Numerical Computations

Hamid Sarbazi-Azad &
Samira Hossein Ghorban
Department of Computer Engineering
Sharif University of Technology (SUT)
Tehran, Iran

Chapter's Topics

- Overview of Systems of Linear Equations
- Solution by inverse matrices (Cramer's rule)
- The Gaussian Method
- The Gaussian Compact Method
- The Modification of Crout-Doolittle
- The Method of Principal Elements
- The Scheme of Khaletsky
- Method of successive approximations
  - Jacobi's Method
  - Gauss-Siedel Method

Overview

Solving systems of linear equations may be required for solving problems with different variables in different fields.

Methods of solving systems of linear equations are divided mainly into two groups:

Exact methods: Computations are carried out exactly; thus they yield exact values of the unknowns variables.

Iterative methods: Begin with an initial guess and improve the answer in each iteration.

Systems of Linear Equations

Suppose we have a system of n linear equations in n unknown variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1(n+1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2(n+1)} \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n(n+1)} \end{cases}$$

Denote the matrix of the coefficients, variables, constant term of the above system, respectively, by A, b and x where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \mathrm{b} = \begin{bmatrix} a_{1(n+1)} \\ a_{2(n+1)} \\ \cdots \\ a_{n(n+1)} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}.$$

So, this system can be written in the matrix form of $\boldsymbol{Ax} = \boldsymbol{b}$

Solution by inverse matrices (Cramer's rule)

If matrix A is non-singular, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0$$

then, the system has a unique solution, i.e.,

$$x = A^{-1}\, b.$$

Cramer's rule: The jth component of $x = A^{-1}\, b$ is the ratio

$$x_j = \frac{\det B_j}{\det A}$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & \boldsymbol{b_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \boldsymbol{b_2} & \cdots & a_{2n} \\ \dots & \dots & \cdots & \vdots & \cdots & \dots \\ a_{n1} & a_{n2} & \cdots & \boldsymbol{b_n} & \cdots & a_{nn} \end{bmatrix}$$

Systems of Linear Equations (cont'd)

But Cramer's rule as a method of solving a linear system with n unknowns leads to computing $n + 1$ determinants, which is quite a laborious operation especially when the number n is rather large.

It is shown that the complexity of computing determinant is $\mathcal{O}(n^{\omega})$ for some $\omega \geq 2$.

The fastest matrix-multiplication algorithms (Coppersmith-Winograd) can be used with $\mathcal{O}(n^{\sim 2.376})$ arithmetic operations, but use heavy mathematical tools and are often impractical.

The Gaussian Method

Carl Fredrich Gauss
(1777-1855)

The Gaussian Method

Example.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 1x_2 + 4x_3 = 8 \\ -x_1 + 8x_2 + 2x_3 = 12 \end{cases}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 8 \\ -1 & 8 & 2 & 12 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 + R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 10 & 5 & 18 \end{array}\right] \xrightarrow{R_3 + 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right] \Longrightarrow x_3 = 10, \quad x_2 = -\frac{16}{5}, \quad x_1 = -\frac{88}{5}$$

The Gaussian Method (cont'd)

The most common technique for the solution of systems of linear equations is via an algorithm for the successive elimination of the unknowns. This method is called the Gaussian method.

For the below system of 4 linear equations:

$$\begin{cases} \boldsymbol{a_{11}}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = a_{15} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = a_{25} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = a_{35} \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = a_{45} \end{cases}$$

Let $a_{11} \neq 0$ (the leading element).

Dividing the first equation of the system by a_{11}, we get

$$x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15}$$

The Gaussian Method (cont'd)

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = a_{25} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = a_{35} \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = a_{45} \end{cases}$$

Using the previous equation, eliminate the unknown x_1 from the second, third, and fourth equations of the system.

We get a system consisting of three equations as:

$$\begin{aligned} a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + a_{24}^{(1)}x_4 &= a_{25}^{(1)} \\ a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + a_{34}^{(1)}x_4 &= a_{35}^{(1)} \\ a_{42}^{(1)}x_2 + a_{43}^{(1)}x_3 + a_{44}^{(1)}x_4 &= a_{45}^{(1)} \end{aligned}$$

$$a_{ij}^{(1)} = a_{ij} - a_{i1}b_{1j} \; (i = 2,3,4;\; j = 2,3,4,5)$$

The Gaussian Method (cont'd)

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ 0x_1 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + a_{24}^{(1)}x_4 = a_{25}^{(1)} \\ 0x_1 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + a_{34}^{(1)}x_4 = a_{35}^{(1)} \\ 0x_1 + a_{42}^{(1)}x_2 + a_{43}^{(1)}x_3 + a_{44}^{(1)}x_4 = a_{45}^{(1)} \end{cases}$$

Dividing the first equation of the new system by the leading element $a_{22}^{(1)}$, we get

$$x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)}$$

$$b_{2j}^{(1)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \text{ for } j = 3, 4, 5.$$

The Gaussian Method (cont'd)

Thus

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ 0x_1 + x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)} \\ 0\ x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = a_{35} \\ 0\ x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = a_{45} \end{cases}$$

Eliminating x_2 in the same way that we eliminated x_1, we arrive at the following new system of equations:

$$a_{33}^{(2)}x_3 + a_{34}^{(2)}x_4 = a_{35,}^{(2)}$$

$$a_{43}^{(2)}x_3 + a_{44}^{(2)}x_4 = a_{45,}^{(2)}$$

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)}b_{2j}^{(1)} (i = 3,4;\ j = 3,4,5)$$

The Gaussian Method (cont'd)

Thus

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ 0x_1 + x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)} \\ 0\ x_1 + 0x_2 + a_{33}^{(2)}x_3 + a_{34}^{(2)}x_4 = a_{35,}^{(2)} \\ 0\ x_1 + 0x_2 + a_{43}^{(2)}x_3 + a_{44}^{(2)}x_4 = a_{45,}^{(2)} \end{cases}$$

Dividing the first equation by the leading element $a_{33}^{(2)}$ we get

$$x3 + b_{34}^{(2)}x4 = b_{35}^{(2)}$$

$$b_{3j}^{(2)} = \frac{a_{3j}^{(2)}}{a_{33}^{(2)}} \ (j = 4{,}5)$$

The Gaussian Method (cont'd)

Thus

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ 0x_1 + x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)} \\ 0\ x_1 + 0x_2 + x_3 + b_{34}^{(2)}x4 = b_{35}^{(2)} \\ 0\ x_1 + 0x_2 + a_{43}^{(2)}x_3 + a_{44}^{(2)}x_4 = a_{45,}^{(2)} \end{cases}$$

We eliminate x_3 from the second equation. We get the equation

$$a_{44}^{(3)}x_4 = a_{45}^{(3)}$$

$$a_{4j}^{(3)} = a_{4j}^{(2)} - a_{43}^{(2)}b_{3j}^{(2)} \text{ for } (j = 4, 5)$$

$$\begin{cases} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15} \\ 0x_1 + x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)} \\ 0\ x_1 + 0x_2 + x_3 + b_{34}^{(2)}x4 = b_{35}^{(2)} \\ 0\ x_1 + 0x_2 + 0x_3 + a_{44}^{(3)}x_4 = a_{45}^{(3)} \end{cases}$$

The Gaussian Method (cont'd)

We have reduced system to an equivalent system with a triangular matrix (left) that can easily be solved (right):

$$\begin{aligned} x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 &= b_{15} \\ x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 &= b_{25}^{(1)} \\ x_3 + b_{34}^{(2)}x_4 &= b_{35}^{(2)} \\ a_{44}^{(3)}x_4 &= a_{45}^{(3)} \end{aligned}$$

$$\begin{aligned} x_4 &= a_{45}^{(3)} / a_{44}^{(3)} \\ x_3 &= b_{35}^{(2)} - b_{34}^{(2)}x_4 \\ x_2 &= b_{25}^{(1)} - b_{24}^{(1)}x_4 - b_{23}^{(1)}x_3 \\ x_1 &= b_{15} - b_{14}x_4 - b_{13}x_3 - b_{12}x_2 \end{aligned}$$

The Gaussian Method (cont'd)

How many separate arithmetical operations does elimination require, for n equations in Type equation here.

$$\begin{cases} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots & = & \vdots \\ a_{n1}x_1 & + & \cdots & + & a_{nn}x_n & = & b_n \end{cases} \qquad \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right]$$

Left Side	$(n^2 - n) + ((n-1)^2 - (n-1)) + \ldots + 1 = \frac{n^3 - n}{3}$
Right Side	$(n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}$
Solution	$1 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$
Total	$\frac{n^3-n}{3} + n^2 + n \simeq \frac{1}{3}n^3$

The Gaussian Method (cont'd)

Example.

$$2.0x_1 + 1.0x_2 - 0.1x_3 + 1.0x_4 = 2.7,$$
$$0.4x_1 + 0.5x_2 + 4.0x_3 - 8.5x4 = 21.9,$$
$$0.3x_1 - 1.0x_2 + 1.0x_3 + 5.2x4 = -3.9,$$
$$1.0x_1 + 0.2x_2 + 2.5x_3 - 1.0x4 = 9.9,$$

Dividing the first equation of the system by $a_{11} = 2$, we get

$$x_1 + 0.5\, x_2 - 0.05\, x_3 = 1.35$$

The Gaussian Method (cont'd)

Compute the coefficients $a_{ij}^{(1)}$ and form the new system. For $i = 2$, we have

$$a_{22}^{(1)} = a_{22} - a_{21}b_{12} = 0.5 - 0.4 \times 0.5 = 0.3,$$

$$a_{23}^{(1)} = a_{23} - a_{21}b_{13} = 4 + 0.4 \times 0.05 = 4.02,$$

$$a_{24}^{(1)} = a_{24} - a_{21}b_{14} = -8.5 - 0.4 \times 0.5 = -8.7,$$

$$a_{25}^{(1)} = a_{25} - a_{21}b_{15} = 21.9 - 0.4 \times 1.35 = 21.36.$$

For $i = 3$ and 4 computations are performed in a similar way.

The Gaussian Method (cont'd)

Thus, we get a system in three unknowns:

$$0.3x_2 + 4.02x_3 - 8.7x_4 = 21.36,$$
$$-1.15x_2 + 1.015x_3 + 5.05x_4 = -4.305,$$
$$-0.3x_2 + 2.55x_3 - 1.5x_4 = 8.55.$$

Dividing the first equation of the obtained system by $a_{22}^{(1)} = 0.3$ we get

$$x_2 + 13.4\, x_3 - 29.00\, x_4 = 71.20$$

$$b_{23}^{(1)} = 13.40; \quad b_{24}^{(1)} = -29.00; \quad b_{25}^{(1)} = 71.20$$

The Gaussian Method (cont'd)

Compute the coefficients $a_{ij}^{(2)}$ and form the new system. For $i = 3$, we have

$$a_{33}^{(2)} = a_{33}^{(1)} - a_{32}^{(1)} b_{23}^{(1)} = 1.015 + 1.15 \times 13.40 = 16.425,$$

$$a_{34}^{(2)} = a_{34}^{(1)} - a_{32}^{(1)} b_{24}^{(1)} = 5.05 - 1.15 \times 29.00 = -28.300,$$

$$a_{35}^{(2)} = a_{35}^{(1)} - a_{32}^{(1)} b_{25}^{(1)} = -4.305 + 1.15 \times 71.20 = 77.575.$$

For $i = 4$ computations are performed in a similar way. We get a system in two unknowns:

$$16.425x_3 - 28300x_4 = 77.575,$$

$$6.570x_3 - 10.200x_4 = 29.910.$$

The Gaussian Method (cont'd)

Dividing the first equation of the obtained system by $a_{33}^{(2)} = 16.425$ we get:

$$x_3 - 1.7229\,x_4 = 4.72298$$

$$b_{34}^{(2)} = -1.72298; \qquad b_{35}^{(2)} = 4.72298$$

Find the coefficients $a_{4j}^{(3)}$:

$$a_{44}^{(3)} = a_{44}^{(2)} - a_{43}^{(2)} b_{34}^{(2)} = -10.200 + 6.570 \times 1.72298 = 1.11998,$$

$$a_{45}^{(3)} = a_{45}^{(2)} - a_{43}^{(1)} b_{35}^{(2)} = 29.910 - 6.570 \times 4.72298 = -1.11998$$

Write one equation in one unknown:

$$1.11998\,x_4 = -1.11998$$

The Gaussian Method (cont'd)

The equivalent system is

$$\begin{aligned} x1 + 0.5x_2 - 0.05x_3 + 0.5x_4 &= 1.35 \\ x_2 + 13.40x_3 - 29.00x_4 &= 71.20 \\ x_3 - 1.72298x_4 &= 4.72298 \\ 1.11998x_4 &= -1.11998 \end{aligned}$$

Reverse procedure:

$$\begin{gathered} x_4 = -1.00000, \\ x_3 = 4.72298 - 1.72298 = 3.00000, \\ x_2 = 71.20 - 13.40 \times 3 + 29.0 = 2.00000, \\ x_1 = 1.35 - 0.5 \times 2 + 0.05 \times 3 + 0.5 = 1.00000. \end{gathered}$$

The Gaussian Method (cont'd)

The computations are checked by so-called "check sums".

For each $1 \le j \le n$, let

$$a_{i(n+2)} = \sum_{j=1}^{n} a_{ij} + a_{i(n+1)}$$

The system equation: $\boldsymbol{Ax} = \boldsymbol{b}$

$$\text{b} = \begin{bmatrix} a_{1(n+1)} \\ a_{2(n+1)} \\ \dots \\ a_{n(n+1)} \end{bmatrix}$$

After finding the value of unknown variables $x_1, \dots, x_n$, we obtain

$$\sum_{j=1}^{n} a_{ij}(x_j+1) = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} + a_{i(n+1)} = a_{i(n+2)}.$$

for each $1 \le i \le n$. Then in the absence of errors in the computations, the sums of elements of the rows of the matrix of original system including the constant terms serves as a check on this direct procedure!!!

The Gaussian Compact Method

Statue of Gauss at his birthplace, Brunswick

The Gaussian Compact Method

If computations are performed on a key computer by means of the scheme of unique division, then much time is spent on recording intermediate results. The Gaussian compact scheme provides an economical method of recording.

Let us consider the sequence of forming this scheme for previous system. All the results of computations we shall enter into one table.

The Gaussian Compact Method (cont'd)

$n = 5$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	$a_{i6} = a_{i(n+2)} = \sum_{j=1}^{5} a_{ij}$
I	1						
	2						
	3						
	4						
II	2						
	3						
	4						
III	3						
	4						
	4						
IV							
V							

The Gaussian Compact Method (cont'd)

Consider a system of 4 equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = a_{15} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = a_{25} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = a_{35} \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = a_{45} \end{cases}$$

Direct procedure: Write down the coefficients of the given system in four rows and five columns in the following table:

	i	a_{i1}	A_{i2}	A_{i3}	a_{i4}	a_{i5}	$a_{i6} = \sum_{j=1}^{5} a_{ij}$
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum_{j=1}^{5} a_{1j}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum_{j=1}^{5} a_{2j}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum_{j=1}^{5} a_{3j}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum_{j=1}^{5} a_{4j}$

The Gaussian Compact Method (cont'd)

Divide all the numbers of the first row by a_{11} and enter the results into the fifth row of the given system of equations:

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	$\mathbf{b_{12}}$	$\mathbf{b_{13}}$	$\mathbf{b_{14}}$	$\mathbf{b_{15}}$	$\mathbf{b_{16} = a_{16}/a_{11}}$

$b_{1j} = \frac{a_{1j}}{a_{11}}$ for $j = 2, 3, 4, 5$

$\frac{a_{16}}{a_{11}} = \frac{a_{11} + \sum_{j=2}^{5} a_{1j}}{a_{11}} = 1 + \sum_{j=2}^{5} b_{1j}$

Compute $\sum \boldsymbol{b_{1j}}$ and carry out a check. If the computations are performed with a constant number of decimal digits, then should not differ by more than one unit of the last retained digit; otherwise, check the operations of the last step.

The Gaussian Compact Method (cont'd)

Using this formula, compute the coefficients

$$a_{ij}^{(1)} = a_{ij} - a_{i1}b1j \quad (i = 2, 3, 4; \qquad j = 2, 3, 4, 5)$$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{15}	$a_{16}/a_{11} = b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$

Make a check. The sum of the elements of each row

$\sum a_{ij}^{(1)} (i = 2,3,4)$ must not differ from $\boldsymbol{a_{i6}^{(1)}}$ by more than one unit of the last retained digit.

The Gaussian Compact Method (cont'd)

Divide all elements of the first row of Section II by $\boldsymbol{a_{22}^{(1)}}$ and write down the results in the fourth row of last part:

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{16}	$a_{16}/a_{11} = b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$
			1	$b_{23}^{(1)}$	$b_{24}^{(1)}$	$b_{25}^{(1)}$	$a_{26}^{(1)}/ a_{22}^{(1)}=b_{26}^{(1)}$

Check.

The Gaussian Compact Method (cont'd)

Using this formula, compute the coefficients. Then check.

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)} b_{2j}^{(1)} \qquad (i = 3, 4; \qquad j = 3, 4, 5)$$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{16}	$a_{16}/a_{11} = b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$
			1	$b_{23}^{(1)}$	$b_{24}^{(1)}$	$b_{25}^{(1)}$	$a_{26}^{(1)}/\ a_{22}^{(1)}=b_{26}^{(1)}$
III	3			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
	4			$a_{43}^{(2)}$	$a_{44}^{(2)}$	$a_{45}^{(2)}$	$a_{46}^{(2)}$

Check.

The Gaussian Compact Method (cont'd)

Divide the elements of the first row of Section III by $a_{33}^{(2)}$ and write down the results in the third row of Section III . Make a check.

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{16}	$a_{16}/a_{11} = b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$
			1	$b_{23}^{(1)}$	$b_{24}^{(1)}$	$b_{25}^{(1)}$	$a_{26}^{(1)}/\ a_{22}^{(1)}=b_{26}^{(1)}$
III	3			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
	4			$a_{43}^{(2)}$	$a_{44}^{(2)}$	$a_{45}^{(2)}$	$a_{46}^{(2)}$
	4			1	$b_{34}^{(2)}$	$b_{35}^{(2)}$	$a_{36}^{(2)}/\ a_{33}^{(2)}=b_{36}^{(2)}$

The Gaussian Compact Method (cont'd)

Compute $a_{4j}^{(3)}$ and enter the results in Section IV.

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{15}	$a_{16}/a_{11} = b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$
			1	$b_{23}^{(1)}$	$b_{24}^{(1)}$	$b_{25}^{(1)}$	$a_{26}^{(1)}/a_{22}^{(1)}=b_{26}^{(1)}$
III	3			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
	4			$a_{43}^{(2)}$	$a_{44}^{(2)}$	$a_{45}^{(2)}$	$a_{46}^{(2)}$
	4			1	$b_{34}^{(2)}$	$b_{35}^{(2)}$	$a_{36}^{(2)}/a_{33}^{(2)}=b_{36}^{(2)}$
IV					$a_{44}^{(3)}$	$a_{45}^{(3)}$	$a_{46}^{(3)}$

The Gaussian Compact Method (cont'd)

Reverse procedure

1. Write down the unities in Section V as is indicated in Table.

2. Compute x_4.

3. For computing the values x_3, x_2, x_1 use only the rows of Sections I, II, III containing units, beginning with the last.

Thus, to compute x_3 multiply x_4 by $b_{34}^{(2)}$ and subtract the result from $b_{35}^{(2)}$. The units put in Section V help us to find the corresponding coefficients for x_i for $(i = 3, 2, 1)$ in the marked rows.

$$x_3 = b_{35}^{(2)} - b_{34}^{(2)} x_4$$

The Gaussian Compact Method (cont'd)

Reverse procedure (cont'd)

4. Compute x_2 for which purpose use the elements of the marked row of Section II:

$$x_2 = b_{25}^{(1)} - b_{24}^{(1)}\, x_4 - b_{23}^{(1)}\, x_3$$

5. Compute x_1 for which purpose use the elements of the marked row of Section I:

$$x_1 = b_{15} - b_{14}x_4 - b_{13}\, x_3 - b_{12}x_2$$

- In the check scheme the reverse procedure is carried out in a similar way. The solutions of this scheme must differ from those of the given scheme by 1.

$$\overline{x}_i = x_i + 1.\ (i = 1, 2, 3, 4)$$

$$\sum_{j=1}^{n} a_{ij}(x_j+1) = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} + a_{i(n+1)} = a_{i(n+2)}.$$

The Gaussian Compact Method (cont'd)

Example.

Direct procedure

I.

$$\left.\begin{aligned}
2.0x_1 + 1.0x_2 - 0.1x_3 + 1.0x_4 &= 2.7,\\
0.4x_1 + 0.5x_2 + 4.0x_3 - 8.5x_4 &= 21.9,\\
0.3x_1 - 1.0x_2 + 1.0x_3 + 5.2x_4 &= -3.9,\\
1.0x_1 + 0.2x_2 + 2.5x_3 - 1.0x_4 &= 9.9,
\end{aligned}\right\}$$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1						
	2						
	3						
	4						

The Gaussian Compact Method (cont'd)

2. Compute the sums of the coefficients along the row.

$$\sum_{i=1}^{5} a_{1j} = 2.0 + 1.0 - 0.1 + 1.0 + 2.7 = 6.6$$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30

3. Divide all the numbers of the first row by $a_{11} = 2.0$ and enter the results into the fifth row of Section I.

4. Checking: computing the sum of the first five numbers obtained in (3), we get 3.30, which completely coincides with the number obtained in the last column.

The Gaussian Compact Method (cont'd)

5. Using this formula, compute the coefficients

$$a_{ij}^{(1)} = aij - ai_1 b_{1j} \quad (i = 2, 3, 4; \qquad j = 2, 3, 4, 5)$$

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i1}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30
II	2		0.30	4.02	-8.70	21.36	16.98
	3		-1.15	1.015	5.05	-4.305	0.610
	4		-0.30	2.55	-1.50	8.55	9.300

6. Make a check.

The Gaussian Compact Method (cont'd)

7. Divide all elements of the first row of Section II by $a_{22}^{(1)}$=0.3 and write down the results in the fourth row of Section II. Then Check.

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30
II	2		0.30	4.02	-8.70	21.36	16.98
	3		-1.15	1.015	5.05	-4.305	0.610
	4		-0.30	2.55	-1.50	8.55	9.300
			1	13.40	-29.00	71.20	56.60

The Gaussian Compact Method (cont'd)

8. Using this formula, compute the coefficients and check.

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)} b_{2j}^{(1)} \quad (i = 3, 4; \quad j = 3, 4, 5)$$

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30
II	2		0.30	4.02	-8.70	21.36	16.98
	3		-1.15	1.015	5.05	-4.305	0.610
	4		-0.30	2.55	-1.50	8.55	9.300
			1	13.40	-29.00	71.20	56.60
III	3			16.425	-28.300	77.575	65.700
	4			6.570	-10.200	29.910	26.280

The Gaussian Compact Method (cont'd)

9. Divide the elements of the first row of Section III by $a_{33}^{(2)} = 16.425$ and write down the results third row of Section III. Then make a check.

Section	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30
II	2		0.30	4.02	-8.70	21.36	16.98
	3		-1.15	1.015	5.05	-4.305	0.610
	4		-0.30	2.55	-1.50	8.55	9.300
			1	13.40	-29.00	71.20	56.60
III	3			16.425	-28.300	77.575	65.700
	4			6.570	-10.200	29.910	26.280
	4			1	-1.72298	4.72298	4.00000

The Gaussian Compact Method (cont'd)

10. Compute $a_{4j}^{(3)}$ and enter the results in Section IV. Then make a check.

Section	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	2.0	1.0	-0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	-8.5	21.9	18.3
	3	0.3	-1.0	1.0	5.2	-3.9	1.6
	4	1.0	0.2	2.5	-1.0	9.9	12.6
		1	0.50	-0.05	0.50	1.35	3.30
II	2		0.30	4.02	-8.70	21.36	16.98
	3		-1.15	1.015	5.05	-4.305	0.610
	4		-0.30	2.55	-1.50	8.55	9.300
			1	13.40	-29.00	71.20	56.60
III	3			16.425	-28.300	77.575	65.700
	4			6.570	-10.200	29.910	26.280
	4			1	-1.72298	4.72298	4.00000
IV					1.11998	−1.11998	0

The Gaussian Compact Method (cont'd)

Reverse procedure

Following the sequence of operations in the reverse procedure, we get the values of the unknowns as

$$x_4 = -1.0000;$$
$$x_3 = 3.00000;$$
$$x_2 = 2.0000;$$
$$x_1 = 1.0000;$$

The solution of the check system:

$$\bar{x}_4 = 0.00000;$$
$$\bar{x}_3 = 4.00000;$$
$$\bar{x}_2 = 3.00000;$$
$$\bar{x}_1 = 2.00000;$$

The Modification of Crout-Doolittle

Alan Mathison Turing
(1912-1954)

The Modification of Crout-Doolittle

By the Gaussian method, we obtain

$$x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15}$$
$$x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)}$$
$$x_3 + b_{34}^{(2)}x_4 = b_{35}^{(2)}$$
$$a_{44}^{(3)}x_4 = a_{45}^{(3)}$$

which gives us the value of $x_1, \dots, x_4$:

$$x_4 = a_{45}^{(3)} / a_{44}^{(3)}$$
$$x_3 = b_{35}^{(2)} - b_{34}^{(2)}x_4$$
$$x_2 = b_{25}^{(1)} - b_{24}^{(1)}x_4 - b_{23}^{(1)}x_3$$
$$x_1 = b_{15} - b_{14}x_4 - b_{13}x_3 - b_{12}x_2$$

The Modification of Crout-Doolittle

Direct procedure

1. Write the coefficients of the system a_{ij} for $i = 1, 2, 3, 4;\ j = 1, 2, 3, 4, 5$ in Section I of Table.

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$

2. Sum the coefficients of each row and enter the results in the sum column as $a_{i6}\ (i = 1,2,3,4)$

The Modification of Crout-Doolittle (cont'd)

3. Find the numbers m_{i1} for $i = 2,3,4$ and write them down in section II.

$$m_{i1} = \frac{a_{i1}}{a_{11}}$$

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
	m_{21}					
	m_{31}					
	m_{41}					

The Modification of Crout-Doolittle (cont'd)

4. Compute the coefficients $a_{2j}^{(1)}$ for $j = 2, 3, 4, 5, 6$ by

$$a_{2j}^{(1)} = a_{2j} - m_{21}a_{1j}$$

and write them down in section II.

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
	m_{21}					
	m_{31}					
	m_{41}					
II		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$

5. Checking: the sum $\sum_{j=2}^{5} a_{2j}^{(1)}$ should not differ from $a_{26}^{(1)}$ by more than one unit of the last retained digit.

The Modification of Crout-Doolittle (cont'd)

6. Find the numbers $\boldsymbol{m}_{i2}$ for $i = 3,4$:

$$m_{i2} = \frac{(a_{i2} - m_{i1}a_{12})}{a_{22}^{(1)}}$$

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11} a_{21} a_{31} a_{41}	a_{12} a_{22} a_{32} a_{42}	a_{13} a_{23} a_{33} a_{43}	a_{14} a_{24} a_{34} a_{44}	a_{15} a_{25} a_{35} a_{45}	$\sum a_{1j} = a_{16}$ $\sum a_{2j} = a_{26}$ $\sum a_{3j} = a_{36}$ $\sum a_{4j} = a_{46}$
II	m_{21} m_{31} m_{41}					
III		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
IV		m_{32} m_{42}				

The Modification of Crout-Doolittle (cont'd)

7. Compute the coefficients $a_{3j}^{(1)}$ for $j = 3, 4, 5, 6$ by the formula $a_{3j}^{(2)} = a_{3j} - m_{31}a_{1j} - m_{32}a_{2j}^{(1)}$ and write them down in section V.

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11} a_{21} a_{31} a_{41}	a_{12} a_{22} a_{32} a_{42}	a_{13} a_{23} a_{33} a_{43}	a_{14} a_{24} a_{34} a_{44}	a_{15} a_{25} a_{35} a_{45}	$\sum a_{1j} = a_{16}$ $\sum a_{2j} = a_{26}$ $\sum a_{3j} = a_{36}$ $\sum a_{4j} = a_{46}$
II	m_{21} m_{31} m_{41}					
III		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
IV		m_{32} m_{42}				
V			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$

The Modification of Crout-Doolittle (cont'd)

8. Checking: compare $\sum_{j=3}^{5} a_{3j}^{(2)}$ with the number $a_{36}^{(2)}$

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11} a_{21} a_{31} a_{41}	a_{12} a_{22} a_{32} a_{42}	a_{13} a_{23} a_{33} a_{43}	a_{14} a_{24} a_{34} a_{44}	a_{15} a_{25} a_{35} a_{45}	$\sum a_{1j} = a_{16}$ $\sum a_{2j} = a_{26}$ $\sum a_{3j} = a_{36}$ $\sum a_{4j} = a_{46}$
II	m_{21} m_{31} m_{41}					
III		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
IV		m_{32} m_{42}				
V			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$

The Modification of Crout-Doolittle (cont'd)

9. Find the numbers

$$m_{43} = \frac{\left(a_{43} - m_{41}a_{13} - m_{42}a_{23}^{(1)}\right)}{a_{33}^{(2)}}$$

10. Find the coefficients $a_{4j}^{(3)}$ for $j = 4, 5, 6$ by the following formula and write them down in section VII.

$$a_{4j}^{(3)} = a_{4j} - m_{41}a_{1j} - m_{42}a_{2j}^{(1)} - m_{43}a_{3j}^{(2)}$$

11. Checking: the sum $a_{44}^{(3)} + a_{45}^{(3)}$ with the number $a_{46}^{(3)}$.

The Modification of Crout-Doolittle (cont'd)

Reverse procedure:

We successively find the numbers x_4, x_3, x_2, x_1 by the formulas.

$$a_{44}^{(3)}x_4 = a_{45}^{(3)}$$
$$a_{33}^{(2)}x_3 + a_{34}^{(2)}x_4 = a_{35}^{(2)}$$
$$a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + a_{24}^{(1)}x_4 = a_{25}^{(1)}$$
$$a_{11}x_1 + a_{12}x_2 - a_{13}x_3 - a_{14}x_4 = a_{15}$$

The Modification of Crout-Doolittle (cont'd)

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j} = a_{16}$
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j} = a_{26}$
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j} = a_{36}$
	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j} = a_{46}$
II	m_{21}					
	m_{31}					
	m_{41}					
III		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
IV		m_{32}				
		m_{42}				
V			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
VI			m_{43}			
VII				$a_{44}^{(3)}$	$a_{45}^{(3)}$	$a_{46}^{(3)}$
VIII					x_4	$\overline{x_4}$
					x_3	$\overline{x_3}$
					x_2	$\overline{x_2}$
					x_1	$\overline{x_1}$

$$\sum_{j=1}^{n} a_{ij}(x_j+1) = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} + a_{i(n+1)} = a_{i(n+2)}.$$

The Method of Principal Elements

Richard Courant
(1888-1972)

The Method of Principal Elements

Example. Solve a linear system of equations where

$$\begin{cases} 0.0003x_1 + 1.566x_2 = 1.569 \\ 0.3454x_1 - 2.436x_2 = 1.018 \end{cases}$$

Solution.

$$\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -2.436 & 1.018 \end{array}\right] \to \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1804 & -1018 \end{array}\right] \to$$

$$x_2 = \frac{-1805}{-1804} = 1.001, \quad x_1 = \frac{0.001}{0.0003} = 3.333.$$

$$\left[\begin{array}{cc|c} 0.3454 & -2.436 & 1.018 \\ 0.0003 & 1.566 & 1.569 \end{array}\right] \to \left[\begin{array}{cc|c} 0.3454 & -2.436 & 1.018 \\ 0.0003 & 1.566 & 1.569 \end{array}\right] \to$$

$$x_2 = \frac{1.568}{1.568} = 1, \qquad x_1 = \frac{3.454}{0.3454} = 10$$

The Method of Principal Elements

Example. Solve a linear system of equations where

$$\begin{cases} 10^{-5}x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 2 \end{cases}$$

Solution.

$$\left[\begin{array}{cc|c} 10^{-5} & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - 10^5 R_1} \left[\begin{array}{cc|c} 10^{-5} & 1 & 1 \\ 0 & (1-10^5) & 2-10^5 \end{array}\right] \rightarrow$$

$$x_2 = \frac{2-10^5}{1-10^5} = 1 + \frac{1}{1-10^5} = 0.9999899999 \approx 1$$

$$x_1 = 1 - \left(1 + \frac{1}{1-10^5}\right) = \frac{1}{1-10^5} = -0.0000100001 \approx 0$$

The Method of Principal Elements

Example (cont'd).

$$\begin{cases} 10^{-5}x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 2 \end{cases}$$

Solution (cont'd).

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-5} & 1 & 2 \end{array}\right] \xrightarrow{R_2 - 10^{-5} R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1-10^{-5} & 1-2\times 10^{-5} \end{array}\right] \rightarrow$$

$$x_2 = \frac{1-2\times 10^{-5}}{1-10^{-5}} = 1 - \frac{10^{-5}}{1-10^{-5}}, = 1 - 0.0000100001 \approx 1$$

$$x_1 = 2 - 1 - 0.0000100001 \approx 1$$

The Method of Principal Elements

Example (cont'd).

$$\begin{cases} 10^{-5}x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 2 \end{cases}$$

As a result:

$$\left[\begin{array}{cc|c} 10^{-5} & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2-10^5\times R_1} \left[\begin{array}{cc|c} 10^{-5} & 1 & 1 \\ 0 & (1-10^5) & 2-10^5 \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-5} & 1 & 2 \end{array}\right] \xrightarrow{R_2-10^{-5}\times R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1-10^{-5} & 1-2\times 10^{-5} \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Method of Principal Elements (cont'd)

Consider a linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1} \end{cases}$$

Write the augmented rectangular matrix consisting of the coefficients of the system

$$M = \begin{pmatrix} a_{11} \dots a_{1q} \dots a_{1n} a_{1,n+1} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{p1} \dots a_{pq} \dots a_{pn} a_{p,n+1} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{n1} \dots a_{nq} \dots a_{nn} a_{n,n+1} \end{pmatrix}$$

The Method of Principal Elements (cont'd)

Direct procedure

Choose a nonzero (as a rule, the numerically largest) element a_{pq} (of matrix M not belonging to the column of constant terms $(q \neq n+1)$) this element being called the principal element.

Compute the multipliers

$$mi1 = \frac{a_{iq}}{a_{pq}} \quad (i \neq p)$$

The row of M with index p which contains the principal element is called the principal row.

The Method of Principal Elements (cont'd)

1. From each ith non-principal row subtract term wise the principal row multiplied by m_i.
2. Obtain a new matrix in which all the elements of the qth column (with the exception of a_{pq}) are equal to zero.
3. Discarding this column and the principal row, we obtain a new matrix M_1 with the number of rows and columns diminished by unity.
4. Repeat these operations with matrix M_1 to get matrix M_2, and so on. Thus, we obtain a sequence of matrices $M, M_1, \dots, M_{n-1}$
5. Combine into a system all the principal rows beginning with the last.
6. After an appropriate replacement they form a triangular matrix which is equivalent to the initial one.

The Method of Principal Elements (cont'd)

Reverse procedure

Solving the system with the obtained matrix of coefficients, we find, step by step, the values of the unknowns x_i $(i = 1, 2, \ldots, n)$

All the above described computations can be arranged in one table, which will be similar to the Gaussian compact scheme, with a check provided for each stage of computations.

The Method of Principal Elements (cont'd)

Example.

$$\begin{aligned}
1.1161x_1 + 0.1254x_2 + 0.1397x_3 + 0.1490x_4 &= 1.5471,\\
0.1582x_1 + 1.1675x_2 + 0.1768x_3 + 0.1871x_4 &= 1.6471,\\
0.1968x_1 + 0.2071x_2 + 1.2168x_3 + 0.2271x_4 &= 1.7471,\\
0.2368x_1 + 0.2471x_2 + 0.2568x_3 + 1.2671x_4 &= 1.8471.
\end{aligned}$$

Solution.

Direct procedure: Write down the coefficients of the system $\boldsymbol{a_{ij}}$, for $i =$ 1, 2, 3, 4, 5 and $j =$ 1, 2, 3, 4, 5 in the first section of Table

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1		1.1161	0.1254	0.1397	0.1490	1.5471	
	2		0.1582	1.1675	0.1768	0.1871	1.6471	
	3		0.1968	0.2071	1.2168	0.2271	1.7471	
	4		0.2368	0.2471	0.2568	1.2671	1.8471	

The Method of Principal Elements (cont'd)

2. Compute the sums of the coefficients along the row. $\sum = \boldsymbol{ai_6}$

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	1.2671	1.8471	3.58490

3. Find the principal element. In the given system it will be the coefficient

$$a_{44} = 1.26710 \; (p = 4, q = 4).$$

4. Find the numbers $m_i \; (i = l, 2, 3)$. To this end divide the elements of the column a_{i4} by a_{44} and write down the results in the column m_i Section I.

$$m_1 = \frac{a_{14}}{a_{44}} = \frac{0.14900}{1.26710} = 0.11759; \qquad m_2 = \frac{a_{24}}{a_{44}} = \frac{0.18710}{1.26710} = 0.14766;$$

$$m_3 = \frac{a_{34}}{a_{44}} = \frac{0.22710}{1.26710} = 0.17923$$

The Method of Principal Elements (cont'd)

5. Compute the coefficients of the new matrix. From each row $i \; (i = 1, 2, 3)$ subtract the principal row multiplied by the corresponding element m_i.

$$a_{ij}^{(1)} = aij - m_i a_{4j} \; (i = 2,3,4; \qquad j = 1,2,3,5,6)$$

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	1.2671	1.8471	3.58490
II	1		1.08825	0.09634	0.10950		1.32990	2.62399
	2		0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	1.17077		1.41604	2.90398

6. Make a check.

The Method of Principal Elements (cont'd)

7. Choose the principal element and underline it. In our case it will be

$$a_{33} = 1.17077.$$

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4	0.09353	0.2368	0.2471	0.2568	<u>1.2671</u>	1.8471	3.58490
II	1	0.11862	1.08825	0.09634	0.10950		1.32990	2.62399
	2		0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	<u>1.17077</u>		1.41604	2.90398

8. Divide the elements of the column a_{i3} by $a_{23}^{(1)}$

$$m_1^{(1)} = \frac{a_{13}^{(1)}}{a_{33}^{(1)}} = \frac{0.10950}{1.17077} = 0.09353; \qquad m_2^{(1)} = \frac{a_{23}^{(1)}}{a_{33}^{(1)}} = \frac{0.13888}{1.17077} = 0.11862$$

The Method of Principal Elements (cont'd)

9. Compute the coefficients $a_{ij}^{(2)}$. For this purpose from each line i $(i = 1, 2)$ subtract the principal row multiplied by the corresponding m_i.

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	<u>1.2671</u>	1.8471	3.58490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	<u>1.17077</u>		1.41604	2.90398
III	1		1.07381	0.08111			1.19746	2.35238
	2		0.10492	1.11170			1.20639	2.42301

10. Make a check.

The Method of Principal Elements (cont'd)

11. Choose the principal element and underline it. Now it will be

$$a_{22} = 1.11170.$$

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	$\underline{1.2671}$	1.8471	3.58490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	$\underline{1.17077}$		1.41604	2.90398
III	1		1.07381	0.08111			1.19746	2.35238
	2		0.10492	1.11170			1.20639	2.42301

12. Find

$$m_1^{(2)} = \frac{a_{12}^{(2)}}{a_{22}^{(2)}} = \frac{0.08111}{1.11170} = 0.07296;$$

The Method of Principal Elements (cont'd)

13. Subtract the second (principal) row multiplied by $m_1^{(2)}$ from the first row.

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	$\underline{1.2671}$	1.8471	3.58490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	$\underline{1.17077}$		1.41604	2.90398
III	1	0.07296	1.07381	0.08111			1.19746	2.35238
	2		0.10492	$\underline{1.11170}$			1.20639	2.42301
IV	1		1.06616				1.10944	2.17560

14. Make a check

The Method of Principal Elements (cont'd)

Result:

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	1.2671	1.8471	3.58490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	1.17077		1.41604	2.90398
III	1	0.07296	1.07381	0.08111			1.19746	2.35238
	2		0.10492	1.11170			1.20639	2.42301
IV	1		1.06616				1.10944	2.17560

The Method of Principal Elements (cont'd)

15. Writing out the principal rows of each section, we get a system equivalent to the given one:

$$1.06616x_1 = 1.10944,$$

$$0.10492x_1 + 1.11170x_2 = 1.20639,$$

$$0.15436x_1 + 0.16281x_2 + 1.17077x_3 = 1.41604,$$

$$0.23680x_1 + 0.24710x_2 + 0.25680x_3 + 1.26710x_4 = 1.84710,$$

The Method of Principal Elements (cont'd)

Reverse procedure

The results of computations carried out in realizing the reverse procedure are entered in Section V.

We get in succession:

$$x_1 = \frac{1.10944}{1.06616} = 1.04059;$$

$$x_2 = \frac{1.20639 - 0.10492 \times 1.04059}{1.11170} = 0.98697;$$

$$x_3 = \frac{1.41604 - 0.15436 \times 1.04059 - 0.16281 \times 0.98697}{1.17077} = 0.93505;$$

$$x_4 = \frac{1.84710 - 0.23680 \times 1.04059 - 0.24710 \times 0.98697 - 0.259680 \times 0.93505}{1.26710}$$

$$= 0.88130;$$

The Method of Principal Elements (cont'd)

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}
I	1	0.11759	1.1161	0.1254	0.1397	0.1490	1.5471	3.07730
	2	0.14766	0.1582	1.1675	0.1768	0.1871	1.6471	3.33760
	3	0.17923	0.1968	0.2071	1.2168	0.2271	1.7471	3.59490
	4		0.2368	0.2471	0.2568	1.2671	1.8471	3.58490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	0.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	1.17077		1.41604	2.90398
III	1	0.07296	1.07381	0.08111			1.19746	2.35238
	2		0.10492	1.11170			1.20639	2.42301
IV	1		1.06616				1.10944	2.17560
V	1		1				1.04059	2.04059
	2			1			0.98697	1.98697
	3				1		0.93505	1.93505
	4					1	0.88130	1.88130

The Scheme of Khaletsky

The Scheme of Khaletsky

Consider a system of linear equations written in matrix notation as $Ax = b$ where $A = (a_{ij})$ is a square matrix $i, j = 1, 2, \ldots, n$ and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ \cdots \\ a_{n,n+1} \end{bmatrix}$$

are column vectors.

Represent matrix A in the form of a product $A = BC$, where

$$B = \begin{bmatrix} b_{11} & 0 & 0 \ldots 0 \\ b_{21} & b_{22} & 0 \ldots 0 \\ \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & b_{n3} \ldots b_{n,n} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & c_{12} \ldots & c_{1n} \\ 0 & 1 \;\; \ldots & c_{2n} \\ \ldots & \ldots & \ldots \\ 0 & 0 \;\; \ldots & 1 \end{bmatrix}$$

The Scheme of Khaletsky (cont'd)

Then the elements b_{ij} and c_{ij} are determined from the formulas

$$b_{i1} = a_{i1},$$

$$b_{ij} = a_{ij} - \sum_{k=1}^{j-1} b_{ik}c_{kj} \ (i \geq j > 1)$$

and

$$c_{1j} = a_{1j}/b_{11},$$

$$c_{ij} = \frac{1}{b_{ii}}\left(a_{ij} - \sum_{k=1}^{i-1} b_{ik}c_{kj} \ (1 < i < j)\right)$$

The Scheme of Khaletsky (cont'd)

Whence the desired vector x may be computed from the chain of equations $By = b, Cx = y$.

Since the matrices B and C are triangular, systems are solved, namely:

$$y_1 = a_{1,n+1}/b_{11},$$

$$y_i = \left(a_{i,n+1} - \sum_{k=1}^{i-1} b_{ik}y_k\right) \quad \text{for } (i > 1)$$

and

$$x_n = y_n,$$

$$x_i = y_i - \sum_{k=i+1}^{n} c_{ik}x_k \ (i < n)$$

The Scheme of Khaletsky (cont'd)

Example.

$$3x_1 + x_2 - x_3 + 2x_4 = 6,$$
$$-5x_1 + x_2 + 3x_3 - 4x4 = -12,$$
$$2x_1 + x_3 - x_4 = 1,$$
$$x_1 - 5x_2 + 3x_3 - 3x4 = 3.$$

Solution.

1. Write down the matrix of the coefficients of the system, its constant terms and the check sums in Section I of Table .

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	3	1	-1	2	6	11
	-5	1	3	-4	-12	-17
	2	0	1	1	1	3
	1	-5	3	-3	3	-1

The Scheme of Khaletsky (cont'd)

2. Transfer the elements of the column x_i from Section I to Section II since

$$b_{i1} = a_{i1}; \quad i = 1, 2, 3, 4$$

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	3	1	-1	2	6	11
	-5	1	3	-4	-12	-17
	2	0	1	1	1	3
	1	-5	3	-3	3	-1
II	3					
	-5					
	2					
	1					

The Scheme of Khaletsky (cont'd)

3. Divide all elements of the first row of Section I by the element $a_{11} = b_{11}$. We have:

$$c_{12} = \frac{1}{3} = 0.333333, \quad c_{13} = -\frac{1}{3} = -0.333333, \quad c_{14} = \frac{2}{3} = 0.666667,$$
$$c_{15} = \frac{6}{3} = 2, \quad c_{16} = \frac{11}{3} = 3.666667$$

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	**3**	**1**	**-1**	**2**	**6**	**11**
	-5	**1**	**3**	**-4**	**-12**	**-17**
	2	**0**	**1**	**1**	**1**	**3**
	1	**-5**	**3**	**-3**	**3**	**-1**
II	**3 | 1**	**0.333333**	**-0.333333**	**0.666667**	**2**	**3.666667**
	-5					
	2					
	1					

The Scheme of Khaletsky (cont'd)

4. Fill in the column x_2 of Section II beginning with the second row. We determine b_{j2} :

$$b_{22} = a_{22} - b_{21}c_{12} = 1 - \left(-5 \times \tfrac{1}{3}\right) = \tfrac{8}{3} = 2.6666667 \ldots$$

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	**3**	**1**	**-1**	**2**	**6**	**11**
	-5	**1**	**3**	**-4**	**-12**	**-17**
	2	**0**	**1**	**1**	**1**	**3**
	1	**-5**	**3**	**-3**	**3**	**-1**
II	**3 | 1**	**0.333333**	**-**	**0.666667**	**2**	**3.666667**
	-5	**2.666667**	**0.333333**			
	2	**0.666667-**				
	1	**5.333333**				

The Scheme of Khaletsky (cont'd)

5. Fill in the second row of Section II determining c_{2j} for $j = 3, 4, 5, 6$

$$c_{23} = \frac{1}{b_{22}}(a_{23} - b_{21}c_{13}) = \frac{3}{8}\left(3 - 5 \times \frac{1}{3}\right) = \frac{1}{2} = 0.5 \ \ldots$$

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	3	1	-1	2	6	11
	-5	1	3	-4	-12	-17
	2	0	1	1	1	3
	1	-5	3	-3	3	-1
II	3 | 1	0.333333	-0.333333	0.666667	2	3.666667
	-5	2.666667|1	0.5	-0.25	-0.75	0.5
	2	0.666667-				
	1	5.333333				

The Scheme of Khaletsky (cont'd)

6. Fill in the column x_3, computing its elements b_{33} and b_{43}

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	3	1	-1	2	6	11
	-5	1	3	-4	-12	-17
	2	0	1	1	1	3
	1	-5	3	-3	3	-1
II	3 | 1	0.333333	-0.333333	0.666667	2	3.666667
	-5	2.666667|1	0.5	-0.25	-0.75	0.5
	2	0.666667-	2			
	1	5.333333	6			

The Scheme of Khaletsky (cont'd)

7. Proceed analogously until Section II is filled in completely.

We thus get a staircase arrangement in Section II:

	x_1	x_2	x_3	x_4		$\sum a_{ij}$
I	3	1	-1	2	6	11
	-5	1	3	-4	-12	-17
	2	0	1	1	1	3
	1	-5	3	-3	3	-1
II	3 | 1	0.333333	-0.333333	0.666667	2	3.666667
	-5	2.666667 | 1	0.5	-0.25	-0.75	0.5
	2	0.666667-	2 | 1	-1.25	-1.75	-2
	1	5.333333	6	2.5 | 1	3	4

The Scheme of Khaletsky (cont'd)

8. Determine y_i and x_i $(i = 1, 2, 3, 4)$, and enter them in Section III:

$$y_1 = \frac{a_{15}}{b_{11}} = \frac{6}{3} = 2;$$

$$y_2 = \frac{(a_{25} - b_{21}y_1)}{b_{22}} = \frac{-12 + 5 \times 5}{2.666667} = -0.75;$$

$$y_3 = \frac{(a_{35} - b_{31}y_1 - b_{32}y_2)}{b_{33}} = \frac{1 - 2 \times 2 - 0.666667 \times 0.75}{2} = -1.75;$$

$$y_4 = \frac{(a_{45} - b_{41}y_1 - b_{42}y_2 - b_{43}y_3)}{b_{44}} = (3 - 2 - 5.333333 \times 0.75 + 6 \times 1.75) = 3;$$

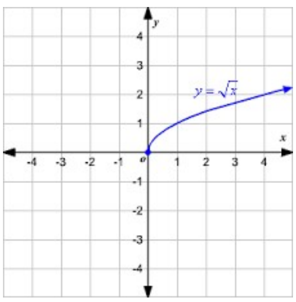
The Scheme of Khaletsky (cont'd)

8'. Determine y_i and x_i $(i = 1, 2, 3, 4)$, and enter them in Section III:

$$x_4 = y_4 = 3;$$
$$x_3 = y_3 - C_{34}x_4 = -1.75 + 1.25 \times 3 = 2;$$
$$x_2 = y_2 - C_{23}x_3 - C_{24}x_4 = -0.75 - 0.5 \times 2 + 0.25 \times 3 = -1;$$
$$x_1 = y_1 - C_{12}x_2 - C_{13}x_3 - C_{14}x_4 = (2 + 0.333333 + 0.333333 \times 2 - 0.666667 \times 3) = 1.$$

9. Intermediate checking is done by means of the I column, which is involved in the same operations as is the Σ column of constant terms.

The Square-Root Method



The Square-Root Method

The square-root method is used for solving a linear system

$$Ax = B$$

where $A = [a_{ij}]$ is a symmetric matrix, i.e.,

$$a_{ij} = a_{ji} \text{ for } i, j = 1,2, \ldots n.$$

If A is non-singular, then the LU decomposition is unique. Consequently, if A is symmetric, then $\mathrm{L} = U^T$, so $A = U^T U$

$$\text{where} \quad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \vdots & \ddots & u_{(n-1)n} \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}.$$

The Square-Root Method(cont'd)

Multiplying together the matrices U^T and U and equating the product to the matrix A, we get:

$$a_{ij} = \left(U^T U\right)_{\mathrm{ij}} = \sum_{k=1}^{n} u_{ki} u_{kj}$$

Consequently,

$$u_{11} = \sqrt{a_{11}}, \qquad u_{1j} = \frac{a_{1j}}{u_{11}} \quad (2 < j \le n)$$

$$\sum_{k=1}^{i-1} u_{ik}^2 + u_{ii}^2 = a_{ii} \Rightarrow u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ik}^2} \; (1 < i \le n)$$

$$\sum_{k=1}^{i} u_{ki} u_{kj} = a_{ij} \Rightarrow u_{ii} u_{ij} = a_{ii} - \sum_{k=1}^{i-1} u_{ki} u_{kj} \Rightarrow$$

$$u_{ij} = \frac{a_{ii} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \quad (i < j)$$

$$u_{ij} = 0 \quad (i > j)$$

The Square-Root Method(cont'd)

Note that we consider that A is non-singular matrix, then $\det A \neq 0$.

Since

$$\det A = \det U^T \det U = (u_{11} \times \cdots \times u_{nn})^2 \neq 0$$

we have $u_{ii} \neq 0$ for each $1 \leq i \leq n$,

On finding the matrix U, we have $Ax = U^T U x = b$.

Thus, we can replace the system by two equivalent systems with triangular matrices:

$$U^T y = b \text{ and } Ux = y$$

The Square-Root Method(cont'd)

Write new systems in the expanded form:

$$U^T y = b \implies \begin{cases} u_{11}y_1 & = b_1 \\ u_{12}y_1 + u_{22}y_2 & = b_2 \\ \vdots & \\ u_{1n}y_1 + u_{2n}y_2 + \cdots + u_{nn}y_n & = b_n \end{cases}$$

$$Ux = y \implies \begin{cases} u_{11}x_1 + u_{12}y_2 + \cdots + u_{1n}x_n = y_1 \\ u_{22}x_2 + \cdots + u_{2n}x_n = y_2 \\ \vdots \\ u_{nn}x_n = y_n \end{cases}$$

The Square-Root Method(cont'd)

We successively find

$$y_1 = \frac{b_1}{t_{11}}$$

$$y_i = \frac{b_i - \sum_{k=1}^{i-1} t_{ki} y_k}{t_{ii}} \qquad (1 < i \le n)$$

$$x_n = \frac{y_n}{t_{nn}}$$

$$x_i = \frac{y_i - \sum_{k=i+1}^{n} t_{ik} x_k}{t_{ii}} (1 < i \le n)$$

The Square-Root Method(cont'd)

Example.

$$\begin{cases} 1x_1 + 3x_2 - 2x_3 + 0x_4 - 2x_5 = 0.5 \\ 3x_1 + 4x_2 - 5x_3 + 1x_4 - 3x_5 = 5.4 \\ -2x_1 - 5x_2 + 3x_3 - 2x_4 + 2x_5 = 0.5 \\ 0x_1 + 1x_2 - 2x_3 + 5x_4 + 3x_5 = 7.5 \\ -2x_1 - 3x_2 + 2x_3 + 3x_4 + 4x_5 = 3.3 \end{cases}$$

Solution. Write down the coefficients of the system a_{ij}, for $i, j = 1, 2, 3, 4, 5$ in the first section of the table

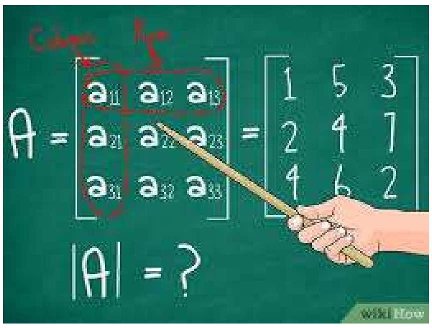
	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}	S_i
I	1	3	-2	0	-2	0.5	0.5
	3	4	-5	1	-3	5.4	5.4
	-2	-5	3	-2	2	0.5	1
	0	1	-2	5	3	7.5	14.5
	-2	-3	2	3	4	3.3	7.3

The Square-Root Method(cont'd)

Find matrix U:

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	b_i
I	1.00	0.42	0.54	0.66	0.30
	0.42	1.00	0.32	0.44	0.50
	0.54	0.32	1.00	0.22	0.70
	0.66	0.44	0.22	1.00	0.90
	u_{i1}	u_{i2}	u_{i3}	u_{i4}	y_i
II	*1*	3	-2	*0*	−2
		2.2361 i	0.4472 i	-0.4472 i	-1.3416 i
			0.8944 i	2.0125 i	1.5653 i
				3.0414	2.2194
					0.1643 i

Computing Determinants



Computing Determinants

If we can use of the Gaussian method, successfully, then the determinant of matrix A is equal to the product of the leading elements of the corresponding Gaussian scheme.

$$\Delta = \det A = a_{11} a_{22}^{(1)} \ \dots \ a_{nn}^{(n-1)}$$

Note that by using the method we obtain $A = LU$ where L is a square lower unit triangular matrix, and U a rectangular matrix.

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Computing Determinants (cont'd)

$$\Delta = \begin{vmatrix} 1.1161 & 0.1254 & 0.1397 & 0.1490 \\ 0.1582 & 1.1675 & 0.1768 & 0.1871 \\ 0.1968 & 0.2071 & 1.2168 & 0.2271 \\ 0.2368 & 0.2471 & 0.2568 & 1.2671 \end{vmatrix}$$

The given determinant is equal to the determinant of the system solved in a example by the Gaussian method with the principal element chosen. Forming the product of the leading elements, we get the required value of the determinant

$$\Delta = \underbrace{\mathbf{1.26710}}_{a_{44}} \times \underbrace{\mathbf{1.17077}}_{a_{33}^{(1)}} \times \underbrace{\mathbf{1.11170}}_{a_{22}^{(2)}} \times \underbrace{1.06616}_{a_{11}^{(3)}} = \mathbf{1.75829}$$

Computing Determinants (cont'd)

If A is a symmetric matrix, then it is advisable to use the square-root method for evaluating the determinant of this matrix.

Thus

$$A = U^T U$$

where

$$T = \begin{pmatrix} u_{11} & u_{12} \dots & u_{1n} \\ 0 & u_{22} \dots & u_{2n} \\ \dots & \dots \dots \dots & \dots \\ 0 & 0 \quad \dots & u_{nn} \end{pmatrix}.$$

Then

$$det\ A = det\ U^T \det U = (detU)^2 = (u_{11}\ u_{22} \dots\ u_{nn})^2.$$

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Computing Determinants (cont'd)

$$\det A = \begin{vmatrix} 1.00 & 0.42 & 0.54 & 0.66 \\ 0.42 & 1.00 & 0.32 & 0.44 \\ 0.54 & 0.32 & 1.00 & 0.22 \\ 0.66 & 0.44 & 0.22 & 1.00 \end{vmatrix}$$

$$\begin{aligned} \det A &= (\underbrace{1.00}_{u_{11}} \times \underbrace{0.90752}_{u_{22}} \times \underbrace{0.83537}_{u_{33}} \times \underbrace{0.70560}_{u_{44}})^2 \\ &= (0.53492)^2 \\ &= 0.28614 \end{aligned}$$

Computing the Elements of an Inverse Matrix by the Gaussian Method

Wilhelm Jordan
(1842-1899)

Computing the Elements of an Inverse Matrix

Definition. A square matrix A is called non-singular if its determinant $\det A$ is non-zero.

Theorem. Every nonsingular matrix has an inverse.

Sketch of proof. It is shown that $(adj\ A)A = A(adj\ A) = (\det A)\ I$.

Computing the Elements of an Inverse Matrix (cont'd)

Let A is non-singular and $AA^{-1} = I_n$ where

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}.$$

Consider

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix} \quad (1 \leq i \leq n) \qquad I = [e_1 \quad \cdots \quad e_n].$$

Thus, $AA^{-1} = A\,[x_1 \quad \cdots \quad x_n]=[Ax_1 \quad \cdots \quad Ax_n] = [e_1 \quad \cdots \quad e_n]$.

Consequently, we get n systems with n equations totally in n^2 unknowns x_{ij}.

Computing the Elements of an Inverse Matrix (cont'd)

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	$a_{i5,\text{I}}$	$a_{i5,\text{II}}$	$a_{i5,\text{III}}$	$a_{i5,\text{IV}}$
I	a_{11} a_{21} a_{31} a_{41} 1	a_{12} a_{22} a_{32} a_{42} b_{12}	a_{13} a_{23} a_{33} a_{43} b_{13}	a_{14} a_{24} a_{34} a_{44} b_{14}	1 0 0 0 b_{15}	0 1 0 0 0	0 0 1 0 0	0 0 0 1 0
II		$a_{22}^{(1)}$ $a_{32}^{(1)}$ $a_{42}^{(1)}$ 1	$a_{23}^{(1)}$ $a_{33}^{(1)}$ $a_{43}^{(1)}$ b_{23}	$a_{24}^{(1)}$ $a_{34}^{(1)}$ $a_{44}^{(1)}$ b_{24}	$a_{25}^{(1)}, \text{I}$ $a_{35}^{(1)}, \text{I}$ $a_{45}^{(1)}, \text{I}$ b_{25}, I	1 0 0 b_{25}, II	0 1 0 0	0 0 1 0
III			$a_{33}^{(2)}$ $a_{43}^{(2)}$ 1	$a_{34}^{(2)}$ $a_{44}^{(2)}$ b_{34}	$a_{35}^{(2)}$ $a_{45}^{(2)}$ b_{35}, I	$a_{35}^{(2)}, \text{II}$ $a_{45}^{(2)}, \text{II}$ b_{35}, II	1 0 b_{25}, III	0 1 0
IV				$a_{44}^{(3)}$ 1	$a_{45}^{(3)}$ b_{45}, I	$a_{45}^{(3)}, \text{II}$ b_{45}, II	$a_{45}^{(3)}, \text{III}$ b_{45}, III	1 b_{25}, IV
V					x_{41} x_{31} x_{21} x_{11}	x_{42} x_{32} x_{22} x_{12}	x_{43} x_{33} x_{23} x_{13}	x_{44} x_{34} x_{24} x_{14}

Computing the Elements of an Inverse Matrix (cont'd)

Find the inverse matrix for the matrix

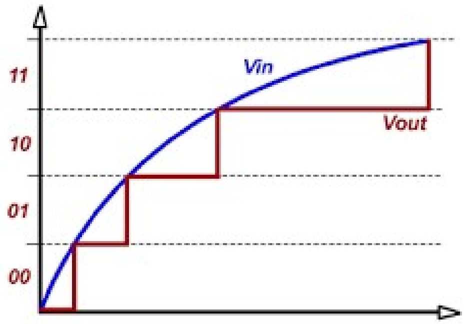
$$A=\begin{bmatrix} 1.8 & -3.8 & 0.7 & -3.7 \\ 0.7 & 2.1 & -2.6 & -2.8 \\ 7.3 & 8.1 & 1.7 & -4.9 \\ 1.9 & -4.3 & -4.9 & -4.7 \end{bmatrix}$$

The computations are given in the table. The last column of the table consists of the sums of elements for each row.

Computing the Elements of an Inverse Matrix (cont'd)

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	$a_{i5,\mathrm{I}}$	$a_{i5,\mathrm{II}}$	$a_{i5,\mathrm{III}}$	$a_{i5,\mathrm{IV}}$
I	1.8	−3.8	0.7	−3.7	1	0	0	0
	0.7	2.1	-2.6	-2.8	0	1	0	0
	7.3	8.1	1.7	-4.9	0	0	1	0
	1.9	-4.3	-4.9	-4.7	0	0	0	1
	1	-2.11111	0.38889	−2.05556	0.55556	0	0	0
II		3.57778	−2.87222	−1.36111	−0.38885	1	0	0
		23.51110	-1.13890	10.10559	-4.05551	0	1	0
		-0.28889	-5.63889	-0.79444	-1.05554	0	0	1
		1	-0.80279	-0.38043	-0.10868	0.27950	0	0
III			17.73577	19.04992	−1.50032	−6.57135	1	0
			-5.87081	-0.90434	-1.08694	0.08074	0	1
			1	1.07411	-0.08459	-0.37108	0.05638	0
IV				5.40155	-1.58355	-2.09780	0.33100	1
V				1	-0.29316	-0.38837	0.06128	0.18513
					0.23030	0.04607	-0.00944	-0.19885
					-0.03533	0.16873	0.01573	-0.08920
					-0.21121	-0.46003	0.16284	0.26956

The Method of Successive Approximations



Method of successive approximations

An Iterative technique to solve the $n \times n$ linear system $A\boldsymbol{x} = b$ starts with an initial approximation $\boldsymbol{x}^{(0)}$ to the solution $\boldsymbol{x}$ and generates a sequence of vectors $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$.

The process stops if the approximations "stabilize", i.e., if the difference between successive approximations becomes negligible.

One of the possible stopped criterion is to iterate until

$$\frac{\left\|\boldsymbol{x}^{(p+1)} - \boldsymbol{x}^{(p)}\right\|}{\left\|\boldsymbol{x}^{(p+1)}\right\|}$$

Is smaller than some prescribed tolerance.

Method of successive approximations

The Jacobia and the Gauss-Seidel iterative methods are classical methods.

Since the time required for sufficient accuracy exceeds that required time for direct techniques such as Gaussian elimination, iterative techniques are seldom used for solving linear system of small dimension.

For system equation with large number of equations and a high percentage of 0 entries, the iterative techniques are efficient in terms of both computer storage and computation.

Jacobi's Method

Given a linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{24}x_4 = b_2 \\ \vdots + \vdots + \cdots + \vdots = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + a_{n4}x_4 = b_n \end{cases}$$

Assuming that the diagonal coefficients are zero, $a_{ii} \neq 0$ for all $1 \leq i \leq n$.

$$\begin{aligned} x_1 &= \beta_1 + b_{12}x_2 + b_{13}x_3 + \cdots + \alpha_{1n}x_n \\ x_2 &= \beta_2 + b_{21}x_1 + b_{23}x_3 + \cdots + b_{24}x_4 \\ &\vdots \\ x_n &= \beta_n + b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{n(n-1)}x_{n-1} \end{aligned}$$

where $\beta_i = \frac{b_i}{a_{ii}}$ and $b_{\mathrm{ij}} = -\frac{a_{ij}}{a_{ii}}$ for all $i \neq j$.

Jacobi's Method (cont'd)

So we can write the given system equation as

$$\boldsymbol{x} = \beta + B\boldsymbol{x}$$

where

$$\mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \text{ and } \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

We take $\boldsymbol{x}^{(0)} = \beta$, then we consecutively construct the column matrices

$$\boldsymbol{x}^{(1)} = \beta + B\boldsymbol{x}^{(0)}.$$

as a first approximation, then

$$\boldsymbol{x}^{(2)} = \beta + B\boldsymbol{x}^{(1)}$$

And second approximation and generally speaking any $k+1$ the approximation is computed from $\boldsymbol{x}^{(k+1)} = \beta + B\boldsymbol{x}^{k}$.

Jacobi's Method (cont'd)

Consequently, the Jacobi method is written in the form

$$\boldsymbol{x}^{(k+1)} = \beta + B\boldsymbol{x}^{k}.$$

For solving the equation system $A\boldsymbol{x} = b$.

The coefficient matrix A can be split into its diagonal and off-diagonal parts (as summation of one strictly lower-triangular part of A, and the strictly upper-triangular part of A. That means:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{D} - \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix}}_{L} - \underbrace{\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{U}$$

The equation $A\boldsymbol{x} = b$, or $(D - L - U)\boldsymbol{x} = b$, is then transformed into $D\boldsymbol{x} = b + (L + U)\boldsymbol{x}$. If D^{-1}exists, i.e., $a_{ii} \neq 0$ for each i, then

$$\boldsymbol{x} = \underbrace{D^{-1}b}_{\beta} + \underbrace{D^{-1}(L+U)}_{B}\,\boldsymbol{x}$$

Jacobi's Method (cont'd)

We have

$$\lim_{k\to\infty} \boldsymbol{x}^{(k+1)} = \beta + B \lim_{k\to\infty} \boldsymbol{x}^{(k)}$$

If the sequence of approximations $x^0, \dots, x^k, \dots$ has a limit, then $\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{x}$ and $\boldsymbol{x} = \beta + B\boldsymbol{x}$.

In the other words, the solution of the system is a fixed point for the function $G: \mathbb{R}^n \to \mathbb{R}^n$ such that $G(\boldsymbol{x}) = \beta + B\boldsymbol{x}$

Thus, the formulation of the approximations is as follows

$$x_i^{(0)} = \beta_i$$

$$x_i^{(k+1)} = \beta_i + \sum_{j=1}^{n} b_{ij} x_j^{(k)}$$

for $1 \le i \le n$. When we have n equations in n unknowns, then $\alpha_{ii} = 0$.

Jacobi's Method (cont'd)

Example. Solve the system

$$\begin{cases} 20.9x_1 + 1.2x_2 + 2.1x_3 + 0.9x_4 = 21.70 \\ 1.2x_1 + 21.2x_2 + 1.5x_3 + 2.5x_4 = 27.46 \\ 2.1x_1 + 1.5x_2 + 19.8x_3 + 1.3x_4 = 28.76 \\ 0.9x_1 + 2.5x_2 + 1.3x_3 + 32.1x_4 = 49.72 \end{cases}$$

Solution. Reduce the above system to the following form:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n)$$

$$\cdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1})$$

$$x_1 = \frac{1}{20.9}(21.70 - 1.2x_2 - 2.1x_3 - 0.9x_4),$$

$$x_2 = \frac{1}{21.2}(27.46 - 1.2x_1 - 1.5x_3 - 2.5x_4),$$

$$x_3 = \frac{1}{19.8}(28.76 - 2.1x_1 - 1.5x_2 - 1.3x_4),$$

$$x_4 = \frac{1}{32.1}(49.72 - 0.9x_1 - 2.5x_2 - 1.3x_3).$$

Jacobi's Method (cont'd)

1st iteration:

$$x_1^{(1)} = \frac{1}{20.9}(21.70 - 1.560 - 3.045 - 1.395) = 0.75,$$

$$x_2^{(1)} = \frac{1}{21.2}(27.46 - 1.248 - 2.175 - 3.875) = 0.95,$$

$$x_3^{(1)} = \frac{1}{19.8}(28.76 - 2.184 - 1.950 - 2.015) = 1.14,$$

$$x_4^{(1)} = \frac{1}{32.1}(49.72 - 0.936 - 3.250 - 1.885) = 1.36$$

Jacobi's Method (cont'd)

2nd iteration:

$$x_1^{(2)} = \frac{16.942}{20.9} = 0.8106, \qquad x_2^{(2)} = \frac{21.450}{21.2} = 1.0118,$$

$$x_3^{(2)} = \frac{23.992}{19.8} = 1.2117, \qquad x_4^{(2)} = \frac{45.188}{32.1} = 1.4077,$$

3rd iteration:

$$x_1^{(3)} = \frac{16.67434}{20.9} = 0.7978, \qquad x_2^{(3)} = \frac{21.15048}{21.2} = 0.9977,$$

$$x_3^{(3)} = \frac{23.71003}{19.8} = 1.1975, \qquad x_4^{(3)} = \frac{44.88575}{32.1} = 1.3983,$$

Jacobi's Method (cont'd)

4th iteration:

$$x_1^{(4)} = \frac{16.7295}{20.9} = 0.8004, \qquad x_2^{(4)} = \frac{21.2106}{21.2} = 1.0005,$$

$$x_3^{(4)} = \frac{23.7703}{19.8} = 1.2005, \qquad x_4^{(4)} = \frac{44.9510}{32.1} = 1.4003$$

Calculate the moduli of the differences of the values of $x_i^{(k)}$ for $k = 3$ and $k = 4$:

$$\left|x_1^{(3)} - x_1^{(4)}\right| = 0.0026, \qquad \left|x_2^{(3)} - x_2^{(4)}\right| = 0.0028,$$

$$\left|x_3^{(3)} - x_3^{(4)}\right| = 0.0030, \qquad \left|x_4^{(3)} - x_4^{(4)}\right| = 0.0020,$$

Jacobi's Method (cont'd)

Since all of them exceed the pre-assigned number $\varepsilon = 10^{-3}$, the process of iteration is continued. We get for $k = 5$:

$$x_1^{(5)} = \frac{16.71808}{20.9} = 0.7999, \qquad x_2^{(5)} = \frac{21.19802}{21.2} = 0.9999,$$

$$x_3^{(5)} = \frac{23.75802}{19.8} = 1.1999, \qquad x_4^{(5)} = \frac{44.93774}{32.1} = 1.3999,$$

Find the moduli of the differences of the values of $x_i^{(k)}$ for $k = 4$ and $k = 5$ $\left\|\boldsymbol{x}^{(5)}\right\| = 2.24479$:

$$\left|x_1^{(4)} - x_1^{(5)}\right| = 0.0005, \qquad \left|x_2^{(4)} - x_2^{(5)}\right| = 0.0006,$$

$$\left|x_3^{(4)} - x_3^{(5)}\right| = 0.0006, \qquad \left|x_4^{(4)} - x_4^{(5)}\right| = 0.0004.$$

Jacobi's Method (cont'd)

They are less than the given number ε therefore we take the following as the solution:

$$x_1 \approx 0.7999, \quad x_2 \approx 0.9999, \quad x_3 \approx 1.1999, \quad x_4 \approx 1.3999$$

In accordance with previous estimate the errors of these values should not exceed 1/3×0.0006 = 0.0002.

For comparison we give the exact values of the unknowns:

$$x_1 = 0.8, \qquad x_2 = 1.0, \qquad x_3 = 1.2, \qquad x_4 = 1.4$$

The Seidel Method

Philipp Ludwig von Seidel
(1821-1896)

The Gauss-Seidel Method

The **Gauss-*Seidel* *method*** is a certain modification of the method of simple iteration. The principal idea behind it is that in computing the $(k+1)$th approximation of the unknown x_i for $i > 1$, the earlier computed $(k+1)$th approximations of the unknowns $x_1, x_2, \dots, x_{i-1}$ are taken into account.

The Seidel Method and the Jacobi Method

Jacobi method	***Seidel method***
In its $(p+1)$-th iteration, the value of $x_1^{(p)}, \dots, x_n^{(p)}$ are substituted into every equation of the re-arranged system simultaneously to obtain $\boldsymbol{x}^{(p+1)}$	This method differs from the Jacobi Method in which immediately after a new $x_i^{(p+1)}$value is obtained from the ith equation, it is used I n place of the old value in successive substitutions.

The Seidel Method (cont'd)

Example. Solve the following system of equations by the Seidel method:

$$\begin{cases} 10x_1 + x_2 + x_3 = 12 \\ 2x_1 + 10x_2 + x_3 = 13 \\ 2x_1 + 2x_2 + 10x_3 = 14 \end{cases}.$$

Solution. Reduce the system, to a form convenient for iteration:

$$\begin{cases} x_1 = 1.2 - 0.1x_2 - 0.1\,x_3 \\ x_2 = 1.3 - 0.2x_1 - 0.1x_3 \\ x_3 = 1.4 - 0.2\,x_1 - 0.2\,x_2 \end{cases}.$$

Let $x^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_3^{(k)} \\ x_3^{(k)} \end{bmatrix}$. For the zeroth approximations of the roots take $x^{(0)} = \begin{bmatrix} 1.2 \\ 0 \\ 0 \end{bmatrix}$.

Then $x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_3^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1.2 - 0.1 \times 0 - 0.1 \times 0 \\ 1.3 - 0.2 \times 1.2 - 0.1 \times 0 \\ 1.4 - 0.2 \times 1.2 - 0.2 \times 1.6 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \\ 0.948 \end{bmatrix}$.

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The Seidel Method (cont'd)

We obtain $x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_3^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \\ 0.948 \end{bmatrix}$. Now,

$$x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_3^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.2 - 0.1 \times 1.6 - 0.1 \times 0.948 \\ 1.3 - 0.2 \times 0.9992 - 0.1 \times 0.948 \\ 1.4 - 0.2 \times 0.9992 - 0.2 \times 1.00536 \end{bmatrix} = \begin{bmatrix} 0.9992 \\ 1.00536 \\ 0.999098 \end{bmatrix}$$

By continuing this process, we obtain:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	1.2000	0.0000	0.0000
1	1.200	1.0600	0.9480
2	0.9992	1.0054	0.991
3	0.9996	1.0001	1.0001
4	1.0000	1.0001	1.0001
5	1.0000	1.0000	1.0001

The exact values of the roots are
$x_1 = 1, x_2 = 1, x_3 = 1$

The Seidel Method (cont'd)

Simple Iteration vs. Seidel:

- Seidel method *usually* improves the rate of convergence (not always!)
- As a programmer, Seidel method consumes less memory:
  - Seidel: The old value of a variable can be overwritten as soon as a new value is obtained.
  - Simple: All values from the last iteration must be kept.
  - Result: In Simple Iteration, twice as much storage is needed.
- However, Simple Iteration is perfectly suited to parallel programming, whereas the Gauss-Seidel method is not.

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The Seidel Method (cont'd)

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General Iteration Methods

To study the convergence of general iteration techniques, we need to analyze the formula

$$\boldsymbol{x}^{(p+1)} = \boldsymbol{c} + T\boldsymbol{x}^{(p)}$$

for $p = 1,2,\ldots$ where $\boldsymbol{x}^{(0)}$ is arbitrary.

Definition. The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|$$

where λ is an eigenvalue of A.

General Iteration Methods (cont'd)

Theorem. For any $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\boldsymbol{x}^{(p)}\}_{p=0}^{\infty}$ defined by

$$\boldsymbol{x}^{(p)} = \boldsymbol{c} + T\boldsymbol{x}^{(p-1)}$$

for each $p \geq 1$, converges to the unique solution of

$$\boldsymbol{x} = \boldsymbol{c} + T\boldsymbol{x}$$

if and only if $\rho(T) < 1$.

General Iteration Methods (Just for fun!)

We present a proof for ($\Longleftarrow$). For this proof, we need some Lemmas.

Lemma. For an $n \times n$ matrix A, the following statements are equivalent:

(i) $\rho(A) < 1$.

(ii) $\lim_{n\to\infty} A^n \boldsymbol{x} = 0$, for every $\boldsymbol{x}$.

Lemma. If $\rho(A) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j .$$

General Iteration Methods (Just for fun!)

Lemma. If $\rho(A) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j .$$

Proof. If λ is an eigenvalue for T, then $1 - \lambda$ is an eigenvalue for $I - T$. $(Tx = \lambda x \rightarrow (I - T)x = (1 - \lambda)x.)$

Let

$$S_m = I + T + T^2 + \cdots + T^m.$$

Then

$$(I - T)\mathrm{S_m} = \left(I + T + T^2 + \cdots + T^m\right) - \left(T + T^2 + \cdots + T^{m+1}\right) = \mathrm{I} - \mathrm{T}^{m+1}.$$

Since $\rho(A) < 1$, we have $\lim_{n\to\infty} T^m = 0$. Thus

$$\lim_{m\to\infty} (I - T)\mathrm{S_m} = \lim_{m\to\infty} \mathrm{I} - \mathrm{T}^{m+1} \Rightarrow (I - T)\sum_{j=0}^{\infty} T^j = I.$$

General Iteration Methods (Just for fun!) (cont'd)

Proof.($\Longleftarrow$) Assume $\rho(T) < 1$. Then,

$$\begin{aligned}\boldsymbol{x}^{(p)} &= \boldsymbol{c} + T\boldsymbol{x}^{(p-1)} \\ &= \boldsymbol{c} + T\left(\boldsymbol{c} + T\boldsymbol{x}^{(p-2)}\right) \\ &= (T + I)\boldsymbol{c} + T^2\boldsymbol{x}^{(p-2)}) \\ &\vdots \\ &= (T^{p-1} + \cdots + T + I)\boldsymbol{c} + T^p\boldsymbol{x}^{(0)}.\end{aligned}$$

Since $\rho(T) < 1$, we have $\lim_{p\to\infty} T^p\boldsymbol{x}^{(0)} = 0$. Thus

$$\lim_{p\to\infty} \boldsymbol{x}^{(p)} = \left(\sum_{j=0}^{\infty} T^j\right)\boldsymbol{c} + \lim_{p\to\infty} T^p\boldsymbol{x}^{(0)}$$

$$\lim_{p\to\infty} \boldsymbol{x}^{(p)} = (I - T)^{-1}\boldsymbol{c}$$

Hence, the sequence $\{\boldsymbol{x}^{(p)}\}_{p=0}^{\infty}$ converge to the vector

$$x = (I - T)^{-1}\boldsymbol{c}.$$

General Iteration Methods (cont'd)

Definition. The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \geq \sum_{j\neq i} a_{ij}$$

hold for each $i = 1, 2, \dots, n$. A diagonally dominant matrix is said to be **strictly diagonally dominant** when the above inequality is strict for each n, i.e.,

$$|a_{ii}| > \sum_{j\neq i} a_{ij}$$

hold for each $i = 1, 2, \dots, n$.

General Iteration Methods (cont'd)

Theorem. If A is strictly diagonally dominant, i.e.,

$$|a_{ii}| > \sum_{j \neq i} a_{ij}$$

hold for each $i = 1, 2, \dots, n$, then for <u>any</u> choice of $\boldsymbol{x}^{(0)}$, both the Jaconi and Gauss-Seidel methods give sequences $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $Ax = b$.

General Iteration Methods (cont'd)

Definition. Let A be a $n \times n$ matrix. The norm of it is defined by vector norm as follows:

$$\|A\| = \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\|.$$

Note that for any $\boldsymbol{z} \neq \boldsymbol{0}$, the vector $\boldsymbol{x} = \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}$ is a unit vector. Hence

$$\max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| = \max_{\boldsymbol{z} \neq \boldsymbol{0}} \left\| A\left(\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}\right) \right\| = \max_{\boldsymbol{z} \neq \boldsymbol{0}} \frac{\|A\boldsymbol{z}\|}{\|\boldsymbol{z}\|}.$$

General Iteration Methods (cont'd)

Theorem. If $\|T\| < 1$, then the sequences $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ that converges the for any choice of $\boldsymbol{x}^{(0)}$, to a vector $x \in \mathbb{R}^n$, and the following error bounds hold:

$$\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\| \leq \frac{\|\boldsymbol{T}\|^k}{1 - \|\boldsymbol{T}\|} \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\|.$$

General Iteration Methods (cont'd)

Remark 1. No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.

Remark 2. Under the conditions
($a_{i\,j} \leq 0$, for each $i = j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$):

- one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method.
- when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

General Iteration Methods (cont'd)

Remark. For a $n \times n$ matrix $A, \rho(A) \leq \|A\|$.

Remark. The rate of convergence of an iterative technique depends on the spectral radius of the matrix associated with the method. It is shown that

$$\left\|x^{(k)} - x\right\| \leq \rho(A)^k \left\|x^{(0)} - x\right\|.$$

Remark. One way to select a procedure to accelerate convergence is to choose a method whose associated matrix has minimal spectral radius.

Successive Over-Relaxation (SOR)

Let

$$T_\omega = (D - \omega L)^{-1}[(1-\omega)D + \omega U]$$

$$c_\omega = \omega(D - \omega L)^{-1} b$$

$$\boldsymbol{x}^{(k)} = c_\omega + T_\omega \boldsymbol{x}^{(k-1)}$$

For $0 < \omega < 1$.

This method accelerate the convergence for linear system that are convergent by the Gauss-Seidel thechnique.

ANY QUESTIONS?