Numerical Computations

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Chapter's Topics

- Overview
- Bisection Methods
- False Position Method
- Newton-Raphson Method
- Fixed Point Method
- Secant Method

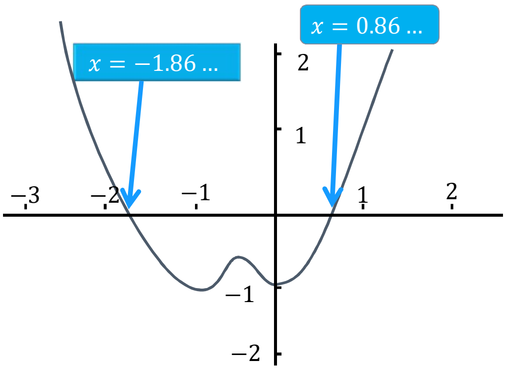
Overview

In this section, we are going to introduce some general methods for computing the root of equation $f(x) = 0$.

Overview: Plotting to Find Roots

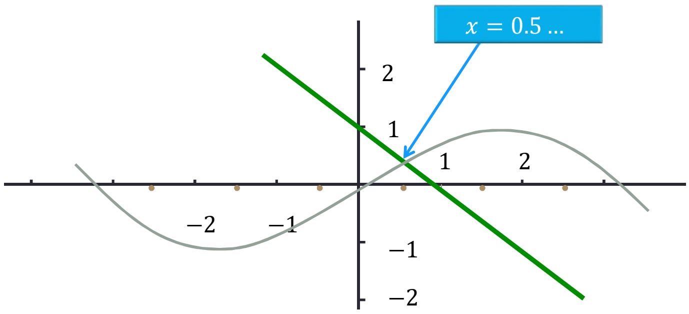
Plot the graph of $f(x)$ and find its intersection with the x-axis.

Example. Finding root(s) for $f(x) = x^4 + 2x^3 - x - 1$.



Overview: Plotting to Find Roots (cont'd)

Example. Finding root(s) for $f(x) = \sin x + x - 1.$



Overview: Plotting to Find Roots (cont'd)

Problem. It is hard to plot $f(x)$.

Solution. Numerical Methods

Overview: Numerical Methods

Our target is to find a root x for function f, i.e.,

$$f(x) = 0.$$

First step: Initially guess an x_0 or an interval (x_0, x_1).

Second (iterative) step: Get the sequence $x_0, x_1, x_2, \dots$.

Third step: Rate of convergence of the sequence $x_0, x_1, x_2, \dots$.

Overview: Numerical Methods (cont'd)

Suppose a sequence of real numbers $x_0, x_1, x_2, \dots$ which converge to some point x.

Question. How fast the number are converging to x?

Definition. If $\lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|^p} = \lambda$ for $0 < \lambda < 1$, then p is called the convergence degree. For convenient, let

$$e(x_0) = |x - x_0| = e_0,$$
$$\vdots$$
$$e(x_i) = |x - x_i| = e_i$$

The convergence degree is p, if

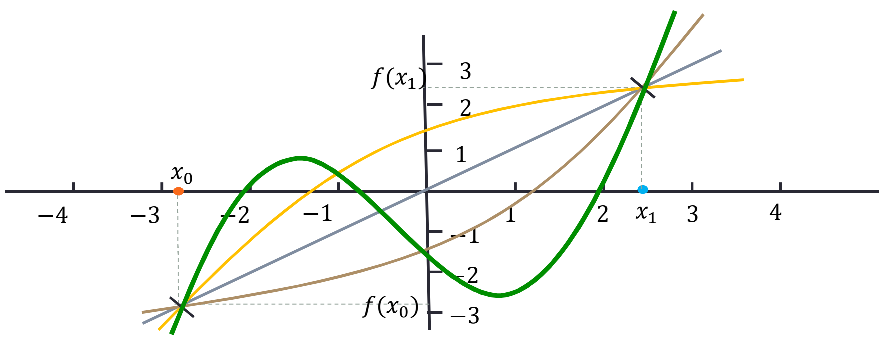
$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^p} = \lambda$$

Bolzano's Theorem

Bernhard Placidus Johann Nepomuk Bolzano
(Bernard Bolzano)
(1781–1848)

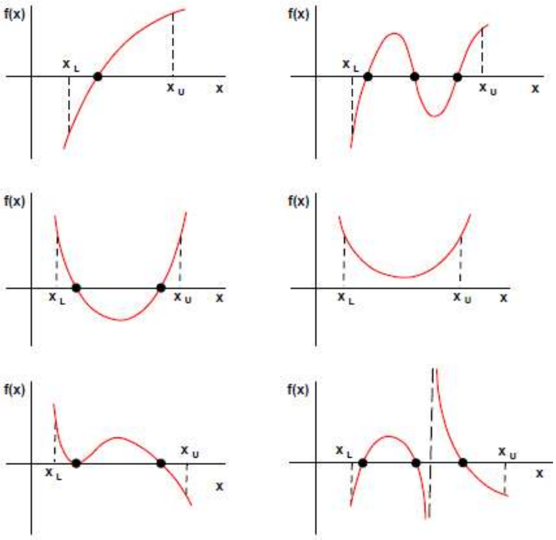
Bolzano's Theorem (1817)

Theorem. Let $f(x)$ be a **continuous** function on the interval $[a, b]$ such that $f(x_0) < 0$ and $f(x_1) > 0$ for some $x_0, x_1 \in [a, b]$. Then it must be 0 at some point.



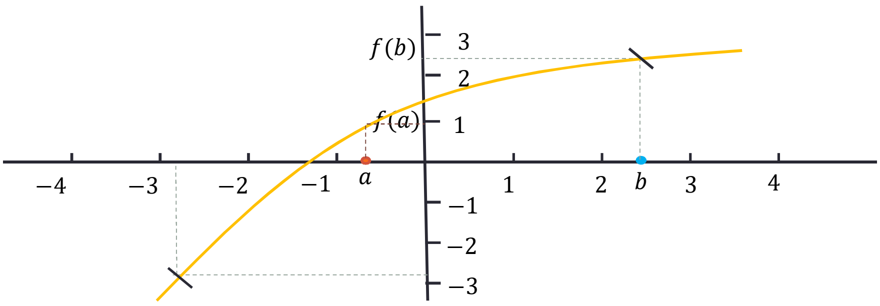
Bolzano's Theorem (cont'd)

Theorem. Let $f(x)$ be a **continuous** function on the interval $[a, b]$ such that $f(x_0) < 0$ and $f(x_1) > 0$ for some $x_0, x_1 \in [a, b]$. Then it must be 0 at some point.

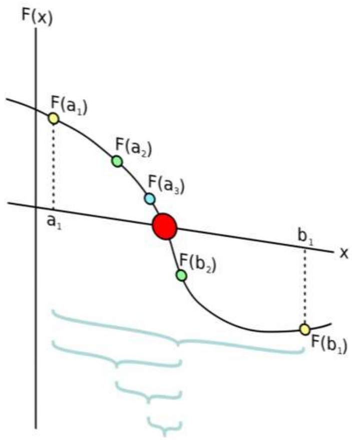


Just for Fun: Intermediate Value Theorem

Theorem. Suppose that $f(x)$ be a real continuous function on the interval $[a, b]$ such that $f(a) \neq f(b)$. For all γ between $f(a)$ and $f(b)$, there is $a < c < b$ for which $f(c) = \gamma$.



Bisection Method



Bisection Method

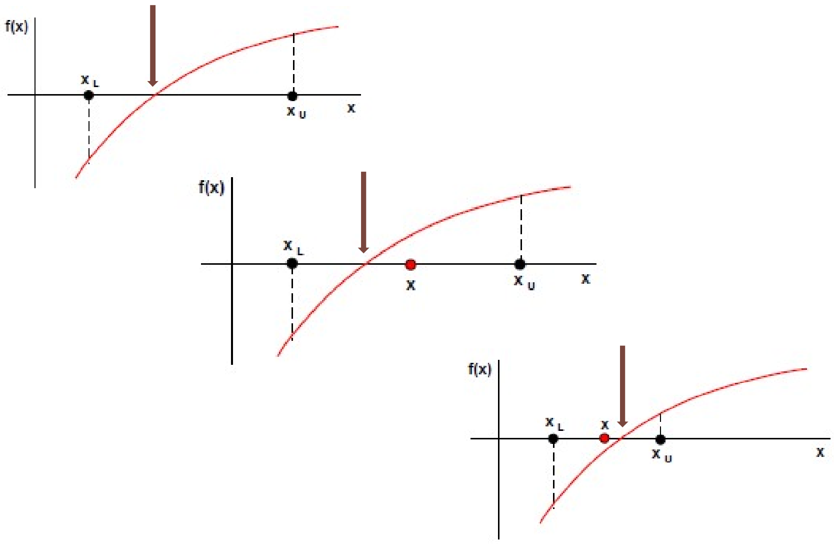
One of the simplest and most reliable iterative methods for the solution of nonlinear equations is the ***Bisection method.***

This method relies on the fact that if

- $f(x)$ is real and continuous in the interval $[a, b]$,
- $f(a)$ and $f(b)$ are of opposite signs, that is,
$$f(a) \times f(b) < 0.$$

Then, by Bolzano's Theorem, there is at least one real root between a and b. Note that there may more than one root in this interval.

Bisection Method (At One Glance)



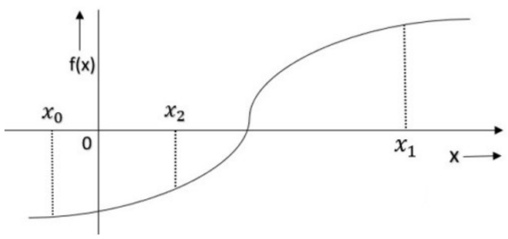
Bisection Method (cont'd)

Let $x_0 = a$, $x_1 = b$ and $x_2 = \frac{x_1+x_2}{2}$.

Now we have one the following conditions:

- $f(x_2) = 0$, so x_0 is a root.
- $f(x_0)f(x_2) < 0$, then there is a root between x_0 and x_2.
- $f(x_1)f(x_2) < 0$, then there is a root between x_1 and x_2.

As a result, we can retrieve the part of $[\mathrm{a}, \mathrm{b}]$ contains the root by testing the sign of the function at midpoint.

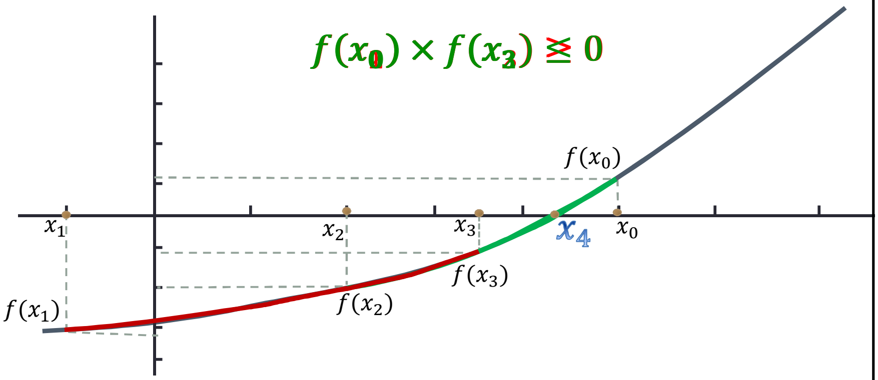


(c) H Sarbazi-Azad & S Hossein Ghorban Numerical Computations - Chapter# 2+: Root-Finding Methods 17

Bisection Method (cont'd)

Example. Find a root for $f(x) = 0.25e^2x - 1.5$.

Solution. Consider $x_0 = 1 \; and \; x_1 = -0.2$



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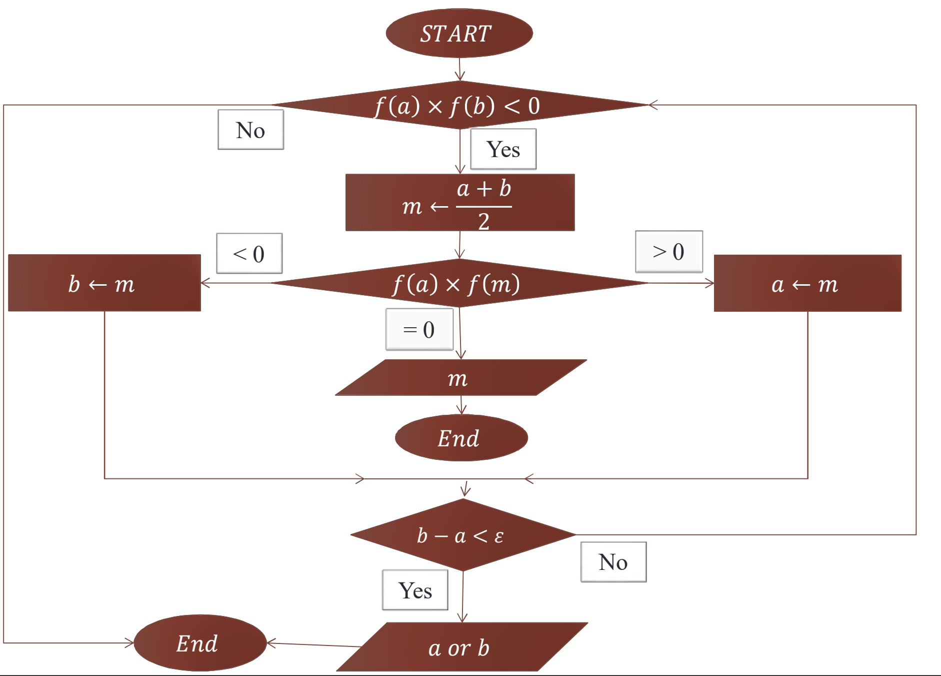


Bisection Method (cont'd)

Example. Find the square root of 11.

Solution. $f(x) = x^2 - 11$.

Initial guesses: $3^2 = 9 < 11, 4^2 = 16 > 11 \rightarrow x_l = 3, x_u = 4$.

Iteration no.	x	f(x)
1	3.5	1.25
2	3.25	-0.4375
3	3.375	0.390625
4	3.3125	-0.02734375
5	3.34375	0.180664062
6	3.328125	0.076416015

Bisection Method (cont'd)

Advantages

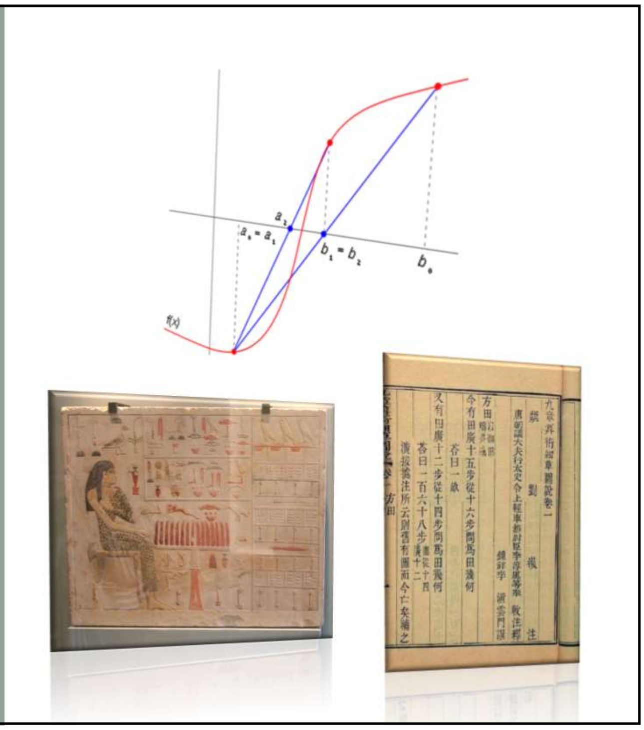
- The Bisection method is always convergent (under the necessary requirements).
- This method is very simple
- There is no need to compute the exact value of $f(x)$. It is only required to know whether f is positive or negative.
- Number of the iterations does not relate to the function.

Disadvantages

- Its convergence rate is very slow.

Remark. Typically this method is used to get an initial estimation for using faster methods such as Newton-Raphson which requires an initial estimation.

False Position Method



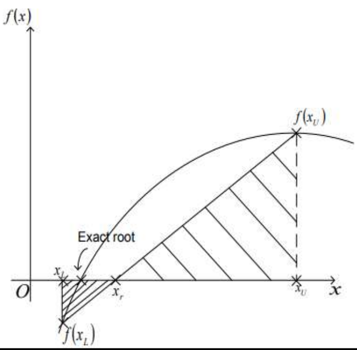
False Position Method

Same as the bisection method, we should identify proper values of x_l (lower bound value) and x_u(upper bound value) for the current bracket, such that

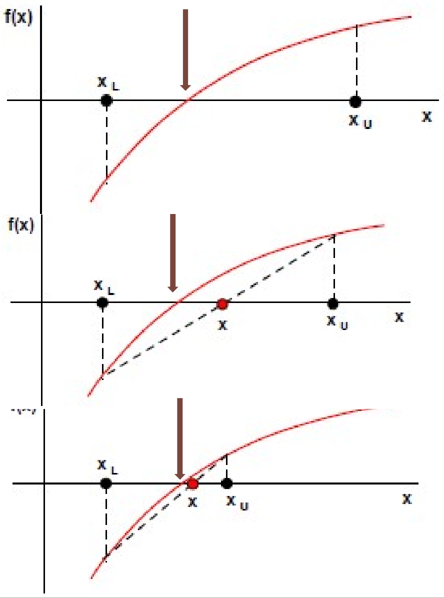
- $f(x)$ is real and continuous in the interval $[x_l, x_u]$,
- $f(x_l) \times f(x_u) < 0$.

The idea of the ***False position method*** is to

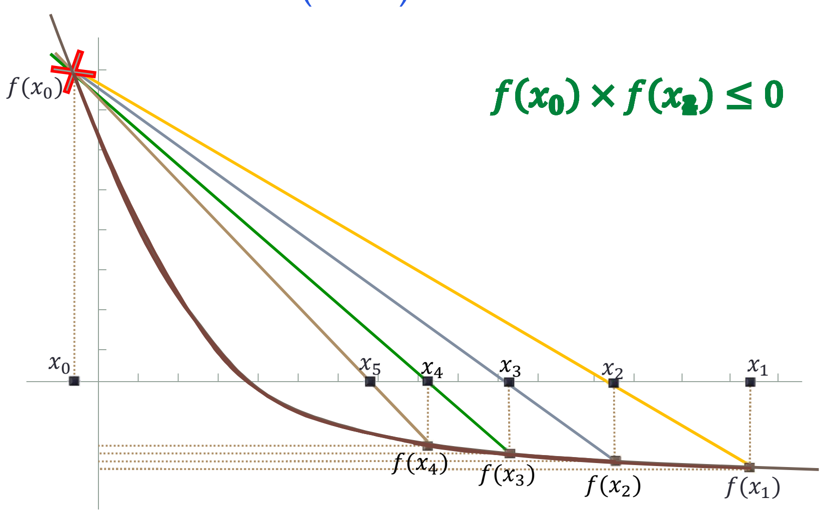
- connect the points $(x_l, f(x_l))$ and $(x_u, f(x_u))$ with a straight line, and
- find the solution of the linear equation connecting the endpoints.



False Position Method (At One Glance)



False Position Method (cont'd)



Why will the algorithm eventually converge to a root?

False Position Method (cont'd)

Example. Solve $2x - \ln x = 7$.

Solution. Let $f(x) = 2x - \ln x - 7$.

Step 1. $\begin{cases} f(4) = -0.38629 = f(x_l) \\ f(4.5) = 0.49592 = f(x_u) \end{cases}$. Let $x_0 = x_l$ and $x_1 = x_u$

Step 2. Find the line between $(x_0, f(x_0))$ and $(x_1, f(x_1))$:

$$\frac{y - f(x_1)}{x - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

So, the root of this line:

$$\frac{0 - 0.49592}{x - 4.5} = \frac{0.49592 + 0.38629}{0.5} \Rightarrow 0.88221(x - 4.5) = -0.24796.$$

Thus $x = 4.281066$. Let $x_2 = x$.

False Position Method (cont'd)

Suppose that sequence approximations $x_0, x_1, x_2, \ldots, x_n$ are calculated.

Now, we find the root of the line between $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$. That means

$$\frac{0 - f(x_n)}{x - x_n} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

Thus, we have

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$

Termination condition: $|x_{n+1} - x_n| < \varepsilon$.

False Position Method (cont'd)

Let α be the exact value of the root of equation $f(x) = 0$,

i.e., $f(\alpha) = 0$.

Let $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$ where e_n and e_{n+1} are errors involved in n^{th} and $(n+1)^{th}$ approximations respectively.

From previous equations $\left(x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)\right)$:

$$\alpha + e_{n+1} = \alpha + e_n - \frac{e_n - e_{n-1}}{f(\alpha + e_n) - f(\alpha + e_{n-1})} f(\alpha + e_n)$$

$$\rightarrow e_{n+1} = e_n - \frac{e_n f(\alpha + e_n) - e_{n-1} f(\alpha + e_n)}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

False Position Method (cont'd)

Using Taylor series around α, we have

$$f(\alpha + h) = f(\alpha) + hf'(\alpha) + \frac{h^2}{2!} f''(\alpha) + \cdots$$

Now consider $h = e_{n-1}$ and $h = e_n$. So

$$e_{n+1} = e_n - \frac{e_n f(\alpha + e_n) - e_{n-1} f(\alpha + e_n)}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

$$= e_n + \frac{e_{n-1}[f(\alpha) + e_n f'(\alpha) + \frac{(e_n)^2}{2!} f''(\alpha) + \cdots] - e_n[f(\alpha) + e_{n-1} f'(\alpha) + \frac{(e_{n-1})^2}{2!} f''(\alpha) + \cdots]}{[f(\alpha) + e_n f'(\alpha) + \frac{(e_n)^2}{2!} f''(\alpha) + \cdots] - [f(\alpha) + e_{n-1} f'(\alpha) + \frac{(e_{n-1})^2}{2!} f''(\alpha) + \cdots]}$$

As $f(\alpha) = 0$:

$$e_{n+1} = e_n + \frac{e_{n-1}[e_n f'(\alpha) + \frac{(e_n)^2}{2!} f''(\alpha) + \cdots] - e_n[e_{n-1} f'(\alpha) + \frac{(e_{n-1})^2}{2!} f''(\alpha) + \cdots]}{[e_n f'(\alpha) + \frac{(e_n)^2}{2!} f''(\alpha) + \cdots] - [e_{n-1} f'(\alpha) + \frac{(e_{n-1})^2}{2!} f''(\alpha) + \cdots]}$$

False Position Method (cont'd)

Simplifying the last equation:

$$e_{n+1} = \frac{\frac{e_{n-1}e_n}{2} f''(\alpha) + \cdots}{f'(\alpha) + (\frac{e_n + e_{n-1}}{2}) f''(\alpha) + \cdots}$$

P = degree of Convergence

Neglecting high powers: $e_{n+1} = \frac{e_n e_{n-1}}{2!} \frac{f''(\alpha)}{f'(\alpha)}$ (2)

Let $e_{n+1} = ce_n^p$, where c is a constant and $p > 0$ (3)

So, $e_n = ce_{n-1}^p$ or $e_{n-1} = c^{-\frac{1}{p}} e_n^{\frac{1}{p}}$ (4)

Substituting (3) and (4) in (2) :

$$ce_n^p = \frac{c^{-\frac{1}{p}}}{2!} e_n^{1+\frac{1}{p}} \frac{f''(\alpha)}{f'(\alpha)} \rightarrow p = 1 + \frac{1}{p} \text{ and } c = \frac{c^{-\frac{1}{P}}}{2!} \frac{f''(\alpha)}{f'(\alpha)}$$

False Position Method (cont'd)

$$p = 1 + \frac{1}{p} \text{ and } c = \frac{c^{-\frac{1}{P}}}{2!} \frac{f''(\alpha)}{f'(\alpha)}$$

Calculating p will give us the degree of convergence.

Solving $p = 1 + \frac{1}{P}$ will give us $p = \frac{1+\sqrt{5}}{2} = 1.618$ ■*

* Proof obtained from *Computer Based Numerical and Statistical Techniques* by Manish Goyal.

False Position Method (cont'd)

Example. Find the square root of 11.
Solution. $f(x) = x^2 - 11$.
Initial guesses: $3^2 = 9 < 11, 4^2 = 16 > 11 \rightarrow x_l = 3, x_u = 4$.

Bisection Method

Iteration no.	x	f(x)
1	3.5	1.25
2	3.25	-0.4375
3	3.375	0.390625
4	3.3125	-0.02734375
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False Position Method

Iteration no.	x	f(x)
1	3.28571429	-0.1040816
2	3.31372549	-0.0192234
3	3.31635389	-0.0017969
4	3.31659949	-0.0001678
5	3.31662243	-0.0000157
6	3.31662457	-0.0000015

False Position Method (cont'd)

Advantages

- Always converges to the answer.
- It converges faster than Bisection method.

Disadvantages

- It is more difficult than bisection method, because the number of operations is more.
- If all of x's (or most of them) are in one side of the root, this method converges slower.

Isaac Newton
(1642 –1726)

Joseph Raphson
(1648-1715)

Newton-Raphson Method

Newton-Raphson Method

In this method, we analyze the behavior of $f(x)$ by using its derivative, $f'(x)$.

Let x_0 be an approximate solution of $f(x) = 0$, and f be differentiable around a neighborhood around x_0.

By Taylor' series, we have:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!} f"(\gamma) \qquad x_0 < \gamma < x.$$

Newton-Raphson Method (cont'd)

Suppose that α is the root of $f(x)$, then

$$0 = f(x_0) + (\alpha - x_0)f'(x_0) + \frac{(\alpha - x_0)^2}{2!}f''(\gamma).$$

Assume that $f'(x_0) \neq 0$, thus,

$$-\frac{f(x_0)}{f'(x_0)} = (\alpha - x_0) + \frac{(\alpha - x_0)^2}{2!}\frac{f''(\gamma)}{f'(x_0)}.$$

If x_0 is near <u>enough</u> to α, then the term $\frac{(\alpha-x_0)^2}{2!}\frac{f''(\gamma)}{f'(x_0)}$ is ignorable.

Thus,

$$\alpha \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

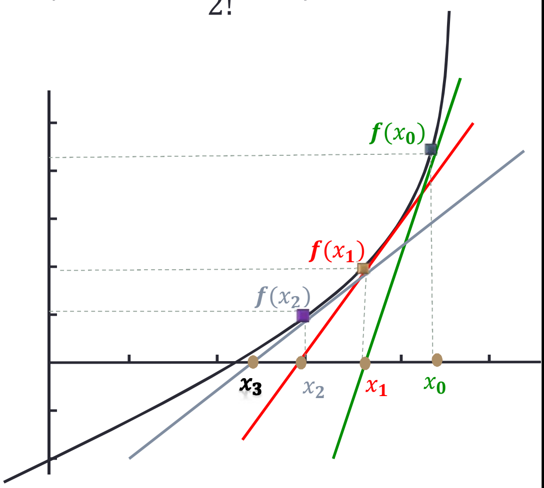
That means $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is an approximation of the root of $f(x)$.

Newton-Raphson Method (cont'd)

$$0 = f(x_{i-1}) + (\alpha - x_{i-1})f'(x_{i-1}) + \frac{(\alpha - x_{i-1})^2}{2!}f''(\gamma).$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$



Newton-Raphson Method (cont'd)

Initial value: x_0

sequence approximations $x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Termination condition: $|x_n - x_{n-1}| < \varepsilon$

Newton-Raphson Method (cont'd)

Rate of Convergence: Let α be the exact value of the root of equation $f(x) = 0$, i.e., $f(\alpha) = 0$.
On substituting $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$ in

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

we obtain

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

Using Taylor expansion:

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)} = e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \cdots}$$

$$= e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \cdots} = \frac{e_n f'(\alpha) + e_n^2 f''(\alpha) + \cdots - e_n f'(\alpha) - \frac{e_n^2}{2} f''(\alpha) - \cdots}{f'(\alpha) + e_n f''(\alpha) + \cdots}$$

$$\approx \frac{\frac{e_n^2}{2} f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)} = \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \frac{f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)} = \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \frac{\mathbf{1}}{\mathbf{1} + e_n \frac{f''(\alpha)}{f'(\alpha)}}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[\mathbf{1} - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots \right] = \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^2}{2} \left[\frac{f''(\alpha)}{f'(\alpha)} \right]^2 + \cdots$$

Newton-Raphson Method (cont'd)

$$e_{n+1} = \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^2}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 + \cdots$$

$$\frac{e_{n+1}}{e_n^2} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{1}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 + \cdots$$

$$\frac{e_{n+1}}{e_n^2} \approx \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \to e_{n+1} \approx \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}e_n^2$$

So

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{2f'(\alpha)}. \blacksquare$$

Degree of Convergence

Newton-Raphson Method (cont'd)

Example. $x^4 - x = 10.$

Solution. By Newton-Raphson's formula, setting $f(x) = x^4 - x - 10.$

We get

$$x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 10}{4x_n^3 - 1},$$

$x_0 = 2,$
$x_1 = 1.871,$
$x_2 = 185578,$
$x_3 = 1.855585,$
$x_4 = 1.85558452522.$

Newton-Raphson Method (cont'd)

Example. $x^4 - x = 10$.

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$$x_0 = 2,$$
$$x_1 = 1.871,$$
$$x_2 = 185578,$$
$$x_3 = 1.855585,$$
$$x_4 = 1.85558452522.$$

Newton-Raphson Method (cont'd)

Example. $e^{-x} = \sin x$.

Solution. By Newton-Raphson's formula, setting $f(x) = e^{-x} - \sin x$.

We get

$$x_{n+1} = x_n + \frac{e^{-x_n} - \sin x_n}{e^{-x_n} + \cos x_n},$$

$$x_0 = 0.6,$$
$$x_1 = 0.5885,$$
$$x_2 = 0.58853274,$$

Newton-Raphson Method (cont'd)

Example. Find the square root of 11.

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Newton-Raphson Method

iter	x	f(x)
0	3	-2
1	3.33333333	0.1111111
2	3.31666667	0.0002778
3	3.31662479	0.0000000

Newton-Raphson Method (cont'd)

Does Newton-Raphson Method Always Work? No!

Example. Let $f(x) = sgn\, x\sqrt{|x|}$.

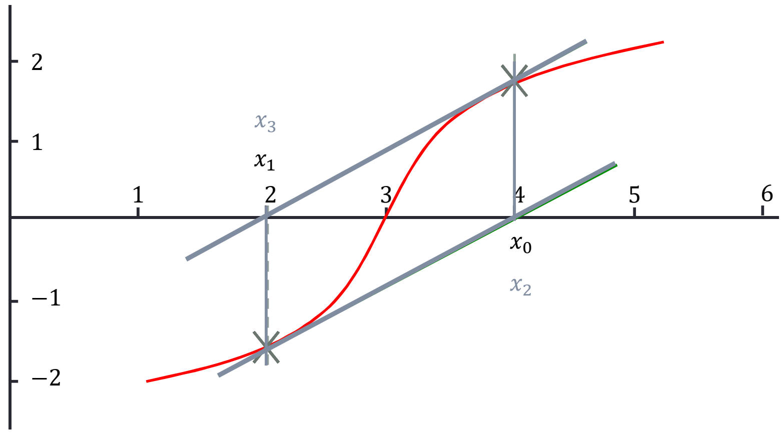
Solution. We have $|x| = x sgn\, x$. So

$$f'(x) = [sgn\, x\sqrt{x sgn\, x}]' = sgn\, x \frac{1}{2\sqrt{x sgn\, x}} sgn\, x = \frac{1}{2\sqrt{|x|}}.$$

Thus

$$x_{n+1} = x_n - \frac{sgn\, x\sqrt{|x|}}{\frac{1}{2\sqrt{|x|}}} = x_n - 2 sgn\, x\left(\sqrt{|x|}\right)^2$$
$$= x_n - 2|x| sgn\, x = x_n - 2x_n = -x_n.$$

Newton-Raphson Method (cont'd)



Newton-Raphson Method (cont'd)

Example. Calculate $\sqrt{5}$ by using Newton-Raphson method

Solution. Newton-Raphson is a good method for calculating $\sqrt[k]{a}$

$$x = \sqrt{5} \rightarrow x^2 = 5$$

$$f(x) = x^2 - 5 = 0 \rightarrow f'(x) = 2x$$

$$x_{i+1} = x_i - \frac{x_i^2 - 5}{2x_i}$$

$$x_{i+1} = \frac{x_i^2 - 5}{2x_i}$$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{5}{x_i}\right)$$

i	x_i
0	2
1	2.25
2	2.236111111
3	2.236067978
4	2.236067978

$$x_{i+1} = \frac{1}{k}\left[(k-1)x_i - a x_i^{1-k}\right]$$

Newton-Raphson Method (cont'd)

Advantages

- It converges very fast (if converge!)

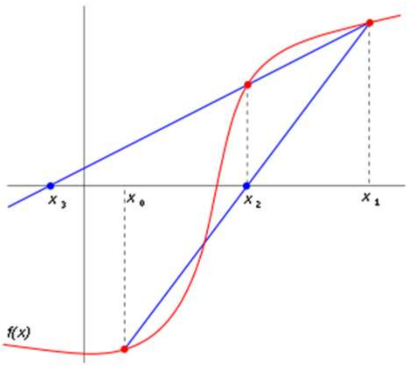
Disadvantages

- No guarantee to converge to answer.
- Calculating derivation may be hard.
- All calculations and convergence of this method strictly depend on the function $f(x)$ and the initial value of x_0.

Important note

- Selecting initial approximation x_0 is very important so it is better to approximate x_0 by plotting $f(x)$.

Secant Method



Secant Method

Although the Newton method is fast, it has a disadvantage.
In each iteration, we should calculate the values of f and f'.

In this method, the value of $f'(x)$ is estimated, i.e.,

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

Thus by substituting in the Newton's formula, we have

$$x_{n+1} = x_n - \left(\frac{h}{f(x_n) - f(x_n - h)}\right) f(x_n)$$

If let $x_n - h = x_{n-1}$, then we obtain

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_n - h)}\right) f(x_n)$$

Secant Method (cont'd)

Secant method is very similar to False-Position method
except that Bolzano's theorem is not needed to be checked.

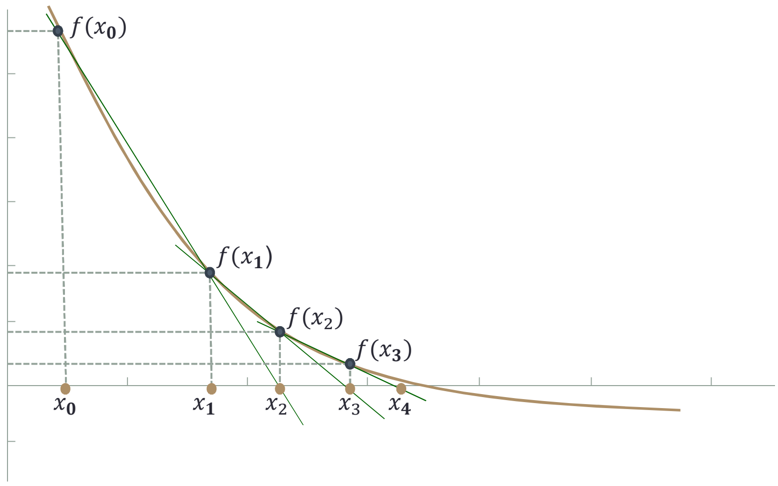
$$x_{n+1} = \frac{x_{n-1} \times f(x_n) - x_n \times f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Initial approximations: interval (x_0 , x_1)

Termination condition: $|x_n - x_{n-1}| < \varepsilon$

Degree of convergence: $p = \frac{1+\sqrt{5}}{2}$

Secant Method (cont'd)



Secant Method (cont'd)

Example. Find the square root of 11.

Solution. $f(x) = x^2 - 11$.

Bisection Method

Iteration no.	x	f(x)
1	3.5	1.25
2	3.25	-0.4375
3	3.375	0.390625
4	3.3125	-0.02734375
5	3.34375	0.180664062
6	3.328125	0.076416015

False Position Method

Iteration no.	x	f(x)
1	3.28571429	-0.1040816
2	3.31372549	-0.0192234
3	3.31635389	-0.0017969
4	3.31659949	-0.0001678
5	3.31662243	-0.0000157
6	3.31662457	-0.0000015

Newton-Raphson Method

iter	x	f(x)
0	3	-2
1	3.33333333	0.1111111
2	3.31666667	0.0002778
3	3.31662479	0.0000000

Secant Method

iter	x	f(x)
-1	2	-7
0	3	-2
1	3.4	0.56
2	3.3125	-0.0273438
3	3.31657356	-0.0003398
4	3.31662482	0.0000002

Secant Method (cont'd)

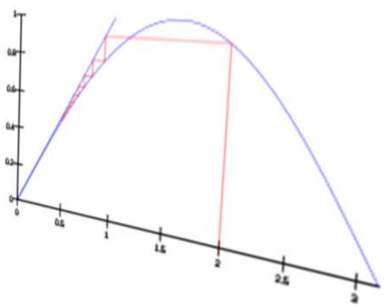
Advantages

- It converges faster than the Bisection method.
- Bolzano's theorem is not needed.

Disadvantages

- It is harder than the Bisection method.
- Sometimes it does not converge.
- If the initial guess is not close to the answer then there is no guarantee it converges to the answer.

Fixed Point Method



Fixed Point Method

Definition. Let $g(x)$ be a real function on $[a, b]$. If there is a point $\alpha \in [a, b]$ such that $g(\alpha) = \alpha$, then α is called a fixed point for $g(x)$.

Example. Let $g(x) = x^2 - 3x + 4$, we have $g(2) = 2$, so 2 is a fixed point for $g(x)$.

Let α be a root for $f(x)$. In Fixed point method, we first rewrite the equation $f(x) = 0$ in the form of

$$x = g(x)$$

In such a way that a fixed point of $x = g(x)$ is a solution of $f(x) = 0$.

Fixed Point Method (cont'd)

Fixed point method's steps:

- Initial approximation: x_0
- Consider the recursive process $x_{n+1} = g(x_n)$.

Question. Under what assumptions on g and x_0, does this method converge? When does the sequence $x_0, x_1, x_2, \ldots$ obtained from the iterative process $x_{n+1} \approx g(x_n)$ converge ($\lim_{n\to\infty} x_{n+1} - g(x_n) = 0$?

Response. Fixed point Theorems!

Fixed Point Method (cont'd)

Theorem. Let $g : [a,b] \rightarrow [a,b]$ be a differentiable function such that

$$|g'(x)| \leq \alpha < 1 \text{ for all } x \in [a,b].$$

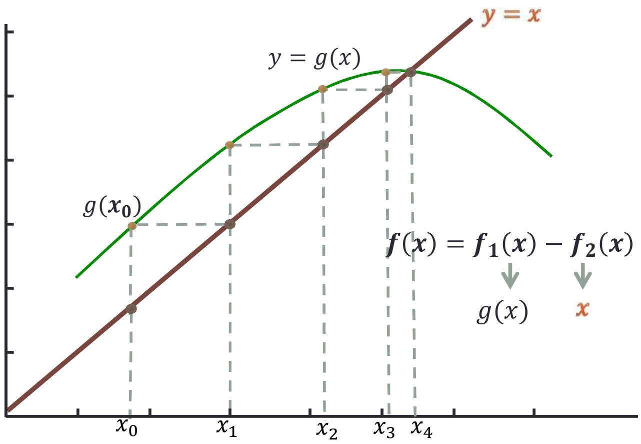
Then g has exactly one fixed point $\xi \in [a,b]$ and the sequence $x_0, x_1, x_2, \ldots$ defined by the iterative process

$$x_{n+1} = g(x_n)$$

with a starting point $x_0 \in [a,b]$, converges to ξ.

Fixed Point Method (cont'd)

How does it work?



Fixed Point Method (cont'd)

Example. Solve $f(x) = 3x - 2e^{-x} = 0$

Solution. We have $f(0) = -2 < 0$ and $f(1) = 3 - \frac{2}{e} > 0$. Thus, $f(x)$ has at least one root in $[0,1]$. The plots of $y = e^{-x}$ and $y = \frac{3}{2}x$ show that this equation has one root.

Let $g(x) = \frac{2}{3}e^{-x}$ which is a differentiable function on $[0,1]$ and $|g'(x)| = \frac{2}{3}e^{-x} \leq \frac{2}{3} < 1$ for all $x \in [a,b]$.

So let $x_0 = 0.5$, then by $x_{n+1} = \frac{2}{3}e^{-x_n}$, we can find an approximation for above equation.

Fixed Point Method (cont'd)

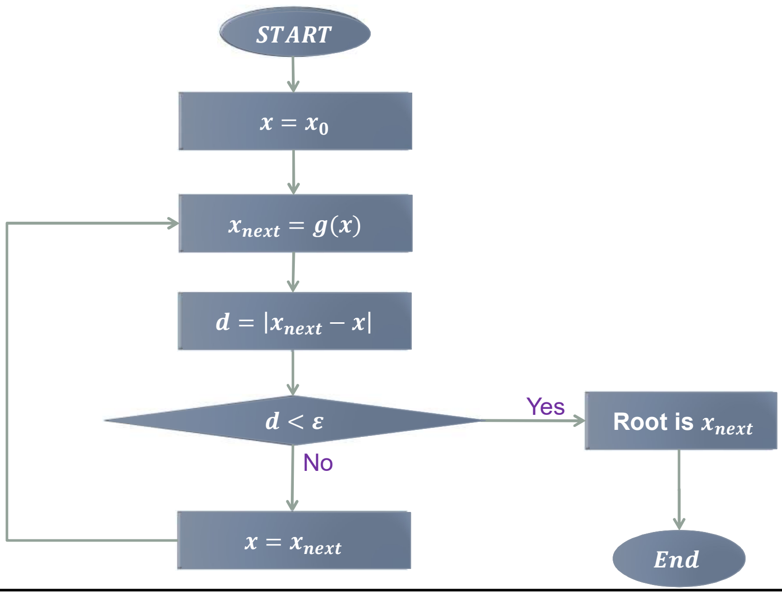
Solution (cont'd). $x_0 = 0.5$, then by $x_{n+1} = \frac{2}{3}e^{-x_n}$, we obtain:

n	x_n	n	x_n
1	0.40435	6	0.43299
2	0.44494	7	0.43238
3	0.42724	8	0.43264
4	0.43487	9	0.43253
5	0.431157	10	0.43258

Thus,

$$|x_{10} - x_9| < 0.0001\,.$$

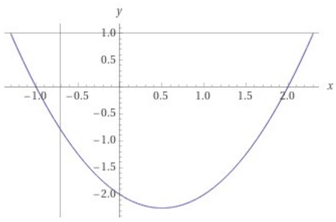
Fixed Point Method (cont'd)



Fixed Point Method (cont'd)

Example. Solve $f(x) = x^2 - x - 2 = 0$.

Solution. If consider $g(x) = x^2 - 2$, then $g'(x) = 2x$. The convergence condition, $|g'(x)| < 1$, mentions $-.05 < x < 0.5$ while plot of $x^2 - x - 2$ is as follows:



But if consider $g(x) = \sqrt{2+x}$, we will have $g'(x) = \frac{1}{2\sqrt{2+x}} \leq \frac{2}{5}$ for all $x > 0$ or $g(x) = 1 + \frac{2}{x}$, so $g'(x) = \frac{-1}{x^2} > 1$ for all $x > \frac{3}{2}$.

Fixed Point Method (cont'd)

Advantages

- None!

Disadvantages

- No guarantee to converge to answer.
- Convergence conditions for this method are hard to be checked.
- The degree of convergence for this method is linear and equal to one.
- All calculations and convergence of this method strictly depend on the function $g(x)$ and initial value x_0.

ANY QUESTIONS?