1 on $\mathbb{P}^2_K$.The associated vector field is

$$v = \left(\frac{2\alpha - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha}{3}\right) y\partial_y + \left(\frac{-1 - \alpha}{3}\right) z\partial_z$$

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By iteration

$$v^p = \left(\frac{2\alpha^p - 1}{3}\right) x\partial_x + \left(\frac{2 - \alpha^p}{3}\right) y\partial_y + \left(\frac{-1 - \alpha^p}{3}\right) z\partial_z$$

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If $\alpha \notin \mathbb{F}_p$:

$$\Delta_{\mathcal{F}} = \{x = 0\} + \{y = 0\} + \{z = 0\}.$$

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Proposition

[a] *Let $\mathcal{F}$ be a non-p-closed foliation on $\mathbb{P}^2_{\mathrm{k}}$ and $C \subset \mathbb{P}^2_{\mathrm{k}}$ an algebraic invariant curve*

- *If C is $\mathcal{F}$-invariant then* $\mathrm{ord}_C(\Delta_{\mathcal{F}}) > 0$*;*

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[a] W.Mendson - **Foliations on smooth algebraic surface in positive characteristic**

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Proposition (J.V. Pereira)

[a] *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$ and suppose that $\deg(\mathcal{F}) < p - 1$. Then, $\mathcal{F}$ has an invariant algebraic curve.*

[a] J. V. Pereira - **Invariant Hypersurfaces for Positive Characteristic Vector Fields**

Example

Let $\mathcal{C}_p$ be the foliation on $\mathbb{P}^2_K$ defined by the 1-form:

$$\omega = zx^{p-1}dx + zy^{p-1}dy - (x^p + y^p)dz.$$

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$$i_{v^p}\omega = \frac{i_{\tilde{v}^p}\omega}{(xy)^p} = \frac{zx^{p-1}\tilde{v}(x) - zy^{p-1}\tilde{v}(y)}{(xy)^p} = \frac{zx^p y^{p^2} - zy^p x^{p^2}}{(xy)^p} = zx^p y^p (y^{p-1} - x^{p-1})^p$$

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The p-divisor of $\mathcal{C}_p$ is given by:

$$\Delta_{\mathcal{C}_p} = \{z = 0\} + p\{y^{p-1} - x^{p-1} = 0\}$$

and thus $\{z = 0\}$ is the **unique algebraic solution** of $\mathcal{F}$.

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Corollary

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Problem

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_K$.

- *What is the structure of the p-divisor?*
- *How many invariant algebraic curves does $\mathcal{F}$ have?*

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Theorem (Brunella-Nicolau)

[a] *A **codimension one foliation** on a smooth projective variety X is p-closed if and only if it has infinitely many invariant algebraic hypersurfaces.*

[a] Brunella, Nicolau - **Sur les hypersurfaces solutions des équations de Pfaff**

Jouanolou

Theorem

[a] *Let K be an algebraically closed field of characteristic $p > 0$. Let $d \in \mathbb{Z}_{>0}$ such that*

- $p < d$ *and* $p \not\equiv 1 \mod 3$*;*
- $d^2 + d + 1$ *is prime.*

Then the Jouanolou foliation $\mathcal{F}_d$ has an irreducible p-divisor or

$$\Delta_{\mathcal{F}_d} = C + pR$$

with $\deg(C) = pl + d + 2$, $l > 0$ *and R is not $\mathcal{F}_d$-invariant.*

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Consequence: The Jouanolou foliation $\mathcal{F}_d$ has a unique invariant algebraic curve.

Reduction **mod** p

Fix $K = \mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree d defined by the projective 1-form:

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1}$$

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$$\omega = yzdx - \alpha xzdy + (\alpha - 1)xydz$$

for some $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. Then the associated algebra is $\mathbb{Z}[\alpha, \alpha^{-1}]$.

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Example: For the Jouanolou foliation, $A, B, C \in \mathbb{Z}[x, y, z]$, so that $\mathbb{Z}[\mathcal{F}_d] = \mathbb{Z}$.

Reduction $\mathbf{mod}\ p$

Fact: For each maximal ideal $\mathfrak{p} \in \mathbf{Spm}(\mathbb{Z}[\mathcal{F}])$, the residue field $\mathbb{F}_{\mathfrak{p}} = \mathbb{Z}[\mathcal{F}]/\mathfrak{p}$ is finite, in particular of characteristic $p > 0$.

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Denote by $\omega_{\mathfrak{p}}$ the 1-form over $\overline{\mathbb{F}}_{\mathfrak{p}}$ obtained by reducing modulo $\mathfrak{p}$ the coefficients appearing in A, B, C. This defines a nonzero element of $\mathrm{H}^0(\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \Omega^1_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}} \otimes \mathcal{O}_{\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}}(d+2))$, and $\omega_{\mathfrak{p}}$ defines a foliation on $\mathbb{P}^2_{\overline{\mathbb{F}}_{\mathfrak{p}}}$:

$$\omega_{\mathfrak{p}} = A dx + B dy + C dz \mod \mathfrak{p}$$

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$$\omega_{\mathfrak{p}} = Adx + Bdy + Cdz \mod \mathfrak{p}$$

Definition

*The foliation defined by $\omega_{\mathfrak{p}}$ is denoted by $\mathcal{F}_{\mathfrak{p}}$ and called the **reduction modulo** p **of** $\mathcal{F}$.*

Reduction **mod** p

Natural question:

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Suppose an abstract property $\boldsymbol{P}$ *holds for* $\mathcal{F}_{\mathfrak{p}}$ *for infinitely many (or almost all) primes* $\mathfrak{p} \in \boldsymbol{Spm}(\mathbb{Z}[\mathcal{F}])$. *What can we say about* $\mathcal{F}$*?*

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When $\mathbb{Z}[\mathcal{F}] = \mathbb{Z}$, the notions **infinitely many primes** and **almost all primes** are the usual ones.

Reduction **mod** p

The property **P** can be:

- the existence of $\mathcal{F}_{\mathfrak{p}}$-invariant curves;

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Proposition

Let $\mathcal{F}$ be a degree d foliation on $\mathbb{P}^2_{\mathbb{C}}$ and suppose that $\mathcal{F}_{\mathfrak{p}}$ has an invariant algebraic curve of degree less than $\boldsymbol{h}$ for almost all primes $\mathfrak{p}$. Then, $\mathcal{F}$ has an invariant algebraic curve of degree less than $\boldsymbol{h}$.

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Idea: The set $S(\mathcal{F}, K, d)$ of degree d foliations on $\mathbb{P}^2_K$ that have algebraic curves of degree $\leq h$ is an algebraic variety over K. In particular, $S(\mathcal{F}, \mathbb{C}, d) \neq \varnothing$ if and only if $S(\mathcal{F}, \overline{\mathbb{F}}_{\mathfrak{p}}, d) \neq \varnothing$ for almost all primes $\mathfrak{p}$.

Part III: A criterion via reduction **mod** 2

Algebraic solutions via reduction **mod** 2

Goal: use reduction modulo p to prove non-algebraicity of holomorphic foliations

[7] W. Mendson - **Arithmetic aspects of the Jouanolou foliation**

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Theorem

[a] *Let $\mathcal{F}$ be a non-dicritcal foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form*

$$\omega = Adx + Bdy + Cdz \qquad A, B, C \in K[x, y, z]$$

where K is a number field. If $\Delta_{\mathcal{F}_2}$ is irreducible then $\mathcal{F}$ has no algebraic solutions.

[a] J. P. Figueredo, W. Mendson - **Non-algebraicity of foliations via reduction modulo 2**

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[a] J. P. Figueredo, W. Mendson - **Non-algebraicity of foliations via reduction modulo 2**

Idea: use the fact that if C is a $\mathcal{F}$-invariant then $C \otimes \mathbb{F}_2$ is not a 2-factor.

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The Jouanolou foliation on $\mathbb{P}^2_{\mathbb{C}}$ of odd degree has no algebraic solutions.

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Idea: The Jouanolou foliation is **non-dicritical** and has good reduction **mod** 2. If $d \equiv 1$ **mod** 2 then its 2-divisor is **irreducible**[7].

[7] W. Mendson - **Arithmetic aspects of the Jouanolou foliation**

Foliations with a unique algebraic invariant curve

Theorem (over $\mathbb{C}$)

[a] *Let $d \in \mathbb{Z}_{>1}$ be an odd integer, and define*

$$f(d) = d^2 + d + 1, \qquad s(d) = d^2 + \frac{d+3}{2}, \qquad h(d) = \frac{d^2+d+2}{2}, \qquad g(d) = \frac{d+1}{2}.$$

Let $\mathcal{F}_d$ be the foliation defined by the 1-form on $D_+(z)$:

$$\mathcal{F}_d : \quad \omega = (x + ay^{g(d)} + by^{h(d)} + cy^{s(d)})\, dx - y^{f(d)}\, dy,$$

*where $a, b, c \in \mathbb{Z}$ are such that $abc \not\equiv 0 \pmod 2$. Then $l_\infty = \{z = 0\}$ is the **only algebraic invariant curve** of $\mathcal{F}_d$.*

[a] J. P. Figueredo, W. Mendson — **Non-algebraicity of foliations via reduction modulo 2**

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Note that

- $f(d) > s(d) > h(d) > g(d)$;
- $\mathcal{F}_d$ has degree $f(d)$ and l_∞ is invariant.

Characteristic 2

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where $a, b, c \in \mathbb{Z}$ satisfy $abc \not\equiv 0 \pmod 2$. Then $\mathcal{F}_d$ is not 2-closed and has 2-divisor given by

$$\Delta_{\mathcal{F}_d} = dl_\infty + (f(d) - 1)\{y = 0\} + C,$$

where C is an irreducible curve of degree $2d^2 + d + 3$.

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Main difficulty: verify the irreduciblity of C.

Examples

- $d = 1$: the foliation has degree 3 and is given by:

$$v = y^3\partial_x + (x + ay + by^2 + cy^3)\partial_y$$

Its 2-divisor is given by

$$\Delta_{\mathcal{F}_1} = \{z = 0\} + 2\{y = 0\} + \{aby^3 + axy + b^2y^4 + by^5 + x^2 + xy^3 + y^4 = 0\}$$

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- $d = 3$: the foliation has degree 13 and is given by:

$$v = y^{13}\partial_x + (x + ay^2 + by^7 + cy^{12})\partial_y$$

The 2-divisor is:

$$\Delta_{\mathcal{F}_3} = 3\{z = 0\} + 12\{y = 0\} + \{a^2y^4 + aby^9 + bxy^7 + by^{19} + x^2 + y^{24} + y^{14} = 0\}$$

$\mathbb{F}_2 \Longrightarrow \mathbb{C}$

- **Step 1:** Suppose, by contradiction, that $\mathcal{F}_d$ has an algebraic curve D of degree e in $D_+(z)$.

[8] Carnicer - **The Poincare problem in the nondicritical case**

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- **Step 1:** Suppose, by contradiction, that $\mathcal{F}_d$ has an algebraic curve D of degree e in $D_+(z)$.
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- **Step 5:** Therefore, we obtain:

$$d+2 \geq \deg(C) \geq \deg(Q) = 2d^2 + d + 3,$$

which leads to $2d^2 \leq -1$, a contradiction.

[8] Carnicer - **The Poincare problem in the nondicritical case**

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- *Understand the* p*-divisor for foliations admitting a unique singularity.*

Obrigado ;-)