Climate Modeling in Differential Equations

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MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop—through a community of users and developers—a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

1. Introduction

Some exercises and problems in this Module use short Mathematica notebooks, which are reproduced in the **Appendix** and can be accessed in electronic form at

`www.mathclimate.org/education/climateandODE`

as well as at this *Journal*'s Web page for supplementary materials at

`http://www.comap.com/product/periodicals/supplements.html`

Introductory background to climate physics and fleshing out of the material below can be found in Walsh and McGehee [2013], as well as in other articles in the same special issue of the *College Mathematics Journal* dedicated to the Mathematics of Planet Earth. For further information and items of potential interest, please click on the `Education` link at `www.mathclimate.org`.

2. Global Average Temperature Models

We can make a basic model of the temperature of the Earth by assuming that it receives incoming insolation (solar energy) but radiates some of it back into space. The global mean temperature T can be modeled by the *energy balance equation (EBM)* [Kaper and Engler 2013, 16]

$$R\,\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4. \tag{1}$$

The first term on the right is incoming heat absorbed by the Earth and its atmosphere system. The second term is heat radiating out as if the Earth were a blackbody with all of the *outgoing longwave radiation (OLR)* escaping to space.

- T (K, kelvins) is the average temperature in the Earth's *photosphere* (upper atmosphere, where the energy balance occurs in this model) (1 kelvin = 1°C);
- t (years) is time;
- R (W-yr/m^{2}K) is the averaged heat capacity of the Earth/atmosphere system (heat capacity is the amount of heat required to raise the temperature of an object or substance 1 kelvin (= 1° C));
- Q (W/m^2) is the annual global mean incoming solar radiation (or *insolation*) per square meter of the Earth's surface;
- α (dimensionless) is planetary *albedo* (reflectivity), and
- σ (W/m^{2}K^4) is a constant of proportionality, the *Stefan-Boltzmann constant*.

Note that **(1)** is an autonomous ordinary differential equation (ODE), meaning that the expression for the derivative does not explicitly involve the independent variable t.

Values for the parameters are:

- $R = 2.912$ W-yr/m^{2}K) [Ichii et al. 2003, Table 1];
- $Q = 342$ W/m^2 [Kaper and Engler 2013, 17],
- $\alpha = 0.30$ [Kaper and Engler 2013, 17], and
- $\sigma = 5.67 \times 10^{-8}$ W/m^{2}K^4.

Exercises

1. Verify that both sides of **(1)** have the same units.

2. What unstated assumptions does the model make?

3. (may be omitted if you are not familiar with Mathematica's `Dsolve`) Use the Mathematica command `DSolve` to try to find an analytic solution to **(1)** with initial condition $T(0) = 0$ in terms of unevaluated parameters R, Q, α, and σ.

4. (may be omitted if you are not familiar with Mathematica's `NDsolve`)

a) For the given values for the parameters R, Q, α, and σ, use the Mathematica command `NDSolve` to find a numerical solution to **(1)** with initial condition $T(0) = 0$ over the time interval $[0, 1000000000]$ (the first billion years). What do you observe?

b) Plot the solution from part **a)**.

5. Find the equilibrium value T^* for the solution to **(1)**.

6. Derive the value of R from the values given in the table of Ichii et al. [2003] for the heat capacity of the Earth's surface, 4.69×10^{23} J K^{-1}, and the area of the Earth, 5.101×10^{14} m^2. (Careful: Note that 1 W = 1 J/s, and pay attention to the units.)

7. The Earth's atmosphere can be incorporated into the simple model **(1)** by including an OLR *emissivity factor* ϵ:

$$R\,\frac{dT}{dt} = Q(1-\alpha) - \epsilon\sigma T^4.$$

The value $\epsilon = 1$ yields an atmosphere completely transparent to OLR, as in **(1)**.

For the given values of the other parameters, what value of ϵ gives an equilibrium global mean temperature $T^* = 288.4\,(= 15.4°\text{C} \approx 59.7°\text{F})$— our current annual global average temperature? (Recall that 273 K = 0°C = 32°F.)

8. Suppose that we model the OLR in a global surface temperature model via a linear term of the form $A + BT$, with B a positive constant (as in Graves et al. [1993]), thereby arriving at the energy balance equation

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT). \qquad \textbf{(2)}$$

In this model, the global average surface temperature T is given in °C, as in Graves et al. [1993, 20]. (The values of the parameters R, Q, and α are not affected by this change of scale for T.)

a) Explain, in terms of the model, the requirement that $B > 0$.

b) Find the general solution of equation **(2)**. What is the behavior of solutions over time?

c) From satellite measurements, the best current estimates for the parameters A and B are $A = 202$ W/m^2 and $B = 1.90$ W/m^2°C [Graves et al. 1993].

(i) Calculate the Earth's average surface temperature T^* at equilibrium from **(2)**. Why might you expect this value to be fairly close to 15.4°C, the Earth's current annual global average surface temperature?

(ii) Repeat part **(i)** but for data from 1979 for the Northern Hemisphere: $A = 203.3$ W/m^2 and $B = 2.09$ W/m^2°C) [Kaper and Engler, 20].

(iii) How does the magnitude of T^* vary with the parameters A and B? Discuss this magnitude in the context of the OLR term in the model.

(iv) Assume that the albedo of any ice-free surface is $\alpha_w = 0.32$ [Budyko 1979]. Find T^* in the case where the Earth is ice-free, that is, with α in **(2)** replaced by α_w. With the temperature governed by **(2)**, would ice ever form again if the Earth were to become ice-free? (Assume that ice forms when $T < T_c = -10$°C.)

(v) Assume that the albedo of ice is $\alpha_s = 0.62$. Compute T^* in the *snowball Earth* state, that is, with α in **(2)** replaced by α_s, indicating that the planet is completely ice-covered. With the temperature governed by **(2)**, would ice ever melt if the Earth were in a "snowball" state?

3. Bifurcation

The notion of a *bifurcation* plays an essential role when modeling with ODEs. The following is a more open-ended problem for which you will need to use the Mathematica notebook `QBifurcation.nb`. (This program and others mentioned below are in the **Appendix** and also at the *Journal*'s Website for supplementary material).

Exercise

9. Consider the autonomous ODE

$$R\,\frac{dT}{dt} = Q\big(1-\alpha(T)\big) - \epsilon\sigma T^4 = f(T), \tag{3}$$

where we take $\epsilon = 0.6$ as the *atmospheric emissivity factor*.

Since heat capacity R plays no role in the qualitative behavior of solutions, we set $R = 1\ \mathrm{J/m^2{}^\circ C}$.

Planetary albedo is a measure of the extent to which insolation is reflected back into space, with larger albedo values corresponding to greater reflectivity (and so less absorption). Ice, for example, has a higher albedo than water.

Suppose that the albedo function for the Earth is given by **(4)** below, a slight variant of that used by Kaper and Engler [2013, 18]:

$$\alpha(T) = 0.5 + 0.2\tanh\big(0.1(265 - T)\big), \tag{4}$$

so that albedo depends explicitly on temperature.

a) Find $\lim_{T\to\infty}\alpha(T)$ and interpret the limit in the context of the model. Repeat for $T \to -\infty$.

b) The insolation Q has changed significantly over the Earth's history. For example, 3.5 billion years ago insolation was less than 80% of its current value. We are thus interested in the ways in which equilibrium solutions to **(3)** change as Q varies (a question arising in the study of *paleoclimate*).

To gain insight into this problem, use the *Mathematica* file

`QBifurcation.nb`

to plot $f(T)$ for a variety of Q-values in the interval $[270, 450]$.

Describe what happens to the number of equilibrium solutions as Q increases from 270 to 450. You should submit one plot with at least five representative $f(T)$ graphs.

c) Via part **(b)**, we see that bifurcations of equilibrium points occur in model **(3)** as Q varies. Possible bifurcations occur when both $f(T) = 0$ and $f'(T) = 0$.

(i) Show that $f(T) = 0$ implies that

$$Q = Q(T) = \frac{\epsilon\sigma T^4}{1-\alpha(T)}. \tag{5}$$

(ii) Compute $f'(T)$, set the result equal to 0, and substitute **(5)** for Q to find

$$-\epsilon\sigma T^3 \left(\frac{T}{1-\alpha(T)}\,\alpha'(T)+4\right)=0. \qquad \textbf{(6)}$$

(iii) Use the `FindRoot` command set up in `QBifurcation.nb` to solve **(6)** for T. You should find two positive solutions $T_1 < T_2$.

(iv) Compute the corresponding values $Q_1 = Q(T_1)$ and $Q_2 = Q(T_2)$. Discuss the behavior of the model in small Q-intervals centered at Q_1 and Q_2. Interpret this behavior in terms of the role played by Q in this model.

d) With the aid of `ParametricPlot` in `QBifurcation.nb`, create a bifurcation diagram of equilibrium solutions versus the parameter Q. Be sure to include representative phase lines in your plot (e.g., as in Blanchard et al. [2012, 99]).

e) Using your bifurcation plot from part **(d)**, determine the long-term behavior of solutions when $Q = 280$ W m^{-2}.

The equilibrium surface temperature when $Q = 280$ is $T = 223$ K $= -50$°C. Hence, for this value of Q and the atmospheric emissivity factor $\epsilon = 0.6$, the surface of the Earth would be covered with ice.

This conclusion contradicts the fact that 3.5 billion years ago, at a time when Q was less than 280 W/m^2, liquid water is known to have existed on the Earth's surface. This contradiction, known as the *faint young Sun paradox* [Wikipedia 2016], has been explained by positing that the atmosphere contained more greenhouse gases early in Earth's history. Wolf and Toon [2013] suggest that a three-dimensional model of the Earth can resolve the paradox [Netburn 2013].

Thus, setting $Q = 280$ but decreasing ϵ to $\epsilon = 0.5$, determine the behavior of solutions to the model. In particular, does a "runaway snowball Earth" episode occur?

4. Sample Exam Question on Bifurcations

Consider the autonomous ODE

$$\frac{dT}{dt} = E_{\text{in}}(T) - E_{\text{out}}(T), \qquad \textbf{(7)}$$

where

T is the average photosphere temperature of the planet, while

$E_{\text{out}}(T) = \epsilon\sigma T^4$ is energy that leaves the Earth/atmosphere system.

Recall that the parameter ϵ is an emissivity factor, allowing us to incorporate the effect of the atmosphere on climate. For example, as the concentration of an atmospheric greenhouse gas such as CO_2 increases, less outgoing longwave radiation escapes to space, corresponding to a *decrease* in ϵ.

Similarly, as the concentration of CO_2 decreases, more outgoing longwave radiation escapes to space, corresponding to an *increase* in ϵ.

We thus treat the parameter ϵ as a proxy for greenhouse gases, although an increase in the former corresponds to a decrease in the latter (and vice versa).

Recall that $E_{\text{in}}(T)$ is the incoming solar radiation absorbed by the Earth/atmosphere system. If the temperature was very cold, much of the planet's surface would be covered with ice. This would imply that the planetary albedo would be large, with much less absorbed insolation. Thus, $E_{\text{in}}(T)$ would be low for smaller T-values. Conversely, a warm planet has little ice cover, leading to greater absorbed insolation and a larger $E_{\text{in}}(T)$.

We thus consider the qualitative graphs of **Figure 1** for $E_{\text{in}}(T)$ and $E_{\text{out}}(T)$, respectively. Assume that the lower branch of $E_{\text{in}}(T)$ corresponds to very cold temperatures, while the upper branch of $E_{\text{in}}(T)$ represents very warm temperatures.

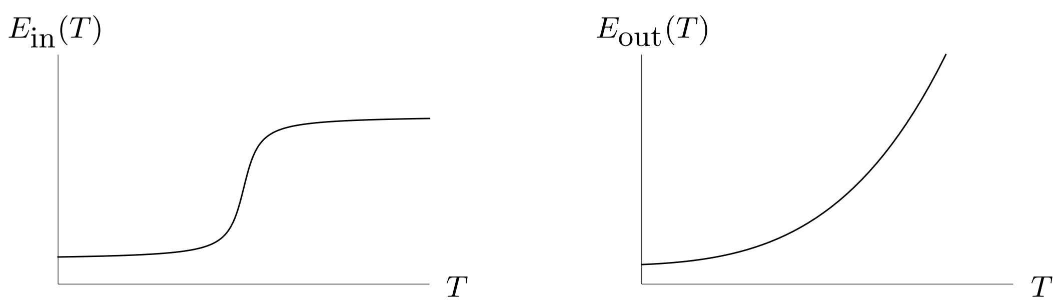


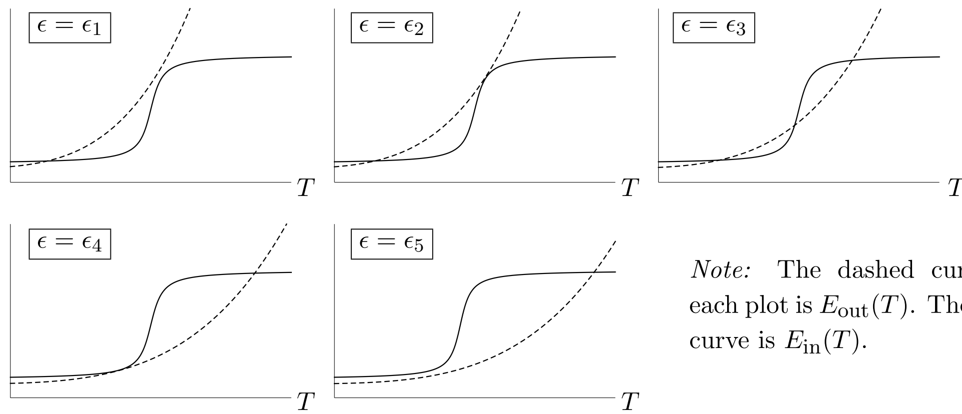
Figure 1. Qualitative graphs of **Figure 1** for $E_{\text{in}}(T)$ and $E_{\text{out}}(T)$.

The plots of **Figure 2** correspond to five decreasing ϵ-values, with $\epsilon_{i+1} < \epsilon_i,\ i = 1, 2, 3, 4.$

a) What does a point of intersection of the solid and dashed curves above represent in terms of ODE **(7)**?

b) Draw the phase line corresponding to ODE **(7)** for each of the above ϵ-values, moving left to right as ϵ decreases from ϵ_1 to ϵ_5.

c) In a brief paragraph, discuss the bifurcation that occurs at $\epsilon = \epsilon_2$ in the context of the model, particularly in terms of the concentration of greenhouse gases such as CO_2.



Note: The dashed curve in each plot is $E_{\text{out}}(T)$. The solid curve is $E_{\text{in}}(T)$.

Figure 2. Qualitative graphs of **Figure 1** for $E_{\text{in}}(T)$ and $E_{\text{out}}(T)$, for five decreasing values of ϵ.

5. Project: Surface Temperature–Ice Sheet Coupled Model

5.1 Introduction

Recall the global climate model of **Exercise 2**:

$$R\,\frac{dT}{dt} = Q(1-\alpha) - (A + BT), \qquad \textbf{(2)}$$

where

- $T = T(t)$ (°C) is the global average surface temperature,
- R is the globally averaged heat capacity of the Earth's surface,
- Q is the incoming solar radiation (or *insolation*),
- α is the globally averaged surface albedo (a measure of the reflectivity of the surface), while
- A and B are empirically determined parameters.

The term $Q(1-\alpha)$ represents energy from the Sun into the system, while $A + BT$ models the outgoing longwave radiation, emitted by the Earth-atmosphere system back into space.

In **Exercise 2b**, you showed that every solution of **(2)** converges to the unique equilibrium solution

$$T^* = \frac{1}{B}\big(Q(1-\alpha) - A\big).$$

In this project, you investigate the next natural ("simplest") energy balance model to consider: one that depends both on time and on latitude (but not on longitude).

5.2 A Latitude-Dependent Model

In seminal papers, M. Budyko [1969] and W. Sellers [1969] independently introduced energy-balance models in which the surface temperature depends on latitude and time. In each model, the temperature is assumed constant on a given latitude circle.

With θ as latitude, the variable $y = \sin\theta$ is convenient. For example, you are invited to show that the surface area 4π of a sphere of radius 1 centered at the origin is given by

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \, d\omega,$$

where ω is the standard polar angle. Thus the differential $dy = \cos\theta \, d\theta$ of y will play a role when integrating a function (such as the temperature function T) over the surface of the planet.

We refer to y as the "latitude" in what follows, trusting that no confusion will arise on the reader's part.

For the Budyko-Sellers model, the temperature function, also called a *distribution* or *profile* and denoted $T = T(t, y)$ (°C), is the annual average surface temperature at latitude y.

We assume further that $T(t, y)$ is symmetric across the Equator. It thus suffices to consider $y \in [0, 1]$, with $y = 0$ the Equator ($\theta = 0°$) and $y = 1$ the North Pole ($\theta = 90°$).

Recall the definition of the average value of a function defined on an interval $[a, b]$; in a nice exercise, and as portended by the preceding discussion, one can show that the global annual average temperature is given simply by

$$\overline{T} = \overline{T}(t) = \int_0^1 T(t, y) \, dy. \tag{8}$$

5.2.1 Glaciers

Glaciers are incorporated into this model via an adjustment to the albedo function.

We assume that ice exists above a given latitude $y = \eta$, while no ice exists below η. The parameter η is referred to as the *ice line*, with the albedo now a function $\alpha_\eta(y) = \alpha(y, \eta)$ depending on y and the position of the ice line η.

Since ice is more reflective than water or land, the albedo will be larger for latitudes above the ice line. A simple albedo function satisfying these specifications is

$$\alpha(y, \eta) = \begin{cases} \alpha_1, & y < \eta; \\ \alpha_2, & y > \eta; \\ \frac{1}{2}(\alpha_1 + \alpha_2), & y = \eta, \end{cases} \tag{9}$$

where $\alpha_1 < \alpha_2$.

5.2.2 Distribution of Insolation

A second adjustment concerns the distribution of insolation. The tropics receive more energy from the sun on an annual basis than do the polar regions.

This difference is taken into account by modeling the energy absorbed by the surface via the term

$$Qs(y)\big(1-\alpha(y,\eta)\big),$$

where $s(y)$ is the distribution of insolation over latitude, normalized so that

$$\int_0^1 s(y)\,dy = 1. \tag{10}$$

While $s(y)$ can be computed explicitly from astronomical principles, it is uniformly approximated to within 2% by the polynomial $1.241 - 0.723y^2$. So henceforth we set

$$s(y) = 1.241 - 0.723y^2.$$

Note that $s(y)$ is largest at the Equator and decreases monotonically to a minimum at the North Pole.

5.2.3 Meridional Heat Transport

A final adjustment concerns *meridional heat transport*, encompassing physical processes such as the heat flux carried by the circulation of the ocean and the fluxes of water vapor and heat transported via atmospheric currents.

We focus on Budyko's model, in which the meridional transport term is simply $C(T-\overline{T})$, with $\overline{T}$ as in **(8)** and C (W/m^2 °C) a positive empirical constant. Budyko's model is then

$$R\,\frac{\partial T(t,y)}{\partial t} = Qs(y)\big(1-\alpha(y,\eta)\big) - (A+BT) - C\left(T-\overline{T}\right). \tag{11}$$

The final term models the simple idea that warm latitudes (relative to the global mean temperature) lose heat energy through transport, while cooler latitudes gain heat energy. We assume throughout that each of A, B, C, and Q is strictly positive.

Equation **(11)** is a *partial differential equation*, since $T = T(t,y)$ is a function of two variables. Equilibrium solutions of **(11)** are now *functions* $T^* = T^*(y)$ satisfying

$$Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A+BT^*(y)\big) - C\big(T^*(y) - \overline{T^*}(y)\big) = 0.$$

(Note that $\frac{\partial}{\partial t}T^*(y) = 0$, that is, T^* is a function of only y.)

Given albedo function **(9)**, one can show that any equilibrium solution $T^*(y)$ is a piecewise quadratic function, with the discontinuity occurring at $y = \eta$. Three such equilibrium solutions (which depend on the position of the ice line η) are plotted in **Figure 3**.

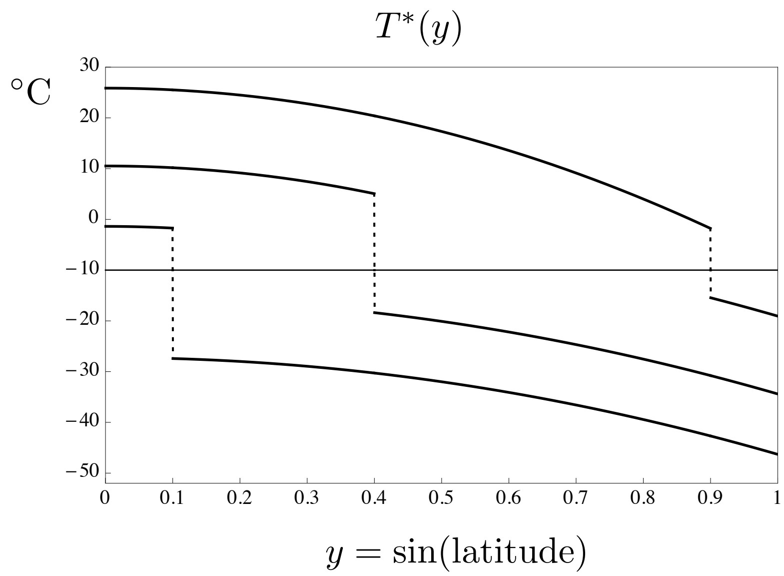


Figure 3. Equilibrium solutions $T^*(y)$ of equation **(11)** are piecewise quadratic.
Top: $\eta = 0.1$. Middle: $\eta = 0.4$. Bottom: $\eta = 0.9$.

5.2.4 Legendre Polynomials

Recall Legendre's differential equation

$$(1 - y^2)\phi'' - 2y\phi' + \gamma(\gamma + 1)\phi = 0, \text{ where } \phi = \phi(y). \tag{12}$$

Given a natural number n, the *Legendre polynomial* $p_n(y)$ is the polynomial solution of ODE **(12)** with $\gamma = n$ that also satisfies $p_n(1) = 1$. The first few Legendre polynomials are

$$p_0(y) = 1, \quad p_1(y) = y, \quad p_2(y) = \tfrac{1}{2}(3y^2 - 1), \quad \text{and} \quad p_3(y) = \tfrac{1}{2}(5y^3 - 3y).$$

As with Taylor and Fourier series, many functions can be expanded in terms of the Legendre polynomials. For example, if $g : [-1, 1] \to \mathbb{R}$ is piecewise continuous, then g has an expansion

$$g(y) = \sum_{n=0}^{\infty} a_n p_n(y), \quad a_n \in \mathbb{R}, \tag{13}$$

that converges at every point at which g is continuous [Lebedev 1965, Section 4.7].

5.3 Putting It All Together

Note that $T(t, y)$ is an even function of y, due to the assumption of symmetry across the Equator. Thus, an expansion **(13)** for T in the variable y requires only the even Legendre polynomials. If we keep finitely many terms in this expansion, we have a polynomial approximation of T in terms of its y-variable.

We have seen this before: Recall that the insolation distribution function is well approximated by

$$\begin{aligned} s(y) &= 1.241 - 0.723y^2 \\ &= 1 \cdot 1 + (-0.482)\left(\tfrac{1}{2}(3y^2 - 1)\right) \\ &= s_0 p_0(y) + s_2 p_2(y), \end{aligned}$$

where we have expressed $s(y)$ as a linear combination of the first two even Legendre polynomials, with $s_0 = 1$ and $s_2 = -0.482$.

We proceed similarly with T; given the discontinuity at $y = \eta$ in **(11)**, we express $T(t, y)$ piecewise as

$$T(t,y) = \begin{cases} U(t,y), & y < \eta; \\ V(t,y), & y > \eta; \\ \big(U(t,\eta) + V(t,\eta)\big)/2, & y = \eta, \end{cases} \qquad \textbf{(14)}$$

where

$$\begin{aligned} U(t,y) &= u_0(t)p_0(y) + u_2(t)p_2(y), \\ V(t,y) &= v_0(t)p_0(y) + v_2(t)p_2(y). \end{aligned} \qquad \textbf{(15)}$$

Note that the coefficients of p_0 and p_2 are now functions of t, rather than constants, since U and V are functions of both t and y.

5.4 Project Problems

1. a) Show **(11)** can be written piecewise as

$$R\frac{\partial}{\partial t}U(t,y) = \begin{cases} Qs(y)(1-\alpha_1) - (A + BU) - C\left(U - \overline{T}\right), & 0 \le y < \eta \\ Qs(y)(1-\alpha_2) - (A + BV) - C\left(V - \overline{T}\right), & \eta < y \le 1. \end{cases} \qquad \textbf{(16)}$$

b) Let $P_2(\eta) = \int_0^\eta p_2(y)dy = \frac{1}{2}(\eta^3 - \eta)$. Show that the global average surface temperature is

$$\overline{T} = \overline{T}(\eta) = \eta u_0 + (1-\eta)v_0 + P_2(\eta)(u_2 - v_2), \qquad \textbf{(17)}$$

where we have suppressed the explicit dependence on t.

c) Assuming that the temperature at the ice line is the average of the left- and the right-hand limits of the temperature profile at η as in **(14)**, show that

$$T(\eta) = \frac{1}{2}(u_0 + v_0) + \frac{1}{2}(u_2 + v_2)p_2(\eta). \tag{18}$$

2. Suppose that U and V are given by **(15)**.

a) Substitute these expressions for U and V into system **(16)** and equate coefficients of $p_0(y)$ and $p_2(y)$ to arrive at the 4-dimensional system of ODEs

$$\begin{aligned} R\dot{u}_0 &= Q(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}(\eta), \\ R\dot{v}_0 &= Q(1-\alpha_2) - A - (B+C)v_0 + C\overline{T}(\eta), \\ R\dot{u}_2 &= Qs_2(1-\alpha_1) - (B+C)u_2, \\ R\dot{v}_2 &= Qs_2(1-\alpha_2) - (B+C)v_2. \end{aligned} \tag{19}$$

What would solutions of system **(19)** tell you about $T(t, y)$?

b) Use the the change of variables $w = \frac{1}{2}(u_0 + v_0)$ and $z = u_0 - v_0$ to show that system **(19)** becomes

$$\begin{aligned} R\dot{w} &= Q(1-\alpha_0) - A - (B+C)w + C\overline{T}(\eta), \\ R\dot{z} &= Q(\alpha_2 - \alpha_1) - (B+C)z, \\ R\dot{u}_2 &= Qs_2(1-\alpha_1) - (B+C)u_2, \\ R\dot{v}_2 &= Qs_2(1-\alpha_2) - (B+C)v_2, \end{aligned} \tag{20}$$

where $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$.

In addition, show that equations **(17)** and **(18)** become

$$\begin{aligned} \overline{T} = \overline{T}(\eta) &= w + \left(\eta - \tfrac{1}{2}\right)z + P_2(\eta)(u_2 - v_2), \\ T(\eta) &= w + \frac{u_2 + v_2}{2}\, p_2(\eta). \end{aligned} \tag{21}$$

3. a) Note that the final three equations in system **(20)** decouple and each is linear! Hence, each of z, u_2, and v_2 can be solved for analytically. Though you do not need to find these analytic solutions, please do describe what happens to each of z, u_2, and v_2 as $t \to \infty$, with justification.

b) We see that, given any initial condition $\big(w(0), z(0), u_2(0), v_2(0)\big)$, as $t \to \infty$ the corresponding solution to system **(20)** satisfies

$$z(t) \to \frac{Q(\alpha_2 - \alpha_1)}{B + C} \equiv z_{\text{eq}}, \tag{22}$$

$$u_2 \to \frac{Qs_2(1-\alpha_1)}{B+C} \equiv u_{2\text{eq}},$$
$$v_2 \to \frac{Qs_2(1-\alpha_2)}{B+C} \equiv v_{2\text{eq}}.$$

Thus, the long-term behavior of solutions to the 4-dimensional system **(20)** reduces to the behavior of w, supposing that $z = z_{\text{eq}}$, $u_2 = u_{2\text{eq}}$, and $v_2 = v_{2\text{eq}}$. Substitute these three values into $\overline{T}(\eta)$ (equation **(21)**) to find

$$\dot{w} = \frac{1}{R}\left\{Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\left[\eta - \frac{1}{2} + s_2P_2(\eta)\right]\right\}, \tag{23}$$

and thereby determine the behavior of $w(t)$ as $t \to \infty$.

Conclude that system **(20)** admits an equilibrium point (depending on η) to which *every* solution converges over time. Interpret this result in terms of the temperature profile **(14)**.

c) Assume that system **(20)** is at equilibrium. Set the parameter values to be as in **Table 1**.

Table 1. Parameter values.

Q	A	B	C	α_1	α_2
343	202	1.9	3.04	0.32	0.62

Use the Mathematica file

`FiniteModel.nb`

to plot the temperature profiles $T(t,y)$ corresponding to this equilibrium point for a variety of η-values. Compare and contrast your plots with those in **Figure 3**.

4. A coupled temperature–ice line model. For any given η, there is an equilibrium solution of system **(20)** with the ice line at η. The temperature at η has been defined as

$$T(\eta) = \frac{1}{2}\left(\lim_{y\to\eta^-} T(t,y) + \lim_{y\to\eta^+} T(t,y)\right).$$

Current data indicate that

ice forms at the ice line (so η decreases) if $T(\eta) < T_c$, and

ice melts at the ice line (so η increases) if $T(\eta) > T_c$,

where the *critical temperature* $T_c = -10°\text{C}$.

Note that the left-most graph in **Figure 3** satisfies $T^*(\eta) < T_c$, while for the middle graph we have $T^*(\eta) > T_c$. In each of these cases, one would expect the ice line to move (equatorward and poleward, respectively).

We incorporate a dynamic ice line into the model by adding the equation

$$\dot{\eta} = \epsilon\big(T(\eta) - T_c\big), \tag{24}$$

where $\epsilon > 0$ is a parameter and $T(\eta)$ is given by **(21)**.

a) Plug $u_{2\text{eq}}$ and $v_{2\text{eq}}$ into **(21)** to get

$$T(\eta) = w + \frac{Qs_2(1-\alpha_0)}{B+C}\, p_2(\eta) \tag{25}$$

(recall that $p(\eta)$ is simply a quadratic polynomial).

b) Explain why the long-term behavior of solutions to the coupled 5-dimensional system comprised of equations **(20)** and **(24)** is completely determined by solutions to the 2-dimensional system

$$\dot{\eta} = \epsilon(T(\eta) - T_c) \tag{26a}$$

$$\dot{w} = \frac{1}{R}\left\{Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\left[\eta - \frac{1}{2} + s_2 P_2(\eta)\right]\right\}, \tag{26b}$$

where $T(\eta)$ is given by **(25)** (note that the right-hand side of equation **(26b)** is just a cubic in η).

c) Use the Mathematica file

`Nullclines.nb`

to plot the nullclines for system **(26a)–(26b)**. How many equilibrium points are there?

d) For each equilibrium point found in **(c)**, plot the corresponding equilibrium temperature profile $T^*(y)$. What can you say about $T^*(\eta)$ in each of these cases?

e) With the aid of the nullclines, determine the type (sink, source, saddle) of each equilibrium point found in **(c)**. With appropriate technology, plot the stable and unstable manifolds of any saddle points.

f) Grand summation! What happens to *all* of the solutions to the coupled 5-dimensional system comprised of equations **(20)** and **(24)** over time? Explain carefully, interpreting your results in terms of the climate model.

6. Solutions to the Exercises

1. Both sides have units W/m^2.

2. The model does not take into account

- the Earth's atmosphere (much less, global warming),
- different insolation at different latitudes,
- variation of insolation with changes in Earth's orbital parameters: obliquity (or tilt) of the spin axis, precession of the spin axis, and eccentricity of the orbit.

3. Good luck with the following!

```
ebm = {T'[t] == (1/R)*(Q*(1 - alpha) - sigma*(T[t])^4),
                T[0] == 0};
soln = DSolve[ebm, T, t]
```

Mathematica will give an answer in terms of an inverse function.

4. a)

```
soln2 = NDSolve[{T'[t] == (1/R)*(Q*(1 - alpha)
                - sigma*(T[t])^4), T[0] == 0},
                T, {t, 0, 1000000000}]
```

b)

```
Plot[T[t] /. soln2, {t, 0, 1000000000},
PlotRange -> {0, 300}, AxesLabel -> {Years, K}]
```

See **Figure S1.** The temperature is tending toward an equilibrium.

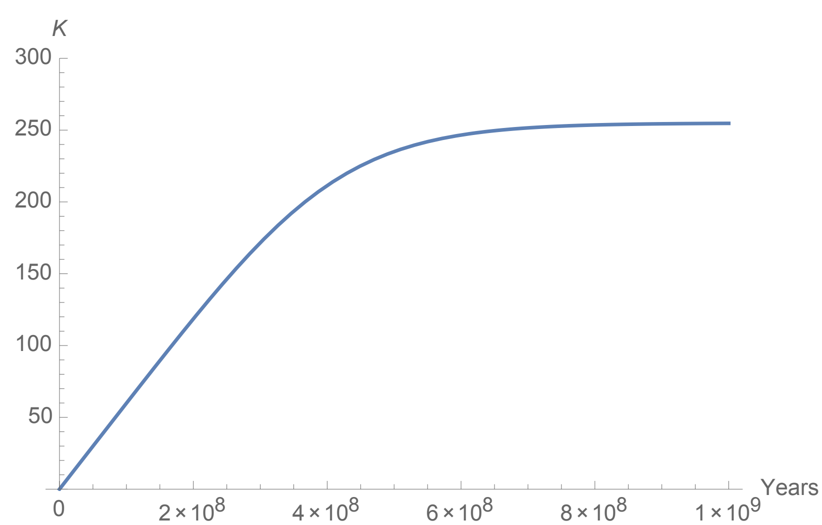


Figure S1. Solution to **Exercise 4b**.

5. Equilibrium occurs when $T' = 0$, or $Q(1-\alpha) = \epsilon\sigma T^4$, hence

$$T^* = \left(\frac{(1-\alpha)Q}{\sigma}\right)^{1/4}.$$

For the given values of the parameters, $T^* = 255\,\mathrm{K} = -18°\mathrm{C} \approx -0.4°\mathrm{F}$. (Recall that 273 K = 0°C = 32°F.)

6.
$$\frac{4.69 \times 10^{23}\ \mathrm{JK}^{-1}}{5.101 \times 10^{14}\ \mathrm{m}^2} = 9.19 \times 10^8\ \mathrm{JK}^{-1}\mathrm{m}^{-2} = 9.19 \times 10^8\ \text{W-sec-K}^{-1}\mathrm{m}^{-2}$$
$$= \frac{9.19 \times 10^8\ \text{W-sec-K}^{-1}\mathrm{m}^{-2}}{3.15 \times 10^8\ \text{sec-yr}^{-1}} = 2.912\ \text{W-yr}/(\mathrm{m}^2\text{-K}).$$

7. At equilibrium, we have $\epsilon = Q(1-\alpha)/(\sigma T^{*4})$. Plugging in the values given yields

$$\epsilon = \frac{342(1-0.3)}{(5.67 \times 10^{-8})(288.4^4)} \approx 0.61,$$

that is, 61% of the OLR would have to escape the Earth's atmosphere and radiate to space.

8. **a)** As temperature increases, so does blackbody radiation via the Stefan-Boltzmann Law. Since $A + BT$ models the radiation that the Earth emits to space, we want this term to increase as T increases (even though this is an empirically-determined term). Hence the requirement that $B > 0$.

b) Note that this equation is linear:

$$T' + \frac{B}{R}T = K, \qquad \text{where } K = \frac{1}{R}\big(Q(1-\alpha) - A\big).$$

Here $\rho(t) = \exp\left(\int (B/R)\,dt\right) = \exp\big((B/R)t\big)$. Multiplying by $\rho(t)$ yields

$$T'e^{(B/R)t} + \frac{B}{R}e^{(B/R)t} = Ke^{(B/R)t}, \text{ that is,} \frac{d}{dt}\left(Te^{(B/R)t}\right) = Ke^{(B/R)t}.$$

Integration yields

$$Te^{(B/R)t} = K\frac{R}{B}e^{(B/R)t} + c,$$

or

$$T(t) = K\frac{R}{B} + ce^{-(B/R)t} = \frac{1}{B}\big(Q(1-\alpha) - A\big) + ce^{-(B/R)t}.$$

Since $B > 0$ and $R > 0$, we see that

$$\lim_{t\to\infty} T(t) = \frac{1}{B}\big(Q(1-\alpha) - A\big)$$

for any $c \in \mathbb{R}$.

c) **(i)** The equilibrium solution T^* is given by

$$\begin{aligned} T^* &= \frac{1}{B}\big(Q(1-\alpha) - A\big) \\ &= \frac{1}{1.9}\big(342(1-0.3) - 202\big) \approx 19.7°\text{C} \approx 67.5°\text{F}. \end{aligned}$$

This temperature is fairly close to the actual average of 15.4°C ≈ 59.7°F. (though a bit warmer). That makes sense, since this model attempts to take into consideration the role our atmosphere plays in determining surface temperature.

(ii) $T^* = \frac{1}{2.09}\big(342(1-0.3) - 203.3\big) \approx 17.3°\text{C} \approx 63.1°\text{F}$, once more slightly warmer.

(iii) (Assuming that $Q(1-\alpha) - A > 0$.) If A decreases, then T^* increases. This makes sense, since a reduction in outgoing longwave radiation $A + BT$ should lead to a warmer planet. Similarly, if A increases, then T^* decreases—an increase in outgoing longwave radiation $A + BT$ leads to a colder planet.

Thus, A can be interpreted as a proxy for atmospheric greenhouse gas concentrations, although when the former increases, the latter decreases, and when the former decreases, the latter increases.

(iv) If $\alpha = \alpha_w = 0.32$, then

$$T^* = \frac{1}{1.9}\big(342(1-0.3) - 202\big) \approx 16°\text{C} \approx 60.8°\text{F}.$$

If the Earth was ice-free at $t = 0$, then $T(0) > -10°$C. From part **(b)**, the solution to this initial value problem (IVP) tends to $T^* \approx 16°$C as $t \to \infty$, implying that the Earth remains ice-free for all time.

(v) If $\alpha = \alpha_s = 0.62$, then

$$T^* = \frac{1}{1.9}\big(342(1-0.62) - 202\big) \approx -37.9°\text{C} \approx -36°\text{F}.$$

If the Earth was covered in ice at $t = 0$, then $T(0) < -10°$C. From part **(b)**, the solution to this IVP tends to $T^* \approx -37.9°$C as $t \to \infty$, implying that the Earth remains in a snowball state for all time.

9. a) $\lim_{T\to\infty}\left[0.5+0.2\tanh\bigl(0.1(265-T)\bigr)\right]=0.5+0.2(-1)=0.3.$ Warmer temperatures imply less ice and snow on the surface of the planet, hence a smaller albedo value.

Similarly,

$$\lim_{T\to-\infty}\left[0.5+0.2\tanh\bigl(0.1(265-T)\bigr)\right]=0.5+0.2(1)=0.7;$$

colder temperatures imply more ice and snow on the surface of the planet, and hence a larger albedo value.

b) Pictured in **Figure S2**, from bottom to top, are plots of $f(T)$ for increasing Q-values; note the sequence of the number of equilibrium points (solutions of $f(T)=0$): 1, 2, 3, 2, 1. Thus, two bifurcations occur, the first from one to three equilibrium points, and the second from three equilibrium points to one.

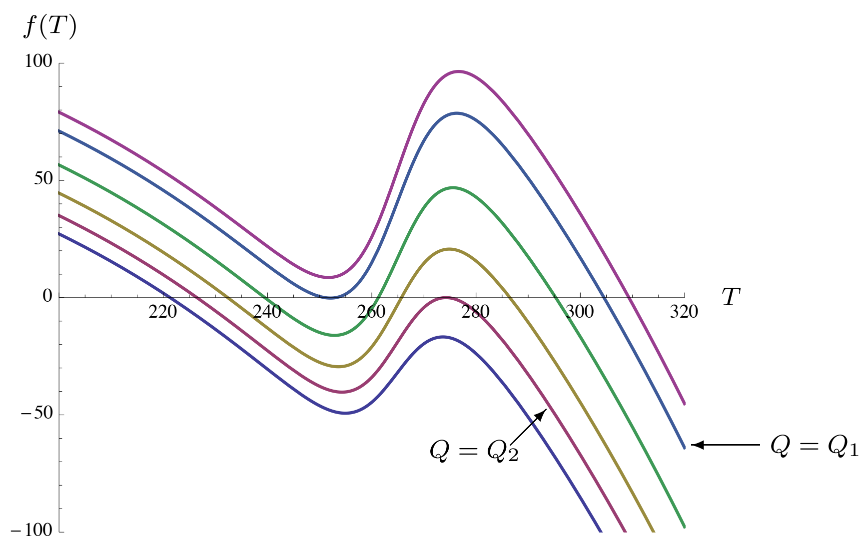


Figure S2. Solution to **Exercise 7b.**

c) (i) Solving $f(T)=0$, we find

$$Q=Q(T)=\frac{\epsilon\sigma T^4}{1-\alpha(T)}. \qquad \textbf{(27)}$$

(ii) We compute $f'(T)=-Q\alpha'(T)-4\epsilon\sigma T^3$. Plugging the above expression in for Q and setting the result equal to 0 gives

$$-\frac{\epsilon\sigma T^4}{1-\alpha(T)}\alpha'(T)-4\epsilon\sigma T^3=-\epsilon\sigma T^3\left(\frac{T}{1-\alpha(T)}\alpha'(T)+4\right)=0. \qquad \textbf{(28)}$$

(iii) Execution of the `FindRoot` command gives the values $T_1 = 252.063\,\mathrm{K}$ and $T_2 = 274.234\,\mathrm{K}$ as positive solutions of equation **(28)**.

(iv) Via equation **(27)**, we find

$$Q_1 = Q(T_1) = 418.7, \qquad Q_2 = Q(T_2) = 298.1.$$

For Q slightly larger than Q_1 (as with the top curve in **Figure S2**), there is but one equilibrium temperature $T = T_3 > 305\,\mathrm{K}$, corresponding to an extremely hot planet. This makes sense in terms of the model, given that Q_1 is very large in this case. Note that $f'(T_3) < 0$, implying that the equilibrium point $T = T_3$ is a sink by the Linearization Theorem for first-order autonomous ODEs [Blanchard et al. 2012, 86].

For Q slightly below Q_1 or slightly above Q_2 (as with the green or gold middle curves in **Figure S2**), there are three equilibrium points $T_1 < T_2 < T_3$. By the Linearization Theorem, we see that $T = T_1$ and $T = T_3$ are sinks (note that $f'(T_1) < 0$ and $f'(T_3) < 0$), while $T = T_2$ is a source (note that $f'(T_2) > 0$). Hence, as Q decreases through Q_1, a second stable equilibrium solution T_1 appears, with T_1 between roughly $T = 225\,\mathrm{K}$ and $T = 250\,\mathrm{K}$. Thus, the temperature of the planet can tend to either the warmer $T = T_3$ world (if $T(0) > T_2$), or to the much colder $T = T_1$ world (if $T(0) < T_2$). This phenomenon is know as *bistability*.

For Q just below Q_2 (as with the bottom curve in **Figure S2**), there is but one equilibrium temperature $T = T_1 < 225\,\mathrm{K}$, corresponding to a snowball Earth. Note that $T = T_1$ is a sink by the Linearization Theorem. Thus, as Q decreases sufficiently, the planet's temperature inexorably tends to $T = T_1$. This makes sense (with current atmospheric conditions given by $\epsilon = 0.6$), since $Q_2 = 298.1$ is far below today's insolation value of $Q = 342\ \mathrm{W/m^2}$.

d) See **Figure S3**.

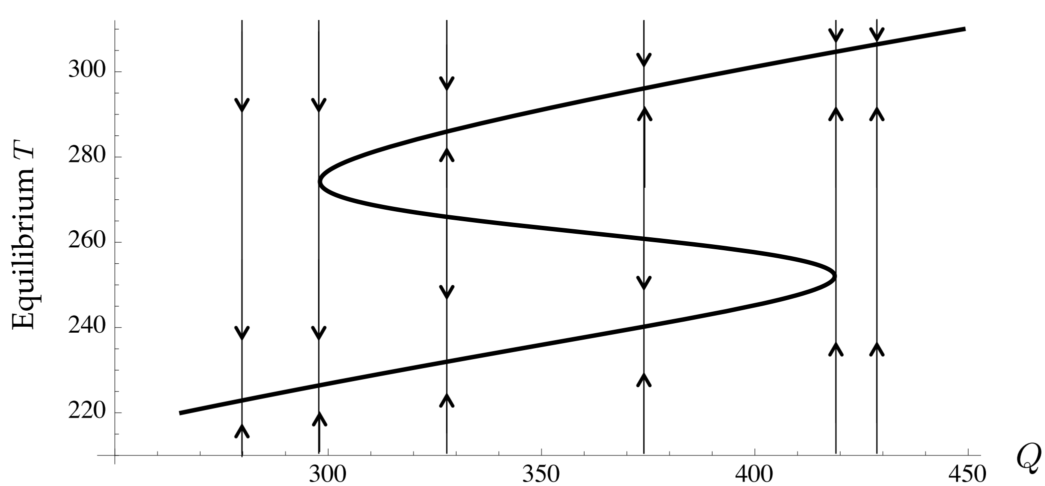


Figure S3. Solution to **Exercise 9d**.

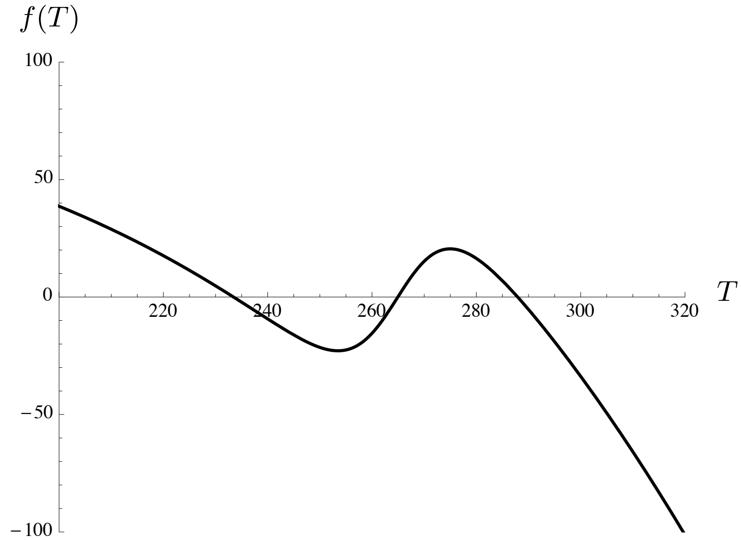


Figure S4. Solution to **Exercise 9e**.

e) From **Figure S3**, we see that temperatures approach the single equilibrium temperature $T = T_1 = 223\,\mathrm{K}$ when $Q = 280$. Thus, with atmospheric transmissivity set to $\epsilon = 0.6$, the Earth would be completely covered with ice for this Q-value.

Suppose that we increase the concentration of atmospheric CO_2, thereby decreasing ϵ to $\epsilon = 0.5$. The corresponding plot of $f(T)$ is shown in **Figure S4**. We see that the decrease in ϵ has warmed the planet up; in particular, as long as $T(0) > T_2 \approx 265\,\mathrm{K}$, the temperature will tend to a very pleasant $T_3 \approx 287\,\mathrm{K}$ over time.

7. Solutions to the Sample Exam Problem

a) Whenever the $E_{\text{in}}(T)$ curve is above the $E_{\text{out}}(T)$ curve, $dT/dt > 0$ and temperature increases. Whenever the $E_{\text{in}}(T)$ curve is below the $E_{\text{out}}(T)$ curve, $dT/dt < 0$ and temperature decreases. A point of intersection $T = T^*$ of these two curves satisfies $E_{\text{in}}(T^*) = E_{\text{out}}(T^*)$, that is, $T = T^*$ is an equilibrium solution.

b) See **Figure S5**.

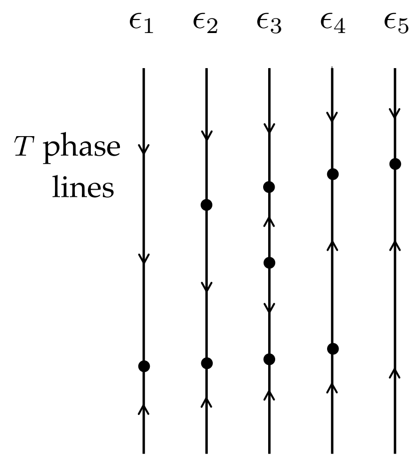


Figure S5. Solution to **Sample Exam Problem 1b**.

c) At $\epsilon = \epsilon_1$, there is a single equilibrium solution $T = T_1$ representing a very cold planet, and $T = T_1$ is a sink.

As ϵ decreases to ϵ_2 (so the concentration of greenhouse gases, such as CO_2, *increases*), an equilibrium solution $T = T_3$ is born.

At $\epsilon = \epsilon_2$, $T = T_3$ is a node and represents a much warmer world.

At $\epsilon = \epsilon_2$, $T = T_1$ remains a sink, and any solution with $T_1 < T(0) < T_3$ will tend to $T = T_1$ over time.

By the time ϵ has decreased to $\epsilon = \epsilon_3$, so that the concentration of CO_2 has continued to increase, $T = T_3$ has become a sink, while a third equilibrium solution $T = T_2$ now exists with $T_1 < T_2 < T_3$. Note that $T = T_2$ is a source. The node when $\epsilon = \epsilon_2$, has split into a source/sink pair as ϵ decreases through $\epsilon = \epsilon_2$.

Thus, as the levels of greenhouse gases increase, a sink equilibrium solution is born that represents a very warm world. Any solution with $T(0) > T_2$ will now approach $T = T_3$ over time.

8. Solutions to the Project Problems

1. We are given

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}), \tag{11}$$

$$T(t,y) = \begin{cases} U(t,y), & y<\eta \\ V(t,y), & y>\eta \\ (U(t,\eta)+V(t,\eta)/2, & y=\eta, \end{cases} \tag{14}$$

and

$$\begin{aligned} U(t,y) &= u_0(t)p_0(y) + u_2(t)p_2(y) \\ V(t,y) &= v_0(t)p_0(y) + v_2(t)p_2(y). \end{aligned} \tag{15}$$

Recall that $p_0(y) = 1$ for all y, $p_2(y) = \frac{1}{2}(3y^2-1)$, and $s(y) = s_0p_0(y) + s_2p_2(y)$, with $s_0 = 1$, $s_2 = -0.482$. We also have the albedo function

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y<\eta; \\ \alpha_2, & y>\eta; \\ \frac{1}{2}(\alpha_1+\alpha_2), & y=\eta, \end{cases} \tag{9}$$

where $\alpha_1 < \alpha_2$.

a) Noting that $T(t,y) = U(t,y)$ for $y<\eta$ and $T(t,y) = V(t,y)$ for $y > \eta$, substitution into **(11)** yields

$$\begin{aligned} R\frac{\partial}{\partial t}U(t,y) &= Qs(y)(1-\alpha_1) - (A+BU) - C(U-\overline{T}), & 0 \le y < \eta \\ R\frac{\partial}{\partial t}V(t,y) &= Qs(y)(1-\alpha_2) - (A+BV) - C(V-\overline{T}), & \eta < y \le 1. \end{aligned} \tag{29}$$

b) Given $P_2(\eta) = \int_0^\eta p_2(y)\,dy = \frac{1}{2}(\eta^3 - \eta)$, we have

$$\begin{aligned} \overline{T} &= \int_0^1 T(t,y)\,dy = \int_0^\eta U(t,y)\,dy + \int_\eta^1 V(t,y)\,dy \\ &= \int_0^\eta \big(u_0 + u_2p_2(y)\big)\,dy + \int_\eta^1 \big(v_0 + v_2p_2(y)\big)\,dy \\ &= \big(u_0\eta + u_2P_2(\eta)\big) + \big(v_0(1-\eta) + v_2(P_2(1) - P_2(\eta)\big) \\ &= \eta u_0 + (1-\eta)v_0 + P_2(\eta)(u_2 - v_2), \text{ as desired.} \end{aligned} \tag{30}$$

c) Via **(14)** and **(15)**, and suppressing the dependence on t, we have

$$T(t,\eta) = \frac{1}{2}\big(U(\eta) + V(\eta)\big) = \frac{1}{2}(u_0 + v_0) + \frac{1}{2}(u_2 + v_2)p_2(\eta). \quad \textbf{(31)}$$

2. a) Upon substitution into **(16)**, we have

$$\begin{aligned} R(\dot{u}_0 p_0 + \dot{u}_2 p_2) &= Qs(y)(1-\alpha_1) - \big(A + B(u_0 p_0 + u_2 p_2)\big) \\ &\qquad - C\big((u_0 p_0 + u_2 p_2) - \overline{T}\big), \\ R(\dot{v}_0 p_0 + \dot{v}_2 p_2) &= Qs(y)(1-\alpha_2) - \big(A + B(v_0 p_0 + v_2 p_2)\big) \\ &\qquad - C\big((v_0 p_0 + v_2 p_2) - \overline{T}\big). \end{aligned} \quad \textbf{(32)}$$

Recall that $s(y) = 1 \cdot p_0 + s_2 p_2$, and note that $A = Ap_0$ and $\overline{T} = \overline{T}p_0$. Equating coefficients of p_0 in the top equation in **(32)** yields

$$R\dot{u}_0 = Q(1-\alpha_1) - (A + Bu_0) - C(u_0 - \overline{T}), \quad \textbf{(33)}$$

while equating coefficients of p_2 in the top equation in **(32)** yields

$$R\dot{u}_2 = Qs_2(1-\alpha_1) - Bu_2 - Cu_2. \quad \textbf{(34)}$$

Equating coefficients of p_0 in the bottom equation in **(32)** yields

$$R\dot{v}_0 = Q(1-\alpha_2) - (A + Bv_0) - C(v_0 - \overline{T}), \quad \textbf{(35)}$$

while equating coefficients of p_2 in the bottom equation in **(32)** yields

$$R\dot{v}_2 = Qs_2(1-\alpha_2) - Bv_2 - Cv_2. \quad \textbf{(36)}$$

Note that equation **(33)–(36)** can be placed in the form

$$\begin{aligned} R\dot{u}_0 &= Q(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}, \\ R\dot{v}_0 &= Q(1-\alpha_2) - A - (B+C)v_0 + C\overline{T}, \\ R\dot{u}_2 &= Qs_2(1-\alpha_1) - (B+C)u_2, \\ R\dot{v}_2 &= Qs_2(1-\alpha_2) - (B+C)v_2. \end{aligned} \quad \textbf{(19)}$$

Solving system **(19)** for $u_0(t)$, $u_2(t)$, $v_0(t)$, and $v_2(t)$ would provide us with expressions for $U(t,y)$ and $V(t,y)$, and hence we would know the temperature profile $T(t,y)$.

b) Given the change of variables $w = \frac{1}{2}(u_0 + v_0)$ and $z = u_0 - v_0$, note that:

$$\begin{aligned} R\dot{w} &= \tfrac{1}{2}(R\dot{u}_0 + R\dot{v}_0) \\ &= \tfrac{1}{2}\big(Q(1-\alpha_1) - A - (B+C)u_0 + C\overline{T} + Q(1-\alpha_2) \\ &\qquad\qquad - A - (B+C)v_0 + C\overline{T}\big) \end{aligned}$$

$$= Q(1-\alpha_0) - A - (B+C)w + C\overline{T}, \tag{37}$$

where $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$.

Similarly,

$$\begin{aligned} R\dot{z} &= R\dot{u}_0 - R\dot{v}_0 \\ &= \big(Q(1-\alpha_1) - A - (B+C)u_0 + C\overline{T}\big) - \big(Q(1-\alpha_2) \\ &\qquad - A - (B+C)v_0 + C\overline{T}\big) \\ &= Q(\alpha_2 - \alpha_1) - (B+C)z. \end{aligned}$$

We see that this change of variables converts system **(19)** into the system

$$R\dot{w} = Q(1-\alpha_0) - A - (B+C)w + C\overline{T}, \tag{37a}$$
$$R\dot{z} = Q(\alpha_2 - \alpha_1) - (B+C)z, \tag{37b}$$
$$R\dot{u}_2 = Qs_2(1-\alpha_1) - (B+C)u_2, \tag{37c}$$
$$R\dot{v}_2 = Qs_2(1-\alpha_2) - (B+C)v_2. \tag{37d}$$

In addition, $w = \frac{1}{2}(u_0 + v_0)$ and $z = u_0 - v_0$ imply that $v_0 = w - \frac{1}{2}z$.

The equation for $\overline{T}$ from **Problem 1b** becomes

$$\begin{aligned} \overline{T} &= \eta u_0 - \eta v_0 + v_0 + P_2(\eta)(u_2 - v_2) \\ &= \eta z + (w - \tfrac{1}{2}z) + P_2(\eta)(u_2 - v_2) \\ &= w + \big(\eta - \tfrac{1}{2}\big)z + P_2(\eta)(u_2 - v_2). \end{aligned} \tag{38}$$

Equation **(31)** becomes

$$T(t,\eta) = w + \frac{u_2 + v_2}{2}\, p_2(\eta). \tag{39}$$

3. a) Our **(37b)** has a unique equilibrium solution $z_{\text{eq}} = Q(\alpha_2 - \alpha_1)/(B+C)$. Moreover, recalling that $B + C > 0$, we realize that z_{eq} is a sink by the Linearization Theorem for first-order ODEs. Given that **(37b)** is linear, we conclude that all solutions to **(37b)** tend to z_{eq} as $t \to \infty$. Similarly, all solutions of equation **(37c)** tend to $u_{2\text{eq}} = Qs_2(1-\alpha_1)/(B+C)$, while all solutions of equation **(37d)** tend to $v_{2\text{eq}} = Qs_2(1-\alpha_2)/(B+C)$, as $t \to \infty$.

b) Given any initial condition $\big(w(0), z(0), u_2(0), v_2(0)\big)$, the corresponding solution approaches the line

$$\Gamma = \{(w, z, u_2, v_2) : z = z_{\text{eq}}, u_2 = u_{2\text{eq}}, v_2 = v_{2\text{eq}}\}, \tag{40}$$

by **Problem 3a**. On Γ, **(38)** becomes

$$\begin{aligned}\overline{T} &= w + (\eta - \tfrac{1}{2})\,\frac{Q(\alpha_2-\alpha_1)}{B+C} + P_2(\eta)\left(\frac{Qs_2(1-\alpha_1)}{B+C} - \frac{Qs_2(1-\alpha_2)}{B+C}\right)\\ &= w + (\eta - \tfrac{1}{2})\,\frac{Q(\alpha_2-\alpha_1)}{B+C} + P_2(\eta)\left(\frac{Qs_2(\alpha_2-\alpha_1)}{B+C}\right)\\ &= w + \frac{Q(\alpha_2-\alpha_1)}{B+C}\,\big(\eta - \tfrac{1}{2} + s_2P_2(\eta)\big). \qquad \textbf{(41)}\end{aligned}$$

Plugging **(41)** into **(37a)**, we have that

$$\begin{aligned}\dot{w} &= \frac{1}{R}\left\{Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\,\big(\eta - \tfrac{1}{2} + s_2P_2(\eta)\big)\right\}\\ &= -\frac{B}{R}\,(w - F(\eta)), \qquad \textbf{(42)}\end{aligned}$$

where $F(\eta)$ is the cubic polynomial

$$F(\eta) = \frac{1}{B}\left[Q(1-\alpha_0) - A + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\,\big(\eta - \tfrac{1}{2} + s_2P_2(\eta)\big)\right]. \qquad \textbf{(43)}$$

The only variable in **(42)** is w; we see then that this is a linear equation in w. Noting that $B > 0$, we have that the unique equilibrium solution

$$\begin{aligned}w_{\text{eq}} = F(\eta) = \frac{1}{B}\bigg[&Q(1-\alpha_0) - A\\ &+ \frac{CQ(\alpha_2-\alpha_1)}{B+C}\,\Big(\eta - \frac{1}{2} + s_2P_2(\eta)\Big)\bigg] \qquad \textbf{(44)}\end{aligned}$$

is a sink by the Linearization Theorem for first-order ODEs.

In summary, every solution to system **(37)** tends to Γ over time. All of the solutions on Γ tend over time to the equilibrium point $H^* = \big(w_{\text{eq}}, z_{\text{eq}}, u_{2\text{eq}}, v_{2\text{eq}}\big)$. Hence, H^* is *globally attracting*, that is, every solution to system **(37)** converges to H^* as $t \to \infty$.

Note that $w = \frac{1}{2}(u_0 + v_0)$ and $z = u_0 - v_0$ imply that $u_0 = w + \frac{1}{2}z$ and $v_0 = w - \frac{1}{2}z$. Thus, independent of initial conditions, $u_0(t) \to w_{\text{eq}} + \frac{1}{2}z_{\text{eq}} \equiv u_{0\text{eq}}$ and $v_0(t) \to w_{\text{eq}} - \frac{1}{2}z_{\text{eq}} \equiv v_{0\text{eq}}$ as $t \to \infty$.

In terms of the model, this result implies the following. Start with any piecewise quadratic function $T(t, y)$ having a discontinuity at η. Then as $t \to \infty$, we have

$$T(t,y) \to T^*(y) = \begin{cases} u_{0\text{eq}} + u_{2\text{eq}}p_2(y), & y < \eta\\ v_{0\text{eq}} + v_{2\text{eq}}p_2(y), & y > \eta. \end{cases} \qquad \textbf{(45)}$$

That is, there is a globally attracting equilibrium temperature profile $T^*(y)$.

Thus, for a given η, *the climate system tends toward an equilibrium temperature distribution that is latitudinally-dependent and has an ice line at* η.

c) Given the parameter values, we plot a few $T(t, y)$ graphs at equilibrium (that is, $T^*(y)$) in **Figure S6**. (Note: These are the same as the graphs plotted in **Figure 3**.)

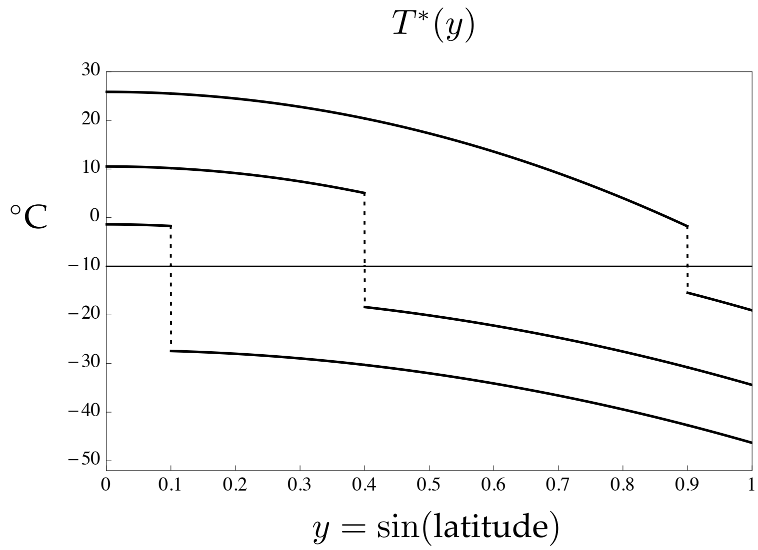


Figure S6. Equilibrium solutions $T^*(y)$ of equation **(11)** are piecewise quadratic. Top: $\eta = 0.1$. Middle: $\eta = 0.4$. Bottom: $\eta = 0.9$.

4. We now consider a dynamic equation for the ice line by adding the equation

$$\dot{\eta} = \epsilon\big(T(t, \eta) - T_c\big), \tag{24}$$

where $\epsilon > 0$ is a parameter and $T(t, \eta)$ is given by **(39)**. We couple **(24)** with system **(37)**.

a) Plugging $u_{2\text{eq}}$ and $v_{2\text{eq}}$ into **(39)** yields

$$\begin{aligned} T(t, \eta) &= w + \frac{1}{2}\left(\frac{Qs_2(1-\alpha_1)}{B+C} + \frac{Qs_2(1-\alpha_2)}{B+C}\right) p_2(\eta) \\ &= w + \frac{Qs_2(1-\alpha_0)}{B+C}\, p_2(\eta), \end{aligned} \tag{46}$$

where $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$. Note that this implies in particular that $\dot{\eta}$ depends on both w and η, that is,

$$\dot{\eta} = \epsilon\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}\, p_2(\eta) - T_c\right) = \epsilon\big(w - G(\eta)\big), \tag{47}$$

where $G(\eta)$ is the quadratic polynomial

$$G(\eta) = -\frac{Q}{B+C}\, s_2(1-\alpha_0)p_2(\eta) + T_c. \tag{48}$$

b) Consider the 5-dimensional system given by **(24)** coupled with system **(37)**. The discussion presented in the solution to **Problem 3a** still holds for this system. Hence, given any initial condition

$$\big(\eta(0), w(0), z(0), u_2(0), v_2(0)\big),$$

the corresponding solution approaches the plane

$$\Lambda = \{(\eta, w, z, u_2, v_2) : z = z_{\text{eq}}, u_2 = u_{2\text{eq}}, v_2 = v_{2\text{eq}}\}. \tag{49}$$

On Λ, the behavior of solutions is governed by the two equations **(47)** and **(42)**. That is, to determine the long-term behavior of solutions to the 5-dimensional system, it suffices to determine the behavior of solutions to the system

$$\begin{aligned} \dot{\eta} &= f(\eta, w) = \epsilon\big(w - G(\eta)\big) \\ \dot{w} &= g(\eta, w) = -\tfrac{B}{R}\big(w - F(\eta)\big), \end{aligned} \tag{50}$$

for $(\eta, w) \in \Lambda$, where we restrict η to be in $[0, 1]$.

c) **Figure S7** shows the nullclines and vector field for system **(50)** when $\epsilon = 1$.

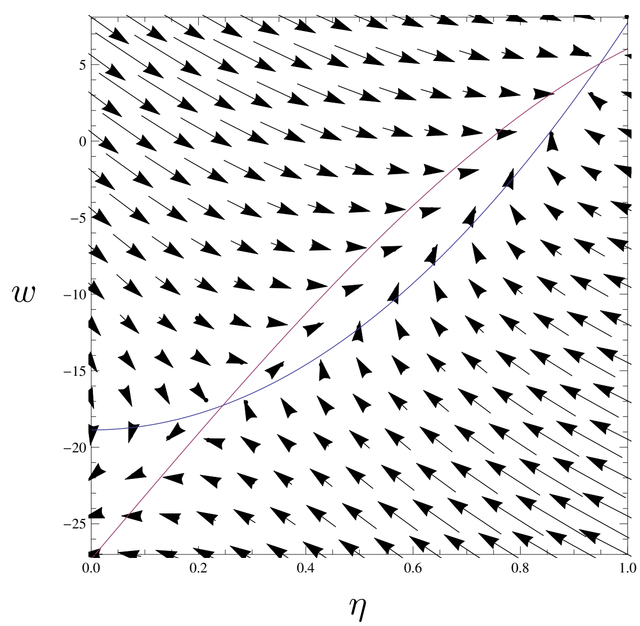


Figure S7. The vector field and nullclines, generated with Mathematica, for system **(50)**.

The η-nullcline is the concave-upward (blue) curve, while the w-nullcline is the concave-downward (red) curve. There are two equilibrium points, (η_1, w_1) and (η_2, w_2), $\eta_1 < \eta_2$, given by the two points of intersection of the nullclines. Mathematica gives $(\eta_1, w_1) \approx (0.246, -17.265)$ and $(\eta_2, w_2) \approx (0.949, 5.080)$.

d) In **Figure S8**, we plot $T^*(y)$ when $\eta = \eta_1$ and when $\eta = \eta_2$. We see that for $i = 1$ and $i = 2$, we get $T^*(\eta_i) = -10°\text{C}$, as it should be, given **(24)** and the fact that the system is at equilibrium. Thus, there are but two ice line positions that satisfy for which the temperature at the ice line at equilibrium equals the critical temperature. One has a very large ice cap ($\eta = \eta_1$), while the other has a small ice cap ($\eta = \eta_2$).

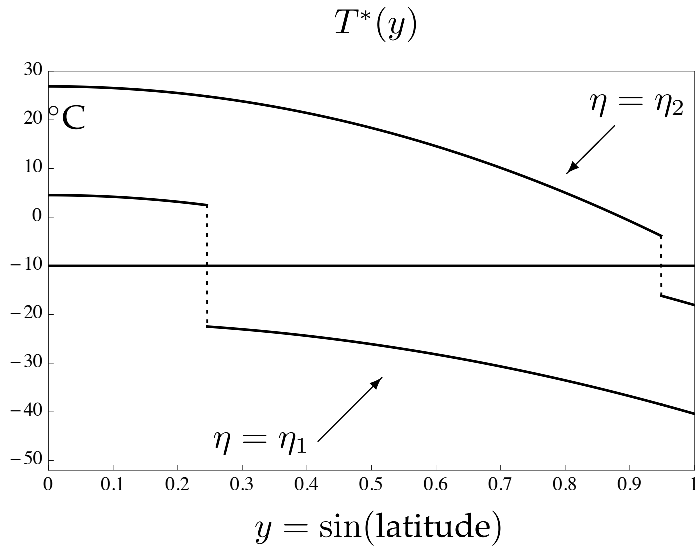


Figure S8. There are two ice line positions for which the temperature at the ice line at equilibrium equals the critical temperature.

e) Via consideration of the (η, w)-vector field plotted above, we see that (η_1, w_1) is a saddle, while (η_2, w_2) is a sink. Alternatively, one can use the free downloadable Java software `pplane` [Polking et al. n.d.] to plot the nullclines and the stable and unstable manifolds (*separatrices*) associated with the saddle (η_1, w_1) (see **Figure S9**).

f) We see that the behavior of solutions is determined by the stable separatrix S of the saddle point (η_1, w_1) (the green concave-upward curve on the left in **Figure S9**). The solution through any (η, w) above S will converge to the sink (η_2, w_2) over time. That is, the ice line converges to a position near the North Pole, similar to what we have today on Earth, while the temperature approaches the corresponding $\eta = \eta_2$ equilibrium $T^*(y)$ in **Figure S9**.

The solution through any (η, w) below S will satisfy $\eta(t) \to 0$, so that the climate system approaches the snowball Earth state.

Including the ice line equation makes for a much improved model!

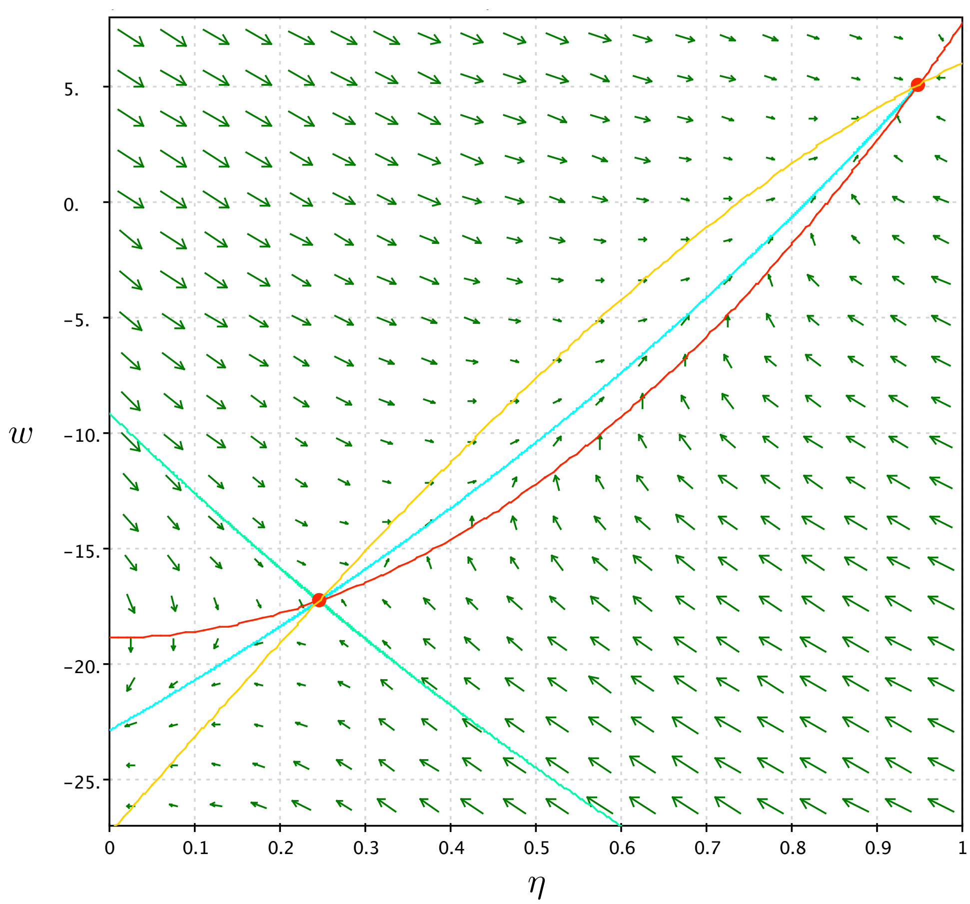


Figure S9. The phase plane for system **(50)** generated with **pplane**.
Concave upward curve on right (red): η-nullcline.
Concave-downward curve (yellow): w-nullcline.
Concave-upward curve on left (green): Stable separatrix for the saddle point.
Almost straight-line curve (cyan): Unstable separatrix for the saddle point.

Appendix: Mathematica Programs

QBifurcation.nb

```
(* This inputs constants and defines the albedo
   function alpha. Be sure to select cells and
   hit "enter."*)
eps = 0.6;
sig = 5.67 * 10^(-8);
k = eps * sig; (* defining k just for convenience *)
alpha[T_] := 0.5 + 0.2 * Tanh[.1 * (265 - T)]

(* This defines the right-hand side of the ODE
   as a function of T and Q *)
Clear[Q]
g[T_, Q_] := Q * (1 - alpha[T]) - k * T^4

(* This plots g for Q-values 200 and 480.
   You can add other plots by adding another g[T,Q]
   (with a specific Q-value) *)
Plot[{g[T, 200], g[T, 480]}, {T, 200, 320},
 AxesOrigin → {200, 0}, PlotRange → {{200, 320}, {-100, 120}},
 AspectRatio → 3 / 4, PlotStyle → Thickness[0.005],
 AxesLabel → {T, "g(T)"}]

(* This command finds a solution to equation (4) on
   the HW handout via Newton's Method,
   with an initial guess of T=TINITIAL
   (which you should supply!).
   To find other solutions, appropriately change
   the initial guess TINITIAL *)
FindRoot[4 + (T / (1 - alpha[T])) *
    (-0.02 * Sech[0.1 * (265 - T)]^2) == 0, {T, TINITIAL}]

(* Bifurcation plot Teq vs. Q *)
q[T_] := eps * sig * T^4 / (1 - alpha[T]) (*eq(3) on HW handout*)
eps = .6;
ParametricPlot[{q[T], T}, {T, 220, 310},
 AxesOrigin → {250, 210}, AxesLabel → {Q, "Equilibrium T"}]
Clear[eps]
```

FiniteModel.nb

```
(*Input various parameters and expressions*)
Q = 343;
A = 202;
B = 1.9;
C1 = 3.04;
alph1 = .32;
alph2 = .62;
alph0 = 0.5 * (alph2 + alph1);
Tc = -10;
P2[eta_] := 0.5 * (eta^3 - eta);
s2 = -0.482;
p2[y_] := .5 * (3 * y^2 - 1);

(*Input the expressions for the equilibrium values*)
zeq = Q * (alph2 - alph1) / (B + C1);
u2eq = Q * s2 * (1 - alph1) / (B + C1);
v2eq = Q * s2 * (1 - alph2) / (B + C1);
weq[eta_] := (1 / B) * (Q * (1 - alph0) - A + C1 * Q * (alph2 - alph1) *
      (eta - .5 + s2 * 0.5 * (eta^3 - eta)) / (B + C1))

(*Compute u0(t) and v0(t) at equilibrium*)
u0eq[eta_] := .5 * (2 * weq[eta] + zeq);
v0eq[eta_] := .5 * (2 * weq[eta] - zeq);

(*Define the equilibrium solution T* in terms of U(t,y) & V(t,y)*)
Tstar[y_, eta_] := If[y < eta, u0eq[eta] + u2eq * p2[y],
  v0eq[eta] + v2eq * p2[y]]

(*Inpute eta and plot T* for that eta value*)
eta = .1;
Plot[{Tstar[y, eta], -10}, {y, 0, 1}, PlotRange -> {{0, 1}, {-52, 30}},
 AxesOrigin → {0, -52}, Frame → True, Exclusions → {0.1, 0.4, 0.9 },
 ExclusionsStyle → Dotted, FrameTicks →
  {  {{-50, -40, -30, -20, -10, 0, 10, 20, 30}, None},
   {{0, 0.1, .2, .3, .4, .5, .6, .7, .8, .9, 1}, None}}, PlotStyle →
  {{Black, Thickness[0.004]}, {Black, Thickness[0.004]},
   {Black, Thickness[0.004]}, {Black, Thickness[0.002]}}]
Clear[eta]
```

Nullclines.nb

```
(* Input various parameters and expressions *)
Q = 343;
A = 202;
B = 1.9;
C1 = 3.04;
alph1 = .32;
alph2 = .62;
alph0 = 0.5 * (alph2 + alph1);
Tc = -10;
P2[eta_] := 0.5 * (eta^3 - eta);
s2 = -0.482;
p2[y_] := .5 * (3 * y^2 - 1);
R = 4.0 * 10^8;

(* Input expressions for the equilibrium values *)
zeq = Q * (alph2 - alph1) / (B + C1);
u2eq = Q * s2 * (1 - alph1) / (B + C1);
v2eq = Q * s2 * (1 - alph2) / (B + C1);
weq[eta_] := (1 / B) * (Q * (1 - alph0) - A +
     C1 * Q * (alph2 - alph1) * (eta - .5 + s2 * 0.5 * (eta^3 - eta)) / (B + C1))

(* Compute u0(t) and v0(t) at equilibrium *)
u0eq[eta_] := .5 * (2 * weq[eta] + zeq);
v0eq[eta_] := .5 * (2 * weq[eta] - zeq);

(* Define the (eta,w) vector field *)
f[eta_, w_] := eps * (w + Q * s2 * (1 - alph0) * p2[eta] / (B + C1) + 10)
g[eta_, w_] := (1 / R) * ( Q * (1 - alph0) - A - B * w + C1 * Q * (alph2 - alph1) *
      (eta - .5 + s2 * P2[eta]) / (B + C1) )

(* This will plot the eta-nullcline and the w-nullcline *)
Plot[{-Q * s2 * (1 - alph0) * p2[eta] / (B + C1) - 10, weq[eta]}, {eta, 0, 1}]

(* This plots the (eta,w) vector field; I had difficulty
   getting the vectors to display nicely*)eps = 1; R = .10;
Show[VectorPlot[{f[eta, w], g[eta, w]},
  {eta, 0, 1 }, {w, -27, 8.1}, VectorScale → {.1, .1},
  VectorStyle → Black, PlotRange → {{0, 1}, {-27, 8.1}}, AspectRatio → 1],
 Plot[{-Q * s2 * (1 - alph0) * p2[eta] / (B + C1) - 10, weq[eta]}, {eta, 0, 1}]
]
Clear[eps, R]
```

Notes for the Instructor

I introduced **(1)** during the second class meeting of the semester in the Fall 2013, Spring 2015, and Fall 2015 offerings of the Differential Equations course at Oberlin College.

I revisited **(1)** when presenting the concept of phase lines and again when I introduced linearization of first-order ODEs at equilibrium points. I invited students to check via either method that the equilibrium value T^* is a sink.

The project concerning a latitudinally-dependent EBM coupled with a dynamic ice sheet [McGehee and Widiasih 2014] was the final assignment of the semester, a two-week group project. My goal was to provide students with an opportunity to experience the extent to which mathematical modeling can be involved—messy at times—and computationally intensive (in addition to enlightening). Due to the amount of work required to complete this project, students were allowed to work with up to two other classmates, with each group submitting one project report.

Assessment

I was interested in learning if the material on climate modeling served to pique students' interest in learning more about mathematical modeling (and the mathematics and science of climate). To that end, students in all three classes filled out brief assessment surveys at the end of the semester. The response scale that I used was

1	2	3	4	5
disagree	somewhat disagree	neutral	somewhat agree	agree

The average response to the statement,

> "The inclusion of material on climate modeling was a positive aspect of this course"

was 3.95 in Fall 2013, 4.5 in Spring 2015, and 4.4 in Fall 2015. The average responses to the statement,

> "The inclusion of material on climate modeling served to increase my desire to learn more about mathematical modeling"

were 3.65, 4.5, and 4.3. During the Spring 2015 and Fall 2015 courses, I made much more of an effort to talk about climate—news items, homework, and exam questions—which I surmise is the reason for the increase in the numbers above.

The average responses to the statement,

> "I would have an interest in taking a mathematics and climate course having Math 234 [the course number] as a prerequisite"

were 3.15, 3.33, and 3.65, fairly close to "neutral." Unfortunately, the statement is somewhat ill-chosen, in that I am unable to deduce whether it is the differential equations prerequisite or the climate models (or both, in part) that spurred students' lukewarm responses to this question. In either case (and as always, it seems, when one tries something new in the classroom), it does indicate that there is room for improvement!

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Acknowledgments

This article arose from exercises on mathematics and climate modeling that I incorporated into my sophomore-level ODE course in the Fall 2013, Spring 2015, and Fall 2015 semesters at Oberlin College

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About the Author

Jim Walsh is currently in his twenty-fifth year in the Mathematics Department at Oberlin College. Jim specializes in applied dynamical systems, with recent focus on the development and analysis of conceptual climate models.