پاسخ تمرین اول محاسبات عددی

سوال 1 . الف) ← خطا از مرتبه 10^{-3} است، پس نشان داد ارقام با این مرتبه یا کمتر، بی معنی است.

$$x \cong 0.39$$

ب) $x = 0.00381 \pm 0.00001 \rightarrow x \simeq 0.0038$

ج) $a = -0.2113 \pm 0.005 \rightarrow x \simeq -0.21$

سوال 2 ← الف) $\delta_a = \frac{\Delta a}{a} = 0.1 \rightarrow \Delta a = 1.35\times10^{-5} \rightarrow a \simeq 1\times10^{-4}$

ب) $\delta_a = \frac{\Delta a}{a} = \frac{2}{100} \rightarrow \Delta a = 11.856 \rightarrow a \simeq 6\times10^{2}$

سوال 3 ← الف) $z = a+b+c \rightarrow \Delta z = \Delta a + \Delta b + \Delta c$

خطای گرد کردن هر عدد را در نظر میگیریم: $\Delta a = 5\times10^{-4},\ \Delta b = 5\times10^{-1},\ \Delta c = 5\times10^{-2}$

$$\Delta z = \Delta a + \Delta b + \Delta c = 0.5505 \rightarrow z = 399.3 \pm 0.6$$

ب) $z = x_1 + x_2 - x_3 \rightarrow \Delta z = \Delta x_1 + \Delta x_2 + \Delta x_3$

$\rightarrow \Delta z = 0.59 \rightarrow z = 19.5 \pm 0.6$

سؤال 4 $\rightarrow z = ab \rightarrow \Delta z = a\Delta b + b\Delta a = 8.6 \times 0.005 + 3.49 \times 0.05$

$\rightarrow \Delta z = 0.2175 \rightarrow z = 30.0 \pm 0.2$

سؤال 5 $\rightarrow z = \frac{a}{b} \rightarrow \delta_z = \delta_a + \delta_b \rightarrow \Delta z = \frac{a}{b}\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$

$\rightarrow \Delta z = 2.11 \times 10^{-4} \rightarrow z = 1.1296 \pm 2.11 \times 10^{-4}$

سؤال 6 $\rightarrow z = \pi r^2 \rightarrow \Delta z = 2\pi r \Delta r \rightarrow \delta_z = \frac{2\pi r \Delta r}{\pi r^2} = \frac{2\Delta r}{r}$

$\Delta z = 37.7 \rightarrow \delta_z = 0.083, \quad z = 4.5 \times 10^1 \pm 0.4 \times 10^1$

سؤال 7 ـ

$P(K=0) = -1.5598$, $P(K=1) = -2.1178$

$P(K=2) = -2.7609$, $P(K=3) = -3.4984$

$P(K=4) = -4.3400$, $P(K=5) = -5.2958$, $P(K=6) = -6.3761$

$P(K=7) = -7.5917$, $P(K=8) = -8.9536$, $P(K=9) = -10.4729$

$P(K=10) = -12.1615$, $P(K=11) = -14.0310$, $P(K=12) = -16.0938$

$P(K=13) = -18.3621$, $P(K=14) = -20.8438$, $P(K=15) = -23.5666$

سؤال 8 ـ

$$f(x) = f(0) + \sum_{i=1}^{n} \frac{f^{i}(0)}{i!} x^{i}$$

چون با تقریب $\varepsilon = 10^{-5}$ خواسته شده، اولین i را پیدا می‌کنیم که به ازای آن

$$\left| \frac{f^{i}(0)}{i!} x^{i} \right| \leq \varepsilon.$$

$f(0) = 1$, $f^{1}(0) = 0$, $f^{2}(0) = -2$, $f^{3}(0) = 0$, $f^{4}(0) = 12$, $f^{5}(0) = 0$

$f^{6}(0) =$ [illegible], $f^{7}(0) = 0$

با محاسبه کردن بسط تیلور، داریم:

$$f(x) = e^{-x^2} = \sum_{K=0}^{\infty} \frac{(-x^2)^K}{K!}$$

اولین K ای که به ازای آن $\left| \frac{(-1.45^2)^K}{K!} \right| < \varepsilon$ است، برابر $K = 13$ است.

پس داریم:

$$\boxed{f(x) \simeq \sum_{K=0}^{13} \frac{(-x^2)^K}{K!}}$$

$f(K=0) = 0.18452$, $f(K=1) = 0.17977$, $f(K=2) = 0.17510$

$f(K=3) = 0.17052$, $f(K=4) = 0.16603$, $f(K=5) = 0.16162$

$f(K=6) = 0.15730$, $f(K=7) = 0.15306$, $f(K=8) = 0.14891$

$f(K=9) = 0.14484$, $f(K=10) = 0.14086$, $f(K=11) = 0.13696$

$f(K=12) = 0.13313$, $f(K=13) = 0.12939$, $f(K=14) = 0.12573$

$f(K=15) = 0.12215$.

سؤال ۹ — برای هر مورد، بسط تیلور را به دست آورده و بسط را از اولین جمله‌ای که کمتر از خطا است، قطع می‌کنیم.

الف)

$$\sin x = \sum_{K=0}^{\infty} \frac{(-1)^K x^{2K+1}}{(2K+1)!} \longrightarrow \frac{\sin(x)}{x} = \sum_{K=0}^{\infty} \frac{(-1)^K x^{2K}}{(2K+1)!}$$

$$\text{for } K \geqslant 3 : \left| \frac{(-1)^K x_{max}^{2K}}{(2K+1)!} \right| < \varepsilon \longrightarrow \boxed{\frac{\sin x}{x} \simeq \sum_{K=0}^{3} \frac{(-1)^K x^{2K}}{(2K+1)!}}$$

ب)

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for } n \geqslant 3 : \left| \frac{x_{max}^{2n+1}}{(2n+1)!} \right| < \varepsilon .$$

$$\longrightarrow \boxed{\sinh x \simeq \sum_{n=0}^{3} \frac{x^{2n+1}}{(2n+1)!}}$$

ج)

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for } n \geqslant 3 : \left| \frac{x_{max}^{2n+1}}{(2n+1)!} \right| < \varepsilon .$$

$$\longrightarrow \boxed{\sinh x \simeq \sum_{n=0}^{3} \frac{x^{2n+1}}{(2n+1)!}}$$

د)

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{x^{-n}}{n!}, \quad \text{for } n \geqslant 7 : \left| \frac{x_{min}^{-n}}{n!} \right| < \varepsilon .$$

$$\longrightarrow \boxed{e^{1/x} \simeq \sum_{n=0}^{7} \frac{x^{-n}}{n!}}$$

سؤال ۱۰ـ الف) از روش Newton-Raphson استفاده می کنیم:

$$y = \frac{1}{x^3} \longrightarrow x^3 = \frac{1}{y} \longrightarrow f(y) = \frac{1}{y} - x^3 = 0 .$$

$$\longrightarrow f'(y) = -\frac{1}{y^2} \longrightarrow y_{n+1} = y_n + \frac{\frac{1}{y_n} - x^3}{1/y_n^{\,2}} = y_n + y_n - y_n^{\,2} x^3$$

$$\longrightarrow \boxed{y_{n+1} = 2y_n - y_n^{\,2} x^3}$$

$$y = \frac{x}{1+x} \to y + yx = x \to y = x(1-y) \to x = \frac{y}{1-y}$$

$$\to f(y) = \frac{y}{1-y} - x = 0 \to f'(y) = \frac{1}{(1-y)^2}$$

$$\to y_{n+1} = y_n - (1-y_n)^2\left(\frac{y_n}{1-y_n} - x\right) = y_n - (1-y_n)y_n + x(1-y_n)^2$$

$$\to \boxed{y_{n+1} = y_n^2 + x(1-y_n)^2}$$

$$y = \frac{\sqrt{x^2+1}}{x} = \frac{1}{x} \times \sqrt{x^2+1} = z \times t \longrightarrow z = \frac{1}{x}, \quad t = \sqrt{x^2+1}$$ (E

$$x = \frac{1}{z} \longrightarrow f(z) = \frac{1}{z} - x \longrightarrow f'(z) = -\frac{1}{z^2}$$

$$\longrightarrow z_{n+1} = z_n + z_n^2\left(\frac{1}{z_n} - x\right) = \boxed{2z_n - z_n^2 x = z_{n+1}}$$

$$t = \sqrt{x^2+1} \longrightarrow \boxed{t_{n+1} = \frac{1}{2}\left(t_n + \frac{x^2+1}{t_n}\right)}$$

$$y = (2x+1)\frac{1}{\sqrt{x}} \longrightarrow y = (2x+1)\, z \longrightarrow z = \frac{1}{\sqrt{x}}$$ ()

$$\longrightarrow x = \frac{1}{z^2} \longrightarrow f(z) = x - \frac{1}{z^2} \longrightarrow f'(z) = \frac{2}{z^3}.$$

$$\longrightarrow z_{n+1} = z_n - \frac{\left(x - \frac{1}{z_n^2}\right)}{2/z_n^3} = z_n - \frac{1}{2}\left(x z_n^3 - z_n\right)$$

$$\longrightarrow \boxed{z_{n+1} = \frac{3}{2} z_n - \frac{1}{2} a z_n^3}$$

$$y=\frac{1}{\sqrt{x(x+1)}} \longrightarrow \boxed{y_{n+1}=\frac{3}{2}y_n-\frac{1}{2}y_n^3(x^2+x)}$$

$$y=\frac{1}{\sqrt[3]{x}} \longrightarrow x=\frac{1}{y^3} \longrightarrow f(y)=\frac{1}{y^3}-x.$$

$$\longrightarrow f'(y)=-\frac{3}{y^4} \longrightarrow y_{n+1}=y_n+\frac{(1/y_n^3-x)}{3/y_n^4}$$

$$\longrightarrow y_{n+1}=y_n+\frac{y_n}{3}-\frac{x y_n^4}{3} \longrightarrow \boxed{y_{n+1}=\frac{4}{3}y_n-\frac{y_n^4 x}{3}}$$

$$y=\sqrt[4]{x} \longrightarrow y^4=x \longrightarrow f(y)=y^4-x \longrightarrow f'(y)=4y^3$$

$$\longrightarrow y_{n+1}=y_n-\frac{y_n^4-x}{4y_n^3}=y_n-\frac{y_n}{4}+\frac{x}{4y_n^3}$$

$$\longrightarrow \boxed{y_{n+1}=\frac{3}{4}y_n+\frac{x}{4y_n^3}}$$

سؤال 11 ـــ الف)

$$\Delta u = \left|\frac{\partial u}{\partial x_1}\right| \Delta x_1 + \left|\frac{\partial u}{\partial x_2}\right| \Delta x_2$$

$$\frac{\partial u}{\partial x_1} = \frac{1}{x_1 + x_2^2} , \quad \frac{\partial u}{\partial x_2} = \frac{2x_2}{x_1 + x_2^2} .$$

$$\longrightarrow \boxed{\Delta u = \frac{\Delta x_1 + 2x_2 \Delta x_2}{x_1 + x_2^2}}$$

ب)

$$\Delta u = \left|\frac{\partial u}{\partial x_1}\right| \Delta x_1 + \left|\frac{\partial u}{\partial x_2}\right| \Delta x_2 + \left|\frac{\partial u}{\partial x_3}\right| \Delta x_3$$

$$\frac{\partial u}{\partial x_1} = \frac{1}{x_3} , \quad \frac{\partial u}{\partial x_2} = \frac{2x_2}{x_3} , \quad \frac{\partial u}{\partial x_3} = -\frac{x_1 + x_2^2}{x_3^2}$$

$$\longrightarrow \boxed{\Delta u = \frac{\Delta x_1}{x_3} + \frac{2x_2 \Delta x_2}{x_3} + \frac{(x_1 + x_2^2)}{x_3^2} \Delta x_3}$$

ج)

$$\Delta u = x_1 \Delta x_2 + x_2 \Delta x_1 + x_1 \Delta x_3 + x_3 \Delta x_1 + x_2 \Delta x_3 + x_3 \Delta x_2$$

$$\longrightarrow \boxed{\Delta u = \Delta x_1 (x_2 + x_3) + \Delta x_2 (x_1 + x_3) + \Delta x_3 (x_2 + x_1)}$$