Numerical Computations

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Approximate solution of ODEs

Ordinary differential equations are of the form:

$$f(x, y, y', y'', \dots) = 0$$

with some initial conditions

$$y(x_0) = y_0$$
$$y'(x_0) = y'_0$$
$$y''^{(x_0)} = y''_0$$
$$\dots.$$

Augustin-Louis Cauchy
(1789-1857)

Cauchy's Problem

Cauchy's problem

Cauchy's problem for the differential equation of the *n*th order

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

consists in finding the function $y = y(x)$ satisfying this equation and the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_0', \qquad \dots \;, \qquad y^{(n-1)}(x_0) = y_0^{(n-1)}$$

where $x_0, y_0, y_0', \dots, y^{(n-1)}$ are the given input numbers.

Cauchy’s problem (cont’d)

Cauchy's problem for a system of differential equations

$$\left.\begin{array}{l} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n), \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n), \\ \quad \dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n) \end{array}\right\}$$

consists in finding the functions $y_1, y_2, \dots, y_n$ satisfying this system and the initial conditions

$$y_1(x_0) = y_{10}, \qquad y_2(x_0) = y_{20}, \qquad \dots \ , \ y_n(x_0) = y_{n0}$$

Cauchy’s problem (cont’d)

A system containing higher-order derivatives and solved with respect to senior derivatives of the required functions by introducing new unknown functions can be reduced to the previous form. So, we have the following system of equations

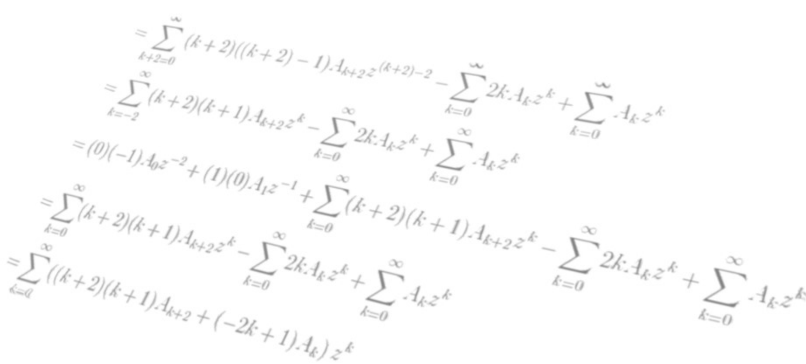
$$\frac{dy}{dx} = y_1, \qquad \frac{dy_1}{dx} = y_2, \qquad \dots \ , \qquad \frac{dy_{n-1}}{dx} = f(x, y_1, y_2, \dots, y_{n-1})$$

It is rather difficult to find the exact solution of Cauchy's problem and it is successfully found only in rare cases; more often we have to solve Cauchy's problem using approximate methods.

Approximate methods for solving ODEs

The approximate methods are divided into two groups:

- Analytic methods yielding the approximate solution of a differential equation in the form of an analytic expression.
- Numerical methods presenting the approximate solution in the form of a table.



Analytic Methods

Integrating Differential Equations with the Aid of Series

Methods for successive differentiation

Suppose the required partial solution $y = y(x)$ can be expanded in a Taylor's series in powers of the difference $x - x_0$:

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

The initial conditions give us directly the values $y^k(x_1)$ for $k = 0, 1, 2, \ldots, n - 1$. The value $y^n(x_0)$ is found from the first equation, substituting $x = x_0$ and using the initial condition:

$$y^{(n)}(x_0) = f(x_0, y_0, y_0', \ldots, y_0^{(n-1)})$$

The values $y^{(n+1)}(x_0)$, $y^{(n+2)}(x_0), \ldots$ are determined successively by differentiating the first equation and substituting $x = x_0$:

$$y^k(x_0) = y_{0k} \; (k = 1,2, \ldots)$$

Methods for successive differentiation (cont'd)

Example: Find the first seven terms of the expansion in a power series of the solution $y = y(x)$ of the equation

$$y'' + 0.1(y')^2 + (1 + 0.1x)y = 0$$

with the initial conditions $y(0) = 1, y'(0) = 2$.

Solution.

A solution of the equation is in the form of the series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots \frac{y^{(n)}(0)}{n!}x^n + \cdots$$

To determine $y''(0)$ solve the given equation with respect to y'':

$$y'' = -0.1(y')^2 - (1 + 0.1x)y$$

Using the initial conditions, we get

$$y'' = -0.1 \times 4 - 1 \times 1 = -1.4$$

Methods for successive differentiation (cont'd)

Solution. (cont'd)

Now differentiate successively both members of the equation with respect to x:

$$y''' = -0.2y'y'' - 0.1(xy' + y) - y',$$
$$y^{(4)} = -0.2(y'y''' + y^{n_2}) - 0.1(xy'' + 2y') - y'',$$
$$y^{(5)} = -0.2\left(y'y^{(4)} + 3y^{n}y'''\right) - 0.1(xy''' + 3y'') - y''',$$
$$y^{(6)} = -0.2(y'y^{(5)} + 4y''y^{(4)} + 3y''') - 0.1\left(xy^{(4)} + 4y'''\right) - y^{(4)}$$

Substituting the initial conditions and the value of $y''(0)$, we find

$$y'''(0) = -1.54, \qquad y^{(4)}(0) = 1.224,$$
$$y^{(5)}(0) = 0.1768, \qquad y^{(6)}(0) = -0.7308.$$

Thus, the required approximate solution is written in the form

$$y(x) \approx 1 + 2x - 0.7x^2 - 0.2567x^3 + 0.051x^4$$
$$+0.00147x^5 - 0.00101x^6$$

Methods for indefinite coefficients

This method is recommended to be used for solving linear differential equations. Assume the second-order equation

$$y'' + p(x)y' + q(x)y = r(x)$$

with the initial conditions $y(0) = y_0,\ y'(0) = y'_0$.

Let us assume that each coefficient of the equation can be expanded in a series in powers of x:

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n \qquad r(x) = \sum_{n=0}^{\infty} r_n x^n$$

We shall look for the solution of the given equation in the form of the series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

c_n are the coefficients to be determined

Methods for indefinite coefficients (cont'd)

Differentiate both members of the equality twice with respect to x:

$$y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substituting the obtained series for y, y', y'', p, q, r into the equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} p_n x^n \times \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} q_n x^n \times \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} r_n x^n$$

Methods for indefinite coefficients (cont'd)

Multiplying the series and equating the coefficients of equal powers of x in the left- and right-hand members of the identity, we get the system

$$\begin{array}{l|c} x^0 & 2c_2 + c_1 p_0 + c_0 q_0 = r_0, \\ x^1 & 3 \times 2c_3 + 2c_2 p_0 + c_1 p_1 + c_1 q_0 + c_0 q_1 = r_1, \\ x^2 & 4 \times 3c_4 + 3c_3 p_0 + 2c_2 p_1 + c_1 p_2 + c_2 q_0 + c_1 q_1 + c_0 q_2 = r_2, \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^n & (n+2)(n+1)c_{n+2} + L(c_{n+1}, c_n, \dots, c_1, c_0) = q_n \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}$$

where $L\,(c_{n+1}, c_n, \dots, c_1, c_0)$ denotes the linear function of the arguments $c_0, c_1 \dots, c_n, c_{n+1}$.

Note: If the initial conditions are given for $x = x_0$, then it is advisable to make the substitution $x - x_0 = t$, thus reducing the problem to the above considered

Methods for indefinite coefficients (cont'd)

Example: Find the solution of the equation

$$y'' + xy' + y = 1 - \cos x\,,$$

satisfying to the initial conditions $y(0) = 0,\ y'(0) = 1.$

Solution. Expand the coefficients of the given equation in power series: $p(x) = -x, \quad q(x) = 1,$

$$r(x) = 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$$

We shall look for the solution of the equation in the form of the series:

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots c_nx^n + \cdots$$
$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots nc_nx^{n-1} + \cdots$$
$$y'' = 2c_2 + 6c_3x + 12c_4x^2 + \cdots + n(n-1)c_nx^{n-2} + \cdots$$

Methods for indefinite coefficients (cont'd)

Solution. (cont'd)

Substituting the obtained series in the equation and equating the coefficients of equal powers of x, we get the system for determining the coefficients c_i:

$$x^0 \,|\, c_0 + 2c_2 = 0,$$
$$x^1 |6c_3 = 0,$$
$$x^2 | -c_2 + 12c_4 = \frac{1}{2},$$
$$x^3 | -2c_3 + 20c_5 = 0,$$
$$x^4 | -3c_4 + 30c_6 = -\frac{1}{24},$$
$$x^5 | -4c_5 + 42c_7 = 0,$$
$$x^6 | -5c_6 + 56c_8 = \frac{1}{720}.$$

From the initial conditions we find

$c_0 = 0\,, c_1 = 1.$

It is easy to note that

$$c_{2x+1} = 0\ (n = 1,2, \dots),\ c_2 = 0, \quad c_4 = \frac{1}{24},\ c_6 = \frac{1}{360}, \quad c_8 = \frac{11}{40320}.$$

We get an approximate solution of the problem in the form

$$y(x) \approx x + \frac{x^4}{24} - \frac{x^6}{360} + \frac{11x^8}{40320}.$$

Methods of successive approximations

Consider Cauchy's problem for the first-order differential equation

$$y' = f(x, y)$$

with the initial condition

$$y(x_0) = y_0.$$

The method of successive approximations consists in that the solution of $y(x)$ is obtained as the limit of a sequence of the functions $y_0(x)$, which are found by the recurrence formula

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(x, y_{n-1}(x)\big)dx.$$

Methods of successive approximations (cont'd)

It is proved that if the right-hand member of $f(x, y)$ in some closed rectangle $R\{|x - x_0| \le a,\ |y - y_0| \le b\}$ satisfies Lipshits' condition with respect to y:

$$|f(x, y_1) - f(x, y_2)| \le N\,|y_1 - y_2|, \qquad N = constant$$

irrespective of the choice of the initial function, the successive approximation of $y_n(x)$ converge on some interval $[x_0, x_0 + h]$ to the solution of Cauchy's problems.

If $f(x, y)$ is continuous in a rectangle R, then the error of the approximate solution of $y_n(x)$ on the interval $[x_0, x_0 + h]$ is estimated by the inequality

$$\varepsilon_n = |y(x) - y_n(x)| \le M N^n \frac{(x - x_0)^{n+1}}{(n+1)!}$$

$M = \max_{(x,y)\epsilon R} |f(x, y)|$

$\mathrm{h} = min(a, \frac{b}{M})$

Methods of successive approximations (cont'd)

Example: For the equation $y' = x + 0.1y^2$

with the initial condition $y(0) = 1$ find the approximate solution on the interval $[0,0.2]$ accurate to 10^{-5}.

Solution.

Choose the initial approximation $y_0(x)$ in the form

$$y_0(x) = y_0 + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2.$$

so

$$y_0 = \mathbf{1},$$
$$y'(0) = 0.1y_0^2 = \mathbf{0.1},$$
$$y''(0) = 1 + 0.2y_0y_0' = \mathbf{1.02}.$$

thus

$$y_0(x) = \quad + \quad x + \quad x^2.$$

Methods of successive approximations (cont'd)

Solution. (cont'd)

We find

$$f(x, y_0) = x + 0.1y_0^2 = 0.1 + 1.02x + 0.103x^2 + 0.0102x^3 + 0.0260x^4$$

and compute the first approximation

$$y_1(x) = 1 + \int_0^x (x + 0.1y_0^2)dx$$
$$= 1 + 0.1x + 0.51x^2 + 0.034x^3 + 0.0025x^4 + 0.0052x^5$$

Consider the difference

$$y_1(x) - y_0(x) = 0.034x^3 + 0.0025x^4 + 0.0052x^5$$

at $x = 0.2$, it has the maximum value

$$\max_{[0,0.2]} |y_1(x) - y_0(x)| = 0.00028 > 10^{-5}$$

Methods of successive approximations (cont'd)

Solution. (cont'd)

The pre-assigned accuracy is not yet achieved. Note that in the expression for y_1 the sum of the last two terms does not exceed 10^{-5}, therefore we may put

$$y_1(x) = 1 + 0.1x + 0.51x^2 + 0.034x^3$$

We find

$$f(x, y_1) = x + 0.1y_0^2$$

$$= 0.1 + 1.02x + 0.103x^2 + 0.0170x^3 + 0.0267x^4 + 0.0017x^5 + 0.0001x^6$$

and compute the second approximation

$$y_2(x) = 1 + \int_0^x (x + 0.1y_0^2)dx$$

$$= 1 + 0.1x + 0.51x^2 + 0.034x^3 + 0.0042x^4 + 0.0053x^5$$

estimate the difference

$$y_2(x) - y_1(x) = 0.0042x^4 + 0.0053x^5 < 10^{-5}$$

$$y(x) \approx 1 + 0.1x + 0.51x^2 + 0.034x^3 + 0.0042x^4 + 0.0053x^5$$

Numerical Methods

Euler's Method

Leonhard Euler
(1707-1783)

Euler's method

Consider differential equation $y' = f(x, y)$
with the initial condition $y(x_0) = y_0$.

Having chosen a sufficiently small interval h, let us construct a system of equally spaced points $x_i = x_0 + ih$ $(i = 0, 1, 2, \dots)$.

In Euler's method approximate value of $y(x_i) \approx y_i$ are computed successively by the formula

$$y_{i+1} = y_i + hf(x_i, y_i) \qquad (i = 0,1,2, \dots)$$

The error estimate

$$\varepsilon_n = |x_n(x) - y_n(x)| \le \frac{hM}{2N}[(1 + hN)^n - 1]$$

Euler's method (cont'd)

Example: Applying Euler's method, form on the interval $[0, 1]$ a table of values of the solution of the equation

$$y' = y - \frac{2x}{y}$$

with the initial condition $y(0) = 1$ for $h = 0.2$.

Solution.

The results of computations are given in a table.

The initial values $x_0 = 0$, $y_0 = 1.0000$ (for $i = 0$) are entered in the first row.

Compute $f(x_0, y_0) = 1$ and $\Delta y_0 = hf(x_0, y_0) = 0.2$.

We get

$$y_1 = 1 + 0.2 = 1.2$$

Euler's method (cont'd)

Solution. (cont'd)

i	x_i	y_i	computing $f(x_i, y_i)$		Δy_i	Exact Solution of $y=\sqrt{2x+1}$
			$\frac{2x_i}{y_i}$	$y_i - \frac{2x_i}{y_i}$		
0	0	1.0000	0	1.0000	0.2000	1.0000

Euler's method (cont'd)

Solution. (cont'd)

The values $x_1 = 0.2,\ y_1 = 1.2000$ are entered in the second row (for $i = 1$). Using them to calculate $f(x_1, y_1) = 0.8667$, and $\Delta y_1 = hf(x_1, y_1) = 0.2 \times 0.8667 = 0.1733$, we can write:

$$y_2 = y_1 + \Delta y_1 = 1.2 + 0.1733 = 1.3733$$

For $i = 2, 3, 4, 5$ the computations are carried out analogously.

i	x_i	y_i	computing $f(x_i, y_i)$		Δy_i	Exact Solution of $y=\sqrt{2x+1}$
			$\frac{2x_i}{y_i}$	$y_i - \frac{2x_i}{y_i}$		
0	0	1.0000	0	1.0000	0.2000	1.0000
1	0.2	1.2000	0.3333	0.8667	0.1733	1.1832
2	0.4	1.3733	0.5928	0.7805	0.1561	1.3416
3	0.6	1.5294	0.7846	7458	0.1492	1.4832
4	0.8	1.6786	0.9532	0.7254	0.1451	1.6124
5	1.0	1.8237				1.7320

Absolute error of y_1 is $\varepsilon = 0.0917$; so, the relative error is 5%.

Leonhard Euler
(1707-1783)

Numerical Methods

Modifications of Euler's Method

Modifications of Euler's method

First improved method:

The intermediate values are computed

$$\begin{cases} x_{i+1/2} = x_i + \dfrac{h}{2}, \\ y_{i+1/2} = y_i + \dfrac{h}{2} f_i, \\ f_{i+1/2} = f(x_{i+1/2}, y_{i+1/2}) \end{cases}$$

Then put

$$y_{i+1} = y_i + h\, f_{i+1/2}$$

Modifications of Euler's method (cont'd)

Second improved method:

Determine the "rough approximation"

$$\widetilde{y_{i+1}} = y_i + h\, f_i$$

Then

$$\widetilde{f_{i+1}} = f\left(\widetilde{x_{i+1}}, \widetilde{y_{i+1}}\right) \qquad y_{i+1} = y_i + h\,\frac{f_i + \widetilde{f_{i+1}}}{2}$$

The remainder terms of Euler's first and second improved methods have the order $O(h^3)$ for each spacing .

The error at the point x_n can be estimated with the aid of double and computation:

$$|y_n^* - y(x_n)| \approx \frac{1}{3}\,|y_n^* - y_n|$$

where $y(x)$ is the exact solution of the differential equation.

Modifications of Euler's method (cont'd)

Example: Solve the equation $y' = y - \dfrac{2x}{y}$

with the initial condition $y(0) = 1$, taking $h = 0.2$.

Solution. (The first improved method)

The table is filled in as follows

Write in the first row $i = 0,\ \ x_0 = 0, y_0 = 1$. Compute $f_0(x_0, y_0) = 1$.

Then by first formula we obtain for $x_{1/2} = 0.1$:

$$y_{1/2} = 1 + 0.1 = 1.1$$

Find $f\left(x_{1/2}, y_{1/2}\right) = 0.9182$

and $\Delta y_0 = hf\left(x_{1/2}, y_{1/2}\right) = 0.2 \times 0.9182 = 0.1836.$

So,

$$y_1 = y_0 + \Delta y_0 = 1 + 0.1836 = 1.1836.$$

Modifications of Euler's method (cont'd)

Solution. (cont'd)

i	x_i	y_i	$\frac{h}{2}f_i$	$x_{i+1/2}$	$y_{i+1/2}$	$\Delta y_i = hf_{i+1/2}$
0	0	1	0.1	0.1	1.1	0.1836
1	0.2	1.2000				

Modifications of Euler's method (cont'd)

Solution. (cont'd)

For $i = 1,2,3,4,5$ the computations are carried out analogously.

i	x_i	y_i	$\frac{h}{2}f_i$	$x_{i+1/2}$	$y_{i+1/2}$	$\Delta y_i = hf_{i+1/2}$
0	0	1	0.1	0.2	1.2	0.1867
1	0.2	1.1836	0.0846	0.3	1.2682	0.1590
2	0.4	1.3426	0.0747	0.5	1.4173	0.1424
3	0.6	1.4850	0.0677	0.7	1.5527	0.1302
4	0.8	1.6152	0.0625	0.9	1.6777	0.1210
5	1.0	1.7362				

Modifications of Euler's method (cont'd)

Solution. (The second improved method)

The table is filled in as follows:

- Write in the first row $i = 0,\ x_0 = 0, y_0 = 1$. Compute $f_0(x_0, y_0) = 1$.
- By second formula compute $\widetilde{y_1} = 1 + 0.2 \times 1 = 1.2$.
- Enter $\frac{h}{2} f_0 = 0.1$ and $x_1 = 0.2$, $\widetilde{y_1} = 1.2$.

i	x_i	y_i	$\frac{h}{2} f_i$	x_{i+1}	$\widetilde{y_{i+1}}$	$\frac{h}{2} \widetilde{f_{i+1}}$	$\Delta y_i = \frac{h}{2}(f_i + \widetilde{f_{i+1}})$
0	0	1	**0.1**	**0.2**	**1.2**		

Modifications of Euler's method (cont'd)

Solution. (cont'd)

Find $\frac{h}{2} f(x_1, \widetilde{y_1}) = 0.1\left(1.2 - \frac{0.4}{1.2}\right) = 0.0867$ and

$$\Delta y_0 = \frac{h}{2} f\left(f_0 + \widetilde{f_1}\right) = 0.1 + 0.0867 = 1.1867,$$

So $y_1 = y_0 + \Delta y_0 = 1 + 0.1867 = 1.1867$.

For $i = 1,2,3,4,5$ the computations are carried out analogously.

i	x_i	y_i	$\frac{h}{2} f_i$	x_{i+1}	$\widetilde{y_{i+1}}$	$\frac{h}{2} \widetilde{f_{i+1}}$	$\Delta y_i = \frac{h}{2}(f_i + \widetilde{f_{i+1}})$
0	0	1	0.1	0.2	1.2	**0.0867**	**1.1867**
1	0.2	1.1836	0.0850	0.4	1.3566	0.0767	0.1617
2	0.4	1.3484	0.0755	0.6	1.4993	0.0699	0.1454
3	0.6	1.4938	0.0690	0.8	1.6180	0.0651	0.1341
4	0.8	1.6272	0.0645	1.0	1.7569	0.0618	0.1263
5	1.0	1.7542					

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Leonhard Euler
(1707-1783)

Joseph-Louis Lagrange
(1736-1813)

Numerical Methods

Euler's Method Completed with an Iterative Process

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Euler's method completed with an iterative process

The Euler-Cauchy method of solving the problem can be made still more accurate by applying an iterative process to each value of y_i. Proceeding from the rough approximation

$$y_{i+1}^{(0)} = y_i + hf(x_i, y_i),$$

Form an iterative process

$$y_{i+1}^{(k)} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k-1)})]$$

Iterations are continued until the corresponding decimal digits of two subsequent approximations $y_{i+1}^{(k)}$, $y_{i+1}^{(k+1)}$ coincide. Then we put

$$y_{i+1} \approx y_{i+1}^{(k+1)}$$

As a rule, for a sufficiently small h iterations converge rapidly. If after three-four iterations the necessary number of decimal digits do not coincide, the spacing h must be decreased.

Euler's method completed with an iterative process

Example: Find the value $y(0.1)$ in solving the equation

$$y' = x + y$$

with the initial condition $y(0) = 1$ (within 10^{-4}).

Solution. Take the interval $h = 0.05$.

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05 \times 1 = 1.05$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}(1 + 1.10) = 1.0525,$$

$$y_1^{(2)} = 1 + \frac{0.05}{2}(1 + 1.1025) = 1.05256,$$

The required accuracy has been reached. Rounding the obtained result to four digits, we get

$$y_1 = 1.05256$$

Euler's method completed with an iterative process

Solution. (cont'd)

- $i = 1$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.0526 + 0.05 \times 1.1026 = 1.1077$$

$$y_2^{(1)} = 1.0526 + 0.025(1.1026 + 1.2077) = 1.11036,$$

$$y_2^{(2)} = 1.0526 + 0.025(1.1026 + 1.2104) = 1.11042,$$

$$y_2 = 1.1104$$

For the sake of comparison, let us compute the exact value of $y(0.1)$ by the solution formula

$$y(0.1) = 2e^{0.1} - 1.1 = 1.1103$$

Carl David Tolmé Runge
(1856-1927)

Martin Wilhelm Kutta
(1867-1944)

Numerical Methods

The Runge-Kutta Method

Runge-Kutta method

Successive values for the function are computed as:

$$\begin{cases} y_{i+1} \approx y_i + \Delta y_i, \\ \Delta y_i = \frac{1}{6}\left(K_1^{(i)} + 2K_2^{(i)} + 2K_3^{(i)} + K_4^{(i)}\right) \end{cases}$$

$$\begin{cases} K_1^{(i)} = hf(x_i, y_i), \\ K_2^{(i)} = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_1^{(i)}}{2}\right), \\ K_3^{(i)} = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_2^{(i)}}{2}\right), \\ K_4^{(i)} = hf\left(x_i + \frac{h}{2}, y_i + K_3^{(i)}\right), \end{cases}$$

It is advisable to arrange all the computations according to the computational scheme shown in the table.

Runge-Kutta method (cont'd)

i	x	y	$k = hf(x,y)$	Δy
0	x_0	y_0	$K_1^{(0)}$	$K_1^{(0)}$
	$x_0 + \frac{h}{2}$	$y_0 + \frac{K_1^{(i)}}{2}$	$K_2^{(i)}$	$2K_2^{(i)}$
	$x_0 + \frac{h}{2}$	$y_0 + \frac{K_2^{(i)}}{2}$	$K_3^{(i)}$	$2K_3^{(i)}$
	$x_0 + h$	$y_0 + K_3^{(i)}$	$K_4^{(i)}$	$K_4^{(i)}$
				Δy_0
1	x_1	y_1		

Note: The interval of computation may be changed when passing from one point to another. To check h for the proper choice it is recommended to compute the fraction:

$$\theta = \left| \frac{K_2^{(i)} - K_3^{(i)}}{K_1^{(i)} - K_2^{(i)}} \right|$$

Runge-Kutta method (cont'd)

Example: Using the Runge-Kutta method find to within $5 \times 10^{-1}$the solution of the differential equation

$$y' = \frac{\sinh(0.5y + x)}{1.5} + 0.5y$$

with the initial condition $y(0) = 0$ on the interval $[0,0.2]$.

Solution. To choose the spacing compute the solution for the point $x = 0.1$ both with $h = 0.1$ and $h = 0.05$. When computing with $h = 0.1$ we have:

$$K_1^{(i)} = 0, \qquad K_2^{(i)} = 0.1 \times \frac{\sinh 0.05}{1.5} = 0.003335,$$

$$K_3^{(i)} = 0.1 \times \frac{\sinh(0.05 \times 0.001667 + 0.05)}{1.5} + 0.5 \times 0.001667 = 0.003475,$$

$$K_4^{(i)} = 0.1 \times \frac{\sinh(0.05 \times 0.003475 + 0.1)}{1.5} + 0.5 \times 0.003475 = 0.006969.$$

Runge-Kutta method (cont'd)

Solution. (cont'd)

Hence $\Delta y_0 = 0.003432$ and $y_1 = y_0 + \Delta y_0 = 0.003432$.

Find the value $y(0.1)$ as the result of computation with $h = 0.05$.

x	y	$0.5y$	$0.5y + x$	$sh(0.5y + x)$	$f(x, y)$	$k = hf(x, y)$	Δy
0	0	0	0	0	0	0	
0.025	0	0	0.025	0.02500	0.01667	0.000834	0.000846
0.025	0.000417	0.000208	0.025208	0.02521	0.01702	0.000851	
0.050	0.000851	0.000426	0.000426	0.05045	0.03406	0.001703	
0.050	0.000846	0.000423	0.000423	0.05044	0.03405	0.001702	
0.075	0.001697	0.000848	0.000848	0.07592	0.05146	0.002573	0.002586
0.075	0.002132	0.001066	0.001066	0.07614	0.05183	0.002592	
0.100	0.003438	0.001719	0.001719	0.10190	0.06965	0.003482	
0.100	0.003432						

Runge-Kutta method (cont'd)

Solution. (cont'd)

The obtained results coincide within the limits of the specified accuracy, go on computing both with $h = 0.1$ and $h = 0.2$.

$$h = 0.1$$

x	y	$0.5y$	$0.5y + x$	$sh(0.5y + x)$	$f(x, y)$	$k = hf(x, y)$	Δy
0.10	0.003432	0.001717	0.101717	0.10190	0.06965	0.006965	
0.15	0.006914	0.003457	0.153457	0.15406	0.10617	0.010617	0.010726
0.15	0.008740	0.004370	0.154370	0.15499	0.10770	0.010770	
0.20	0.014202	0.007101	0.207101	0.20858	0.14615	0.014615	
0.20	0.014158						

Runge-Kutta method (cont'd)

Solution. (cont'd)

$$h = 0.2$$

x	y	$0.5y$	$0.5y + x$	$sh(0.5y + x)$	$f(x, y)$	$k = hf(x, y)$	Δy
0.0	0	0	0	0	0	0	
0.1	0	0	0.1	0.10017	0.06678	0.013356	0.014155
0.1	0.006678	0.003339	0.103339	0.10352	0.07235	0.014470	
0.2	0.014470	0.007235	0.207235	0.20872	0.14638	0.029276	
0.2	0.014155						

The comparison of the results obtained for $h = 0.1$ and $h = 0.2$ shows that we may take $y(0.2) = 0.014158$ accurate to 5×10^{-6}.

For further computations the spacing h should be doubled once again.

John Couch Adams
(1819-1892)

Numerical Methods

Adams' Method (Self-study)

Adam’s method

k	x_k	y_k	$\Delta y_k = y_{k+1} - y_k$	$y'_k = f(x_k, y_k)$	$q_k = hy'_k$	$\Delta q_k = q_{k+1} - q_k$	$\Delta^2 q_k$	$\Delta^3 q_k$
0	x_0	y_0	Δy_0	$f(x_0, y_0)$	q_0	Δq_0	$\Delta^2 q_0$	$\Delta^3 q_0$
1	x_1	y_1	Δy_1	$f(x_1, y_1)$	q_1	Δq_1	$\Delta^2 q_1$	$\Delta^3 q_1$
2	x_2	y_2	Δy_2	$f(x_2, y_2)$	q_2	Δq_2	$\Delta^2 q_2$	$\Delta^3 q_2$
3	x_3	y_3	Δy_3	$f(x_3, y_3)$	q_3	Δq_3	$\Delta^2 q_3$	
4	x_4	y_4	Δy_4	$f(x_4, y_4)$	q_4	Δq_4		
5	x_5	y_5	Δy_5	$f(x_5, y_5)$	q_5			
6	x_6	y_6						

$= y(x_1) = y(x_0 + h)$ $= hy'_1 = hf(x_1, y_1)$

Adams' extrapolation formula denote the "predicted" value computed by this formula by y_{k+1}^{pred}

$$\Delta y_k = q_k + \frac{1}{2}\Delta q_{k-1} + \frac{5}{12}\Delta^2 q_{k-2} + \frac{3}{8}\Delta^3 q_{k-3} \quad (k = 3,4,\dots)$$

Adams' interpolation formula denote the value specified by this formula by $y^{corr}{}_{k+1}$

$$\Delta y_k = q_k + \frac{1}{2}\Delta q_k - \frac{1}{12}\Delta^2 q_{k-1} - \frac{1}{24}\Delta^3 q_{k-2}$$

The formulas are of a very high accuracy, yielding an error of order $O(h^4)$.

Adam’s method (cont’d)

Example: Using Adams' method, find on the interval $[0,0.5]$ the solution of the differential equation

$$y' = \frac{\sinh(0.5y + x)}{1.5} + 0.5y$$

with the initial condition $y(0) = 0$ take $h = 0.05$.

Solution.

In previous example the values of the required function were computed by the Runge-Kutta method for $x_1 = 0.05, x_2 = 0.1$.

Let us take advantage of these results and continue the computations by Adam’s formula.

Adam's method (cont'd)

Solution. (cont'd)

k	x_k	y_k	Δy_k	$q_k = f(x_k, y_k)$	Δq_k	$\Delta^2 q_k$	$\Delta^3 q_k$
0	0	0		0	1702	78	7
1	0.05	0.000846		0.001702	1780	85	
2	0.10	0.003432		0.003482	1865		
3	0.15	0.007838	0.006318	0.005347			
4	0.20	0.014156					

Enter the values $x_0 = 0,\ x_1 = 0.05,\ x_2 = 0.1,\ x_3 = 0.15$ and the corresponding values of $y_k (k = 0, 1, 2, 3)$, find $f(x_k, y_k), q_k$.

For $k = 3$ we have

$$\Delta y_3 = 0.005347 + \frac{1}{2} \times 0.001865 + \frac{5}{12} 0.000085 + \frac{3}{8} 0.000007 = 0.006318$$

Compute $y_4 = 0.007838 + 0.006318 = 0.014156$

Adam's method (cont'd)

Solution. (cont'd)

Entering the values x_4, y_4 in the table, we find

$$y_4' = f(x_4, y_4) = \frac{2}{3}\, sinh\, (0.5 \times 0.014156 + 0.2) + 0.5 \times 0.014156 = 0.14612$$

k	x_k	y_k	$0.5y_k$	$0.5y_k + x_k$	$sinh\,(0.5y_k + x_k)$	$f(x_k + y_k)$
4	0.20	0.014156	0.007078	0.207078	0.20856	0.14612

So $q_4 = hy_4' = 0.007306$

Write down the result obtained and compute the differences $\Delta q_3,\ \Delta^2 q_2,\ \Delta^3 q_1$.

k	x_k	y_k	Δy_k	$q_k = f(x_k, y_k)$	Δq_k	$\Delta^2 q_k$	$\Delta^3 q_k$
0	0	0		0	1702	78	7
1	0.05	0.000846		0.001702	1780	85	9
2	0.10	0.003432		0.003482	1865	94	
3	0.15	0.007838	0.6318	0.005347			
4	0.20	0.014156		0.007306	1959		

Adam’s method (cont’d)

Solution. (cont’d)

Compute the corrected value

$$\Delta y_3 = 0.005347 + \frac{1}{2}0.001959 - \frac{1}{12}0.000094 - \frac{1}{24}0.000009 = 0.006318$$

Since the corrected value of Δy_3 coincides with its predicted value, we continue to carry out the computation with the chosen spacing and without resorting to a further correction.

k	x_k	y_k	$0.5y_k$	$0.5y_k + x_k$	$sinh\,(0.5y_k + x_k)$	$f(x_k + y_k)$
4	0.20	0.014156	0.007078	0.207078	0.20856	0.14612
5	0.25	0.022485	0.011242	0.261242	0.26422	0.18739
6	0.30	0.032936	0.016468	0.316468	0.32178	0.23099
7	0.35	0.045628	0.022814	0.378814	0.38151	0.27715
8	0.40	0.060698	0.030349	0.430349	0.44376	0.32619
9	0.45	0.078301	0.039150	0.489150	0.50889	0.37841

Adam’s method (cont’d)

Solution. (cont’d)

k	x_k	y_k	Δy_k	$q_k = f(x_k, y_k)$	Δq_k	$\Delta^2 q_k$	$\Delta^3 q_k$
0	0	0		0	1702	78	7
1	0.05	0.000846		0.001702	1780	85	9
2	0.10	0.003432		0.003482	1865	94	11
3	0.15	0.007838	06318	0.005347	1959	105	11
4	0.20	0.014156	08329	0.007306	2064	116	13
5	0.25	0.022485	10451	0.009370	2180	129	13
6	0.30	0.032936	12692	0.011550	2309	142	17
7	0.35	0.045628	15070	0.013859	2451	159	
8	0.40	0.060698	17603	0.016310	2610		
9	0.45	0.078301	20295	0.018920			
10	0.50	0.098596					

Adam’s method (cont’d)

We can write Adam’s formulas in terms of derivatives of y as

$$y_{i+1}^{pred} = y_i + \frac{h}{24}(55y_i' - 59y_{i-1}' + 37y_{i-2}' - 9y_{i-3}')$$

Using y_{i+1}^{pred} find $y_{i+1}' = f(x_i, y_{i+1}^{pred})$ and carry out correction by Adams's second formula

$$y_{i+1}^{corr} = y_i + \frac{h}{24}(9y_{i+1}' + 19y_i' - 5y_{i-1}' + y_{i-2}')$$

Edward Arthur Milne
(1896-1950)

Numerical Methods

Milne’s Method (self-study)

Milne's method

For prediction Milne's first formula is used

$$y_i^{pred} = y_{i-4} + \frac{4h}{3}(2y'_{i-3} - y'_{i-2} + 2y'_{i-1})$$

Using y_i^{pred} find $y'_i = f(\mathrm{x}_i, y_i^{pred})$and carry out correction by Milne's second formula

$$y_i^{corr} = y_{i-2} + \frac{h}{3}(2y'_{i-2} + 4y'_{i-1} + y'_i)$$

The absolute error ε_t of the more correct value y_i^{corr} is approximately determined by the formula

$$\varepsilon_i \approx \frac{1}{29}\left|y_i^{corr} - y_i^{pred}\right|.$$

If we have to find the required solution accurate to ε, and it turns out that $\varepsilon_i \leq \varepsilon$ then we may put $y_i \approx y_i^{corr}$ and pass to computing y_{i+1}. Otherwise the spacing h should be reduced.

Milne's method (cont'd)

Example: Using Milne's method, find to within 3×10^{-4} on the interval $[0,1]$ the solution of the equation $xy'' + y' + xy = 0$ with the initial condition $y(0) = 1,\ y'(0) = 0.$

Solution.

Transform the equation into a system. In this case it is favorable to use the substitution $xy' = z$. We get the system

$$\begin{cases} y' = \dfrac{z}{x}, \\ z' = -xy \end{cases}$$

with the initial conditions $y(0) = 1,\ z(0) = 0.$

Milne's method (cont'd)

Solution. (cont'd)

Take $h = 0.2$. To get the initial interval make use of the answer to the Problem of Euler's Method Complete With An Iterative Process.

Take $y(x)$ in the form of the interval of this series retaining only 3 terms:

$$y(x) \approx 1 - \frac{x^2}{4} + \frac{x^4}{64} \qquad z'(x) \approx -x + \frac{x^3}{4} - \frac{x^5}{64}$$

at the point $x = 0.6$, we get

$$z(x) \approx -\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{384}$$

Milne's method (cont'd)

Solution. (cont'd)

- Form a table of four-digit values of $y(x)$ and $z(x)$ for $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$.
- Compute $y_i' = \frac{z_i}{x_i}$, $z_i' = -x_i y_i$ $(i = 0, 1, 2, 3)$.
- Compute the predicted "differences y" and "differences z"

$$y_4^{pred} - y_0 = \frac{4h}{3}(2y_1' - y_2' + 2y_3') = -0.1537$$

$$z_4^{pred} - z_0 = \frac{4h}{3}(2z_1' - z_2' + 2z_3') = -0.2950$$

i	x_i	y_i	z_i	y_i'	z_i'	differences y	differences z	
0	0	1	0	0	0			
1	0.2	0.9900	-0.01990	-0.0995	-0.1980			
2	0.4	0.9604	-0.07841	-0.1960	-0.3842			
3	0.6	0.9120	-0.17202	-0.2867	-0.5472			
						-0.1537	-0.2950	Prediction

Milne's method (cont'd)

Solution. (cont'd)

- Compute predicted values of the required functions at point $x_4 = 0.8$:

$$y_4^{pred} = 1 - 0.1537 = 0.8463, \qquad z_4^{pred} = 0 - 0.2950 = -0.2950.$$

- Compute corrected "differences y" and "differences z":

$$y_4^{corr} - y_2 = \frac{h}{3}(y_2' - 4y_3' + y_4') = -0.1141$$

$$z_4^{corr} - z_2 = \frac{h}{3}(z_2' - 4z_3' + z_4') = -0.2167$$

- Compute corrected values of the required functions at point $x_4 = 0.8$:

$$y_4^{corr} = 0.9604 - 0.1141 = 0.8463, \qquad z_4^{corr} = -0.0784 - 0.2167 = -0.2951.$$

i	x_i	y_i	z_i	y_i'	z_i'	differences y	differences z	
0	0	1	0	0	0			
1	0.2	0.9900	-0.01990	-0.0995	-0.1980			
2	0.4	0.9604	-0.07841	-0.1960	-0.3842			
3	0.6	0.9120	-0.17202	-0.2867	-0.5472			
4	0.8	0.8463	-0.2950	-0.3688	-0.6770	-0.1537	-0.2950	Prediction
		0.8463	-0.2951	-0.3689	-0.6770	-0.1141	-0.2167	Correction

Milne's method (cont'd)

Solution. (cont'd)

Since the difference between the predicted and corrected values does not exceed 10^{-4}, we put

$$y_4 = 0.8463, \qquad z_4 = -0.2951$$

and proceed with the calculations for $i = 5$.

i	x_i	y_i	z_i	y_i'	z_i'	differences y	differences z	
0	0	1	0	0	0			
1	0.2	0.9900	-0.01990	-0.0995	-0.1980			
2	0.4	0.9604	-0.07841	-0.1960	-0.3842			
3	0.6	0.9120	-0.17202	-0.2867	-0.5472			
4	0.8	0.8463	-0.2950	-0.3688	-0.6770	-0.1537	-0.2950	Prediction
		0.8463	-0.2951	-0.3689	-0.6770	-0.1141	-0.2167	Correction
5	1.0	0.7652	-0.4400	-0.4400	-0.7652			
		0.7652	-0.4400					

Aleksei Krylov
(1863-1945)

Numerical Methods

The Method of Krylov for Finding the "Initial Interval" (Self-study)

Krylov's method

Adams' and Milne's methods call for the existence of four initial values of the required function. If the right-hand member of the equation

$$y' = f(x, y)$$

is given analytically →

- The initial values can be found by any of the above treated methods
  - the method of successive approximations,
  - the method of power series, or
  - the Runge-Kutta method.

is specified in a tabular form →

- To construct the "initial interval" it is very convenient to apply the method of successive approach, suggested by Krylov and modified by Milne.

Krylov's method (cont'd)

The method under consideration is based on iterative processing of points with the aid of the formulas of Euler, Adams, and Milne.

First approach. By Euler's formula we put

$$\Delta y_0^{(1)} = q_0 = hf(x_0, y_0), \qquad \Delta y_{-1}^{(1)} = q_0$$

$$y_1^{(1)} = y_0 + \Delta y_0^{(1)}, \qquad y_{-1}^{(1)} = y_0 - \Delta y_{-1}^{(1)}$$

$$q_1^{(1)} = hf\left(x_1, y_1^{(1)}\right), \qquad q_{-1}^{(1)} = hf(x, y_{-1}^{(1)})$$

$$\Delta q_0^{(1)} = q_1^{(1)} - q_0, \qquad \Delta q_{-1}^{(1)} = q_0 - q_1^{(1)}, \qquad \Delta^2 q_{-1}^{(1)} = \Delta q_0^{(1)} - \Delta q_{-1}^{(1)}$$

Approch No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
I	-1	x_{-1}						
	0	x_0	y_0		q_0			
	1	x_1						

Krylov's method (cont'd)

Second approach. Specify the values of Δy_0 and Δy_{-1} by Adam's interpolation formula and Milne's correction formula, rejecting all the differences of order higher than the second

$$\Delta y_0^{(2)} = q_0 + \frac{1}{2}\Delta q_0^{(1)} - \frac{1}{12}\Delta^2 q_{-1}^{(1)} \qquad \Delta y_{-1}^{(2)} = q_0 - \frac{1}{2}\Delta q_{-1}^{(1)} - \frac{1}{12}\Delta^2 q_{-1}^{(1)}$$

$$\Delta y_{-1}^{(2)} + \Delta y_0^{(2)} = y_1 - y_{-1} = 2q_0 + \frac{1}{3}\Delta^2 q_{-1}^{(1)} = \frac{1}{3}(q_{-1}^{(1)} + 4q_0 + q_1^{(1)})$$

predict the value of$\Delta y_1^{(2)}$ by Adams' extrapolation formula

$$\Delta y_1^{(2)} = q_1^{(1)} + \frac{1}{2}\Delta q_0^{(1)} + \frac{5}{12}\Delta^2 q_{-1}^{(1)}$$

$$y_{-1}^{(2)} = y_0 - \Delta y_{-1}^{(2)}, \qquad y_1^{(2)} = y_0 + \Delta y_0^{(2)} \qquad y_2^{(2)} = y_1^{(2)} + \Delta y_1^{(2)}$$

$$q_{-1}^{(2)} = hf\left(x, y_{-1}^{(2)}\right), \qquad q_1^{(2)} = hf\left(x_1, y_1^{(2)}\right), \qquad q_2^{(2)} = hf\left(x_2, y_2^{(2)}\right).$$

Approch No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
II	-1	x_{-1}				$\Delta q_{-1}^{(2)}$	$\Delta^2 q_{-1}^{(1)}$	$\Delta^3 q_{-1}^{(2)}$
	0	x_0	y_0		q_0	$\Delta q_0^{(2)}$	$\Delta^2 q_0^{(1)}$	
	1	x_1				$\Delta q_1^{(2)}$		
	2	x_2						

Krylov’s method (cont’d)

Third approach. Now we have a sufficient number of points to continue the computation, but it is necessary to specify the found points by the complete Adam’s formulas.

$$\Delta y_0^{(3)} = q_0 + \frac{1}{2}\Delta q_0^{(2)} - \frac{1}{12}\Delta^2 q_{-1}^{(2)} - \frac{1}{24}\Delta^3 q_{-1}^{(2)}$$

$$\Delta y_1^{(3)} = q_1^{(2)} + \frac{1}{2}\Delta q_1^{(2)} - \frac{1}{12}\Delta^2 q_0^{(2)} - \frac{1}{24}\Delta^3 q_{-1}^{(2)}$$

$$\Delta y_2^{(3)} = q_2^{(2)} + \frac{1}{2}\Delta q_2^{(2)} + \frac{5}{12}\Delta^2 q_0^{(2)} + \frac{3}{8}\Delta^3 q_{-1}^{(2)}$$

$$y_1^{(3)} = y_0 + \Delta y_0^{(3)} \qquad y_2^{(2)} = y_1^{(3)} + \Delta y_1^{(3)} \qquad y_3^{(3)} = y_2^{(3)} + \Delta y_2^{(3)}$$

Approch No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
III	0	x_0	y_0		q_0	$\Delta q_0^{(2)}$	$\Delta^2 q_0^{(1)}$	$\Delta^3 q_0^{(2)}$
	1	x_1			$q_1^{(2)}$	$\Delta q_1^{(2)}$	$\Delta^2 q_1^{(1)}$	
	2	x_2			$q_2^{(2)}$	$\Delta q_2^{(2)}$		
	3	x_3			$q_3^{(2)}$			

Krylov’s method (cont’d)

Example: Find the numerical solution of the equation

$$y' = 2x + y$$

with the initial condition $y(0) = 0.1$, taking $h = 0.1$.

Solution.

Let us determine the numerical solution of the given equation by Adams' method, using Krylov's method for computing the "initial interval".

Krylov's method (cont'd)

Solution. (cont'd)

First approach:

Write down in section I, $x_{-1} = -0.1$, $x_0 = 0, x_1 = 0.1$, and compute

$$\Delta y_{-1}^{(1)} = \Delta y_0^{(1)} = 0.1 \times 0.1 = \mathbf{0.0100}$$

$$y_1^{(1)} = y_0 + \Delta y_0^{(1)} = 0.1 + 0.0100 = \mathbf{0.1100}\,,$$

$$y_{-1}^{(1)} = 0.1 - 0.0100 = \mathbf{0.0900},$$

$$q_1^{(1)} = 0.1\left(2x_1, y_1^{(1)}\right) = 0.1(0.2 + 0.110) = \mathbf{0.0310},$$

$$q_{-1}^{(1)} = 0.1(-0.2 + 0.90) = \mathbf{-0.0110},$$

Approch No.	*i*	*x*	*y*	Δy	*q*	Δq	$\Delta^2 q$	$\Delta^3 q$
I	-1	-0.1				0.0210	0.0000	
	0	0.0	0.1000	0.0100	0.0100	0.0210		
	1	0.1						

Krylov's method (cont'd)

Solution. (cont'd)

Second approach:

$$\Delta y_{-1}^{(2)} = 0.0100 - 0.5 \times 0.0210 = -\mathbf{0.0005},$$

$$\Delta y_0^{(2)} = 0.0100 + 0.5 \times 0.0210 = \mathbf{0.0205},$$

$$\Delta y_1^{(2)} = 0.0310 + 0.5 \times 0.0210 = \mathbf{0.0425}$$

$$y_{-1}^{(2)} = 0.1000 + 0.0005 = \mathbf{0.1005},$$

$$y_1^{(2)} = 0.1000 + 0.0205 = \mathbf{0.1205},$$

$$y_2^{(2)} = 0.1205 + 0.0425 = \mathbf{0.1630}.$$

$$q_{-1}^{(2)} = 0.1(-0.2 + 0.1005) = \mathbf{-0.0100},$$

$$q_1^{(2)} = 0.1(0.2 + 0.1205) = \mathbf{0.0320},$$

$$q_2^{(2)} = 0.1(0.4 + 0.1630) = \mathbf{0.0563}.$$

Approch No.	*i*	*x*	*y*	Δy	*q*	Δq	$\Delta^2 q$	$\Delta^3 q$
II	-1	-0.1				0.0200	0.0020	0.0003
	0	0.0	0.1000		0.0100	0.0220	0.0023	
	1	0.1				0.0243		
	2	0.2						

Krylov's method (cont'd)

Solution. (cont'd)

Third approach:

$$\Delta y_0^{(3)} = 0.0100 + \frac{1}{2}0.0220 - \frac{1}{12}0.0020 = \mathbf{0.0208},$$

$$\Delta y_1^{(3)} = 0.0320 + \frac{1}{2}0.0243 - \frac{1}{12}0.0023 = \mathbf{0.0440},$$

$$\Delta y_2^{(3)} = 0.0563 + \frac{1}{2}0.0243 + \frac{5}{12}0.0023 + \frac{3}{8}0.0003 = \mathbf{0.0695}$$

$$y_1^{(3)} = 0.1000 + 0.0208 = \mathbf{0.1208},$$

$$y_2^{(2)} = 0.1208 + 0.0440 = \mathbf{0.1648},$$

$$y_3^{(3)} = 0.1648 + 0.0695 = \mathbf{0.2343}.$$

Approch No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
III	0	0	0.100		0.0100	0.0221	0.0023	0.0002
	1	0.1			0.0321	0.0	0.0025	
	2	0.2			0.0565	244		
	3	0.3			0.0834	0.0269		

Krylov's method (cont'd)

Solution. (cont'd)

Continuing the computation:

This is the end of computing the "initial interval". The further computations are carried out by Adams' method. First specify y_3:

$$\Delta y_2^{corr} = 0.0565 + \frac{1}{2}0.0269 - \frac{1}{12}0.0025 = 0.0697,$$

$$y_3^{corr} = 0.1648 + 0.0697 = 0.2345,$$

$$\Delta y_3 = 0.0834 + \frac{1}{2}0.0269 + \frac{5}{12}0.0025 + \frac{3}{8}0.0002 = 0.0980,$$

$$y_4 = 0.2343 + 0.0980 = \mathbf{0.3323},$$

Approch No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
IV	4	0.4		0.1295	0.1132	0.0330	0.0034	
	5	0.5	0.4620	0.1641	0.1462	0.036		
	6	0.6	0.6261		0.1826	4		

ANY QUESTIONS?