Mean Field Games and the Invisible Hand

In the context of mean field games (MFGs), the collective minimization of a global functional arises from individual agents solving their own optimization problems. Consider the following system for (u, m) on $\Omega \subset \mathbb{R}^d$ and $[0, T]$:

$$\begin{cases} -\partial_t u(t,x) - \nu \Delta u(t,x) + H(x, Du(t,x), m(t)) = F(x, m(t)), \\ \partial_t m(t,x) - \nu \Delta m(t,x) - \nabla \cdot \big(m(t,x) D_p H(x, Du(t,x), m(t)) \big) = 0. \end{cases}$$

Here:

- u is the value function for individual agents.
- m is the population density.
- $H(x, p, m) = \inf_\alpha \big(L(x, \alpha, m) + p \cdot b(x, \alpha, m) \big)$ is the Hamiltonian.

Global Functional Minimization

At equilibrium, the system minimizes the global functional

$$\mathcal{J}(m) = \int_0^T \int_\Omega \big[L(x, \alpha^*(t,x), m(t))\, m(t,x) + F(x, m(t)) \big]\, dx\, dt + G(m(T)),$$

where $\alpha^*(t,x)$ is the optimal control derived from u via

$$\alpha^*(t,x) = \arg\min_\alpha \big(L(x, \alpha, m(t)) + Du(t,x) \cdot b(x, \alpha, m(t)) \big).$$

Connection to the Invisible Hand

1. Each agent minimizes their own cost functional:

$$J(x_0, \alpha) = \mathbb{E} \left[\int_0^T L(t, X_t, m(t), \alpha_t)\, dt + g(X_T, m(T)) \right].$$

2. The population density $m(t)$ evolves through the Fokker-Planck equation, determined by the collective dynamics of all agents.
3. At equilibrium, individual rational actions minimize $\mathcal{J}(m)$, representing the total cost of the population.

Thus, individual optimization leads to collective optimality, analogous to the invisible hand in economics: decentralized self-interested behavior results in globally efficient outcomes.

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