STATE SPACE ANALYSIS

What is state space analysis?

A state space analysis in control system theory describes the behaviour of a system using a set of state variables, input signals, and output signals. It represents how the system's internal states change over time in response to inputs, leading to corresponding outputs. In simple terms, it's like describing how a system works from the inside, showing how its components interact to produce the observed behaviour.

Before knowing about this concept, we should first know about few terms:

- **System:** A system in control systems refers to any physical or mathematical entity that takes inputs, processes them in some way, and produces outputs. It could be a machine, a device, a process, or even an abstract mathematical model. In simple terms, a system is something that does something based on what you do to it or what you give it.
- **State Variables:** These are variables that describe the internal state of the system at any given time. They represent the system's history and current conditions that affect its future behaviour. For example, in a mechanical system like a pendulum, the state variables could be the position and velocity of the pendulum.
- **Input Signals:** These are external signals or forces applied to the system that influence its behaviour. Inputs can come from sensors, actuators, or external commands. In the pendulum example, an input signal could be a force applied to the pendulum to move it.
- **Output Signals:** These are the observable quantities or responses of the system to the inputs. Outputs can be measured using sensors or by observing the system's behaviour. In the pendulum example, the output signal could be the angle of the pendulum.
- **State Space Equations:** These are mathematical equations that describe how the state variables, inputs, and outputs are related over time. The equations typically take the form of first-order differential equations, representing how the state variables change with respect to time based on the inputs and internal dynamics of the system.
- **State Space Representation:** In a state space model, the state variables, inputs, and outputs are organized into matrices and vectors. The state vector contains all the state variables, the input vector contains the input signals, and the output vector contains the output signals. Matrices are used to represent the relationships between these variables and how they evolve over time.
- **Feed forward matrix:** The feedforward matrix D in a state space model represents the direct influence of system inputs on system outputs without considering the system's internal states. It describes the input-to-output relationship in a linear time-invariant (LTI) system, highlighting the direct impact of inputs on outputs.

How do we obtain a state space model with a simple example?

Let's consider a mass-spring-damper system. Suppose we have a mass *m* attached to a spring with stiffness *k* and a damper with damping coefficient *c*. The displacement of the mass is denoted by *x*(*t*).

1. **Identify State Variables:**
   - State variable $x(t)$: Displacement of the mass.
2. **Write State Equations:**

   The dynamics of the system can be described by the second-order differential equation:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

   Where $F(t)$ is an external force or input to the system.

   Now, let's define the state variables:
   - $x_1(t) = x(t)$: Displacement of the mass.
   - $x_2(t) = \dot{x}(t)$: Velocity of the mass.

   The state equations become:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{m}\left(F(t) - cx_2(t) - kx_1(t)\right) \end{aligned}$$

3. **Define Inputs and Outputs:**
   - Input $u(t) = F(t)$: External force applied to the mass.
   - Output $y(t) = x_1(t)$: Displacement of the mass (same as state variable $x_1(t)$).
4. **Formulate Output Equation:**

   The output equation is simply:

$$y(t) = x_1(t)$$

5. **Organize into State Space Form:**

   Now, we can organize the state equations and output equation into the standard state space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

   Where:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

We obtained the state space model for a mass-spring-damper system by defining state variables, writing state equations, specifying inputs and outputs, formulating the output equation, and organizing everything into the standard state space form with matrices *A*, *B*, *C*, and *D*.

Why do we use this state space analysis?

State space analysis is widely used in control theory and engineering for several reasons:

- **Comprehensive Modelling:** State space models provide a comprehensive and accurate representation of the dynamics of a system. By describing the system in terms of state variables, inputs, outputs, and state equations, state space analysis captures the internal behaviour and interactions of the system components more effectively than other modelling approaches.
- **Flexibility:** State space models are versatile and can represent a wide range of systems, including linear and nonlinear systems, time-varying systems, multi-input multi-output (MIMO) systems, and stochastic systems. This flexibility makes state space analysis applicable to diverse engineering and scientific applications.
- **Control System Design:** State space analysis is fundamental to modern control system design. It enables engineers to design controllers (such as state feedback controllers, observers, and estimators) that can regulate system behaviour, achieve desired performance objectives, and respond to disturbances or changes in operating conditions.
- **System Analysis:** State space analysis provides powerful tools for analysing system stability, controllability, observability, sensitivity, robustness, and transient response. Engineers can use state space models to evaluate system performance, identify critical parameters, and optimize system design.

Overall, state space analysis is a powerful and essential tool in control system engineering, offering benefits such as accurate modelling, design flexibility, system analysis capabilities, simulation capabilities, and support for advanced control strategies. It is widely used in industries such as aerospace, automotive, robotics, process control, and more.

State space analysis came into effect to satisfy the drawbacks of transfer function analysis.

Few advantages of state variable analysis are:

- It can be applied to non-linear system.
- It can be applied to time invariant system.
- It can be applied to Multiple Input Multiple Output (MIMO) system.
- It gives an idea about the internal state of the system.
- It can be performed on any type of systems, and it is very easy to perform state variable analysis on computers.

Where do we use this state space analysis?

State space analysis is used in various engineering fields and applications where precise modelling, control system design, and system analysis are essential. Some common areas where state space analysis is applied include:

- **Control Systems:** State space analysis is fundamental in designing and analysing control systems for industries such as aerospace, automotive, robotics, manufacturing, and process control. It enables engineers to design controllers that regulate system behaviour, achieve performance objectives, and respond to disturbances.
- **Aerospace and Aviation:** State space analysis is used in aircraft and spacecraft control systems, flight dynamics analysis, stability analysis, autopilot design, trajectory planning, and attitude control.
- **Robotics:** State space models are used in robot control, motion planning, path tracking, robot arm control, robot manipulator dynamics, and robot localization and mapping (SLAM).
- **Biomedical Engineering:** State space models are used in physiological systems modelling, medical device control, drug delivery systems, patient monitoring, and neural control systems.
- **Electrical Engineering:** State space analysis is applied in power system control, motor control, renewable energy systems, inverters, power converters, and electrical machine modelling.
- **Mechanical Engineering:** In mechanical systems, state space analysis is used in modelling mechanical vibrations, mechatronic systems, HVAC systems, fluid dynamics control, and structural control.
- **Academic and Research:** State space analysis is taught in control theory courses and is extensively used in academic research for studying system dynamics, stability analysis, nonlinear control, adaptive control, and optimal control.
- **Process Control:** Industries such as chemical engineering, power systems, oil and gas, and manufacturing utilize state space analysis for process control, plant modelling, PID controller design, predictive control, and system optimization.

Conversion of state space model to transfer function:

Given a state space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- $x(t)$ is the state vector.
- $u(t)$ is the input vector.
- $y(t)$ is the output vector.
- A is the state matrix.

- B is the input matrix.
- C is the output matrix.
- D is the feedforward matrix.

Laplace Transform: Apply the Laplace transform to the state space equations to convert them into the Laplace domain:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Where s is a LaPlace variable.

Solve for Transfer Function: Solve the above equations for the output $Y(s)$ in terms of the input $U(s)$ to obtain the transfer function $G(s)$:

$$Y(s) = (C(sI - A)^{-1}B + D)U(s)$$

Therefore, the transfer function $G(s)$ is given by:

$$G(s) = C(sI - A)^{-1}B + D$$

Where 'I' is identity matrix and G(s)= Y(s)/U(s).

This process converts the state space model of a system into its equivalent transfer function representation, which is commonly used for analysis, design and simulation of control system.

Practical application:

Application: Quadcopter System

- **System Description:** A quadcopter has multiple inputs (e.g., rotor speeds) and multiple outputs (e.g., position and orientation). The goal is to design a control system that stabilizes the quadcopter and allows it to follow a desired trajectory.
- **State Variables:** Define the state variables that fully describe the system's dynamics. For a quadcopter, typical state variables may include:

  Position (x, y, z)
  Linear velocity (vx, vy, vz)
  Orientation (roll, pitch, yaw)
  Angular velocity (p, q, r)
- **Dynamics Equations:** Based on the physics of the quadcopter, derive the differential equations that govern its dynamics. These equations relate the state variables, inputs (rotor speeds), and external forces (e.g., wind).

$$\dot{x} = Ax + Bu$$

Here, x is the state vector, u is the input vector (rotor speeds), A is the system matrix, and B is the input matrix.

- **Measurement Equations:** Determine how the outputs (position and orientation) depend on the state variables:

  $$\boldsymbol{Y = Cx + Du}$$

  Here, y is the output vector, C is the output matrix, and D is the feedforward matrix.

With the state space model in hand, engineers can design controllers (such as PID controllers, LQR controllers, or robust controllers) to stabilize the quadcopter, track trajectories, and reject disturbances.

In this example, the state space model captures the MIMO nature of the quadcopter system, where multiple inputs (rotor speeds) affect multiple outputs (position and orientation). This model serves as the basis for control system design and analysis in aerospace applications like quadcopters, drones, and other MIMO systems.