

A note on [KB/2020]

Introduction

Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of separable Hilbert spaces with dense and continuous embeddings. We write $\|\cdot\|_X$ for the norm on X , and $\langle \cdot, \cdot \rangle$ for the duality pairing on $V' \times V$. Let $V_n \subset V$ be nontrivial finite-dimensional subspaces, parameterized by $n \geq 1$. Let $Q_n: V' \rightarrow V_n$ denote the H -orthogonal projection. Let $P_n: V' \rightarrow V_n$ denote the V -orthogonal projection. We assume that

$$\forall v \in V : \quad \|v - P_nv\|_V = o(\|v\|_V) \quad \text{as } n \rightarrow \infty. \quad (1)$$

We abbreviate $\|\cdot\|_V := \|\cdot\|_{\mathcal{L}(V)}$ for the operator norm, for example $\|Q_n\|_V := \|Q_n|_V\|_{\mathcal{L}(V)}$. Recall [S/2006] that for any nontrivial idempotent operator $Q: V \rightarrow V$ we have

$$\|Q\|_V = \|I - Q\|_V. \quad (2)$$

Therefore, for any $v \in V$ and $n \geq 1$,

$$\|v - P_nv\|_V \leq \|v - Q_nv\|_V = \|(I - Q_n)(v - P_nv)\|_V \quad (3)$$

$$\leq \|I - Q_n\|_V \|v - P_nv\|_V = \|Q_n\|_V \|v - P_nv\|_V. \quad (4)$$

First characterization

Consider the statements

$$\sup_{v \in V} \frac{\|v - Q_nv\|_V}{\|v - P_nv\|_V} \rightarrow 1 \quad \text{as } n \rightarrow \infty \quad (5)$$

and

$$\|Q_n\|_V \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (6)$$

Clearly, (6) implies (5). Conversely, (5) implies $\|Q_n\|_V = \|I - Q_n\|_V = (1 + o(1))\|I - P_n\|_V = (1 + o(1))\|P_n\|_V \rightarrow 1$ as $n \rightarrow \infty$. Therefore,

$$(5) \Leftrightarrow (6). \quad (7)$$

Second characterization

It was shown in [A/2013, Lemma 6.2] that, for any $\kappa > 0$, the statements

$$\|Q_n\|_V \leq \kappa^{-1} \quad (8)$$

and

$$\forall v' \in V_n : \quad \sup_{v \in V_n} \frac{\langle v', v \rangle}{\|v\|_V} \geq \kappa \|v'\|_{V'} \quad (9)$$

are equivalent. The latter statement is a measure of self-duality of $V_n \subset V$. In the following, $0 \leq \kappa^*(V_n) \leq 1$ denotes the maximal κ for any closed subspace $V_n \subset V$.

Third characterization (special setting)

The setting of [KB/2020] is atypical for Gelfand triples in that

$$\|\cdot\|_V \sim \|\cdot\|_H \text{ are equivalent norms.} \quad (10)$$

We assume this from now on. In particular, (8) holds uniformly in n with *some* κ , because $c\|Q_nv\|_V \leq \|Q_nv\|_H \leq \|v\|_H \leq C\|v\|_V$, and therefore the arguments in the introduction show that (1) also holds for Q_n . In this context, [KB/2020] propose a condition, which translates to the following: Suppose $\{\varphi_j\}_{j \geq 1}$ is an orthonormal basis for H that is also orthogonal in V . Then:

$$w_j := \|\varphi_j\|_V \rightarrow a \text{ as } j \rightarrow \infty, \text{ for some } a > 0. \quad (11)$$

For each $J \geq 1$, define the subspace $W_J \subset V$ as the span of φ_j with $j \leq J$, and $W_{-J} \subset V$ as the span of those with $j > J$. Those subspace are self-dual, because $\kappa = 1$ in (9) for any V_n that is a linear span of finitely many φ_j , and $P_n = Q_n$ on such subspaces. For this reason alone,

$$(6) \text{ does not imply } (11). \quad (12)$$

What is more, the subspace V_n could be at some angle to all such subspaces. For example, take an arbitrary nonzero sequence $c \in \ell_2(\mathbb{N})$. Define V_1 as the span of $\varphi := \sum_j c_j \varphi_j$. Then

$$\|\varphi\|_{V'}^2 = \sum_j c_j^2 w_j^{-2}, \quad \|\varphi\|_H^2 = \sum_j c_j^2, \quad \|\varphi\|_V^2 = \sum_j c_j^2 w_j^2. \quad (13)$$

Therefore, the κ in (9) is at best

$$\kappa^*(V_1) := \frac{\|\varphi\|_H^2}{\|\varphi\|_{V'} \|\varphi\|_V} \geq \kappa. \quad (14)$$

Suppose only $c_1 \neq 0$ and $c_2 \neq 0$ and $w_2 = w_3 = \dots$. Then, for large w_1 we have $\kappa^*(V_1) \approx \frac{1}{w_1} (\frac{c_1}{c_2} + \frac{c_2}{c_1})$, which can be arbitrarily small. Condition (11) is meant to moderate this behavior. Indeed, condition (11) constrains the spread of w_j , i.e. it is equivalent to

$$\omega_{-J} := \frac{\sup_{j>J} w_j}{\inf_{j>J} w_j} \rightarrow 1 \quad \text{as } J \rightarrow \infty. \quad (15)$$

It follows that $\kappa^*(W_{-J}) \geq 1/\omega_{-J} \rightarrow 1$ as $J \rightarrow \infty$.

References

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