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2.1

Convert to a system of first-order ODEs (in matrix form, if possible):

- $y''(t) + y(t) = 0$,
- $\sum_{k=0}^{n} a_k(t) y^{(k)}(t) = b(t)$,
- $y''(t) + x(t) = 0$, $x''(t) + y(t) = 0$,
- $y''(t) + (y'(t))^2 = y(t)$.

2.2

Let A be an $n \times n$ real matrix. Show that the solutions $x\colon \mathbb{R} \to \mathbb{R}^n$ to $x'(t) = Ax(t)$ form a vector space. What is the dimension of that vector space?

2.3

Write $y''(t) + y(t) = 0$, $y(0) = 0$, as a first-order system of ODEs in matrix form. What are the eigenvalues of that matrix? Solve the ODE.

2.4

The singular value decomposition of a real matrix A is the product $USV^\top$ with orthogonal U and V and a diagonal S (with non-negative *singular values* ordered high-to-low). How does the matrix 2-norm, $\|A\|_2 := \sup\{\|Ax\|_2 : \|x\|_2 = 1\}$, relate to the singular values?

2.5

Find the trace / determinant / eigenvalues / eigenvectors / $\exp(tA)$ / 2-norm of A, and sketch the solutions to $x'(t) = Ax(t)$ with the initial value $x_1(0) = 1$, $x_2(0) = 0$, for

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2.6

If $x_1' = \log(1 - x_2)$ and $x_2' = \log(x_1)$, what happens for $x_1(0) = 1 + \varepsilon_1$ and $x_2(0) = \varepsilon_2$?

1.20

Compute $\exp(tJ)$, $t \in \mathbb{R}$, where J is the $n \times n$ matrix $J_{ij} = \lambda\delta_{i,j} + \delta_{i,j-1}$ with $\lambda \in \mathbb{C}$.