Numerical Computations

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Chapter's Topics

- Overview
- Horner's Scheme
- Taylor's Method / Maclaurin's Method

Overview

Computing the value of function by reducing to a sequence of elementary arithmetic operation since the basic operations of most computer are addition, subtraction, multiplication, and division.

Horner's Scheme

William George Horner
(1786-1837)

Computing the Values of Polynomials: Horner' Scheme

Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

be a real polynomial of degree n.

We want to find the value of $P(x)$ for $x = \xi$:

$$P(\xi) = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n$$

Computing the Values of Polynomials: Horner' Scheme (cont'd)

Represent the following formula:

$$P(\xi) = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_{n-2}\xi^2 + a_{n-1}\xi + a_n$$

$$P(\xi) = \left(a_0\xi^{n-1} + a_1\xi^{n-2} + \cdots + a_{n-2}\xi + a_{n-1}\right)\xi + a_n$$

$$P(\xi) = \left(\left(a_0\xi^{n-2} + a_1\xi^{n-3} + \cdots + a_{n-2}\right)\xi + a_{n-1}\right)\xi + a_n$$

$$\vdots$$

$$P(\xi) = \left(\ldots\left(\left(\left((a_0\xi + a_1)\xi + a_2\right)\xi + \cdots + a_{n-1}\right)\right)\right)\xi + a_n$$

Computing the Values of Polynomials: Horner' scheme (cont'd)

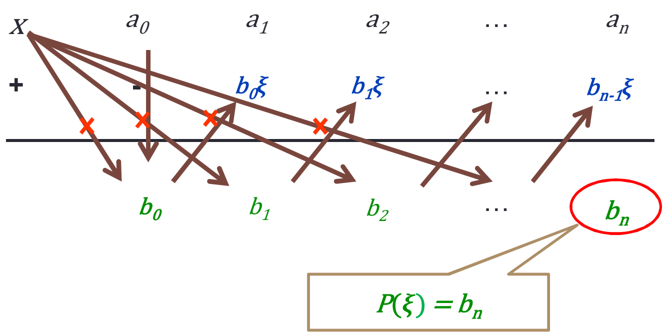
$$P(\xi) = \left(\dots\left(\left(\left((a_0\xi + a_1)\xi + a_2\right)\xi + \dots + a_{n-1}\right)\right)\xi + a_n\right)$$

We then successively compute the following equations:

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= b_0\xi + a_1, \\ b_2 &= b_1\xi + a_2, \\ b_3 &= b_2\xi + a_3, \\ &\vdots \\ b_n &= b_{n-1}\xi + a_n, \end{aligned}$$

and find $b_n = P(\xi)$.

Computing the Values of Polynomials: Horner' scheme (cont'd)



Horner's scheme requires the performing of n multiplications and $n - k$ additions, where k is the number of coefficients a_i equal to zero.

Computing the Values of Polynomials: Horner' Scheme (cont'd)

Example: $P(x) = x^7 - 2x^6 + x^5 - 3x^4 + 4x^3 - x^2 + 6x - 1$

$$f(-1.5) = ?$$

X	1	-2	1	-3	4	-1	6	-1
+		-1.5	5.25	-9.375	18.5625	-33.8438	-52.2657	-87.3985
	1	-3.5	6.25	-12.375	22.5625	-34.8438	58.2657	-88.3985

$$\mathbf{P(-1.5) = -88.3985}$$

Computing the Values of Polynomials: Horner Scheme (cont'd)

Let $Q(x)$ be the quotient on the division of the given polynomial $P(x)$ by the binomial $x - \xi$ where

$$Q(x) = \beta_0 x^{n-1} + \beta_1 x^{n-1} + \cdots + \beta_{n-2} x + \beta_{n-1}.$$

So, there is $\beta_n \in \mathbb{R}$ such that

$$P(x) = Q(x)(x - \xi) + \beta_n.$$

Computing the Values of Polynomials: Horner' Scheme (cont'd)

$$P(x) = Q(x)(x - \xi) + \beta_n,$$

that means

$$P(x) = (\beta_0 x^{n-1} + \beta_1 x^{n-1} + \cdots + \beta_{n-2} x + \beta_{n-1})(x - \xi) + \beta_n$$

So

$$P(x) = \beta_0 x^n + (\beta_1 - \beta_0 \xi) x^{n-1} + (\beta_2 - \beta_1 \xi) x^{n-2} + \cdots + (\beta_{n-1} - \beta_{n-2}\,\xi) x + (\beta_n - \beta_{n-1}\,\xi).$$

Computing the Values of Polynomials: Horner' Scheme (cont'd)

Comparing coefficients of identical powers of the variable x in the two following equations:

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \quad \text{and}$$

$$P(x) = \beta_0 x^n + (\beta_1 - \beta_0 \xi) x^{n-1} + (\beta_2 - \beta_1 \xi) x^{n-2} + \cdots + (\beta_{n-1} - \beta_{n-2}\,\xi) x + (\beta_n - \beta_{n-1}\,\xi),$$

we have

$$\begin{aligned} \beta_0 &= a_0 \\ \beta_1 - \beta_0 \xi &= a_1 \\ &\cdots \\ \beta_{n-1} - \beta_{n-2} \xi &= a_{n-1} \\ \beta_n - \beta_{n-1} \xi &= a_n \end{aligned}$$

$$\begin{aligned} \beta_0 &= a_0 = \boldsymbol{b_0} \\ \beta_1 &= a_1 + \beta_0 \xi = \boldsymbol{b_1} \\ &\cdots \\ \beta_{n-1} &= a_{n-1} + \beta_{n-2} \xi = \boldsymbol{b_{n-1}} \\ \beta_n &= a_n + \beta_{n-1} \xi = \boldsymbol{b_n} \end{aligned}$$

Computing the Values of Polynomials: Horner' Scheme (cont'd)

$$\beta_0 = a_0 = \boldsymbol{b_0} \qquad\qquad \boldsymbol{b_0} = a_0,$$

$$\beta_1 = a_1 + \beta_0\xi = \boldsymbol{b_1}, \qquad\qquad \boldsymbol{b_1} = a_1 + b_0\xi,$$

$$\beta_2 = a_2 + \beta_1\xi = \boldsymbol{b_2}, \qquad \Rightarrow \qquad \boldsymbol{b_2} = a_2 + b_1\xi,$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\beta_{n-1} = a_{n-1} + \beta_{n-2}\xi = \boldsymbol{b_{n-1}}, \qquad\qquad \boldsymbol{b_{n-1}} = a_{n-1} + b_{n-2}\xi$$

$$\beta_n = a_n + \beta_{n-1}\xi = \boldsymbol{b_n} \qquad\qquad \boldsymbol{b_n} = a_n + b_{n-1}\xi$$

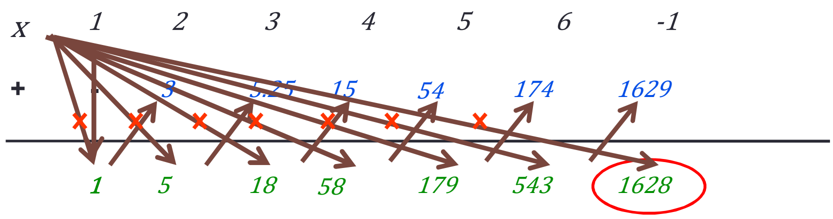
So, without performing the operation of division, we can determine $Q(x)$.

Computing the Values of Polynomials: Horner' Scheme (cont'd)

Example. Given the polynomial

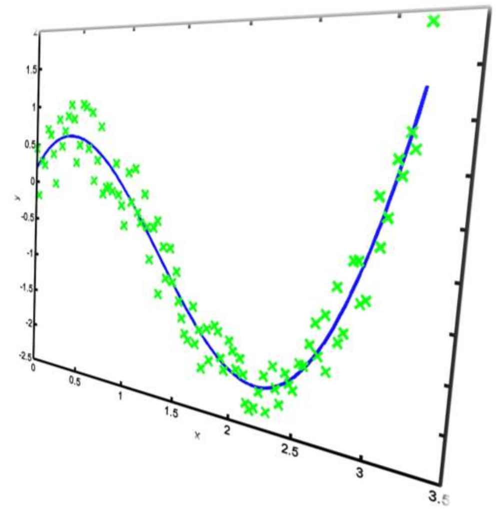
$$P(x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x - 1.$$

Find the quotient on dividing $P(x)$ by the binomial $x - 3$.



$$P(x) = \left(x^5 + 5x^4 + 18x^3 + 58x^2 + 179x + 543\right)(x - 3) + 1628.$$

Polynomial Approximations



Evaluating Some Function with Power Series

Suppose that for function $f(x)$, the values of $f(0)$ and $f^n(0)$ for each $1 \leq n$ are given. Find the series $\sum_{i=0}^{\infty} a_i x^i$ such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

We know that

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\
f'(x) &= \quad +a_1 \ + 2a_2 x^2 + \cdots + n a_n x^n + \cdots \\
&\vdots \\
f^{n-1}(x) &= (\mathrm{n}-1)!\,\mathrm{a_n} + \mathrm{n}(\mathrm{n}-1)\cdots 2 a_n x + \cdots \\
f^n(x) &= n!\, a_n + \cdots
\end{aligned}$$

Maclaurin's Method

Colin Maclaurin
(1698–1746)

Computing the Values of an Analytic Function

Definition. A real function $f(x)$ is called analytic at a point ξ if and only if it Taylor series in some neighborhood $|x-\xi|<R$ of point ξ converges to the function for every x in its domain.

Taylor series:

$$f(x)=f(\xi)+f'(\xi)(x-\xi)+\frac{f''(\xi)}{2!}(x-\xi)^2+\cdots+\frac{f^n(\xi)}{n!}(x-\xi)^n+\cdots$$

Taylor series expansion of the function $f(x)$ about 0 is called **Maclaurin series:**

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^n(0)}{n!}x^n+\cdots$$

Computing the Values of an Analytic Function (cont'd)

If the analytical function $f(x)$ is replaced by Taylor polynomial of order n:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(\xi)}{k!}(x-\xi)^k,$$

then the error resulting from the replacement is

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^k(\xi)}{k!}(x-\xi)^k.$$

The difference is called the ***remainder term***. It is known that there is $0 < \theta < 1$ such that

$$R_n(x) = \frac{f^{(n+1)}(\xi+\theta(x-\xi)}{(n+1)!}(x-\xi)^{n+1}.$$

Computing the Values of an Analytic Function (cont'd)

In other words, we want to show that laced by Taylor polynomial of order n:

$$f(x) = \sum_{k=0}^{n} \frac{f^k(\xi)}{k!}(x-\xi)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\xi)^{n+1},$$

where $c = \xi + \theta(x-\xi)$ for $0 < \theta < 1$.

Theorem. (**Mean Value Theorem**) Suppose that the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), Then there is a point c in the open interval (a, b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Computing the Values of an Analytic Function (cont'd)

Theorem. (**Taylor's Theorem**) Suppose that the function f is real function on $[a, b]$, n is a positive integer, $f', f'', \ldots, f^{(n-1)}$ are continuous on $[a, b]$, $f^{(n)}(x)$ exits for every $x \in (a, b)$. Let $\xi \in (a, b)$. Then for each $x \in (a, b)$, there is a point c between

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\xi)^{n+1}.$$

Computing the Values of an Analytic Function (cont'd)

For $n = 1$, this is the Mean Value Theorem, so we obtain

$$f(x) - f(\xi) = f'(c)(x-\xi).$$

In addition, the theorem shows that f can be approximated by a polynomial of degree n, and $f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\xi)^{n+1}$ let us to estimate the error, if we know bounds on $|f^{(n)}(c)|$.

Computing the Values of an Analytic Function (cont'd)

Sketch of proof.

Our target is to find M such that

$$f(x) = P_n(x) + M(x-\xi)^{n+1},$$

Let

$$g(x) = f(x) - f(\xi) - \frac{x-\xi}{1!}f'(\xi) - \cdots - \frac{(x-\xi)^n}{n!}f^{(n)}(\xi) - M(x-\xi)^{n+1}$$

Thus, we should find M such that $g(x) = 0$.

Computing the Values of an Analytic Function (cont'd)

Sketch of proof (cont'd). Assume $a_i = \frac{f^{(i)}(\xi)}{i!}$, thus

$$g'(x) = f'(x) - \sum_{i=1}^{n} i\,\mathrm{a_i}(\mathrm{x}-\xi)^{i-1} - (n+1)M(x-\xi)^n,$$

$$\vdots$$

$$g^{(n)}(x) = f^{(n)}(x) - n!\,a_n - (n+1)\times\cdots\times 2M(x-\xi)^1$$

$$g^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\,M.$$

So $g(\xi) = g'(\xi) = \cdots = g^{(n)}(\xi) = 0$.

Also, our choice of M shows $g(x) = 0$. By using Mean Value Theorem, we have

$$0 = g(x) - g(\xi) = g'(c_1)(x-\xi) \quad \Longrightarrow g'(c_1) = 0$$

$$0 = g'(c_1) - g'(\xi) = g(\mathrm{c_2})(\mathrm{c_1}-\xi) \quad \Longrightarrow g''(c_2) = 0$$

$$\vdots$$

$$0 = g^n(c_n) - g^n(\xi) = g^{(n+1)}(c_{n+1})(c_n-\xi) \Longrightarrow g^{(n+1)}(c_{n+1}) = 0$$

$$0 = g^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1}) - (n+1)!\,M \Longrightarrow M = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}$$

Computing the Values of an Analytic Function (cont'd)

Remark 1. In many cases, a way to compute the value of a function is expanding the function in a Taylor series / Maclaurin series.

Remark 2. If $f(\xi)$ is known and it is required to find the value $f(\xi + h)$ where h is **small correction**, then

$$f(\xi + h) = f(\xi) + f'(\xi)h + \frac{f''(\xi)}{2!}h^2 + \cdots +$$
$$\cdots + \frac{f^n(\xi)}{n!}h^n + R_n(h),$$

where

$$R_n(h) = \frac{f^{(n+1)}(\xi + \theta h)}{(n+1)!}h^{n+1} \qquad (0 < \theta < 1).$$

Computing the Values of an Analytic Function (cont'd)

Example: Approximate $\sqrt{10}$.

Solution. We have $\sqrt{10} = \sqrt{3^2 + 1}$. Consider $f(x) = x^{\frac{1}{2}}$, we successively obtain

$$f'(x) = \frac{1}{2}x^{\frac{-1}{2}} \qquad f'(9) = \frac{1}{2}9^{\frac{-1}{2}} = \frac{1}{6}$$
$$f''(x) = \frac{-1}{2^2}x^{\frac{-3}{2}} \qquad f''(9) = \frac{-1}{2^2}9^{\frac{-3}{2}} = \frac{-1}{108}$$
$$f'''(x) = \frac{3}{2^3}x^{\frac{-5}{2}} \qquad f'''(9) = \frac{3}{2^3}9^{\frac{-5}{2}} = \frac{1}{648}$$

$$\sqrt{10} = 3 + \frac{1}{6} + \frac{-1}{2 * 108} + \frac{1}{6 * 648} + R_3 = 3.162 + R_3$$

See and correct

$$R_3 = \frac{f^{(4)}(9+\theta)}{4!} = \frac{1}{4!}\frac{-3}{2^3}\frac{5}{2}(9+\theta)^{\frac{-7}{2}}\left(\frac{1}{9}\right)^4 < 0.0000123.$$

Computing the Values of an Analytic Function (cont'd)

Example: Approximate $\sqrt{10}$ by Maclaurin's series.

Solution. We have $\sqrt{10} = \sqrt{3^2+1} = 3\left(1+\frac{1}{9}\right)^{\frac{1}{2}}$.

Consider $f(x) = (1+x)^{\frac{1}{2}}$.

$$f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{2^2}(1+x)^{\frac{-3}{2}} \qquad f''(x) = \frac{-1}{4}$$

$$f'''(x) = \frac{3}{2^3}(1+x)^{\frac{-5}{2}} \qquad f'''(x) = \frac{3}{8}$$

$$\left(1+\frac{1}{9}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}.\frac{1}{9} - \frac{1}{8}.\frac{1}{81} + \frac{1}{16}.\frac{1}{729} + R_3 = 1.05411 + R_3$$

$$R_s = \frac{f^{(4)}(9+\theta)}{4!} = \frac{1}{4!}\frac{-3}{2^3}\frac{5}{2}\left(1+\frac{\theta}{9}\right)^{\frac{-7}{2}} = \frac{10}{1680616}\left(1+\frac{\theta}{9}\right)^{\frac{-7}{2}} < 0.000006.$$

Computing the Values of Exponential Functions

Taylor series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Note that it is known that the interval of convergence is $-\infty < x < \infty$.

So the remainder term for $P_n(x)$ is

$$R_n(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1} \quad 0 < \theta < 1.$$

As a result, for large absolute values of x, the error of replacing e^x with $P_n(x)$ may not be tolerable.

Computing the Values of Exponential Functions (cont'd)

For large absolute values of x, the ordinary procedure is as follows:

$$x = E(x) + q.$$

where $E(x)$ id the largest integer in the number x and $0 \leq q < 1$ is the fractional part of the number. So

$$e^x = e^{E(x)} \times e^q .$$

Also,

$$\begin{cases} e^{E(x)} = \overbrace{e\, e\, \ldots\, e}^{E(x)\ times} & if \quad E(x) \geq 0 \\ e^{E(x)} = \underbrace{\frac{1}{e}\frac{1}{e}\cdots\frac{1}{e}}_{-E(x)\ times} & if \quad E(x) < 0 \end{cases}$$

where

$$e = 2.718281828459045 \ldots$$
$$\frac{1}{e} = 0.367879441171442 \ldots$$

Computing the Values of Exponential Functions (cont'd)

With large moduli of x series is hardly fit for computations

$$\boldsymbol{x = E(x) + q}$$

E (x) is the integral part of x

q its fractional part, $0 \leq q < 1$

$$\boldsymbol{e^x = e^{E(x)} . e^q}$$

$$e^{E(x)} = \underbrace{e.e \ldots e}_{E(x)\ times}, \quad if \ E(x) > 0$$

$$e^{E(x)} = \underbrace{\frac{1}{e}.\frac{1}{e} \ldots \frac{1}{e}}_{-E(x)\ times}, \quad if \ E(x) < 0$$

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}$$

$$0 \leq R_n(q) < \frac{1}{n!\, n} \ q^{n+1}$$

Computing the Values of Exponential Functions (cont'd)

The second factor e^q of the equation $e^x = e^{E(x)} \times e^q$ can be commutated as follows

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}$$

So the remainder term for $P_n(q)$ is bounded by

$$0 \le R_n(q) = \frac{e^{\theta q}}{(n+1)!} q^{n+1} < \frac{3}{(n+1)!} q^{n+1}$$

$$\text{for } \; 0 < \theta < 1.$$

Now, our target is to improve the above upper bound for $R_n(q)$.

Computing the Values of Exponential Functions (cont'd)

We have
$$\begin{aligned} R_n(q) &= \frac{q^{n+1}}{(n+1)!} + \frac{q^{n+2}}{(n+2)!} + \frac{q^{n+3}}{(n+3)!} + \cdots \\ &= \frac{q^{n+1}}{(n+1)!}\left[1 + \frac{q}{n+2} + \frac{q^2}{(n+2)(n+3)} + \cdots\right] \\ &< \frac{q^{n+1}}{(n+1)!}\left[1 + \frac{q}{n+2} + \left(\frac{q}{n+2}\right)^2 + \cdots\right] \end{aligned}$$

$a + ar + ar^2 + \cdots = \frac{a}{1-r}$, $r < 1$

As we know that $1 + \frac{q}{n+2} + \left(\frac{q}{n+2}\right)^2 + \cdots \frac{1}{1 - \frac{q}{n+2}}$,

$\frac{q^n}{n!}$

so, $$\boldsymbol{R_n(q) < \frac{q^{n+1}}{(n+1)!} \times \frac{1}{1 - \frac{q}{n+2}}}.$$

$= \frac{q^n \cdot q}{n!\,(n+1)} \times \frac{n+2}{\underbrace{n+2-q}_{1}}$

Note that $$\frac{n+2}{n+1} < \frac{n+1}{n}.$$

Thus $$R_n(q) < \frac{q^{n+1}}{(n+1)!} \times \frac{1}{1 - \frac{q}{n+2}} < \frac{q^n \cdot q}{n!\,n} = \frac{q^n}{n!} \cdot \frac{q}{n}.$$

Computing the Values of Exponential Functions (cont'd)

Let

$$u_k = \frac{q^k}{k!}.$$

It is convenient to approximate e^x for small x by

$$e^q = u_0 + u_1 + \cdots + u_n + R_n(q)$$

Such that

$$0 < R_n\,(q) < u_n \frac{q}{n}.$$

Note that our goal is to split up the operation into repeating cycles. So

$$u_k = u_{k-1} \cdot \frac{q}{k}$$

Computing the Values of Exponential Functions (cont'd)

Now, our concern is to find the necessary number of terms n respect to the given the residual error ϵ.

We obtained $0 < R_n\,(q) < u_n \frac{q}{n}$. Now, we want to modified it based on the following restriction on n.

Let $$n \geq 2|q| > 0.$$

Thus

$$|R_n(q)| \leq R_n(|q|) < \frac{|q|^{n+1}}{(n+1)!} \cdot \frac{1}{1 - \dfrac{|q|}{n+2}} <$$

$$< \frac{2|q|^{n+1}}{(n+1)!} = \frac{2|x|}{n+1} \cdot \frac{|x|^n}{n!} < |u_n| \leq \epsilon.$$

Therefore

$$|u_n(x)| \leq \epsilon.$$

Computing the Values of Exponential Functions (cont'd)

Example: Find $\sqrt{e}$ within 10^{-5}.

Solution.

$$u_k = u_{k-1} \cdot \frac{\frac{1}{2}}{k} = \frac{u_{k-1}}{2k} \quad (k= 1, 2, \dots, n)$$

$$e^{\frac{1}{2}} = \sum_{k=0}^{n} u_k + R_n\left(\frac{1}{2}\right)$$

$u_0 = 1,$

$u_1 = \frac{u_0}{2} = 0.5000000,$ $\quad u_2 = \frac{u_1}{4} = 0.1250000,$

$u_3 = \frac{u_2}{6} = 0.0208333,$ $\quad u_4 = \frac{u_3}{8} = 0.0026042,$

$u_5 = \frac{u_4}{10} = 0.0002604,$ $\quad u_6 = \frac{u_5}{4} = 0.0000217,$

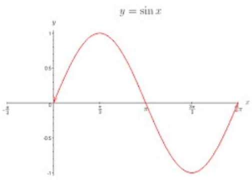
$u_7 = \frac{u_6}{14} = 0.0002604,$ $\quad S_7 = 1.6487212.$

$$\sqrt{e} = 1.64872.$$

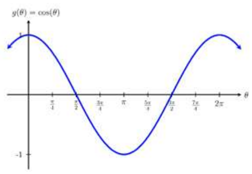
Computing the Values of Sine and Cosine

For computing the value of the functions $\sin x$ and $\cos x$, it is sufficient to know how to compute $\sin x$ and $\cos x$ for the interval $0 \le x \le \frac{\pi}{4}$.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$



$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$



Computing the Values of Sine and Cosine (cont'd)

If $0 \le x \le \frac{\pi}{4}$, we have

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

But if $\frac{\pi}{4} \le x \le \frac{\pi}{2}$

$$\sin x = \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

where

$$z = \frac{\pi}{2} - x \text{ and } 0 \le z \le \frac{\pi}{4}$$

Computing the Values of Sine and Cosine (cont'd)

$$\sin x = \sum^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$u_1 = x$$

$$u_2 = -u_1 \cdot \frac{x^2}{3 \times 2}$$

$$u_3 = -u_2 \cdot \frac{x^2}{5 \times 4}$$

$$\sin x = \sum_{k=1}^{n} u_k + R_n(x)$$

$$u_1 = x, \quad u_{k+1} = -\frac{x^2}{2k(2k+1)} u_k.$$

$$\cos x = \sum_{k=1}^{n} v_k + R_n(x)$$

$$v_1 = 1, \quad v_{k+1} = -\frac{-x^2}{2k(2k-1)} v_k.$$

Computing the Values of Sine and Cosine (cont'd)

Thus

$$sin\, x = u_1(x) + u_2(x) + \cdots + u_n(x) + R_n(x)$$

and

$$|R_n(x)| = \left|\frac{x^{2n+1}}{(2n+1)!}\right| \leq \frac{|x|^{2n+1}}{(2n+1)!} = |u_{n+1}|.$$

Also,

$$cos\, x = v_1(x) + v_2(x) + \cdots + v_n(x) + R_n(x)$$

and

$$|R_n(x)| \leq |v_{n+1}|.$$

Computing the Values of Sine and Cosine (cont'd)

$$degree + \frac{minutes}{60} + \frac{seconds}{3600}$$

Example: Compute $\sin(23°54')$ within 10^{-4}.

Solution. $x = arc\ 23°54'$

$$23 + \frac{54}{60} = 23.9°$$

$$u_1 = x = \ +0.41714,$$

$$u_2 = -\frac{x^2u_1}{2\times 3} = -0.01210,$$

$$u_3 = -\frac{x^2u_2}{4\times 5} = +0.00011,$$

$$u_4 = -\frac{x^2u_3}{6\times 7} = -0.000002,$$

$$radian = \frac{degree}{180} * \pi$$

$$\frac{23.9}{180} \times \pi = 0.41714$$

$$x = arc\ 23°54' = 0.41714$$

$$\boldsymbol{sin\ 23°54' = 0.4052.}$$

Computing the Values of hyperbolic

We know $\sinh x = \frac{e^x - e^{-x}}{2}$.

$$\sinh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right)$$

The following expansion holds for the hyperbolic sine:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

And the following recurrent notation

$$\sinh x = \sum_{k=1}^{n} u_k + R_n(x)$$

$$u_1 = x$$
$$u_2 = \frac{x^3}{3!} = \frac{x^2}{3 \times 2} u_1$$
$$u_3 = \frac{x^5}{5!} = \frac{x^2}{5 \times 4} u_2$$

where

$$u_1 = x, \qquad u_{k+1} = \frac{x^{2k+1}}{(2k+1)!} = \frac{x^2}{2k(2k+1)} u_k$$

Computing the Values of Hyperbolic Sine (Cont'd)

Since $\sinh(-x) = -\sinh x$, we assume that $x > 0$.
So, for $0 \le x \le n$, we have

$$R_n(x) = \frac{x^{2n+1}}{(2n+1)!} + \frac{x^{2n+3}}{(2n+3)!} + \frac{x^{2n+5}}{(2n+5)!} + \cdots$$

$$< \frac{x^{2n+1}}{(2n+1)!}\left[1 + \frac{x^2}{(2n+2)(2n+3)} + \frac{x^4}{(2n+2)^2(2n+3)^2} + \cdots\right]$$

$$< \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{1}{1 - \frac{x^2}{(2n+2)(2n+3)}}$$

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \qquad r < 1$$

$$< \frac{4}{3}\frac{x^{2n+1}}{(2n+1)!} = \frac{4}{3}u_{n+1} = \frac{4}{3}\frac{x^2}{2n(2n+1)}u_n < \frac{4}{3}\cdot\frac{1}{4}u_n = \frac{1}{3}u_n.$$

$$\frac{1}{1 - \frac{x^2}{(2n+2)(2n+3)}} = \frac{1}{\frac{(2n+2)(2n+3) - x^2}{(2n+2)(2n+3)}} \le \frac{1}{\frac{(2n+2)(2n+3) - n^2}{(2n+2)(2n+3)}}$$
$$\le \frac{1}{1 - \frac{n^2}{(2n+2)(2n+3)}} \le \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

So, $R_n(x) < \frac{1}{3}u_n$.

Computing the Values of Hyperbolic Sine (Cont'd)

Example: Find sinh 1.4 within 10^{-5} .

$$R_n(x) < \frac{1}{3}u_n$$

Solution.

$$u_1 = x, \qquad u_{k+1} = \frac{x^2}{2k(2k+1)}u_k$$

$$u_1 = x = 1.4, \qquad u_2 = \frac{x^2 u_1}{2.3} = 0.4573333,$$

$$u_3 = \frac{x^2 u_2}{4.5} = 0.0448187, \qquad u_4 = \frac{x^2 u_3}{6.7} = 0.\mathbf{00}20915,$$

$$u_5 = \frac{x^2 u_4}{8.9} = 0.0000569, \qquad u_6 = \frac{x^2 u_5}{10.11} = 0.\mathbf{00000}10.$$

$$\boldsymbol{sin\ 1.4} = 1.904301.$$

Computing the Values of Hyperbolic Cosine

We know $\cosh x = \frac{e^x + e^{-x}}{2}$.

The following expansion holds for the hyperbolic sine:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

And the following recurrent notation

$$\cosh x = \sum_{k=1}^{n} v_k + R_n(x)$$

where

$$v_1 = 1, \qquad v_{k+1} = \frac{x^{2k}}{2k!} = \frac{x^2}{(2k-1)2k}v_k \quad (k = 1,2,\dots,n-1)$$

Computing the Values of Hyperbolic Cosine (Cont'd)

Since $cosh(-x) = cosh\, x$, we assume that $x > 0$.

So, for $0 \le |x| \le n$, we have

$$R_n(x) = \frac{x^{2n+1}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)!} + \frac{x^{2n+4}}{(2n+4)!} + \cdots$$

$$< \frac{x^{2n}}{(2n)!}\left[1 + \frac{x^2}{(2n+1)(2n+2)} + \frac{x^4}{(2n+2)^2(2n+3)^2} + \cdots\right]$$

$$< \frac{x^{2n}}{(2n)!} \cdot \frac{1}{1 - \frac{x^2}{(2n+1)(2n+2)}}$$

$$< \frac{4}{3}\frac{x^{2n}}{(2n)!} = \frac{4}{3} v_{n+1} = \frac{4}{3}\frac{x^2}{(2n-1)2n} v_n < \frac{4}{3} \cdot \frac{n}{2(2n-1)} v_n < \frac{4}{3} \cdot \frac{1}{2} v_n = \frac{2}{3} v_n.$$

So, $R_n(x) < \frac{2}{3} v_n$.

$$n \ge 1 \Rightarrow 2n - 1 \ge n \Rightarrow \frac{n}{2(2n-1)} \le \frac{n}{2n}$$

Computing the Values of Logarithmic Functions

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

For the logarithmic function $\ln(1 + x)$, we have the following expansion

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots \qquad (1)$$

for each $-1 \le x \le 1$ holds true.

By replacing x with $-x$ in Equation (1), we have

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^n}{n} - \cdots \qquad (2)$$

By subtracting Eq. (2) from Eq. (1), we obtain

$$\ln\frac{1-x}{1+x} = \ln(1-x) - \ln(1+x) = -2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

Computing the Values of Logarithmic Functions (cont'd)

Note that our target was to calculate the value of function $\ln z$ not $\ln(1+x)$ or $\ln\frac{1-x}{1+x}$.

To achieve it, let $\frac{1-x}{1+x} = z$.

So, we get

$$x = \frac{1-z}{1+z},$$

$$z = \frac{1 - \frac{1-z}{1+z}}{1 + \frac{1-z}{1+z}}$$

and hence,

$$\ln z = -2\left(\frac{1-z}{1+z} + \frac{1}{3}\left(\frac{1-z}{1+z}\right)^3 + \frac{1}{5}\left(\frac{1-z}{1+z}\right)^5 + \cdots\right)$$

for each $0 < z < +\infty$.

Computing the Values of Logarithmic Functions (cont'd)

Now, our concern is to estimate the error of approximating logarithmic function with a polynomial of degree n.

Consider x is a positive number. We represent it as

$$x = 2^m \cdot z$$

where m is an integer and $\frac{1}{2} \le z < 1$. Thus $\ln x = m \ln 2 + \ln z$.

For using $\ln z = -2\left(\frac{1-z}{1+z} + \frac{1}{3}\left(\frac{1-z}{1+z}\right)^3 + \frac{1}{5}\left(\frac{1-z}{1+z}\right)^5 + \cdots\right)$,

let $\frac{1-z}{1+z} = \xi$ where $0 \le \xi \le \frac{1}{3}$.

Thus

$$\ln x = m \ln 2 - 2\left(\xi + \frac{\xi^3}{3} + \frac{\xi^5}{5} + \cdots + \frac{\xi^{2n-1}}{2n-1}\right) - \boldsymbol{R_n}(\xi)$$

Note that the total error of $\ln x$ is equal to sum of error of $m \times \ln 2$ and $R_n(\xi)$

Computing the Values of Logarithmic Functions (cont'd)

Now, we want to calculate the error R_n.

$$R_n(\xi) = 2\left(\frac{\xi^{2n+1}}{2n+1} + \frac{\xi^{2n+3}}{2n+3} + \frac{\xi^{2n+5}}{2n+5} + \cdots\right)$$

$$< 2 \cdot \frac{\xi^{2n+1}}{2n+1}\left(\mathbf{1} + \boldsymbol{\xi^2} + \boldsymbol{\xi^4} + \cdots\right)$$

$$< 2 \cdot \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{\xi^2}} \cdot \frac{\xi^{2n+1}}{2n+1}.$$

For $0 < \xi \le \frac{1}{3}$, we have $\frac{\mathbf{2}}{\mathbf{1}-\boldsymbol{\xi^2}} \le \frac{9}{4}$. So,

$$0 < R_n < \frac{\mathbf{9}}{\mathbf{4}} \cdot \frac{\xi^{2n+1}}{2n+1}.$$

Computing the Values of Logarithmic Functions (cont'd)

To summarize:

$$\ln x = m \ln 2 \; - 2\left(\boldsymbol{\xi} + \frac{\boldsymbol{\xi^3}}{\mathbf{3}} + \frac{\boldsymbol{\xi^5}}{\mathbf{5}} + \cdots + \frac{\boldsymbol{\xi^{2n-1}}}{\mathbf{2n-1}}\right) - R_n$$

and

$$0 < R_n < \frac{9}{4} \cdot \frac{\boldsymbol{\xi^{2n+1}}}{\mathbf{2n+1}}.$$

In order to calculate $\ln x$, let

$$u_k = \frac{\xi^{2k-1}}{2k-1} \qquad (k = 1,2,\cdots).$$

$$u_1 = \xi$$
$$u_2 = \frac{\xi^3}{3}$$

Thu,

$$\ln x = m \ln 2 \; - 2\left(\boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3} + \cdots + \boldsymbol{u_n}\right) - \boldsymbol{R_n}$$

where $\ln 2 = 0.6931718\cdots$ .Also

$$0 < R_n < \frac{9}{4} \cdot \frac{\xi^{2n+1}}{2n+1} \le \frac{9}{4} \cdot \frac{\boldsymbol{\xi^2}}{\mathbf{1}} \cdot \frac{\boldsymbol{\xi^{2n-1}}}{\mathbf{2n-1}} \le \frac{1}{4} \cdot u_n \Longrightarrow \boldsymbol{R_n} < \frac{\mathbf{1}}{\mathbf{4}} \cdot \boldsymbol{u_n}$$

Computing the Values of Logarithmic Functions (cont'd)

Example. Find $\ln 3$ within 10^{-5}.

$R_n < \frac{1}{4} \cdot u_n$

Solution. Let $3 = 2^2 \cdot \frac{3}{4} = 2^2 \cdot 0.75$. Hence, $z = 0.75$ and

$$\xi = \frac{1-z}{1+z} = \frac{0.25}{1.75} = \frac{1}{7} = 0.148571.$$

Hence

$$u_1 = \xi = 0.148571$$
$$u_2 = \frac{\xi^3}{3} = 0.0009718$$
$$u_3 = \frac{\xi^5}{5} = 0.\mathbf{0000}119$$
$$u_4 = \frac{\xi^7}{7} = 0.\mathbf{00000}02$$

$$u_1 + u_2 + u_3 + u_4 = 0.1438410$$

$$\ln 3 = 0.6931718 - 2 \cdot (0.1438410) = 1.09861$$

ANY QUESTIONS?