Project Proposal

M.Sc Degree in Computer Science

Hadassah Academic College

Title: Frequency-dependent attenuation in fractional Helmholtz wave equations

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2. Project Details

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Project Name: Frequency-dependent attenuation in fractional Helmholtz wave equations
Location: Hadassah Academic College

2.1. Project description

"By beaming high-frequency sound waves into the body, physicians can translate the "echoes" that bounce off body tissues and organs into "sound you can see," colorful, visual images that provide valuable medical information"[1]. The technique is very cheap (compared to other techniques such as CT (Computed Tomography), MRI (Magnetic Resonance Imaging), not harmful at all (fetal imaging), gives results in real-time (surgery) and is also extremely portable (moving ultrasound installation to the patient's place is possible).

In standard (B-mode) ultrasound the image is created thanks to the information contained in the back-reflected waves (from the body organs). The acoustic stack is emitting and receiving only back-reflected sound waves. Images are constructed on the basis of the assumption of an average *speed of sound* and average *attenuation* that are independent of the location in the anatomy.

The following picture[2] of Figure 1 shows detected masses for given sound speed and attenuation and a biopsy (sample of cells) revealed the nature of the detected mass. From this graph it is obvious that the above two parameters (speed of sound and attenuation) deserve to be measured in order to gain further important information, such as characterizing tissue and segregating malignant tumors from benign ones. As mentioned above, this information is simply not available with B-mode ultrasound devices, and that is the main motivation for using ultrasound tomography. In tomography the anatomy is surrounded with transducers, and waves that pass through it are measured, providing enough information for reconstructing speed of sound and attenuation.

[1] http://www.sdms.org, Society of Diagnostic Medical Sonography.
[2] "Detection and characterization of breast masses with ultrasound tomography: Clinical results", Neb Duric, Peter Littrup, Cuiping Li, Olsi Rama, Lisa Bey-Knight, Steven Schmidt and Jessica Lupinacci Medical Imaging 2009: Ultrasonic Imaging and Signal Processing, edited by Stephen A. McAleavey, Jan D'hooge, Proc. of SPIE Vol. 7265, 1-8

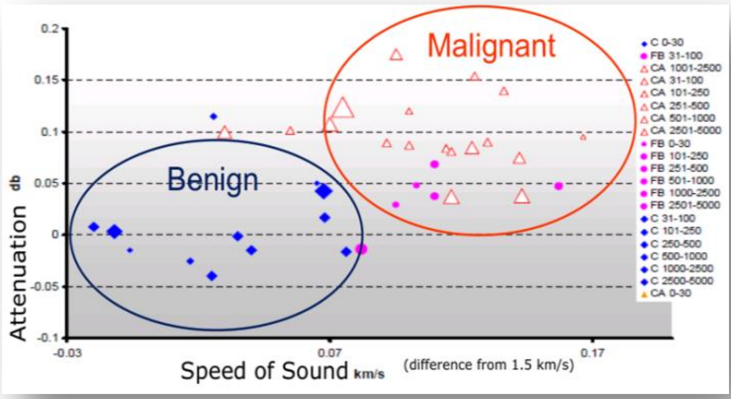

Figure 1: Detection and characterization of breast masses as a function of attenuation and speed of sound

Beside this, there is always the desire to increase resolution on the one hand and to access deeper organs on the other hand. The latter is limited due to attenuation. Indeed, better resolution may be obtained with high frequency, but attenuation also increases with higher frequencies and prevent the signal from reaching deeper in the tissue. In order to better understand this trade-off, a model is required that takes into account the speed of sound as a function of the location (2d or 3d) and the attenuation as a function of the location, as well as a function of the frequency.

The experiments show that attenuation follows a power low with an exponent in the interval of 0 to 2 that may be formulated in the following way [***1***]:

$$\alpha(\mathbf{x},\omega) = \alpha(\mathbf{x})|\omega|^{\gamma(\mathbf{x})} \tag{1}$$

Here $\alpha(\mathbf{x},\omega)$ is the location and frequency dependent attenuation, $\alpha(\mathbf{x})$ is the location dependent and frequency independent attenuation coefficient. The location is denoted by the space parameter $\mathbf{x} \in \mathbb{R}^{d}$ and ω is the angular frequency related to the frequency f ($\omega = 2\pi f$) of the sound waves. $\gamma(\mathbf{x})$ is the location dependent exponent.

A possible model amongst many others [***2***], [***3***], [***4***], [***5***] that satisfies equation *(1)* is given by equation *(2)* below, proposed by [***6***] and [***7***]:

$$\nabla^2 p(\mathbf{x},t) - \frac{1}{c^2(\mathbf{x})}\frac{\partial^2 p(\mathbf{x},t)}{\partial t^2} - \frac{2\alpha(\mathbf{x})}{c(x)\cos\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}\frac{\partial^{\gamma(\mathbf{x})+1}p(\mathbf{x},t)}{\partial t^{\gamma(\mathbf{x})+1}} - \frac{\alpha^2(\mathbf{x})}{\cos^2\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}\frac{\partial^{2\gamma(\mathbf{x})}p(\mathbf{x},t)}{\partial t^{2\gamma(\mathbf{x})}} = 0 \tag{2}$$

Equation *(2)* is a Fractional Partial Differential Equation (FPDE) describing a power-law fractional wave equation.

Note that for $\alpha(\mathbf{x}) = 0$, equation *(2)* reduces to the usual wave equation

$$\nabla^2 p(\mathbf{x},t) - \frac{1}{c^2(\mathbf{x})}\frac{\partial^2 p(\mathbf{x},t)}{\partial t^2} = 0 \tag{3}$$

The approach adopted here is to take the temporal Fourier transform of the pressure wave $p(\mathbf{x},t)$ define as:

$$\hat{p}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} p(\mathbf{x},t)\, e^{-i\omega t} dt \tag{4}$$

This allows us to rewrite *(2)* as:

$$\nabla^2\hat{p}(\mathbf{x},\omega) + \left\{\frac{\omega^2}{c^2(\mathbf{x})} - \frac{2\alpha(\mathbf{x})}{c(\mathbf{x})\, cos\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}(-i\omega)^{\gamma(\mathbf{x})+1} - \frac{\alpha^2(\mathbf{x})}{cos^2\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}(-i\omega)^{2\gamma(\mathbf{x})}\right\}\hat{p}(\mathbf{x},\omega) = 0 \tag{5}$$

Note that for $\alpha(\mathbf{x}) = 0$ equation (5) reduces to the usual Helmholtz equation

$$\nabla^2\hat{p}(\mathbf{x},\omega) + \frac{\omega^2}{c^2(\mathbf{x})}\hat{p}(\mathbf{x},\omega) = 0 \tag{6}$$

Define a refractive index $\mathrm{n}(\mathbf{x},\omega)$ by (7) below

$$n^2(\mathbf{x},\omega) \equiv \frac{1}{c^2(\mathbf{x})} - \frac{2\alpha(\mathbf{x})}{c(\mathbf{x})\, cos\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}(-i)^{\gamma(\mathbf{x})+1}\omega^{\gamma(\mathbf{x})-1} - \frac{\alpha^2(\mathbf{x})}{cos^2\left(\frac{\pi\gamma(\mathbf{x})}{2}\right)}(-i)^{2\gamma(\mathbf{x})}\omega^{2(\gamma(\mathbf{x})-1)} \tag{7}$$

making it possible to rewrite equation (5) as:

$$\nabla^2\hat{p}(\mathbf{x},\omega) + n^2(\mathbf{x},\omega)\omega^2\hat{p}(\mathbf{x},\omega) = 0, \quad \mathbf{x} \in \mathbb{R}^d \tag{8}$$

This is no more than the Helmholtz equation already well known in the literature, but with a slight change that adds an additional degree of complexity: the refractive index is a function of space and the frequency. Methods for solving the frequency independent refractive index case are well studied in the literature. However, there is a need to further understand the role of the frequency dependent refractive index of equation (7) in the solution of equation (8).

At high frequencies, the numerical solution of equation (8) is very difficult, due to the oscillatory nature of the resulting pressure wave $\hat{p}(\mathbf{x},\omega)$. By considering a solution of the form

$$\hat{p}(\mathbf{x},\omega) = A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})} \tag{9}$$

where $A_\omega(\mathbf{x})$ is the amplitude and $T_\omega(\mathbf{x})$ is the travel-time, one obtains the following two coupled equations (see Appendix AAppendix A)

$$-A_\omega(\mathbf{x})\nabla T_\omega(\mathbf{x}) \cdot \nabla T_\omega(\mathbf{x}) + \frac{\nabla^2 A_\omega(\mathbf{x})}{\omega^2} + Re[\, n^2(\mathbf{x},\omega)]A_\omega(\mathbf{x}) = 0 \tag{10}$$

$$A_\omega(\mathbf{x})\nabla^2 T_\omega(\mathbf{x}) + 2\nabla A_\omega(\mathbf{x}) \cdot \nabla T_\omega(\mathbf{x}) + Im\,[n^2(\mathbf{x},\omega)]\omega A_\omega(\mathbf{x}) = 0 \tag{11}$$

Equations (10) and (11) are [two-way] coupled. In cases where $T_\omega(\mathbf{x})$ and $A_\omega(\mathbf{x})$ are relatively non-oscillatory, there is an advantage in solving them instead of solving equation (8) directly. Furthermore, in the high-frequency limit, the middle term $\frac{\nabla^2 A_\omega(\mathbf{x})}{\omega^2}$ of equation (10) can be neglected, and one obtains a frequency-dependent Eikonal Equation that is independent of the amplitude $A_\omega(\mathbf{x})$

$$-\nabla T_\omega(\mathbf{x}) \cdot \nabla T_\omega(\mathbf{x}) + Re\,[n^2(\mathbf{x},\omega)] = 0 \tag{12}$$

Equations (11) and (12) are [one-way] coupled, i.e. the gradient $\nabla T_\omega(\mathbf{x})$ solved for in the Eikonal equation (12) should be inserted in the transport equation (11) for the amplitude $A_\omega(\mathbf{x})$.

2.2. *Project goals*

1. Explore and compare several methods to solve the forward problem of a two-dimensional tomography reconstruction with frequency-dependent fractional wave equation.
2. The simplest method to be studied, will involve a ray approximation to the wave front propagation, namely a solution of the Transport equation coupled to the Eikonal equation (one-way-coupling).
3. **The ultimate method and goal of the project is the solution of the full-wave Helmholtz equation.**
4. Build a tool based on the chosen method for solving the frequency dependent refraction index Helmholtz equation.
5. Conduct numerical simulations to compare and analyze the quality of approximations and algorithms.

2.3. *Specifications and requirements*

This part of the proposal is related to point 4 of the preceding paragraph and aims to describe the tool that is part of the deliverables of the entire project. The tool is by no means a substitute to the real goal of the project and therefore remains secondary in relation to the computation of the Helmholtz equation. It is intended to be a convenient way of synthesizing the simulation cases.

Given the research oriented nature of the project (see preceding paragraph) and the fact that the methods (see the following paragraph) for solving our specific problem need further investigations, the requirements and specifications should be viewed as preliminary and subject to dynamic changes as the project progresses.

However, it is desirable that the following elements of the problem be user-defined:

1. The basic geometry of the grid
2. The parameters of the equation to solve
    a. Single set of parameters
    b. A range of values for a parameter
3. The boundary conditions specifications
4. Visualization capabilities
5. A file format definition to handle export of the result
    a. Input format of files for visualization
    b. Output format of files for saving
6. Error analysis

The purpose of the tool is to help in conveniently handling the simulations, demonstrating them, keeping and analyzing the results and highlighting achievements.

2.4. Background and relevant branches of computer science

1. Algorithms
2. Numerical analysis
    - Error analysis
    - Numerical methods to solve PDE (Partial Differential Equation) with their corresponding boundary and/or initial conditions
3. Object Oriented Programming

2.5. Project complexity

The project has several types of complexity that are embedded in the theoretical field we are dealing with. These include: (i) the physical and mathematical formulations for describing the problem; (ii) the numerical frameworks to be used for computing a solution; and (iii) the particular design.

First of all the problem belongs to physics of waves (acoustic waves) and their propagation. The mathematics that describes the problem leads to differential equations and more particularly to partial differential equations. The latter are a consequence of dealing with the propagation in 2 or 3 dimensions plus the time. In our case, the main equation is even more complex, as it involves fractional derivatives [***8***], i.e. the order of the PDE is not an integer. This means that one is dealing with a non-conventional wave equation. The problem is circumvented by moving to Fourier space, replacing the time derivatives by powers of the frequency as in equation (5). In practice I further simplify the calculations by focusing on a single frequency in each simulation run.

Secondly, the very wide range of ODE (Ordinary Differential Equation) and PDE (Partial Differential Equation) leads to a forest of methods. In both, analytical and numerical challenges are encountered. Even if not all of them are eventually implemented, one has to make rational choices for a robust solution. More particularly, the field of numerical analysis proposes amongst other things some general frameworks to solve PDE accompanied by the sought degree of precision. These include among others, Finite Differences (FD) [***9***] [***10***] [***11***], Finite Elements (FE) [***12***] [***13***] [***14***], spectral methods (SP) [***15***] [***16***] [***17***] [***18***], multi-grid (MG) [***19***] [***20***] [***21***] [***22***]. Each of these methods may be approached as an initial value or boundary value problem.

Last but not least, the concrete implementation of a solution requires some practical insight in object oriented programming and algorithm. In my case, I seek a solution for a problem that, for a part of it, some publication witness of the existence of a numerical solution [***23***] or [***24***]. However, the *degree of precision* needed for high frequency solution is not readily available. I also want to achieve it in a *reasonable computation time*. Furthermore, the constant term k^2, $k^2 = n^2(\mathbf{x}, \omega)\omega^2$ in the simplest Helmholtz equation (8), is replaced by a *frequency and space dependent* term – this constituting a significant challenge here. All these problems require adaptation and therefore technical exploration of the techniques and available packages. To achieve this several packages (e.g. GetFEM++, FEniCS) that provide computational algorithms for solving PDEs have to be understood and implemented.

2.6. Technology that will be used

The project will make intensive use of **Matlab** as the tool of choice both for computation and production of graphical results. If at some point efficient computation is particularly required, for some developed / adapted / modified / improved algorithms, these developments may be performed in a language such as **C/C++/Java** and later on made available to client code Matlab environment as a MEX file.

2.7. Evaluation: How the success of the project will be measured?

- The success of the project will be measured by the quality of the numerical results. In particular, I'll compare the results of special cases to published numerical results. In some special cases, I'll compare the results to closed form solutions of the simplified example.
- An important part of the project relates to analyzing the various approximations, such as the Eikonal and Transport equations, (12) and (11) respectively. Optionally, if time will permit, I'll compare additional approximations, such as fat rays solutions, for example [8] and references therein. A proper analysis of the approximations as compared to the full Helmholtz equation is another measure for the success of the project.

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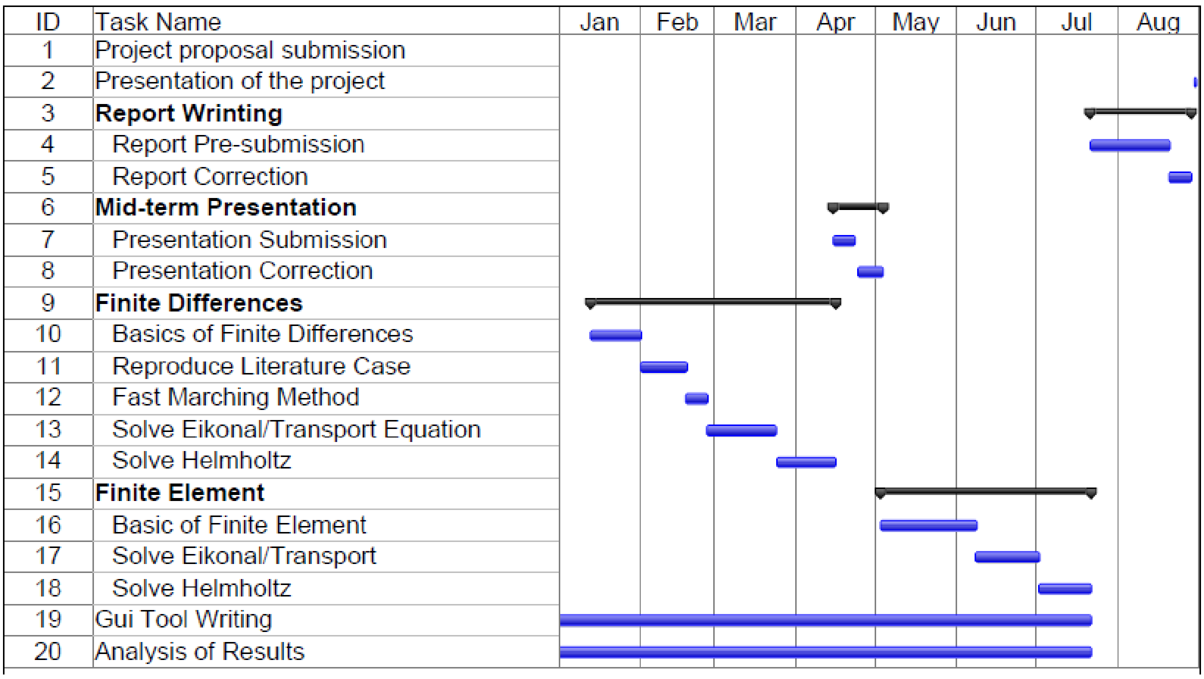
3. Estimated Schedule

Although the main and ultimate goal of the project is to solve the forward problem equation (8), it is also a major concern to acquire progressively a good understanding of the major techniques that allow solving our PDE numerically. The numerical analysis literature shows that Finite Differences and Finite Elements are the first choice methods for numerical computation of diverse kinds of PDEs [***25***]. Therefore, the planned schedule has been oriented to achieve a good grasp on them. However, if time and progress allows it, other methods may be explored (spectral methods, k-wave, multi-grid…). Of course, I also want the results to be as correct as possible (analysis of results) and the code usable (GUI Tool). The schedule tries to reflect this concern. Hereunder, is displayed a coarse-grained schedule in the form of a Gantt diagram. Both the information about the required academic studies tasks and the project implementation itself are displayed in the Gantt diagram hereunder. As the step title is not fully self-explanatory, short explanations of the tasks are provided in the table below.

3.1. *Gantt*



3.2. *Details*

Basics of Finite Differences	I want here to study the implementation of Finite Differences (FD) and reproduce published cases for well-known version of PDE or even some basic ODE. The purpose is to exercise various derivations

	and their representation in terms of finite differences including boundary/initial conditions. The goal is to understand how this approximation influences the solution. The underlying mathematics involves solving a linear algebraic system of the type $Ax = b$, but with a large matrix A, depending on the resolution of the grid. Usually A is sparse, i.e. most of the matrix elements are zeros, the matrix being diagonal or tri-diagonal for example.
Reproduce Literature Case	There are existing publications, for example [***23***], that solve the Helmholtz equation (13) or (14) with a FD scheme. However, in [***23***] the coefficient k in ($\nabla^2 \mathrm{u} + \mathrm{k}^2 \mathrm{u} = 0$) is a constant (usually $k = \omega/c$), which is not our case. I hope that the method is adaptable to my situation.
Solve Helmholtz	In relation to what was said above, the task is here to adapt the solution of the above publications to our problem.
Fast Marching Method	The Eikonal equation (12) that is derived from the Helmholtz equation once the approximation (12) is made, may be solved by the Fast-Marching Method [***26***]. For this step the routine is already coded and available, but needs to be adapted to our current problem. In any case, it should be sufficiently understood in order to be integrated into our main task.
Solve Eikonal/Transport Equation	Solving the Eikonal equation (12) provides us with an input to the Transport equation (11), thereby providing the solution for the full problem of the amplitude and travel time of equation (9) .
Analysis of the results	I distribute this phase along the entire project, as it is always necessity to understand in each step the meanings of the results, and identify the errors involved in these results. The analysis should verify if progress has been achieved as compared to other attempts to solve the problem, and whether one needs to think about other ways or of modifications to a given simulation.
GUI Toole	I propose to build and use the GUI tool for demonstrating the computation and the results. Such a requirement implies that it would be necessary to modify it along the way, as I understand more clearly and precisely what demonstrations are useful and what are the relevant parameters, output, and graphical representation of a simulation.
Finite Elements	The same process may be followed by using the Finite Elements (FE) technique instead of the FD method described above - the principles would remain very similar.

Appendix A

Splitting the Helmholtz Equation

We wish to explain how the two equations (10) and (11) are obtains from equation (8) by plugging equation (9). This will be shown for the three dimensional space, but other dimensionalities are easily understood from it.

Let f be a function of a vector $\mathbf{x} = (x, y, z)$ sufficiently smooth (at least two times derivable):

$$f(\mathbf{x}) = f(x, y, z)$$

The gradient operator in three dimensions is defined as:

$$\nabla \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Applied to our function it gives:

$$\nabla f(\boldsymbol{x}) = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$$

$$\begin{aligned} \nabla^2 f(\boldsymbol{x}) = \nabla \cdot \nabla f(\boldsymbol{x}) &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned} \tag{15}$$

$$\begin{aligned} \nabla f(\boldsymbol{x}) \cdot \nabla g(\boldsymbol{x}) &= \left(\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}\right) \cdot \left(\boldsymbol{i}\frac{\partial g}{\partial x} + \boldsymbol{j}\frac{\partial g}{\partial y} + \boldsymbol{k}\frac{\partial g}{\partial z}\right) \\ &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \end{aligned} \tag{16}$$

Let f be the ansatz solution given in equation (9):

$$f(\boldsymbol{x}) \equiv A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})};$$

We will develop the second derivative term of equation (8):

$$\nabla^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right] = \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial x^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial y^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial z^2}$$

Calculation in one dimension for x:

$$\frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial x^2} = \left[\frac{\partial^2 A_\omega(\mathbf{x})}{\partial x^2} - \omega^2 A_\omega(\mathbf{x})\frac{\partial T_\omega(\mathbf{x})}{\partial x}\frac{\partial T_\omega(\mathbf{x})}{\partial x} + i\omega\left(2\frac{\partial A_\omega(\mathbf{x})}{\partial x}\frac{\partial T_\omega(\mathbf{x})}{\partial x} + A_\omega(x)\frac{\partial^2 T_\omega(\mathbf{x})}{\partial x^2}\right)\right]e^{i\omega T_\omega(\mathbf{x})}$$

Similarly for y and z:

$$\begin{aligned} \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial y^2} &= \left[\frac{\partial^2 A_\omega(\mathbf{x})}{\partial y^2} - \omega^2 A_\omega(\mathbf{x})\frac{\partial T_\omega(\mathbf{x})}{\partial y}\frac{\partial T_\omega(\mathbf{x})}{\partial y} + i\omega\left(2\frac{\partial A_\omega(\mathbf{x})}{\partial y}\frac{\partial T_\omega(\mathbf{x})}{\partial y} + A_\omega(x)\frac{\partial^2 T_\omega(\mathbf{x})}{\partial y^2}\right)\right]e^{i\omega T_\omega(\mathbf{x})} \\ \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial z^2} &= \left[\frac{\partial^2 A_\omega(\mathbf{x})}{\partial z^2} - \omega^2 A_\omega(\mathbf{x})\frac{\partial T_\omega(\mathbf{x})}{\partial z}\frac{\partial T_\omega(\mathbf{x})}{\partial z} + i\omega\left(2\frac{\partial A_\omega(\mathbf{x})}{\partial z}\frac{\partial T_\omega(\mathbf{x})}{\partial z} + A_\omega(\mathbf{x})\frac{\partial^2 T_\omega(\mathbf{x})}{\partial z^2}\right)\right]e^{i\omega T_\omega(\mathbf{x})} \end{aligned}$$

Summing up the three equations one gets

$$\frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial x^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial y^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial z^2} =$$

$$\left[\frac{\partial^2 A_\omega(\boldsymbol{x})}{\partial x^2} - \omega^2 A_\omega(x)\frac{\partial T_\omega(\boldsymbol{x})}{\partial x}\frac{\partial T_\omega(\boldsymbol{x})}{\partial x} + i\omega\left(2\frac{\partial A_\omega(\boldsymbol{x})}{\partial x}\frac{\partial T_\omega(\boldsymbol{x})}{\partial x} + A_\omega(x)\frac{\partial^2 T_\omega(\boldsymbol{x})}{\partial x^2}\right)\right]e^{i\omega T_\omega(x)} +$$
$$\left[\frac{\partial^2 A_\omega(\boldsymbol{x})}{\partial y^2} - \omega^2 A_\omega(x)\frac{\partial T_\omega(\boldsymbol{x})}{\partial y}\frac{\partial T_\omega(\boldsymbol{x})}{\partial y} + i\omega\left(2\frac{\partial A_\omega(\boldsymbol{x})}{\partial y}\frac{\partial T_\omega(\boldsymbol{x})}{\partial y} + A_\omega(x)\frac{\partial^2 T_\omega(\boldsymbol{x})}{\partial y^2}\right)\right]e^{i\omega T_\omega(x)} +$$
$$\left[\frac{\partial^2 A_\omega(\boldsymbol{x})}{\partial z^2} - \omega^2 A_\omega(x)\frac{\partial T_\omega(\boldsymbol{x})}{\partial z}\frac{\partial T_\omega(\boldsymbol{x})}{\partial z} + i\omega\left(2\frac{\partial A_\omega(\boldsymbol{x})}{\partial z}\frac{\partial T_\omega(\boldsymbol{x})}{\partial z} + A_\omega(x)\frac{\partial^2 T_\omega(\boldsymbol{x})}{\partial z^2}\right)\right]e^{i\omega T_\omega(\boldsymbol{x})}$$

Rearranging terms

$$\frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial x^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial y^2} + \frac{\partial^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right]}{\partial z^2}$$
$$= \left\{\begin{array}{c} \left[\dfrac{\partial^2 A_\omega(\boldsymbol{x})}{\partial x^2} + \dfrac{\partial^2 A_\omega(\boldsymbol{x})}{\partial y^2} + \dfrac{\partial^2 A_\omega(\boldsymbol{x})}{\partial z^2}\right] \\ -\omega^2 A_\omega(x)\left[\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial x}\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial x} + \dfrac{\partial T_\omega(\boldsymbol{x})}{\partial y}\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial y} + \dfrac{\partial T_\omega(x)}{\partial z}\dfrac{\partial T_\omega(x)}{\partial z}\right] \\ +i\omega\left[2\left(\dfrac{\partial A_\omega(\boldsymbol{x})}{\partial x}\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial x} + \dfrac{\partial A_\omega(\boldsymbol{x})}{\partial y}\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial y} + \dfrac{\partial A_\omega(\boldsymbol{x})}{\partial z}\dfrac{\partial T_\omega(\boldsymbol{x})}{\partial z}\right) + A_\omega(x)\left(\dfrac{\partial^2 T_\omega(\boldsymbol{x})}{\partial x^2} + \dfrac{\partial^2 T_\omega(\boldsymbol{x})}{\partial y^2} + \dfrac{\partial^2 T_\omega(x)}{\partial z^2}\right)\right]\end{array}\right\} e^{i\omega T_\omega(x)}$$

By inspecting equations (15) and (16) one gets

$$\nabla^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right] = \left\{\begin{array}{c} [\nabla^2 A_\omega(\boldsymbol{x})] \\ -\omega^2 A_\omega(x)[\nabla T_\omega(\boldsymbol{x})\cdot\nabla T_\omega(\boldsymbol{x})] \\ +i\omega\left[2\left(\nabla A_\omega(\boldsymbol{x})\cdot\nabla T_\omega(\boldsymbol{x})\right) + A_\omega(x)\nabla^2 T_\omega(x)\right]\end{array}\right\} e^{i\omega T_\omega(x)}$$

Now the generalized Helmholtz equation is

$$\nabla^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right] + n^2(\mathbf{x},\omega)\omega^2\left[A_\omega(\mathbf{x})e^{i\omega T_\omega(\mathbf{x})}\right] = 0, \quad \mathbf{x}\in\mathbb{R}^d \tag{17}$$

By defining

$$n^2(\boldsymbol{x},\omega) = Re[n^2(\boldsymbol{x},\omega)] + i\, Im[n^2(\boldsymbol{x},\omega)]$$

One arrives at

$$\begin{aligned}\Big\{\nabla^2 A_\omega(\boldsymbol{x}) - \omega^2 A_\omega(x)\nabla T_\omega(\boldsymbol{x})\cdot\nabla T_\omega(\boldsymbol{x}) + Re[n^2(\boldsymbol{x},\omega)]\omega^2 A_\omega(\mathbf{x}) \\ + i\big\{\omega\, 2\left(\nabla A_\omega(\boldsymbol{x})\cdot\nabla T_\omega(\boldsymbol{x})\right) + \omega\, A_\omega(x)\nabla^2 T_\omega(x) \\ + Im[n^2(\boldsymbol{x},\omega)]\omega^2 A_\omega(\mathbf{x})\big\}\Big\} e^{i\omega T_\omega} = 0\end{aligned} \tag{18}$$

By requiring separately that the real part and the imaginary part in equation (18) are equal to zero, one arrives at equations (10) and (11).

The two dimensional case is trivially done in the same way, just omitting the z terms.

4. Approval

Advisor signature: Date: 14 December 2014

Michael Berman