Black–Scholes: derivation

Let S_t be the price of the underlying at time t. Suppose that over a small time period Δt, the price moves with the following probabilities:

$$p_\pm := \mathbb{P}[S_{t+\Delta t} = \gamma_\pm S_t] \quad \text{where} \quad \gamma_\pm := 1 + \mu\Delta t \pm \sigma_\pm\sqrt{\Delta t}. \tag{1}$$

Then, over n such periods with independent price shifts, we have

$$\mathbb{P}[S_{t+n\Delta t} = \gamma_+^k \gamma_-^{n-k} S_t] = \binom{n}{k} p_+^k p_-^{n-k}. \tag{2}$$

The expected price is thus

$$\begin{aligned}
\mathbb{E}[S_{t+n\Delta t}] &= \sum_{k=0}^{n} \binom{n}{k} (\gamma_+ p_+)^k (\gamma_- p_-)^{n-k} S_t & (3)\\
&= (p_+\gamma_+ + p_-\gamma_-)^n S_t & (4)\\
&= (1 + \mu\Delta t + (p_+\sigma_+ - p_-\sigma_-)\sqrt{\Delta t})^n S_t. & (5)
\end{aligned}$$

Now we couple $\Delta t = 1/n$. Define $\sigma^2 := \sigma_+\sigma_-$. We assume that

$$p_+\sigma_+ = p_-\sigma_- \tag{6}$$

so that the following are finite, as $n \to \infty$,

$$\mathbb{E}[S_{t+1}/S_t] \to e^{\mu} \quad \text{and} \quad \mathbb{E}[S_{t+1}^2/S_t^2] \to e^{2\mu+\sigma^2}. \tag{7}$$

This suggests the following expressions that can be used to estimate μ and σ^2:

$$\mu = \log \mathbb{E}[S_{t+1}/S_t] \quad \text{and} \quad \sigma^2 = \log \frac{\mathbb{E}[S_{t+1}^2/S_t^2]}{\mathbb{E}[S_{t+1}/S_t]^2}. \tag{8}$$

Let $V(s,t)$ denote the price of the option at time t when the price of the underlying is s. Consider the portfolio $\Pi = \alpha S + B$, consisting of the underlying S and riskless B. We require $\Pi_{t+\Delta t} = \alpha S_{t+\Delta t} + (1 + r\Delta t)B$, where r is the risk-free rate of return. The two possibilities $S_{t+\Delta t} = \gamma_\pm S_t$ determine α and B, such that

$$\Pi_t = \frac{1}{1 + r\Delta t}\left(q_+ V(\gamma_+ S_t, t + \Delta t) + q_- V(\gamma_- S_t, t + \Delta t)\right) \quad \text{with} \quad q_\pm = \frac{\sigma_\mp \pm (1 + r\Delta t)}{\sigma_+ + \sigma_-}. \tag{9}$$

The no-arbitrage argument gives $V(S_t, t) = \Pi_t$. Thus, we have the relation

$$V(s,t) = \frac{1}{1 + r\Delta t} \sum_{\pm} q_\pm V(\gamma_\pm s, t + \Delta t). \tag{10}$$

Note that

$$(\gamma_+ - 1)q_+ + (\gamma_- - 1)q_- = r\Delta t \tag{11}$$

and

$$(\gamma_+ - 1)^2 q_+ + (\gamma_- - 1)^2 q_- = (\sigma^2 + \mathcal{O}(\sqrt{\Delta t}))\Delta t. \tag{12}$$

Therefore, Taylor expansion of V in the first argument gives

$$\frac{V(s,t) - V(s,t+\Delta t)}{\Delta t} \approx -rV(s,t) + rs\frac{\partial V}{\partial s}(s,t+\Delta t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}(s,t+\Delta t), \tag{13}$$

where the error is $\mathcal{O}(\sqrt{\Delta t})$. Whence ensues the Black–Scholes equation:

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rV = 0. \tag{14}$$

RA, May 20, 2020