Mean Field Games and Optimal Transport

Consider a mean field game (MFG) system on $\Omega \subset \mathbb{R}^d$ and $[0,T]$:

$$\begin{cases} -\partial_t u(t,x) - \nu\Delta u(t,x) + H(x, Du(t,x)) = F(x, m(t)), \\ \partial_t m(t,x) - \nu\Delta m(t,x) - \nabla\cdot\big(m(t,x) D_p H(x, Du(t,x))\big) = 0. \end{cases}$$

If $H(x,p) = \sup_v(p\cdot v - L(x,v))$, (m,u) can be derived from minimizing

$$\mathcal{J}(m,v) \;=\; \int_0^T \int_\Omega [L(x, v(t,x))\, m(t,x) + F(x, m(t,x))]\, dx\, dt + G(m(T))$$

subject to

$$\partial_t m(t,x) + \nabla\cdot(m(t,x)v(t,x)) = 0, \quad m(0) = m_0.$$

Setting $L(x,v) = \frac{1}{2}|v|^2$ and $F, G = 0$, this reduces to the Benamou–Brenier formula for the Wasserstein distance:

$$W_2^2(\mu_0, \mu_1) = \inf_{m,v} \left\{ \int_0^1 \int_\Omega |v|^2 m\, dx\, dt : \partial_t m + \nabla\cdot(mv) = 0,\ m(0) = \mu_0,\ m(1) = \mu_1 \right\}.$$

Thus, certain MFGs can be viewed as dynamic optimal transport problems, with $m(t)$ evolving as a Wasserstein gradient flow.

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