A note on [KB/2020]

Introduction

Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of separable Hilbert spaces with dense and continuous embeddings. We write $\|\cdot\|_X$ for the norm on X, and $\langle\cdot,\cdot\rangle$ for the duality pairing on $V' \times V$. Let $V_n \subset V$ be nontrivial finite-dimensional subspaces, parameterized by $n \geq 1$. Let $Q_n\colon V' \to V_n$ denote the H-orthogonal projection. Let $P_n\colon V' \to V_n$ denote the V-orthogonal projection. We assume that

$$\forall v \in V: \quad \|v - P_n v\|_V = o(\|v\|_V) \quad \text{as} \quad n \to \infty. \tag{1}$$

We abbreviate $\|\cdot\|_V := \|\cdot\|_{\mathcal{L}(V)}$ for the operator norm, for example $\|Q_n\|_V := \|Q_n|_V\|_{\mathcal{L}(V)}$. Recall [S/2006] that for any nontrivial idempotent operator $Q\colon V \to V$ we have

$$\|Q\|_V = \|I - Q\|_V. \tag{2}$$

Therefore, for any $v \in V$ and $n \geq 1$,

$$\begin{aligned} \|v - P_n v\|_V \leq \|v - Q_n v\|_V &= \|(I - Q_n)(v - P_n v)\|_V \qquad (3)\\ &\leq \|I - Q_n\|_V \|v - P_n v\|_V = \|Q_n\|_V \|v - P_n v\|_V. \qquad (4) \end{aligned}$$

First characterization

Consider the statements

$$\sup_{v \in V} \frac{\|v - Q_n v\|_V}{\|v - P_n v\|_V} \to 1 \quad \text{as} \quad n \to \infty \tag{5}$$

and

$$\|Q_n\|_V \to 1 \quad \text{as} \quad n \to \infty. \tag{6}$$

Clearly, (6) implies (5). Conversely, (5) implies $\|Q_n\|_V = \|I - Q_n\|_V = (1 + o(1))\|I - P_n\|_V = (1 + o(1))\|P_n\|_V \to 1$ as $n \to \infty$. Therefore,

$$(5) \Leftrightarrow (6). \tag{7}$$

Second characterization

It was shown in [A/2013, Lemma 6.2] that, for any $\kappa > 0$, the statements

$$\|Q_n\|_V \leq \kappa^{-1} \tag{8}$$

and

$$\forall v' \in V_n: \quad \sup_{v \in V_n} \frac{\langle v', v\rangle}{\|v\|_V} \geq \kappa \|v'\|_{V'} \tag{9}$$

are equivalent. The latter statement is a measure of self-duality of $V_n \subset V$. In the following, $0 \leq \kappa^*(V_n) \leq 1$ denotes the maximal κ for any closed subspace $V_n \subset V$.

Third characterization (special setting)

The setting of [KB/2020] is atypical for Gelfand triples in that

$$\|\cdot\|_V \sim \|\cdot\|_H \text{ are equivalent norms.} \tag{10}$$

We assume this from now on. In particular, (8) holds uniformly in n with *some* κ, because $c\|Q_n v\|_V \leq \|Q_n v\|_H \leq \|v\|_H \leq C\|v\|_V$, and therefore the arguments in the introduction show that (1) also holds for Q_n. In this context, [KB/2020] propose a condition, which translates to the following: Suppose $\{\varphi_j\}_{j\geq 1}$ is an orthonormal basis for H that is also orthogonal in V. Then:

$$w_j := \|\varphi_j\|_V \to a \text{ as } j \to \infty, \text{ for some } a > 0. \tag{11}$$

For each $J \geq 1$, define the subspace $W_J \subset V$ as the span of φ_j with $j \leq J$, and $W_{-J} \subset V$ as the span of those with $j > J$. For any $v \in V'$, write v_J and v_{-J} for the V-orthogonal projections onto those subspaces. By uniform stability of Q_n, for each J, we have from (3):

$$\|(I - Q_n)v_J\|_V \leq o_n^J \|v_J\|_V \quad \text{uniformly in } v \tag{12}$$

where $o_n^J \searrow 0$ as $n \to \infty$. Those subspace are self-dual, because $\kappa = 1$ in (9) for any V_n that is a linear span of finitely many φ_j, and $P_n = Q_n$ on such subspaces. For this reason alone,

$$\text{(6) does not imply (11) .} \tag{13}$$

What is more, the subspace V_n could be at some angle to all such subspaces. For example, take an arbitrary nonzero sequence $c \in \ell_2(\mathbb{N})$. Define V_1 as the span of $\varphi := \sum_j c_j \varphi_j$. Then

$$\|\varphi\|_{V'}^2 = \textstyle\sum_j c_j^2 w_j^{-2}, \quad \|\varphi\|_H^2 = \sum_j c_j^2, \quad \|\varphi\|_V^2 = \sum_j c_j^2 w_j^2. \tag{14}$$

Therefore, the κ in (9) is at best

$$\kappa^\star(V_1) = \frac{\|\varphi\|_H^2}{\|\varphi\|_{V'}\|\varphi\|_V} \geq \kappa. \tag{15}$$

Suppose only $c_1 \neq 0$ and $c_2 \neq 0$ and $w_2 = w_3 = \ldots$. Then, for large w_1 we have $\kappa^\star(V_1) \approx \frac{1}{w_1}(\frac{c_1}{c_2} + \frac{c_2}{c_1})$, which can be arbitrarily small. Condition (11) is meant to moderate this behavior. Indeed, condition (11) constrains the spread of w_j, i.e. it is equivalent to

$$\omega_{-J} := \frac{\sup_{j>J} w_j}{\inf_{j>J} w_j} \to 1 \quad \text{as} \quad J \to \infty. \tag{16}$$

It follows that $\kappa^\star(W_{-J}) \geq 1/\omega_{-J} \to 1$ as $J \to \infty$.
To investigate further, let $T\colon V' \to V$ be the isometry $T\varphi_j := w_j^{-2}\varphi_j$. Note that $TW_J = W_J$. For any $v' \in V'$ we have $\langle v', Tv' \rangle = \|v'\|_{V'}\|Tv'\|_V$, so $v := Tv'$ is the ideal choice in (9).

However, if $v' \in V_n$ then $v := Tv'$ may not be in V_n. Given $v' \in V_n$, construct $v \in V_n$ as follows:

$$\begin{aligned} v &:= Q_n T v'_J + [v' - Q_n v'_J] && (17) \\ &= [Tv'_J - (I - Q_n) T v'_J] + [Tv'_{-J} - (T - I) v'_{-J} + (I - Q_n) v'_J] && (18) \\ &\approx Tv'_J + Tv'_{-J} && (19) \\ &= Tv'. && (20) \end{aligned}$$

For sufficiently large n compared to J, the $(I - Q_n)$ terms are small by (12). For sufficiently large J, the term $(T - I)v'_{-J}$ is small by (16). This shows that

$$\text{(11) implies (6)} . \tag{21}$$

References

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