Numerical Computations

Hamid Sarbazi-Azad &
Samira Hossein Ghorban
Department of Computer Engineering
Sharif University of Technology (SUT)
Tehran, Iran

Chapter's Topics

- Newton's Method for a System of Two Equations
- Solution by inverse matrices (Cramer's rule)
- The Method of Simple Iteration for a System of Two Equations
- Newton's Method Spread to Systems of n Equations in n Unknowns
- Applying the Method of Iteration to Systems of Iteration to Systems of n Equations in n Unknowns

Overview

One of solving a system of nonlinear equations is to approximate the nonlinear system by a system of linear equations.

The principle tool in Chapter2+ was Newton's method.

Newton's method, as modified for systems of equation, is quite costly to apply as same as in equations with one variable. So we try to modify Secant method in this context.

Newton's Method

Isaac Newton
(1643-1727)

Newton's Method for a System of Two Equations

Suppose we are given a system

$$\begin{cases} f(x_1, x_2) = 0 \\ g(x_1, x_2) = 0 \end{cases}$$

We solve the above system by the method of successive approximations. Suppose we have found the p-th approximation $\left(x_1^{(p)}, x_2^{(p)}\right)$ of one of the isolated roots (α, β). Then the exact root of this system can be presented as

$$(\alpha, \beta) = \left(x_1^{(p)} + e_1^{(p)}, x_2^{(p)} + e_2^{(p)}\right)$$

where $e^{(p)} = \left(e_1^{(p)}, e_2^{(p)}\right)$ is error of the root in the p-th iteration.

Newton's Method for a System of Two Equations

By Taylor theorem:

$$0 = f(\alpha, \beta) = f\left(x_1^{(p)} + e_1^{(p)}, x_2^{(p)} + e_2^{(p)}\right)$$

$$\approx f\left(x_1^{(p)}, x_2^{(p)}\right) + e_1^{(p)} \frac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x} + e_2^{(p)} \frac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial y}$$

Similarly,

$$0 \approx g\left(x_1^{(p)}, x_2^{(p)}\right) + e_1^{(p)} \frac{\partial g\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_1} + e_2^{(p)} \frac{\partial g\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_2}.$$

Consequently,

$$\begin{cases} -f\left(x_1^{(p)}, x_2^{(p)}\right) = e_1^{(p)} \dfrac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_1} + e_2^{(p)} \dfrac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_2} \\ -g\left(x_1^{(p)}, x_2^{(p)}\right) = e_1^{(p)} \dfrac{\partial g\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_1} + e_2^{(p)} \dfrac{\partial g\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_2} \end{cases}$$

Newton's Method for a System of Two Equations

The coefficient matrix of the system equation,

$$J(x_1^{(p)}, x_2^{(p)}) = \begin{bmatrix} \dfrac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial f\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_2} \\ \dfrac{\partial g\left(x^{(p)}, y^{(p)}\right)}{\partial x_1} & \dfrac{\partial g\left(x_1^{(p)}, x_2^{(p)}\right)}{\partial x_2} \end{bmatrix}$$

Is called *Jacobi matrix*. Thus, we obtain the following linear system:

$$-\begin{bmatrix} f\left(x_1^{(p)}, x_2^{(p)}\right) \\ g\left(x_1^{(p)}, x_2^{(p)}\right) \end{bmatrix} = J(x_1^{(p)}, x_2^{(p)}) \begin{bmatrix} e_1^{(p)} \\ e_2^{(p)} \end{bmatrix}$$

Review: Solution by inverse matrices (Cramer's rule)

If matrix A is non-singular, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0$$

then, the system has a unique solution, i.e.,

$$x = A^{-1}\, b.$$

Cramer's rule: The jth component of $x = A^{-1}\, b$ is the ratio

$$x_j = \frac{\det B_j}{\det A}$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & \boldsymbol{b_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \boldsymbol{b_2} & \cdots & a_{2n} \\ \dots & \dots & \cdots & \vdots & \cdots & \dots \\ a_{n1} & a_{n2} & \cdots & \boldsymbol{b_n} & \cdots & a_{nn} \end{bmatrix}$$

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Newton's Method for a System of Two Equations

The coefficient matrix of the system equation,

$$J(x_1^{(p)},x_2^{(p)}) = \begin{bmatrix} \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \\ \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \end{bmatrix}$$

Is called *Jacobi matrix*. If

$$\det J(x_1^{(p)},x_2^{(p)}) = \begin{vmatrix} \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \\ \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \end{vmatrix}$$

$$= \frac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1}\frac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} - \frac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2}\frac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1}$$

$$\neq 0$$

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Newton's Method for a System of Two Equations

Then, by Cramer' rule,

$$e_1^{(p)} = \frac{\begin{vmatrix} -f(x_1^{(p)},x_2^{(p)}) & \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \\ -g(x_1^{(p)},x_2^{(p)}) & \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \\ \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \end{vmatrix}} \qquad e_2^{(p)} = \frac{\begin{vmatrix} \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & -f(x_1^{(p)},x_2^{(p)}) \\ \dfrac{\partial g\left(x^{(p)},y^{(p)}\right)}{\partial x_1} & -g(x^{(p)},y^{(p)}) \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial f\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \\ \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_1} & \dfrac{\partial g\left(x_1^{(p)},x_2^{(p)}\right)}{\partial x_2} \end{vmatrix}}$$

Let

$$x^{(p+1)} = x^{(p)} + e_1^{(p)}$$
$$y^{(p+1)} = y^{(p)} + e_2^{(p)}$$

The initial approximations $x^{(p)}$ and $y^{(p)}$ are determined roughly.

Newton's Method for a System of Two Equations

Let $\Delta_x^{(p)} = \begin{vmatrix} f(x^{(p)}, y^{(p)}) & \frac{\partial f(x^{(p)},y^{(p)})}{\partial y} \\ g(x^{(p)}, y^{(p)}) & \frac{\partial g(x^{(p)},y^{(p)})}{\partial y} \end{vmatrix}$ and

$\Delta_y^{(p)} = \begin{vmatrix} \frac{\partial f(x^{(p)},y^{(p)})}{\partial x} & f(x^{(p)}, y^{(p)}) \\ \frac{\partial g(x^{(p)},y^{(p)})}{\partial x} & g(x^{(p)}, y^{(p)}) \end{vmatrix}$. Then

$$x^{(p+1)} = x^{(p)} - \frac{\Delta_x^{(p)}}{\det J(x^{(p)}, y^{(p)})}$$

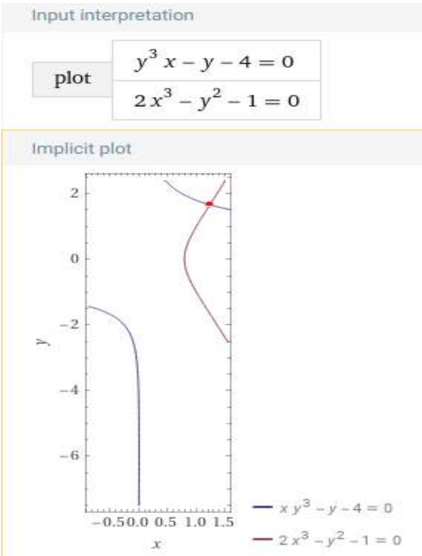
$$y^{(p+1)} = y^{(p)} - \frac{\Delta_y^{(p)}}{\det J(x^{(p)}, y^{(p)})}$$

Newton's Method for a System of Two Equations (cont'd)

Example. Find the real roots of the system

$$\begin{cases} f(x,y) = 2x^3 - y^2 - 1 = 0 \\ g(x,y) = xy^3 - y - 4 = 0 \end{cases}$$

Solution. Find graphically the approximate initial values $x_0 = 1.2$ and $y_0 = 1.7$.



Newton's Method for a System of Two Equations (cont'd)

Solution. (cont'd). Computing the Jacobian at the point

$$\begin{bmatrix} 1.2 \\ 1.7 \end{bmatrix}$$

We have

$$\det J(x, y) = \begin{vmatrix} 6x^2 & -2y \\ y^3 & 3xy^2 - 1 \end{vmatrix}.$$

So Jacbi matrix in initial point is equal to

$$\det J(1.2,1.7) = \begin{vmatrix} 8.64 & -3.40 \\ 4.91 & 9.40 \end{vmatrix} = 97.910$$

Newton's Method for a System of Two Equations (cont'd)

Solution. (cont'd). Continuing this process with the obtained values $x^{(1)}$ and $y^{(1)}$, we get

$$x^{(1)} = 1.2 - \frac{1}{97.910} \begin{vmatrix} -0.434 & -3.40 \\ 0.1956 & 9.40 \end{vmatrix} = 1.2 + 0.0349$$
$$= 1.2349.$$

$$y^{(1)} = 1.7 - \frac{1}{97.910} \begin{vmatrix} 8.64 & -0.434 \\ 4.91 & 0.1956 \end{vmatrix} = 1.7 - 0.0390$$

$$= 1.6610.$$

Continuing this process with the brained values $x^{(2)} = 1.2343$ and $y^{(2)} = 6615.$

and so forth.

Newton's Method for a System of Two Equations (cont'd)

Example. Find the real roots of the system

$$\begin{cases} f(x,y) = \cos\left(0.4y + x^2\right) + x^2 + y^2 - 1.6 = 0 \\ g(x,y) = 1.5x^2 - \dfrac{y^2}{0.36} - 1 = 0. \end{cases}$$

Solution. We graphically find the approximate initial values:

$$x^{(0)} = 1.04 \quad \text{and } y^{(0)} = 1.7.$$

By following formulas,

$$x^{(p+1)} = x^{(p)} - \frac{\Delta_x^{(p)}}{\det J(x^{(p)}, y^{(p)})}$$

$$y^{(p+1)} = y^{(p)} - \frac{\Delta_y^{(p)}}{\det J(x^{(p)}, y^{(p)})}$$

Newton's Method for a System of Two Equations (cont'd)

Solution (cont'd). The following results, listed in the following table, are obtained:

x	1.04	1.03864	f'_y	0.55801	0.56172	J	-1.985549	-1.99889
y	0.47	0.47173	g'_x	3.12	3.11592	$-\frac{\Delta x}{J}$	-0.00136	0.00000
F	-0.00084	0.00000	g	-2.61111	-2.62072			
G	0.00879	0.00002	Δx	-0.00271	0.00001	$-\frac{\Delta y}{J}$	0.00173	0.00000
F'_x	0.09364	0.09483	Δy	+0.00344	0.00000			

Newton's Method for a System of Two Equations (Summary)

Given non-linear equation System: $\begin{cases} f(x_1, x_2) = 0 \\ g(x_1, x_2) = 0 \end{cases}$

Its linearization: $-\begin{bmatrix} f\left(x_1^{(p)}, x_2^{(p)}\right) \\ g\left(x_1^{(p)}, x_2^{(p)}\right) \end{bmatrix} = J(x_1^{(p)}, x_2^{(p)}) \begin{bmatrix} e_1^{(p)} \\ e_2^{(p)} \end{bmatrix}$

If $\det J\left(x^{(p)}, y^{(p)}\right) \neq 0$, then

$$\begin{bmatrix} e_1^{(p)} \\ e_2^{(p)} \end{bmatrix} = -J\left(x_1^{(p)}, x_2^{(p)}\right)^{-1} \begin{bmatrix} f\left(x_1^{(p)}, x_2^{(p)}\right) \\ g\left(x_1^{(p)}, x_2^{(p)}\right) \end{bmatrix}$$

And then, let

$$\begin{bmatrix} x_1^{(p+1)} \\ x_2^{(p+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(p)} \\ x_2^{(p)} \end{bmatrix} + \begin{bmatrix} e_1^{(p)} \\ e_2^{(p)} \end{bmatrix} = \begin{bmatrix} x_1^{(p)} \\ x_2^{(p)} \end{bmatrix} - J\left(x_1^{(p)}, x_2^{(p)}\right)^{-1} \begin{bmatrix} f\left(x_1^{(p)}, x_2^{(p)}\right) \\ g\left(x_1^{(p)}, x_2^{(p)}\right) \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Suppose we are given a system:

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

Suppose we have found the p-th approximation is $x^{(p)} = \begin{bmatrix} x_1^{(p)} \\ \vdots \\ x_n^{(p)} \end{bmatrix}$. By Taylor Theorem around $x^{(p)}$ by the assumption of $0 = f_j\left(x_1^{(p)} + e_1^{(p)}, \dots, x_n^{(p)} + e_n^{(p)}\right)$:

$$0 \approx f_j\left(x_1^{(p)}, \dots, x_n^{(p)}\right) + e_1^{(p)} \frac{\partial f_j\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x} + \dots + e_n^{(p)} \frac{\partial f_j\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x}$$

Newton's Method for a System of n Equations in n Unknowns

The linearization of system

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

around $x^{(p)}$ is:

$$\begin{cases} f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right) + \boldsymbol{e_1^{(p)}} \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} + \dots + \boldsymbol{e_n^{(p)}} \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} = 0 \\ \vdots \\ f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right) + \boldsymbol{e_1^{(p)}} \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} + \dots + \boldsymbol{e_n^{(p)}} \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} = 0 \end{cases}$$

Newton's Method for a System of n Equations in n Unknowns

The obtained linear system:

$$\begin{cases} f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right) + \boldsymbol{e_1^{(p)}} \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} + \dots + \boldsymbol{e_n^{(p)}} \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} = 0 \\ \vdots \\ f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right) + \boldsymbol{e_1^{(p)}} \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} + \dots + \boldsymbol{e_n^{(p)}} \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} = 0 \end{cases}$$

The matrix form of the above system:

$$\begin{bmatrix} \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} & \dfrac{\partial f_2\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_{22}} & \cdots & \dfrac{\partial f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} \\ \dfrac{\partial f_2\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} & \dfrac{\partial f_2\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_2} & \cdots & \dfrac{\partial f_2\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} & \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_1} & \cdots & \dfrac{\partial f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right)}{\partial x_n} \end{bmatrix} \begin{bmatrix} \boldsymbol{e_1^{(p)}} \\ \boldsymbol{e_2^{(p)}} \\ \vdots \\ \boldsymbol{e_n^{(p)}} \end{bmatrix} = - \begin{bmatrix} f_1\left(x_1^{(p)}, \dots, x_n^{(p)}\right) \\ f_2\left(x_1^{(p)}, \dots, x_n^{(p)}\right) \\ \vdots \\ f_n\left(x_1^{(p)}, \dots, x_n^{(p)}\right) \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Let

$$J\left(x_1^{(p)},\dots,x_n^{(p)}\right)=\begin{bmatrix} \frac{\partial f_1\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_1} & \frac{\partial f_2\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_{22}} & \cdots & \frac{\partial f_1\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_n} \\ \frac{\partial f_2\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_1} & \frac{\partial f_2\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_2} & \cdots & \frac{\partial f_2\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_1} & \frac{\partial f_n\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_1} & \cdots & \frac{\partial f_n\left(x_1^{(p)},\dots,x_n^{(p)}\right)}{\partial x_n} \end{bmatrix}$$

and $e^{(p)}=\begin{bmatrix} e_1^{(p)} \\ e_2^{(p)} \\ \vdots \\ e_n^{(p)} \end{bmatrix}$, $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $F\left(x_1^{(p)},\dots,x_n^{(p)}\right)=\begin{bmatrix} f_1\left(x_1^{(p)},\dots,x_n^{(p)}\right) \\ f_2\left(x_1^{(p)},\dots,x_n^{(p)}\right) \\ \vdots \\ f_n\left(x_1^{(p)},\dots,x_n^{(p)}\right) \end{bmatrix}$

Newton's Method for a System of n Equations in n Unknowns

So we obtain:

$$e^{(p)}=-J\left(x_1^{(p)},\dots,x_n^{(p)}\right)^{-1}F\left(x_1^{(p)},\dots,x_n^{(p)}\right).$$

Thus, the new approximation, $x^{(p+1)}$, is obtained by adding $e^{(p)}$ to $x^{(p)}$:

$$x^{(p+1)}=x^{(p)}+e^{(p)}=x^{(p)}-J\left(x_1^{(p)},\dots,x_n^{(p)}\right)^{-1}F\left(x_1^{(p)},\dots,x_n^{(p)}\right)$$

This is called **Newton's method for nonlinear systems**.

To compute and invert the Jacobian matrix is one of weakness in Newton's method.

Newton's Method for a System of n Equations in n Unknowns

Example. Using Newton's method, approximate the positive solution of the system of equations:

$$\begin{cases} x^2 + y^2 + z^2 - 1 = 0 \\ 2x^2 + y^2 - 4z \quad = 0 \\ 3x^2 - 4y + z^2 \quad = 0 \end{cases}$$

proceeding from the initial approximation $x_0 = y_0 = z_0 = 0.5$.

Solution. Let

$$F(x) = \begin{bmatrix} x^2 + y^2 + z^2 - 1 \\ 2x^2 + y^2 - 4z \\ 3x^2 - 4y + z^2 \end{bmatrix}$$

So

$$F\left(x^{(0)}\right) = \begin{bmatrix} 0.25 + 0.25 + 0.25 - 1 \\ 0.50 + 0.25 - 2.00 \\ 0.75 - 2.00 + 0.25 \end{bmatrix} = \begin{bmatrix} -0.25 \\ -1.25 \\ -1.00 \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Solution (cont'd). We obtain

$$J(\boldsymbol{x}) = \begin{bmatrix} 2x & 2y & 2z \\ 4x & 2y & -4 \\ 6x & -4 & 2z \end{bmatrix}$$

where $\boldsymbol{x} = (x, y, z)$. So

$$J\left(\boldsymbol{x}^{(0)}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 3 & -4 & 1 \end{bmatrix} \Rightarrow \det J\left(\boldsymbol{x}^{(0)}\right) = -40.$$

$$J\left(\boldsymbol{x}^{(0)}\right)^{-1} = -\frac{1}{40}\begin{bmatrix} -15 & -5 & -5 \\ -14 & -2 & 6 \\ -11 & 7 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{20} & \frac{1}{20} & -\frac{3}{20} \\ \frac{11}{40} & -\frac{7}{40} & \frac{1}{40} \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Solution (cont'd). 1nd approximation:

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} - J\left(\boldsymbol{x}^{(0)}\right)^{-1} F\left(x^{(0)}\right)$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{20} & \frac{1}{20} & -\frac{3}{20} \\ \frac{11}{40} & -\frac{7}{40} & \frac{1}{40} \end{bmatrix} \begin{bmatrix} -0.25 \\ -1.25 \\ -1.00 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0.375 \\ 0 \\ -0.125 \end{bmatrix} = \begin{bmatrix} 0.875 \\ 0.500 \\ 0.375 \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Solution (cont'd). 2st approximation:

$$F\left(x^{(1)}\right) = \begin{bmatrix} 0.875^2 + 0.500^2 + 0.375^2 - 1 \\ 2 \times 0.875^2 + 0.500^2 - 4 \times 0.375 \\ 3 \times 0.875^2 - 4 \times 0.500 + 0.375^2 \end{bmatrix} = \begin{bmatrix} 0.15625 \\ 0.28125 \\ 0.43750 \end{bmatrix}.$$

$$J\left(x^{(1)}\right) = \begin{bmatrix} 2 \times 0.875 & 2 \times 0.500 & 2 \times 0.375 \\ 4 \times 0.875 & 2 \times 0.500 & -4 \\ 6 \times 0.875 & -4 & 2 \times 0.375 \end{bmatrix} = \begin{bmatrix} 1.750 & 1 & 0.750 \\ 3.500 & 1 & -4 \\ 5.250 & -4 & 0.750 \end{bmatrix}.$$

$$\det J\left(x^{(1)}\right) = \begin{vmatrix} 1.750 & 1 & 0.750 \\ 3.500 & 1 & -4 \\ 5.250 & -4 & 0.750 \end{vmatrix} = \begin{vmatrix} 1.750 & 1 & 0.750 \\ 1.750 & 0 & -4.750 \\ 12.250 & 0 & 3.750 \end{vmatrix} = -64.75.$$

$$J^{-1}\left(x^{(1)}\right) = -\frac{1}{64.75} \begin{bmatrix} -15.25 & -3.75 & -4.75 \\ -23.625 & -2.6250 & 9.625 \\ -19.25 & 12.25 & -1.75 \end{bmatrix}.$$

Newton's Method for a System of n Equations in n Unknowns

Solution (cont'd). 2nd approximation:

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} - J\left(\boldsymbol{x}^{(1)}\right)^{-1} F\left(x^{(0)}\right)$$

$$= \begin{bmatrix} 0.875 \\ 0.500 \\ 0.375 \end{bmatrix} + \frac{1}{64.75} \begin{bmatrix} -15.25 & -3.75 & -4.75 \\ -23.625 & -2.6250 & 9.625 \\ -19.25 & 12.25 & -1.75 \end{bmatrix} \begin{bmatrix} 0.15625 \\ 0.28125 \\ 0.43750 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0.08519 \\ 0.00338 \\ 0.00507 \end{bmatrix} = \begin{bmatrix} 0.78981 \\ 0.49662 \\ 0.36993 \end{bmatrix}$$

Newton's Method for a System of n Equations in n Unknowns

Solution (cont'd). In 2nd approximation:

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 0.78981 \\ 0.49662 \\ 0.36993 \end{bmatrix}$$

3nd approximation:

$$\boldsymbol{x}^{(3)} = \begin{bmatrix} 0.78521 \\ 0.49662 \\ 0.36992 \end{bmatrix} \qquad F\left(\boldsymbol{x}^{(3)}\right) = \begin{bmatrix} 0.00001 \\ 0.00004 \\ 0.00005 \end{bmatrix}$$

Confining ourselves to the third approximation, we obtain

$$x = 0.7852$$

$$y = 0.4966$$

$$z = 0.3699$$

Just for Fun!

By **Newton's method for nonlinear systems**, we have:

$$x^{(p+1)} = x^{(p)} - J\left(x_1^{(p)}, \dots, x_n^{(p)}\right)^{-1} F\left(x_1^{(p)}, \dots, x_n^{(p)}\right)$$

Let

$$G(x) = x - J(x)^{-1}F(x)$$

Thus, the vetor equation $F(x) = \mathbf{0}$ has a solution if and only if $G(x)$ has a fixed point, i.e., $G(x) = x$.

Theorem Let $D = \{(x_1, x_2, \dots, x_n)^t \mid a_i \le x_i \le b_i$, for each $i = 1, 2, \dots, n\}$ for some collection of constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$. Suppose $\mathbf{G}$ is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$. Then $\mathbf{G}$ has a fixed point in D.

Moreover, suppose that all the component functions of $\mathbf{G}$ have continuous partial derivatives and a constant $K < 1$ exists with

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \le \frac{K}{n}, \quad \text{whenever } \mathbf{x} \in D,$$

for each $j = 1, 2, \dots, n$ and each component function g_i. Then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $\mathbf{x}^{(0)}$ in D and generated by

$$\mathbf{x}^{(k)} = G(\mathbf{x}^{(k-1)}), \quad \text{for each } k \ge 1$$

converges to the unique fixed point $\mathbf{p} \in D$ and

$$\left\|\mathbf{x}^{(k)} - \mathbf{p}\right\|_\infty \le \frac{K^k}{1-K}\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_\infty.$$

Quasi-Newton Method

Quasi-newton Methods

A **significant weakness** of Newton's method for solving systems of nonlinear equations is the need, at each iteration,

- to evaluate the n^2 partial derivatives of the n component functions of F.
- to determine a Jacobian matrix and solve an $n \times n$ linear system that involves this matrix.
- to calculate inverse of Jacobian.

we can use finite difference approximations to the partial derivatives. For example,

$$\frac{\partial f_j}{\partial x_k} \approx \frac{f_j\left(x^{(p)} + e_k h\right) - f_j(x^{(p)})}{h}$$

The Method of Simple Iteration

The Method of Simple Iteration for a System of Two Equations (cont'd)

- We can then use the following iterative equations:

$$x_{n+1}= \varphi_1(x_n, y_n)$$
$$y_{n+1} = \varphi_2(x_n, y_n)$$

for $n = 0,1,2, \ldots$,where (x_0, y_0) is some initial approximation.

The Method of Simple Iteration for a System of Two Equations (cont'd)

Theorem. Let in some closed neighborhood $R(a \leq x \leq A, b \leq y \leq B)$ there be one and only one solution $x = \varepsilon, y = \eta$ for the new system. If

1. The functions $\varphi_1(x_n, y_n)$ and $\varphi_2(x_n, y_n)$ are defined and continuously differentiable in R,
2. The initial approximations x_0, y_0 and all successive approximations $x_n, y_n (n = 1, 2, \ldots)$ belong to R, and
3. The following inequalities are fulfilled in R

$$\left|\frac{\partial \varphi_1}{\partial x}\right| + \left|\frac{\partial \varphi_1}{\partial y}\right| \leq q_1 < 1,$$
$$\left|\frac{\partial \varphi_2}{\partial x}\right| + \left|\frac{\partial \varphi_2}{\partial y}\right| \leq q_2 < 1.$$

The Method of Simple Iteration for a System of Two Equations (cont'd)

Theorem. (cont'd) Error estimation of the nth approximation is given by the inequality

$$|\varepsilon - x_n| + |\eta - y_n| \leq \frac{M}{1-M}\left(|x_n - x_{n-1}| + |y_n - y_{n-1}|\right)$$

***M* is the greatest number of q_1 and q_2**

Convergence of the method of iteration is considered as good if $M < 1/2$, and $M/(1-M) < 1$, so that if the first three decimal digits coincide in two successive approximations, then the error of the last approximation does not exceed 0.001.

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The Method of Simple Iteration for a System of Two Equations (cont'd)

Example. Find the real roots of the system

$$x^3 + y^3 - 6x + 3 = 0,$$
$$x^3 - y^3 - 6y + 2 = 0.$$

Solution. For applying the method of iteration rewrite the given system as:

$$x = \frac{(x^3+y^3)}{6} + \frac{1}{2} = \varphi_1(x, y),$$
$$x = \frac{(x^3-y^3)}{6} + \frac{1}{3} = \varphi_2(x, y).$$

Consider the square $0 \leq x \leq 1$, $0 \leq y \leq 1$. If point (x_0, y_0) is situated in this square, then we have

$$0 < \varphi_1(x_0, y_0) < 1 \text{ and } 0 < \varphi_2(x_0, y_0) < 1$$

The Method of Simple Iteration for a System of Two Equations (cont'd)

Solution. (cont'd) Since $0 < \frac{(x_0^3+y_0^3)}{6} < \frac{1}{3}, -\frac{1}{6} < \frac{(x_0^3-y_0^3)}{6} < \frac{1}{6}$, the sequence (x_k, y_k)

remains in a square for any choice of the point (x_0, y_0).

Moreover, the points (x_k, y_k) remain in the rectangle

$$\frac{1}{2} < x < \frac{5}{6}, \frac{1}{6} < y < \frac{1}{2}$$

$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

Homework: Why?

The Method of Simple Iteration for a System of Two Equations (cont'd)

Solution. (cont'd) For the points of this rectangle we have

$$\left|\frac{\partial\varphi_1}{\partial x}\right| + \left|\frac{\partial\varphi_1}{\partial y}\right| = \frac{x^2}{2} + \frac{y^2}{2} \le \frac{\frac{25}{36}+\frac{1}{4}}{2} = \frac{34}{72} < 1,$$

$$\left|\frac{\partial\varphi_2}{\partial x}\right| + \left|\frac{\partial\varphi_2}{\partial y}\right| = \frac{x^2}{2} + \left|-\frac{y^2}{2}\right| \le \frac{34}{72} < 1.$$

There exists the unique solution in the indicated rectangle and it can be found by the method of iteration.

Putting $x_0 = \frac{1}{2}, y_0 = \frac{1}{2}$, we have

$$x_1 = \frac{(\frac{1}{8}+\frac{1}{8})}{6} + \frac{1}{2} = 0.542, \qquad x_2 = \frac{0.19615}{6} + \frac{1}{2} = 0.533$$

$$y_1 = \frac{(\frac{1}{8}-\frac{1}{8})}{6} + \frac{1}{3} = 0.333, \qquad y_2 = \frac{0.1223}{6} + \frac{1}{3} = 0.354$$

The Method of Simple Iteration for a System of Two Equations (cont'd)

Solution. (cont'd) Continuing this process, we get

$$x_3 = 0.533, \qquad x_4 = 0.532$$
$$y_3 = 0.351, \qquad y_4 = 0.351$$

Since here $q_1 = q_2 = \frac{34}{72} < 0.5$, the coincidence of the first two decimal points means that the required accuracy is achieved.

Thus, we can take ε = 0.532, η= 0.351.

The Method of Simple Iteration for a System of Two Equations

Constructing iterative functions for the system

We recommend the following method for transforming the system of equations to the new form with the previous conditions fulfilled. Put

$$\varphi_1(x_n, y_n) = x + \alpha\, F_1(x,y) + \beta\, F_2(x,y),$$
$$\varphi_2(x_n, y_n) = y + \gamma\, F_1(x,y) + \delta\, F_2(x,y) \qquad (\alpha\delta \neq \beta\gamma),$$

Find the coefficients $\alpha, \beta, \gamma, \delta$ as approximate solutions of the following system of equations:

(Homework: Prove it)

$$1 + \alpha \frac{\partial F_1(x_0, y_0)}{\partial x} + \beta \frac{\partial\, F_2(x_0, y_0)}{\partial x} = 0,$$
$$\alpha \frac{\partial F_1(x_0, y_0)}{\partial y} + \beta \frac{\partial\, F_2(x_0, y_0)}{\partial y} = 0,$$
$$\gamma \frac{\partial F_1(x_0, y_0)}{\partial x} + \delta \frac{\partial\, F_2(x_0, y_0)}{\partial x} = 0,$$
$$1 + \gamma \frac{\partial F_1(x_0, y_0)}{\partial y} + \delta \frac{\partial\, F_2(x_0, y_0)}{\partial y} = 0,$$

The Method of Simple Iteration for a System of Two Equations

Example. Choose suitable iterative functions $\varphi_1(x,y)$ and $\varphi_2(x,y)$ for the system of equations

$$x^2+y^2-1=0,$$
$$x^3-y=0.$$

for $x_0=0.8,\ y_0=0.55$.

Solution.

We shall look for the functions φ_1 and φ_2 in the form

$$\varphi_1(x,y)=x+\alpha(x^2+y^2-1)+\beta\,(x^3-y),$$
$$\varphi_2(x,y)=y+\gamma(x^2+y^2-1)+\delta(x^3-y),$$

To determine the parameters $\alpha,\beta,\gamma,\delta$ from the system. We have

$$\frac{\partial F_1}{\partial x}=2x,\quad \frac{\partial F_1(x_0,y_0)}{\partial x}=1.6,\quad \frac{\partial F_1}{\partial y}=2y,\quad \frac{\partial F_1(x_0,y_0)}{\partial y}=1.1,$$
$$\frac{\partial F_2}{\partial x}=3x^2,\quad \frac{\partial F_2(x_0,y_0)}{\partial x}=1.92,\quad \frac{\partial F_2}{\partial y}=-1,\quad \frac{\partial F_2(x_0,y_0)}{\partial y}=-1$$

The Method of Simple Iteration for a System of Two Equations

Solution.(cont'd)Hence, we get the system

$$1+1.6\alpha+1.92\beta=0,$$
$$1.1\alpha-\beta=0,$$
$$1.6\gamma+1.92\delta=0,$$
$$1+1.1\gamma+\delta=0,$$

Solving it, we obtain

$$\alpha\approx-0.3,\quad \gamma\approx-0.5,\quad \beta\approx-0.3,\quad \delta\approx 0.4,$$

Thus,

$$\varphi_1(x,y)=x-0.3(x^2+y^2-1)-0.3(x^3-y),$$
$$\varphi_2(x,y)=y-0.5(x^2+y^2-1)+0.4(x^3-y),$$

Newton's Method for a System of n Equations in n Unknowns

Consider a nonlinear system of equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ \dots &\dots \\ f_n(x_1, x_2, \dots, x_n) &= 0, \end{aligned}$$

The totality of the arguments $x_1, x_2, \dots, x_n$ may be considered as an n-dimensional vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

The totality of functions $f_1, f_2, \dots, f_n$ is also an n-dimensional vector

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{pmatrix}$$

Newton's Method for a System of n Equations in n Unknowns (cont'd)

Therefore, the system is briefly written as:

$$f(x) = 0$$

- New system is solved by the method of successive approximations.
- Suppose the pth approximation of one of the isolated roots $x = (x_1, x_2, \dots, x_n)$ of the above vector equation is found.

$$x^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)})$$

Then, the exact root of this equation can be represented in the form

$$x = x^{(p)} + \varepsilon^{(p)}$$

$$\varepsilon^{(p)} = (\varepsilon_1^{(p)}, \varepsilon_2^{(p)}, \dots, \varepsilon_n^{(p)})$$

The Method of Iteration to a System of n Equations in n Unknowns

Let there be given a system of nonlinear equations of a special form:

$$\begin{aligned} x_1 &= \varphi_1(x_1, x_2, \dots, x_n), \\ x_2 &= \varphi_2(x_1, x_2, \dots, x_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots. \\ x_n &= \varphi_n(x_1, x_2, \dots, x_n), \end{aligned}$$

where the functions $\varphi_1, \varphi_2, \dots, \varphi_n$ are real, defined and continuous in some neighborhood ω of an isolated solution $(x_1, x_2, \dots, x_n)$ of this system; in vector notation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \longleftarrow \quad x = \varphi(x) \quad \longrightarrow \quad \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \dots \\ \varphi_n(x) \end{pmatrix}$$

The Method of Iteration to a System of n Equations in n Unknowns (cont'd)

- The vector root is sometimes found by the method of iteration

$$x^{(p+1)} = \varphi(x^{(p)}) \qquad (p=0,1,\dots)$$

The Method of Iteration to a System of n Equations in n Unknowns (cont'd)

If a system of equations is given in the general form

$$f(x) = 0$$

where $f(x)$ is the vector function defined and continuous in the neighborhood w of the isolated vector root x, then it can be written in the equivalent form of

$$x = \varphi(x)$$

Iterative vector function
$\varphi(x) = x + Af(x)$
$A = -w^{-1}(x^{(0)})$

$$\boldsymbol{x}^{(p+1)} = \boldsymbol{x}^{(p)} + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}^{(p)}) \qquad (p=0,1,\ldots)$$

The Method of Iteration to a System of n Equations in n Unknowns (cont'd)

Example. Use simple iterative to find a solution to the nonlinear system

$$3x - \cos(yz) - \frac{1}{2} = 0,$$
$$x^2 - 81(y + 0.1)^2 + \sin z + 1.06 = 0,$$
$$e^{-xy} + 20z + \frac{10\pi - 3}{3} = 0$$

With initial approximations $x_0 = 0.1, y_0 = 0.1, z_0 = -0.1$

Solution.

$$x = \frac{1}{3}\cos(yz) + \frac{1}{6},$$
$$y = \frac{1}{9}\sqrt{x^2 + \sin z + 1.06} - 0.1,$$
$$z = -\frac{1}{20}e^{-xy} - \frac{10\pi - 3}{60}$$

The Method of Iteration to a System of n Equations in n Unknowns (cont'd)

Solution.(cont'd)

$$\varphi_1(x,y,z) = \frac{1}{3}\cos(yz) + \frac{1}{6},$$
$$\varphi_2(x,y,z) = \frac{1}{9}\sqrt{x^2 + \sin z + 1.06} - 0.1,$$
$$\varphi_3(x,y,z) = -\frac{1}{20}e^{-x} - \frac{10\pi - 3}{60}.$$

$$x_{i+1} = \frac{1}{3}\cos(y_i z_i) + \frac{1}{6},$$
$$y_{i+1} = \frac{1}{9}\sqrt{(x_i)^2 + \sin z_i + 1.06} - 0.1,$$
$$z_{i+1} = -\frac{1}{20}e^{-x_i y_i} - \frac{10\pi - 3}{60}.$$

The Method of Iteration to a System of n Equations in n Unknowns (cont'd)

i	0	1	2	3	4
x_i	0.1	0.49998333	0.49997747	0.50000000	0.50000000
y_i	0.1	0.02222979	0.00002815	0.00000004	0.00000000
z_i	-0.1	-0.52304613	-0.52359807	-0.52359877	-0.52359878

ANY QUESTIONS?