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Simple Climate Models to Illustrate How Bifurcations Can Alter Equilibria and Stability

Christopher M. Herald, Satoko Kurita, and Aleksey S. Telyakovskiy

Department of Mathematics and Statistics, University of Nevada, Reno, NV

Abstract: Climate change is, by its nature, a truly interdisciplinary topic. While college level science classes now frequently include exposure to climate change issues, not all science majors, math majors and future math K-12 teachers are likely to see climate issues in the course of their studies. Here we present one self-contained topic that can be presented to those students without requiring too much additional explanation about climate change issues. This case study also can serve to illustrate the rather sophisticated concept of a "tipping point" to a diverse science audience without advanced training in dynamical systems. We consider the effect of solar radiation on the size of ice caps, and show that small changes in solar radiation can cause major irreversible changes in the size of ice caps. We present two sets of exercises that students can be asked to work on their own, after the overall conceptual model has been presented during the class. This material was inspired by the experience gained by one of the authors in teaching Interdisciplinary Modeling: Water-Related Issues and Climate Change course in summer 2012. It was used by the authors in a Math 420/620 Mathematical Modeling class at the University of Nevada, Reno in Fall 2012.

Keywords: *Tipping point, climate change, ice caps, bifurcation, mathematical modeling*

Climate change has recently become one of the most pressing scientific challenges facing society, as changes in various ecosystems, ocean systems, and atmospheric systems have been observed and documented (Weart 2008). In 1988, the Intergovernmental Panel on Climate Change (IPCC) was formed by two UN organizations to gather scientific information on the effect of human activity on climate and its potential impact on the environment and society. The Norwegian Nobel Committee highlighted the importance of understanding man-made climate change in its announcement (Nobel Peace Prize 2007) that the 2007 Nobel Peace Prize would be shared by the IPCC and Al Gore, stating that "Extensive climate changes may alter and threaten the living conditions of much of mankind. They may induce large-scale migration and lead to greater competition for the earth's resources. . . . Action is necessary now, before climate change moves beyond man's control."

Some of man's effects on the climate are gradual, but many models point to the possibility of abrupt changes in the near future (National Research Council 2002), which could be far more devastating for society and the environment, because there could be far less time to adapt. Our goal is to present the topic of climate change, particularly the subtle but crucial concept of a tipping point, to students with limited background in this area. By tipping point, we mean the phenomenon wherein small changes in circumstance in a dynamical system cause dramatic changes in its behavior.

In interdisciplinary modeling, researchers and students with diverse backgrounds, training, and perspectives combine their expertise in order to analyze one complex problem, but taking multiple effects simultaneously into account. Often in such situations, mathematics is treated as a black box that provides solutions to a system of differential equations. The system is usually highly complex and nonlinear, making it necessary to rely on the results of the numerical simulations. Typically, the parameter values are usually not known precisely, and there is uncertainty about the form of the functional expression of many processes. Thus results of numerical simulations should be viewed cautiously, and additional analysis is required.

One of the authors has had first-hand experience participating in an interdisciplinary modeling course, Interdisciplinary Modeling: Water-Related Issues and Climate Change, in summer 2012. At the end of the course, the students from many varied backgrounds completed projects modeling multiple phenomena. Often they used the Stella software to simplify the mathematical solution of the differential equations, with little appreciation of the dangers of blindly trusting results from numerical simulation. We want to emphasize that results of black box calculations should be evaluated critically.

In this paper we consider one mathematical phenomenon, of which people doing interdisciplinary modeling must be aware – bifurcation and tipping points. In linear systems, our intuition that small changes in parameters will result in small changes in the outcome is valid. For nonlinear systems, this may not be so; it is often the case that solutions are highly sensitive to small changes in parameters.

We present a case study with two sets of exercises developed for an interdisciplinary modeling class. We used it in teaching the mathematical modeling class at the University of Nevada, Reno. The course is geared towards STEM majors and future high school mathematics teachers. These new exercises do not require extensive background in climate change models or sophisticated techniques to analyze complicated dynamics. Thus, we hope to bring some of the compelling questions of climate change to a broader audience of students.

With a little tailoring, these exercise sets may be equally valuable in bringing calculus-literate science and engineering majors into a fruitful exploration of the relevant dynamical system behavior as well. Depending upon the mathematical backgrounds of the students in an interdisciplinary course on climate change, the dynamical models explored here could be approached using qualitative theory of differential equations, graphical methods, or ready-made packages for solving differential equations. After investigating this toy model of a tipping point, there is room to have broader discussions of other feedback loops that make up components of the complicated earth climate system, such as permafrost melting, methane hydrates, and cloud cover.

There are many great challenges in modeling climate change and its effects, largely because the earth's climate involves so many interconnected factors. Some of the most important current debates have to do with the most fundamental properties of climate behavior and how changes in the amount of greenhouse gases in the atmosphere, ice cap size, etc., will affect the climate. For example, might relatively small changes in these conditions alter the climate so that it is no longer in (or near) a stable equilibrium? We now investigate this question with a very simple model, with minimal prerequisite background.

It has been documented that temperatures have been rising in recent decades, and it is clear that ice packs and glaciers in various parts of the world are receding, very likely as a result of the increased temperatures. Current climate trends are expected to lead to continued melting of the ice caps, resulting in an 18 to 59 cm increase in sea levels over the next century (Intergovernmental Panel on Climate Change AR4, SYR 2007). These projections do not take into account all possible feedback loops between ice sheet melting and climate change (Jarvis 2013). Such rises in ocean levels can logically be expected to cause dramatic flooding in many coastal areas, virtual destruction of certain low-lying island nations, and radical changes in many ecological and climate systems around the globe.

Let's leave aside the questions of whether the human race has the technology, political will, political structures, and determination necessary to radically cut our CO_2 emissions, or even take bold steps such as wider implementation of carbon sequestration to draw significant amounts of existing CO_2 out of the atmosphere. At best, implementing such changes would take time. What changes are needed to avert disaster, and by what deadline? Is it conceivable that, after the current disturbances in greenhouse gas levels, the climate will change dramatically, even if we manage to return the CO_2 levels to the old values?

Climate change discussions often refer to a tipping point, a point at which the dynamical system governing the climate is altered, causing the equilibria to change dramatically. Even though the system remained for a long time in an equilibrium or oscillatory pattern, the behavior after passing the tipping point can be very different. Read, for example, Hansen (2008).

The expected, familiar behavior of physical phenomena governed by dynamical systems is that there are equilibria, perhaps some stable and others unstable. Because of occasional disturbances to the system from outside influences, we don't expect to find a natural process resting in an unstable equilibrium, for example, like a bowling ball perched on a pointed mountain peak. Over time, if outside factors change gradually, we tend to expect the equilibria in a system to change gradually. But in some cases, a small change in a parameter in the system can cause a bifurcation, that is, a change in the set of equilibria; and the result can be a transition to a very different equilibrium.

In this study, we will explore two fairly simple models of certain aspects of the climate system to illustrate that the presumption that gradual change in parameters leads to gradual change in equilibria can be incorrect. We illustrate instances where a gradual change in outside factors (which are parameters in our dynamical system) can give rise to sudden dramatic shifts in the behavior of the dynamical system. Furthermore, we will explore whether resetting a parameter value back to its old value necessarily leads the system to return to its old stable equilibrium. For example, if gradual change in certain climactic parameters like greenhouse gas levels reach a point where some dramatic change begins to happen, will backing off those parameters save the system from catastrophic change?

Modeling Ice Cap Size as a Function of Solar Radiation

Let us consider the effects of solar radiation level on ice cap size. Roughly speaking, solar radiation affects temperatures (by different amounts at different latitudes), and those temperatures affect the formation or melting of polar ice caps. But, in addition, the existence or nonexistence of ice caps in certain regions affects the absorption and reflection of solar radiation. This is what is known as a feedback loop. (Similar feedback loops occur when temperature alters the amount of water vapor or cloud cover in the atmosphere, which then affects radiation absorption and cooling of the planet).

We consider solar radiation level not because changes in solar radiation are thought to be a primary driver of the global warming being observed, but as a proxy for other forms of warming, without the complications of multiple feedback loops in our model. We start with some comments on a differential equation to illustrate how small changes in a coefficient can have a profound effect on the behavior of solutions.

Preliminary Remarks on ODEs

We start with a brief discussion of some properties of differential equations. The parameters in a differential equation can affect how many equilibria there are, and what stability properties they have. Behavior of nonlinear equations, which often appear in applications, are more complex than that of linear equations. Consider, for example, the differential equation:

$$\frac{dx}{dt} = -(x-a)(x-b)(x-c) \text{ when } a < b < c.$$

Since $f(x) = -(x-a)(x-b)(x-c)$ is positive on $(-\infty,a)$ and (b,c) and negative on (a,b) and (c,∞), we see that there are three equilibria: $x = a$ and $x = c$ are stable, but $x = b$ is unstable.

Next, consider the same differential equation, but with $a = b < c$. Then $f(x)$ is positive on $(-\infty,b)$ and on (b,c), but negative on (c,∞) so $x = c$ is stable but $x = b$ is semistable.

Finally, consider the differential equation:

$$\frac{dx}{dt} = -((x-b)^2+\epsilon)(x-c).$$

When $\epsilon = 0$, this agrees with the previous case, but as soon as $\epsilon > 0$, there is only one equilibrium, $x = c$, and it is stable. This illustrates that varying a parameter can cause bifurcations, sudden changes in the number and/or stability properties of equilibria. In climate-related applications, it is an illustration of the tipping point concept.

North's Ice Cap Model

Gerald North has studied a model for average sea-level temperature T as a function of x = sin(latitude) and time t, which takes into account warming from solar radiation at different latitudes, diffusion of heat from warm areas to cold areas, and the albedo (reflectiveness) of ice cap and ice-free regions (North 1984).

The evolution over time of the sea-level temperature may be modeled by a partial differential equation:

$$\frac{\partial \mathrm{T}}{\partial t} = D\frac{\partial}{\partial x}\left((1\text{-}x^2)\frac{\partial \mathrm{T}}{\partial x}\right) \text{-}A - BT{+}QS(x)a(T).$$

In this formula, Q is the solar constant divided by 4, measured in W/m^2, D, A, B are empirical constants,

$$a(t) = \begin{cases} 0.38 \text{ for } T < \text{-}10°\mathrm{C} \\ 0.38 \text{ for } T < \text{-}10°\mathrm{C}' \end{cases}$$

represents the co-albedo (so a higher $a(T)$ value represents lower reflectivity and greater heating through absorption of sunlight), and $S(x)$ is the mean annual sunlight distribution.

The methods required to solve this partial differential equation go beyond the scope of the present analysis, but we will derive from this equation a simpler model for the size of the ice cap, and how it evolves over time.

We can obtain an ordinary differential equation (ODE) describing the temperature in the steady state regime by setting $\delta T/\delta t$, namely:

$$\text{-}D\ \frac{d}{dx}\left((1\text{-}x^2)\ \frac{d\mathrm{T}}{dx}\right) {+}A + BT = QS(x)a(T).$$

This model contains the following feedback loop: the co-albedo in a region is determined by whether it is covered with ice, and this is determined by whether or not $T(x) < \text{-}10°$, but this co-albedo affects the solar heating.

Solutions to the ODE for the Northern Hemisphere satisfying appropriate boundary conditions (dT/dx = 0 at $x = 0$ and $x = 1$, in order to extend to symmetric solutions on the sphere) are potential steady-state temperature distributions. For each such temperature distribution, the lowest value of x above which $T(x) <$ -10° is an equilibrium ice cap size.

Even solving this nonlinear ODE with boundary conditions requires techniques beyond the scope of many Mathematical Modeling courses, so we

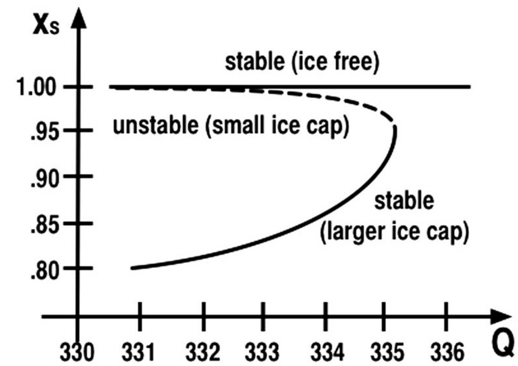

Figure 1. Ice cap size versus solar constant. Line $x_s = 1$ corresponds to the stable solution with no ice cap present; dotted curve corresponds to an unstable small ice cap; solid curve corresponds to a stable ice cap of larger size.

will rely on the numerical work done in (North 1984), and focus our attention on the equilibrium ice cap sizes, determined by Q, the average amount of solar energy received by a square meter of the earth's upper atmosphere at the equator. We normally think of Q as constant, but it might change with a deformation of the earth's orbit, or with changes in the power released by the sun. In terms of global warming considerations, we can view an increase in the value of Q as a proxy for an increase in greenhouse gas levels, which increases the absorption or retention of solar energy.

North's analysis leads to the surprising conclusion (Figure 1) that the equilibrium ice cap size is not uniquely determined by Q. Figure 1 qualitatively matches the results of his numerical simulations. Namely, for a certain range of values of the Q parameter in the model, there are three different steady state ice cap sizes. In this diagram, x_s denotes sine of the latitude of the ice cap boundary. Note also that the middle equilibrium, in the range of Q values where there are three, is unstable; the other two are stable. For example, when $Q = 332$, there are three steady state x_s values. One is $x_s = 1$, meaning that there is no ice cap all the way up to latitude 90°; this one is stable, in that if there was a very small ice cap around the pole, it would melt away. The second x_s equilibrium is slightly smaller, around 0.99 (meaning a small, but nonempty ice cap), but this is unstable; if the ice cap is a little smaller, it shrinks to nothing, and if it is larger, it grows. Finally, there is a third stable equilibrium $x_s \approx 0.8$, corresponding to latitude ≈

55°. As Q increases toward 335, the two lower x_s equilibrium values get closer together, and they ultimately meet and disappear around $Q = 335$.

A Cubic Model for Ice Cap Size

We now shift our attention to a simpler dynamical model, where the ice cap size, measured by x_s = sin(latitude of boundary), satisfies a differential equation involving the parameter Q, and has three similar equilibria for $332 < Q < 335$ (with the middle one unstable, the others, stable), and around $Q = 335$, the lower equilibria come together and disappear, leaving only the stable equilibrium at $x_s = 1$. The primary simplification is that we treat the ice cap size as one quantity whose rate of change depends on its current value and Q, rather than having the ice cap size and the entire temperature distribution $T(x)$ interdependent on one another.

This model is given by:

$$\frac{dx_s}{dt} = (1\text{-}x_s)\left(Q\text{-}355 + 3 \cdot \left(\frac{x_s\text{-}0.89}{0.09}\right)^2\right)$$

Assume here that t measures time in centuries. We next present a sample set of student exercises pertaining to this cubic model.

1. Figure 2 shows an implicit plot of the equilibrium x_s values for a range of Q values, created using a computer algebra system without attention to using sufficiently fine resolution. Compare this with the conclusions you reach using algebra. How should the correct figure look? What qualitative differences are there between the behavior of the equilibria in this model and the equlibria approaching each other asymptotically in Figure 1?
2. Assume $Q = 333$. Verify by simulating with a few initial values that there is a stable equilibrium near $x_s = 0.82$.
3. Still assuming that $Q = 333$, what sort of simulation scheme would allow you to find the unstable equilibrium near 0.99? (Hint: think about running time backward.) Carry this process out.
4. Assume that Q has remained around 333 for a long time and x_s has settled into the stable equilibrium found in Part 2, but then external factors suddenly cause Q to jump up to 336. Assuming it stays up this high from then on, what happens to x_s, and what does this imply about the ice caps?

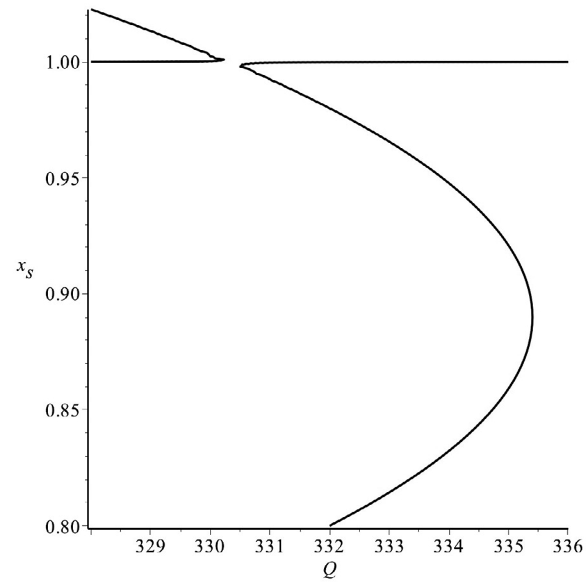

Figure 2. Ice cap size versus Q (implicit plot from cubic model).

5. Assume that Q has been 333 and x_s has been at the stable equilibrium from Part 2, but Q then increases to 336 for a finite time interval of M years, after which it decreased to $Q = 333$. Use simulation to determine what happens if M = 50, M = 100, M = 200. Explain in English what your conclusions are for these scenarios.

This model was developed by approximating the $x_s < 1$ equilibria (Figure 1) by a parabola passing through xs = 0.80 and $x_s = 0.98$ for $Q = 332$ and having a vertex at $Q = 335$. The factor $(x_s - 1)$ adds another equilibrium at $x_s = 1$, independent of Q.

Parts 1 through 5 of the problem provide opportunities for students to explore with numerical techniques what the stability means for the different equilibria in this dynamical system, which has Q as a parameter. Parts 4 and 5 get at the heart of the question of whether small parameter changes can cause catastrophic climate change. We note that students can do the numerical simulations with Euler's method or other standard techniques.

A Second Model for Ice Cap Size

The cubic model in the previous section does an adequate job fitting North's model when $332 \leq Q$, but this model also had the property that the upper and middle equilibria crossed when Q dropped below a certain level. We next discuss a slight modification of this differential equation which displays some different features as Q is varied:

$$\frac{dx_s}{dt} = -0.003+(1-x_s)\left(Q-355 + 3 \cdot \left(\frac{x_s-0.89}{0.09}\right)^2\right)$$

In the ranges $332 < Q < 335$, this model again provides a pair of equilibria similar to the two branches of a parabola, as well as a third equilibrium near $x_s \approx 1$. But in this model, the $x_s \approx 1$ and the unstable equilibrium cancel each other out and disappear when $Q < 331.5$. The set of equilibria for a range of Q values is illustrated in Figure 3. In this model, assume that time t is again measured in centuries.

Here, again, we present a series of student exercises pertaining to this model.

1. Assuming $Q = 334$, locate the three equilibria using numerical solvers. Linearize to determine the stability of each.
2. Use a computer algebra system to reproduce the same graph as Figure 3.
3. In the region illustrated in Figure 3, where is dx_s/dt positive and where is it negative? Figure 1 indicates the stability or instability of the equilibria corresponding to different parts of the curves. On your copy of Figure 3, identify which parts of the curve correspond to stable equilibria and which correspond to unstable equilibria.
4. If the solar constant has been $Q = 333$ for some time, and the ice cap size is at the lower x_s equilibrium, but then Q is raised suddenly to 336 for 200 years, and then it is returned to 333, what will happen to the ice caps?
5. Suppose that the solar constant has been $Q = 333$ for some time, and the ice caps have been at the lower x_s equilibrium, but then Q is raised suddenly to 336 for 135 years. At this point, suppose that through massive technical advances, the human race develops the ability to decrease Q. How far would we have to decrease Q, at least on a temporary basis, in order to restore the ice caps to their historical size? Explain and illustrate with diagrams of some kind.

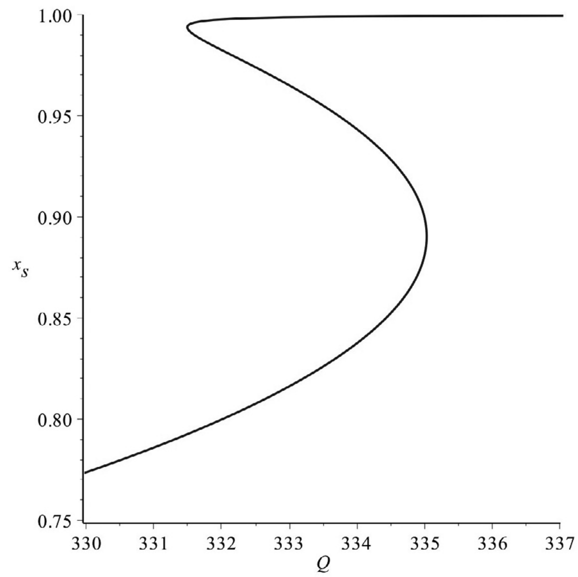

Figure 3. A second nonlinear model of ice cap size versus solar constant.

Conclusion and Discussion

Our goal with these two sample problem sets is to increase the awareness of climate change issues among students who may not have any other exposure to climate change in their coursework. In order to be successful in conveying the main ideas, we ask students to read the paper by North (1984) beforehand; it is often the case that math and math education students do not read a single scientific research paper during their undergraduate studies. Reading an actual research paper allows them to see that important information is presented not only in textbooks but also in research publications, and that with their mathematical background, they are in a position to understand some of the research literature. During the class dedicated to climate change we explain in detail the key equations of North's paper and our further set of simplifying assumptions.

With these climate projects we try to convey the point that, in a nonlinear system, small changes in parameters can lead to major changes in the behavior and, moreover, changes can be irreversible. The climate on Earth is highly a nonlinear system. We presented a simple model that uses a nonlinear ordinary differential equation with a bifurcation point; for multiple examples of such systems see Strogatz (1994). Similar in spirit to our model is a conceptual model describing circulation in Atlantic Ocean saltwater versus freshwater input (Figure 8.4 in Broecker 2010). Again, under certain conditions, the system may be close enough to a bifurcation point that even a small change in the amount of incoming freshwater to the ocean can completely alter the circulation pattern.

We note that similar energy climate models were considered in various papers. We refer to just a few of them (Cahalan and North 1979; Drazin and Griffel 1977; North 1990). In general, the modern energy climate models were developed independently by Budyko and Sellers, and the current model is also an example of an energy balance climate model. While it would be great to present more of Budyko's original work to the students, the time constraints in our modeling class required a more restricted focus. More comprehensive treatment of climate energy models likely will fit better into the curriculum in atmospheric sciences or various interdisciplinary programs. We have at least succeeded in conveying to students outside these disciplinary areas that even small changes can have extreme effects on the climate.

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Authors Bios and Contact Information

Christopher M. Herald is a Professor of Mathematics at the University of Nevada, Reno. His research specializations are topology and applied mathematics. He has taught various math courses, including Partial Differential Equations, Mathematical Modeling and Stochastic Processes. His publications include articles in mathematical journals, as well as interdisciplinary work in science and engineering journals, including two recent papers discussing modeling of pollution particles published in *Aerosol Science and Technology*. He can be reached at herald@unr.edu.

Satoko Kurita is a Visiting Lecturer in Mathematics at the University of Nevada, Reno. Her research specialization is applied mathematics and flow through porous media. She has taught different college level math classes. Also she taught Mathematical Modeling and Engineering Statistics. Some of her work is related to modeling groundwater flows and was published in *Advances in Water Resources*. She can be reached at skurita@unr.edu.

Aleksey S. Telyakovskiy is an Associate Professor of Mathematics at the University of Nevada, Reno. His research specialization is applied mathematics and flow through porous media. He has taught numerous courses on applied and computational mathematics, in addition to Mathematical Modeling and Engineering Statistics. He has also team-taught the course Interdisciplinary Modeling: Water-Related Issues and Changing Climate. He has published in various mathematical journals, and has published a number of studies related to groundwater flows in *Advances in Water Resources*. He can be reached at alekseyt@unr.edu.

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