Numerical Computations

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Overview

The Taylor polynomial is designed to approximate a given function f well at one point.

In this chapter, we are going to talk about polynomial approximation where the agreement with f is not all focused at only one point, but is spread over a number of points.

Chapter's Topics

- What is interpolation?
- Linear Interpolation
- Polynomial Interpolation (Lagrange's Method)
- Accuracy of Interpolation
- Nevill's Method
- Divided Difference

What is Interpolation ?

Assume that results of a census of the population which was taken every 10 years from 1950 to 2000, are listed as follows:

Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422

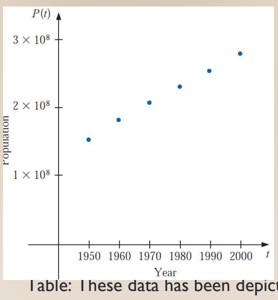


Table: These data has been depicted

In reviewing these data, it might be asked to provide a reasonable estimate of the population, for instance in 1975, or even in the year 2020.

Using a function that fits the given data might predict the population, say, in 1975 or 2020.

The process is called ***interpolation***.

The Problem of Interpolation

Let a function $y = f(x)$ be given by a table:

$$y_0 = f(x_0),\ \ y_1 = f(x_1)\ \ ,\ldots.\ \ ,y_n = f(x_n).$$

The problem of interpolation is to find the polynomial

$$p(x) = p_n(x)$$

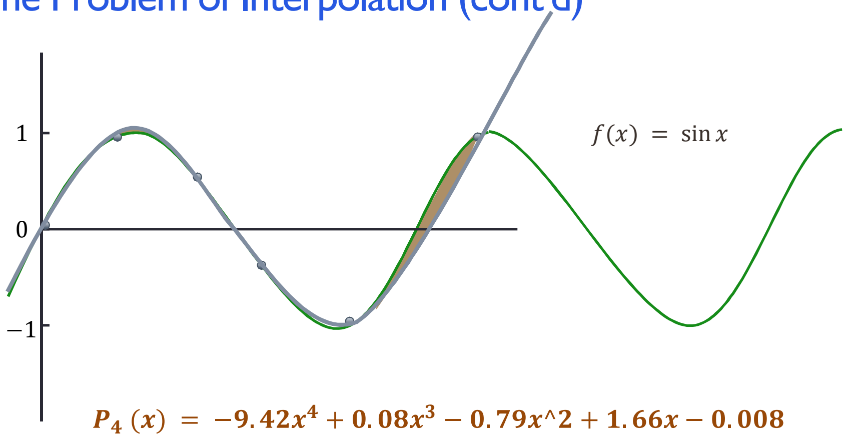
of degree at most n, such that

$$p(x_i) = f(x_i)$$

for $i = 0,1,2,\ldots,n$.

Geometrically, this means that one has to find an algebraic curve of the form $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ that passes through the given set of points (x_i, y_i), for $i = 0,1,\ldots,n$.

The Problem of Interpolation (cont'd)



$$P_4(x) = -9.42x^4 + 0.08x^3 - 0.79x\text{^}2 + 1.66x - 0.008$$

Lagrange's Method

Joseph-Louis (Giuseppe Luigi), comte de Lagrange
(1736-1813)

Linear Interpolation

The linear interpolation polynomial is a simple case where the values of $f(x)$ at two points, say x_0 and x_1 ,are given.

The problem is to obtain a polynomial of degree one which passes through the distinct points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

The unique straight line which passes through theses points is:

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

which can be written as:

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

Linear Interpolation (cont'd)

Let $L_0(x) = \frac{x-x_1}{x_0-x_1}$ and $L_1(x) = \frac{x-x_0}{x_1-x_0}$.

The linear Lagrange Interpolating Polynomial through $(x_0, f(x_1))$ and $(x_1, f(x_1))$ is:

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

Note that:

$$L_0(x_0) = 1, L_0(x_1) = 0, L_1(x_0) = 0, L_1(x_1) = 1$$

which implies that:

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0)$$

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1)$$

Linear Interpolation (cont'd)

Remark. Without loss of generality, suppose that $x_0 < x_1$. Let $f(x)$ be a continuous on $[x_0, x_1]$ and differentiable in (x_0, x_1).

By the mean value theorem, there $y \in (x_0, x_1)$ such that:

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = f'(y).$$

Hence, the first-degree polynomial $p_1(x)$ may be expressed as follows:

$$P_1(x) = f(x_0) + (x - x_0)f'(y).$$

Linear Interpolation (cont'd)

Remark (cont'd). If x_0 is kept fixed and let x_1 tend to x_0 ,then:

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

And consequently, $P_1(x)$ becomes precisely the Taylor Polynomial.

Note that the main purpose of constructively $P_1(x)$ is to give an approximation to $f(x)$ for each x between x_0 and x_1 .

Polynomial Interpolation

Now, our aim is to generalize the result of the last section to the case where the values of $f(x)$ are given at $n+1$ distinct points $x_0, x_1, \dots, x_n$.

We are supposed to find a polynomial

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

such that

$$P_n(x_0) = f(x_0), \dots, P_n(x_n) = f(x_n).$$

Consequently we might think about $P_n(x)$ as the following form:

$$\begin{aligned} P_n(x) = \; & c_0(x-x_1)(x-x_2)\cdots(x-x_n)f(x_0) \\ & + c_1(x-x_0)(x-x_2)\cdots(x-x_n)f(x_1) \\ & + \cdots \\ & + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_n) \end{aligned}$$

Polynomial Interpolation (cont'd)

For each $0 \le i \le n$, let:

$$L_i(x) = c_i(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n).$$

As a result we must determine c_i such that:

$$L_i(x) = \begin{cases} 1, & x = x_i \\ 0, & x \neq x_i. \end{cases}$$

Since $L_i(x_i) = 1$, we have:

$$c_i = \frac{1}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}.$$

Polynomial Interpolation (cont'd)

Therefore, we can write:

$$L_i(x) = \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

Thus, we have:

$$P_n(x) = L_0(x)f(x_0) + \cdots + L_n(x)f(x_n)$$

which for each $0 \le i \le n$, we have:

$$P_n(x_i) = f(x_i)$$

Polynomial Interpolation (cont'd)

Claim. The polynomial $P(x)$ is a unique polynomial which agrees with the supposed requirement.

Proof. Consider any polynomial $Q_n(x)$ of degree at most n such that

$$Q_n(x_i) = f(x_i).$$

Consider polynomial:

$$Q_n(x) - p_n(x) = T_n(x).$$

Thus, $T_n(x_i) = 0$ for $0 \le i \le n$.

$T_n(x)$ is zero in $n+1$ points. But a polynomial of degree at most n has no more than n zero points unless it is zero at any points. So:

$$T_n(x) = 0 \quad \Longrightarrow \quad Q_n(n) = p_n(n)$$

Polynomial Interpolation (cont'd)

Example. Write the Lagrange's interpolation polynomial for the function $f(x)$ given in tabular form below:

i	0	1	2	3
x_i	0	0.1	0.3	0.5
$f(x_i)$	-0.5	0	0.2	1

Solution. We have

$$L_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

$$p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

Polynomial Interpolation (cont'd)

Solution (cont'd).

$$L_0(x) = \frac{(x-0.1)(x-0.3)(x-0.5)}{(-0.1)(-0.3)(-0.5)} = -\frac{x^3 - 0.9x^2 + 0.23x - 0.015}{0.015}$$
$$L_2(x) = \frac{x(x-0.1)(x-0.5)}{0.3 \times 0.2(-0.2)} = -\frac{x^3 - 0.6x^2 + 0.05x}{0.012}$$
$$L_3(x) = \frac{x(x-0.1)(x-0.3)}{0.5 \times 0.4 \times 0.2} = \frac{x^3 - 0.4x^2 + 0.03x}{0.04}.$$

$$p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

$$= \frac{x^3 - 0.9x^2 + 0.23x - 0.015}{0.015} \times 0.5 - \frac{x^3 - 0.6x^2 + 0.05x}{0.012} \times 0.2$$
$$+ \frac{x^3 - 0.4x^2 + 0.03x}{0.04} \times 1 = \frac{125}{3}x^3 - 30x^2 + \frac{73}{12}x - 0.5$$

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Accuracy of interpolation

In this section the aim is examining the accuracy of the interpolating polynomial $P_n(x)$ as an approximation of $f(x)$.

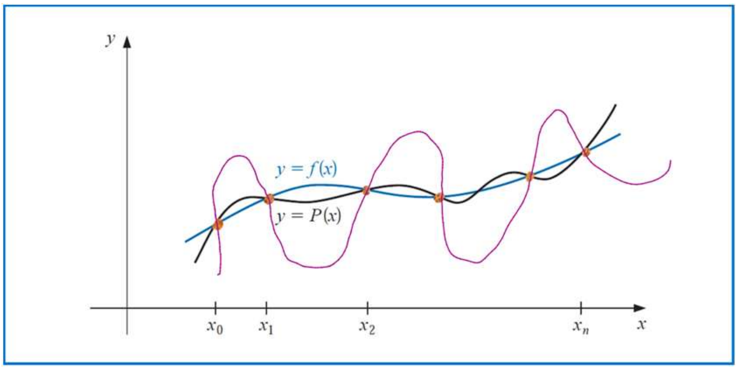
In other words, the remainder term or a bound for error must be calculated.

Note that it is not possible to examine the exact value of

$$R_n(x) = f(x) - P_n(x).$$

Accuracy of interpolation (cont'd)

Based on the knowledge of the values of $f(x)$ at $x_0, x_1 \dots x_n$, as it is shown in the below figure, we are free to draw any curve which passes through these $n+1$ points.



Accuracy of interpolation (cont'd)

Hence we might face to arbitrarily large $f(x) - P_n(x)$ values at any x except for $x_0, x_1, \dots, x_n$ where $f(x) - P_n(x) = 0$.

Thus, we require more information about $f(x)$. The error estimation is obtainable in term of the $(n+1)^{th}$ derivative of f, if exists.

Accuracy of interpolation (cont‘d)

Theorem. Let $x_0, x_1, \dots, x_n$ be distinct points in the interval $[a, b]$. If $f, f^1, \dots, f^n$ exist and be continuous on $[a, b]$ and f^{n+1} exists on (a, b), then for each $x \in [a, b]$, there exists a number $\mu(x)$ (generally unknown) between $x_0, x_1, \dots, x_n$ and hence in (a, b), such that

$$R_n(x) = f(x) - P_n(x) = \frac{f^{n+1}(\mu(x))}{(n+1)!}(x - x_0)\cdots(x - x_n)$$

Accuracy of interpolation (cont‘d)

Proof. Let

$$g(x) = f(x) - P_n(x) + \mu(x - x_0) \dots (x - x_n)$$

where μ is any constant.

$g(x)$ have zeros at the $n + 1$ points $x_0, x_1, \dots, x_n$.

We wish to estimate the error at point $x = \alpha \in [a, b]$ and $\alpha \neq x_0, x_1, \dots, x_n$, by choosing μ such that $g(\alpha) = 0$. So

$$0 = f(\alpha) - P_n(\alpha) + \mu(\alpha - x_0) \dots (\alpha - x_n)$$

Thus:

$$\mu = -\frac{f(\alpha) - P_n(\alpha)}{(\alpha - x_0)\dots(\alpha - x_n)}.$$

Accuracy of interpolation (cont‘d)

Proof. With $\mu = -\frac{f(\alpha)-P_n(\alpha)}{(\alpha-x_0)\ldots(\alpha-x_n)}$, we have

$$g(x) = f(x) - P_n(x) - \frac{f(\alpha)-P_n(\alpha)}{(\alpha-x_0)\ldots(\alpha-x_n)}(x-x_0)\ldots(x-x_n).$$

Consequently, $g(x)$ is zero at the $n+2$ distinct points $x_0, x_1, \ldots, x_n$ and $x = \alpha$ at not less than $n+2$ points.

By using Mean value theorem, $g'(x)$ has at least $n+1$ zeros. Then, we apply the mean value theorem on g'', and so $g''(x)$ has at least n zeros.

By repeating this process, we obtain $g^{n+1}(x)$ has at least one zero, say $\mu(\alpha)$.

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Accuracy of interpolation (cont‘d)

Proof. Differentiatly $n+1$ times gives:

$$g^{n+1}(x) = f^{n+1}(x) - \frac{(n+1)!}{(\alpha-x_0)\ldots(\alpha-x_n)}\left(f(\alpha) - P_n(\alpha)\right)$$

Thus:

$$0 = g^{n+1}(\mu(\alpha)) = f^{n+1}(\mu(\alpha)) - \frac{(n+1)!}{(\alpha-x_0)\ldots(\alpha-x_n)}\left(f(\alpha) - P_n(\alpha)\right)$$

$$\Downarrow$$

$$f(\alpha) - P_n(\alpha) = (\alpha - x_0)\cdots(\alpha - \mathrm{x}_n)\frac{f^{(n+1)}(\mu(\alpha))}{(n+1)!}$$

Accuracy of interpolation (cont'd)

Example. Suppose $ln\, 2.1 = 0.7419$ and $ln\, 2.2 = 0.7885$.

i. By interpolation on these two values, estimate $ln\, 2.14$.

ii. Estimate the error in linear interpolation

Solution.

i) By interpolation

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$P_1\,(2.14) = \left(\frac{2.14-2.2}{2.1-2.2}\right) \times 0.7419 + \left(\frac{2.14-2.1}{2.2-2.1}\right) \times 0.7885$$

$$= \quad 0.76054.$$

Only for comparison, $ln\, 2.14 = 0.76081$ to the decimal places

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Accuracy of interpolation (cont'd)

Solution. (cont'd)

ii)

$$f(x) - P_1(x) = (x - x_0)(x - x_1)\frac{f''(\mu(x))}{2!}.$$

Also,

$$f(x) = \ln x\,, f'(x) = \frac{1}{x}, f''(x) = \frac{-1}{x^2}.$$

Thus:

$$f(2.14) - P_1(2.14) = \frac{-(0.04)(-0.06)}{2\mu(2.14)^2}.$$

Since $2.1 \le \mu(2.14) \le 2.2$, thus the error is between 0.00024 and 0.00025.

Nevilles Method

In this section, we talk about Neville's method in which interpolating polynomial approximations are recursively generated.

The theorem, as follows, describes this method for recursively generating language polynomial approximations.

Nevilles Method (cont'd)

Theorem. Assume that the value of f are given at $n+1$ points $x_0, x_1, \dots, x_n$.
Let x_j and x_i be two distinct numbers in the set $\{x_0, x_1, \dots, x_n\}$ then

$$P_n(x) = \frac{(x - x_j)P_{0,1,\dots,j-1,j+1,\dots,n}(x) \; - (x - x_i)P_{0,1,\dots,i-1,i+1,\dots,n}(x)}{(x_i - x_j)}$$

where $P_n(x)$, $P_{0,1,\dots,j-1,j+1,\dots,n}(x)$ and $P_{0,1,\dots,i-1,i+1,\dots,n}(x)$ are Lagrange polynomial of degree n, $n-1$, $n-1$, respectively such that these polynomial interpolates $f(x)$ at the points $x_0, x_1, \dots, x_n$, and $x_0, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$, and $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, respectively.

Nevilles Method (cont'd)

Proof. Let $Q \equiv P_{0,1,\ldots,j-1,j+1,\ldots,n}$ and $\bar{Q} \equiv P_{0,1,\ldots,i-1,i+1,\ldots,n}$ and

$$p(x) = \frac{(x - x_j)Q(x) - (x - x_i)\bar{Q}(x)}{(x_i - x_j)}.$$

We have

i. $\deg(Q) \le n-1,\ \deg(\bar{Q}) \le n-1, \deg(p) \le n.$

ii. $\bar{Q}(x_j) = f(x_j)$ (why?) implies that

$$p(x_j) = \frac{(x_j - x_j)Q(x_j) - (x_j - x_i)\,\bar{Q}(x_j)}{(x_i - x_j)} = \bar{Q}(x_j) = f(x_i).$$

iii. $Q(x_i) = f(x_i)$ implies $p(x_i) = f(x_i)$.

iv. For each $x_r \in \{1, \ldots, n\} \backslash \{i, j\}$, $Q(x_r) = \bar{Q}(x_r) = f(x_r)$, and $p(x_r) = f(x_r)$.

By uniqueness of Lagrange polynomial of degree at most n which agrees with f at $x_0, x_1, \ldots, x_n$,we have $p(x) = P_n(x)$.

Nevilles Method (cont'd)

Example. The interpolating polynomial at 5 points x_0, x_1, x_2, x_3, x_4 can be generated recursively as follows:

$$P_4(x) = P_{0,1,2,3,4} = \frac{(x - x_4)P_{0,1,2,3}(x) - (x - x_0)P_{1,2,3,4}(x)}{(x_4 - x_0)}$$

x_0	P_0				
x_1	P_1	$P_{0,1}$			
x_2	P_2	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

Nevilles Method (cont'd)

Remark. The interpolating polynomial at 5 points x_0, x_1, x_2, x_3, x_4 can be generated recursively as follows:

$$P_4(x) = P_{0,1,2,3,4} = \frac{(x - x_4)P_{0,1,2,3}(x) - (x - x_0)P_{1,2,3,4}(x)}{(x_4 - x_0)}$$

$$P_{0,1,2,3}(x) = \frac{(x - x_3)P_{1,2,3}(x) - (x - x_0)P_{0,1,2}(x)}{(x_3 - x_0)}$$

$$P_{1,2,3,4}(x) = \frac{(x - x_4)P_{1,2,3}(x) - (x - x_1)P_{2,3,4}(x)}{(x_4 - x_1)}$$

Nevilles Method (cont'd)

Remark(cont'd).

$$P_{0,1,2}(x) = \frac{(x - x_2)P_{0,1}(x) - (x - x_0)P_{1,2}(x)}{(x_2 - x_0)}$$

$$P_{1,2,3}(x) = \frac{(x - x_3)P_{1,2}(x) - (x - x_1)P_{2,3}(x)}{(x_3 - x_1)}$$

$$P_{2,3,4}(x) = \frac{(x - x_4)P_{2,3}(x) - (x - x_2)P_{3,4}(x)}{(x_4 - x_2)}$$

Nevilles Method (cont'd)

Remark(cont'd).

$$P_{0,1}(x) = \frac{(x - x_0)P_1 - (x - x_1)P_0}{x_1 - x_0}$$

$$P_{1,2}(x) = \frac{(x - x_1)P_2 - (x - x_2)P_1}{x_2 - x_1}$$

$$P_{2,3}(x) = \frac{(x - x_2)P_3 - (x - x_3)P_2}{x_3 - x_2}$$

$$P_{3,4}(x) = \frac{(x - x_3)P_4 - (x - x_4)P_3}{x_4 - x_3}$$

where $P_i = f(x_i)$ for $0 \leq i \leq 4$.

Divided Differences

Suppose the nth Lagrange polynomial $P_n(x)$ whose values in $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ are equal to values of $f(x)$, i.e., $P_n(x_i) = f(x_i)$ for $0 \leq i \leq n$.

We know that this polynomial is unique. In this section, our aim is to express $P_n(x)$ based on the divided differences of f with respect to $x_0, x_1, \ldots, x_n$. This algebraic representation is useful in certain situations.

Divided Differences

As you may remember, the following viewpoint is used to construct a Lagrange polynomial.

$$P_n(x) = c_0(x-x_1)(x-x_2)\cdots(x-x_n)f(x_0)$$
$$+ c_1(x-x_0)(x-x_2)\cdots(x-x_n)f(x_1)$$
$$+\cdots$$
$$+ c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_n)$$

To express $P_n(x)$ based on the divided differences of $f(x)$ with respect to $x_0, x_1, \dots, x_n$, the following expression is used

$$P_n(x) = a_0 + (x-x_0)a_1 + (x-x_0)(x-x_1)a_2 + \cdots + (x-x_0)(x-x_1)\dots(x-x_{n-1})a_n$$

Divided Differences (cont'd)

By substituting $x = x_0, x_1, \dots, x_n$ in turn into $P_n(x)$, we have

$$f(x_0) = a_0$$
$$f(x_1) = a_0 + (x_1-x_0)a_1$$
$$f(x_2) = a_0 + (x_2-x_0)a_1 + (x_2-x_0)(x_2-x_1)a_2$$
$$\vdots$$
$$f(x_n) = a_0 + (x_n-x_0)a_1 + (x_n-x_0)(x_n-x_1)a_2 + \dots + (x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})a_n$$

Divided Differences (cont'd)

Consequently, values for $a_0, a_1, a_2, \ldots, a_n$ are determined uniquely. We obtain

$$a_0 = f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Now, we define the *divided-difference* notation. Denote $f[x_i]$ as the value of f at x_i: $f[x_i] = f(x_i)$

Divided Differences (cont'd)

Recursively, the remaining divided differences. Denote the first divided differences of f with respect to x_i and x_{i+1} by $f[x_i, x_{i+1}]$ and define as follows:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The second divided difference, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Divided Differences (cont'd)

So, after finding the $(k-1)$th divided differences as follows:

$$f[x_i, x_{i+1}, \dots, x_{i+k-1}],$$

$$f[x_{i+1}, x_{i+2}, \dots, x_{i+k-1}, x_{i+k}].$$

The kth divided difference relative to x is denoted as:

$$f[x_i, x_{i+1}, \dots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

The above process ends with the single nth divided difference

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

Divided Differences (cont'd)

Thus

$$a_0 = f[x_0]$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_n - x_0} = \frac{f[x_1] - f[x_0]}{x_n - x_0} = f[x_0, x_1]$$

$$f[x_2] - f[x_0] = (x_2 - x_0) f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1) a_2$$

$$a_2 = \frac{f[x_2] - f[x_0] - (x_2 - x_0) f[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f[x_2] - f[x_0]}{(x_2 - x_0)(x_2 - x_1)} - \frac{f[x_0, x_1]}{x_2 - x_1}$$

Divided Differences (cont'd)

$$a_2 = \frac{f[x_2] - f[x_0]}{(x_2 - x_0)(x_2 - x_1)} - \frac{f[x_0, x_1]}{x_2 - x_1}$$

$$= \frac{f[x_0, x_2]}{x_2 - x_1} - \frac{f[x_0, x_1]}{x_2 - x_1}$$

$$= \frac{f[x_2, x_0] - f[x_0, x_1]}{x_2 - x_1}$$

$$= f[x_0, x_1, x_2]$$

Inductively, $a_k = f[x_0, x_1, x_2, \dots, x_k]$ for $k = 0, 1, \dots, n$ thus

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, x_2, \dots, x_k](x - x_0)(x - x_1)(x - x_2) \dots (x - x_{k-1})$$

Divided Differences (cont'd)

Remark. It is shown that the value of $f[x_0, x_1, \dots, x_k]$ is independent of the order of the numbers $x_0, x_1, \dots, x_k$.

The following table despite the outline of the generation of the divided difference.

Divided Differences (cont'd)

x	$f(x)$	First divided differences	Second Divided differences
x_0	$f[x_0]$		
		$f[x_0,x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$	
			$f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$
x_1	$f[x_1]$		
		$f[x_1,x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$	
			$f[x_1,x_2,x_3] = \dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$
x_2	$f[x_2]$		
		$f[x_2,x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$	
			$f[x_2,x_3,x_4] = \dfrac{f[x_3,x_4] - f[x_2,x_3]}{x_4 - x_2}$
x_3	$f[x_3]$		
		$f[x_3,x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$	
			$f[x_3,x_4,x_5] = \dfrac{f[x_4,x_5] - f[x_3,x_4]}{x_5 - x_3}$
x_4	$f[x_4]$		
		$f[x_4,x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$	
x_5	$f[x_5]$		

Divided Differences (cont'd)

Second divided differences	Third divided differences
$f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$	
	$f[x_0,x_1,x_2,x_3] = \dfrac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0}$
$f[x_1,x_2,x_3] = \dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$	
	$f[x_1,x_2,x_3,x_4] = \dfrac{f[x_2,x_3,x_4] - f[x_1,x_2,x_3]}{x_4 - x_1}$
$f[x_2,x_3,x_4] = \dfrac{f[x_3,x_4] - f[x_2,x_3]}{x_4 - x_2}$	
	$f[x_2,x_3,x_4,x_5] = \dfrac{f[x_3,x_4,x_5] - f[x_2,x_3,x_4]}{x_5 - x_2}$
$f[x_3,x_4,x_5] = \dfrac{f[x_4,x_5] - f[x_3,x_4]}{x_5 - x_3}$	

Divided Differences (cont'd)

Example. The upward velocity of a rocket is given as a function of time in as following table.

x (s)	$f(x)$ (m/s)
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Divided Differences (cont'd)

Determine the value of the velocity at $t = 16$ seconds with third order polynomial interpolation using divided difference polynomial method.

Solution.

$$\boldsymbol{f(t) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +}$$
$$\boldsymbol{f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)}$$
$$\boldsymbol{= 227.04 + 27.148(x - 10) + 0.7532(x - 10)(x - 15)}$$
$$\boldsymbol{+ 0.2316(x - 10)(x - 15)(x - 20)}$$
$$\boldsymbol{f(16) = 388.87}$$

Divided Differences (cont'd)

x_i	$f[x_i]$	$f[x_i, x_j]$	$f[x_0, x_1, x_2]$		
$x_0 = 10$					
	$f[x_0] = 227.04$				
$x_1 = 15$		$f[x_0, x_1] = 27.148$			
			$f[x_0, x_1, x_2] = 0.753$		
	$f[x_1] = 362.78$			$f[x_0, x_1, x_2, x_3] = 0.232$	
$x_2 = 20$		$f[x_1, x_2] = 30.914$			
			$f[x_1, x_2, x_3] = 1.336$		$f[x_0, x_1, x_2, x_3, x_4] = -0.041$
	$f[x_2] = 517.35$				
$x_3 = 22.5$		$f[x_2, x_3] = 34.248$		$f[x_1, x_2, x_3, x_4] = -0.078$	
			$f[x_2, x_3, x_4] = 0.743$		
	$f[x_3] = 602.97$				
$x_4 = 30$		$f[x_3, x_4] = 39.826$			
	$f[x_4] = 901.67$				

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Newton's Divided-Difference Formula

The divided difference method on equally spaced points is due to Isaac Newton.

Let $x_j = x_0 + jh$, $0 \le j \le n$ where $h > 0$ denoted the equal spacing between the distinct points $x_0, x_1, \dots, x_n$.

Thus the distinct points are represented by parameters x_0, h_1 and $n + 1$.

Newton's Divided-Difference Formula

In this setting, finite differences are defined as follows:

- Finite differences of first order $\Delta^1 f_i = f_{i+1} - f_i$
- Finite differences of second order $\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$
- Finite differences of kth order $\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$

where $f_i = f(x_i)$ for each $0 \leq i \leq n$.

Newton's Divided-Difference Formula (cont'd)

We seek the polynomial:

$$P_n[x] = f[x_0] + (x - x_0) f[x_0, x_1]$$

$$+ (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots$$

$$+ (x - x_0) \dots (x - x_{n-1}) f[x_0, x_1, \dots, x_n]$$

Newton's Divided-Difference Formula (cont'd)

We have

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{h} = \frac{\Delta f_i}{h}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{\Delta f_{i+1} - \Delta f_i}{x_{i+2} - x_i} = \frac{\Delta^2 f_i}{2h^2}$$

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{\frac{\Delta^2 f_{i+1}}{2h^2} - \frac{\Delta^2 f_i}{2h^2}}{x_{i+3} - x_i} = \frac{\Delta^2 f_{i+1} - \Delta^2 f_i}{(3h)(2h^2)} = \frac{\Delta^3 f_i}{3!\, h^2}$$

$$\vdots$$

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{\Delta^k f_i}{k!\, h^k}$$

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Newton's Divided-Difference Formula (cont'd)

Thus

$$P_n(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2!\, h^2}(x - x_0)(x - x_1) + \dots + \frac{\Delta^n f_0}{n!\, h^n}(x - x_0) \dots (x - x_{n-1})$$

Newton's Divided-Difference Formula (cont'd)

Let $\frac{x-x_0}{h} = q$. Thus

$$x - x_0 = hq$$

$$x - x_1 = x - x_0 - h = hq - h = (q-1)h$$

$$x - x_2 = x - x_0 - 2h = hq - 2h = (q-2)h$$

$$\vdots$$

$$x - x_{n-1} = x - x_0 - (n-1)h = hq - (n-1)h = (q-n+1)h$$

Newton's Divided-Difference Formula (cont'd)

As a result:

$$P_n(x_0 + qh) = f_0 + q\Delta f_0 + \frac{q(q-1)}{2!}\Delta^2 f_0 + \ \dots + \frac{q(q-1)\dots(q-n+1)}{n!}\Delta^n f_0$$

Newton's Divided-Difference Formula (cont'd)

Divided differences with equally spaced points are expressed in the following table:

x	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

Newton's Divided-Difference Formula (cont'd)

In this relation, the interpolating polynomial error is equal to:

$$R(x_0 + qh) = h^{n+1} \frac{q(q-1)\dots(q-n)}{(n+1)!} f^{(n+1)}(\eta)$$

Where η is some internal point of the least interval containing all the points $x_i, 0 \le i \le n$ and the point $x_0 + qh$.

Remark. It is shown that if there is an additional point x_{n+1}, we can use the following formula for practical computation

$$R(x_0 + qh) \approx \frac{\Delta^{n+1} f_0}{(n+1)!} q(q-1)\dots(q-n)$$

Newton's Divided-Difference Formula (cont'd)

Example. Given a table of the values of the function $y = \log x$, find $\log 1001$.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1000	3.0000000	0.0043214	-0.0000426	0.0000008
1010	3.0043214	0.0042788	-0.0000418	0.0000009
1020	3.0086002	0.0042370	-0.0000409	0.0000008
1030	3.0128372	0.0041961	-0.0000401	
1040	3.0170333	0.0041560		
1050	3.0211893			

$$f(x) = f_0 + q\Delta f_0 + \frac{q(q-1)}{2!}\Delta^2 f_0 + \cdots + \frac{q(q-1)(q-2)}{3!}\Delta^3 f_0$$

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Newton's Divided-Difference Formula (cont'd)

Solution.

For $x = 1001$ we have $q = 0.1$. Thus

$$\log 1001 = 3.0000000 + 0.1 \times 0.0043214 + \frac{0.1 \times 0.9}{2} 0.0000426$$
$$+ \frac{0.1 \times 0.9 \times 1.9}{6} 0.0000008 = 3.0004341$$

$$R_3(x) = h^4 \frac{q(q-1)(q-2)(q-3)}{4!} f^4(\varepsilon)$$

where $1000 < \varepsilon < 1030$

Since $f(x) = \log x$, we have $f^{(4)}(x) = -\frac{3!}{x^4} \log e$

$$|f^{(4)}(\varepsilon)| = \frac{3!}{(1000)^4} \log e$$

For $h = 10$ and $q = 0.1$, we finally get

$$|R_3(1001)| < \frac{0.1 \times 0.9 \times 1.9 \times 2.9 \times 10^4 \log e}{4 \times (1000)^4} \approx 0.5 \times 10^{-9}$$

ANY QUESTIONS?