Problem setup

There are three water containers. Containers #1 and #2 have outflows into #3. Container #2 is constantly fed through a tap. Container #3 is being drained through an opening. What are the water levels in the containers?

Notation

At any given time, for the container $k = 1, 2, 3$,

- the water level is h_k [m],
- the water volume is V_k [m^3],
- the container capacity is $V_k^{\max}$ [m^3],
- the outflow is $y_k \geq 0$ [m^3/s] and the inflow is $x_k \geq 0$ [m^3/s].

ODE

This is sufficient to formulate the following ODEs. As long as no container is maximally filled, we have

$$\tfrac{d}{dt} V_1 = -y_1, \tag{1a}$$
$$\tfrac{d}{dt} V_2 = -y_2 + x_2, \tag{1b}$$
$$\tfrac{d}{dt} V_3 = -y_3 + y_1 + y_2. \tag{1c}$$

More generally, if a container is maximally filled then the water volume is allowed to decrease but not increase, thus (the first min is redundant)

$$\tfrac{d}{dt} V_1 = \min\{0, -y_1\}, \tag{2a}$$
$$\tfrac{d}{dt} V_2 = \min\{0, -y_2 + x_2\}, \tag{2b}$$
$$\tfrac{d}{dt} V_3 = \min\{0, -y_3 + y_1 + y_2\}. \tag{2c}$$

Here we anticipate the physically meaningful setup that an empty container yields no outflow, while a full container simply spills over water that is lost rather than flowing into one of the other containers.

Assumptions

The following are reasonable assumptions but subject to discussion.

- All free water surfaces are in contact with the atmosphere and the temperature is room temperature throughout.
- Container k has a constant horizontal cross-section of area A_k [m^2].
- The tap supply of container #2 is at a constant rate $x_2 \geq 0$ [m^3/s].
- We neglect the initiation time when any valves are opened, i.e. we start with a fully developed flow.
- The flows y_1 and y_2 are unregulated flows due to geometry and gravity only. The opening is merely an orifice or at most a short pipe pointing straight down out of the bottom of the container and of cross-section a_k [m^2] that is relatively thin compared to the container dimensions.
- Water density is constant at ρ [kg/m^3]. The graviational acceleration is g [m/s^2].
- Friction losses at walls and due to viscosity are ignored. Standard engineering assumptions of incompressible flow, in particular there is (negligible flow and) static pressure of ($g\rho h_k$ + 1atm) in the container at the orifice and 1atm just outside the orifice; by the Bernoulli equation of flow, this translates into an outflow velocity v_k given by $\frac{1}{2}\rho v_k^2 = g\rho h_k$. In reality, the flow is smaller due to "geometrical conditions" (cf. discharge coefficient).

  The same outflow velocity is obtained if we simply assume that the water column above the orifice falls straight through the opening without any friction while being "refilled" from the top water layer of the container.

Closure of the ODE system

The ODE system can now be supplemented and fully specified (up to initial conditions) with the following equations ($k = 1, 2, 3$):

$$\text{water volume}: \quad V_k = A_k h_k, \tag{3}$$

$$\text{outflow}: \quad y_k = a_k v_k = a_k \sqrt{2 g h_k}. \tag{4}$$