Boundary value problems (DRAFT October 8, 2019)

Introduction

Let $I = (a, b)$ be a non-trivial bounded interval. An important class of differential equations (historically, theoretically, didactically and practically) are of the form

$$-(pu')' + qu = f \qquad \text{on} \qquad I \tag{1}$$

where $u\colon I \to \mathbb{R}$ is the unknown function, while p, q and f are known, sufficiently smooth functions. Typically,

$$p \geq 0 \text{ is non-negative.} \tag{2}$$

Example. Consider $u'' + a_1 x' + a_0 x = g$. Set $p(s) := \exp(\int_a^s a_1(\tau)d\tau)$ and $q := -pa_0$.

To streamline the notation we introduce the differential operator L,

$$Lu := -(pu')' + qu. \tag{3}$$

Thus, Eqn. (1) becomes

$$Lu = f. \tag{4}$$

The equation (1) is linear in u and involves two derivatives. From the ODE theory we expect that the homogeneous equation $L\varphi = 0$ has two (linearly independent) solutions φ_1 and φ_2, forming a so-called fundamental system. If, moreover, φ_0 is a particular solution to the original non-homogeneous equation (4) then so is any combination,

$$L(\varphi_0 + c_1\varphi_1 + c_2\varphi_2) = L\varphi_0 + c_1 L\varphi_1 + c_2 L\varphi_2 = g \qquad \forall c_1, c_2 \in \mathbb{R}. \tag{5}$$

The solution space is two-dimensional, parameterized by the two constants c_1 and c_2. To fully specify the solution, two additional constraints are needed.

The operator L is self-adjoint (almost)

The particular form of L is significant for the following reason, called "Lagrange identity" in a more general context.

Observation. For any two smooth functions u and v,

$$uLv - vLu = -u(pv')' + v(pu')' = (p(u'v - uv'))'. \tag{6}$$

The significance of the observation is further seen by defining the bilinear form

$$B(u, v) := \int_I (Lu)(x)v(x)dx. \tag{7}$$

Flipping L onto v using (6), we have the identity

$$B(u,v) = B(v,u) + \int_I (p(u'v - uv'))'dx, \tag{8}$$

which says that B is almost symmetric but for the last anti-symmetric term. However, if we restrict u and v to the *vector space* of functions that vanish on the boundary of I (for example), the bilinear form *is* symmetric. This unleashes operator-theoretic tools that form the basis of the *finite element method* for boundary value problems like (4).

Boundary conditions I

In *initial value problems*, we specify two conditions on one end of the interval, such as $u(a) := u_0$ and $u'(a) := u_1$. Such equations typically model evolution over time.

In *boundary value problems*, we specify conditions at both ends of the interval, called boundary conditions. For example,

$$\text{homogeneous Dirichlet condition at } a: \qquad D_a u := u(a) \stackrel{!}{=} 0 \tag{9a}$$

$$\text{homogeneous Neumann condition at } b: \qquad N_b u := u'(b) \stackrel{!}{=} 0. \tag{9b}$$

More generally, so-called Robin boundary conditions can be imposed:

$$H_a u := \alpha_0 u(a) + \alpha_1 u'(a) \stackrel{!}{=} \eta_a \tag{10a}$$

$$H_b u := \beta_0 u(b) + \beta_1 u'(b) \stackrel{!}{=} \eta_b \tag{10b}$$

where α_i, β_i and η_x are constants. Note that H_a and H_b are *linear operators* that take a function and return a real number (if defined). In more general boundary value problems posed on d-dimensional domains, the boundary operator returns a function (called *trace* of u) on the $(d-1)$-dimensional boundary.

For the candidate solution in the form $u = \varphi_0 + c_1\varphi_1 + c_2\varphi_2$ we can write the set of boundary conditions in matrix-vector form,

$$\begin{pmatrix} H_a\varphi_1 & H_a\varphi_2 \\ H_b\varphi_1 & H_b\varphi_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \eta_a \\ \eta_b \end{pmatrix} - \begin{pmatrix} H_a\varphi_0 \\ H_b\varphi_0 \end{pmatrix}, \tag{11}$$

providing conditions on the unknonwn coefficients c_1 and c_2. If the determinant of the matrix is non-zero, these coefficients are uniquely determined (but otherwise there may be no solution, one solution or infinitely many solutions).

Example. Take $p := 1$ and $q := -(\pi k)^2$, where $k \in \mathbb{N}$, and $g = 0$, on the interval $(0,1)$ subject to the homogeneous Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 0$. There is no need for the particular solution, in other words $\varphi_0 = 0$. Determine the solution.

Boundary conditions II

In practice, that is in algorithms and software packages for boundary value problems, there are other ways to impose the (linear) boundary conditions (10). A relatively easy way is to decompose $u = \tilde{u} + u_0$ where u_0 is *any* function that satisfies the boundary conditions (10). Then, setting $\tilde{f} := f - Lu_0$, the function $\tilde{u}$ satisties the *homogeneous* boundary conditions $H.\tilde{u} = 0$ and solves the non-homogeneous boundary value problem

$$L\tilde{u} = \tilde{f}. \tag{12}$$

Example. On $I = (-1, 1)$, take

$$Lu := -u'' + u = 0, \quad \text{with} \quad u(\pm 1) = \pm 1. \tag{13}$$

Then $u_0(x) := x$ satisfies the boundary conditions. Therefore, if $\tilde{u}$ solves

$$(L\tilde{u})(x) \stackrel{!}{=} -(Lu_0)(x) = -x \quad \forall x \in (-1, 1), \quad \text{with} \quad \tilde{u}(\pm 1) = 0, \tag{14}$$

then $u := \tilde{u} + u_0$ solves the original problem.

In this way, we do not need the fundamental system $\{\varphi_1, \varphi_2\}$, which is in general hard to find anyway. In the following we therefore focus on homogeneous boundary conditions.

Green's function

We consider boundary value problems $Lu = f$ (4) on an interval I with homogeneous boundary conditions. The Green's function is a device for representing solutions as a superposition of the system responses to localized impulses. This also works for linear evolution and boundary value problems in higher dimensions. It is frequently used in some applications such as electromagnetics and control systems.

Let δ denote the Dirac functional at 0 that formally verifies, for all smooth f,

$$\int_I f(y)\delta(y - x)dy = f(x). \tag{15}$$

Example. Let H be the Heaviside function, $H(x) = 1_{x>0}$. Show that

$$\int_{\mathbb{R}} f(y)H'(y - x)dy = f(x) \quad \forall f \in C^1(\mathbb{R}). \tag{16}$$

The Green's function to the operator L is a scalar-valued function $G\colon I \times I \to \mathbb{R}$ of two variables such that

$$L[G(\cdot, y)](x) = \delta(y - x) \quad \forall x \in I, \tag{17}$$

where we agree that L acts on the first variable of G. Now set

$$u(x) := \int_I G(x, y)f(y)dy. \tag{18}$$

Then, bravely exchanging differentiation in x and integration in y,

$$(Lu)(x) \overset{(18)}{=} \int_I L[G(\cdot,y)](x)f(y)dy \overset{(17)}{=} \int_I \delta(y-x)f(y)dy \overset{(15)}{=} f(x) \quad \forall x \in I. \tag{19}$$

Remark. If $x \mapsto G(x,y)$ satisfies the homogeneous boundary conditions then so does (18).

Green's function for $Lu = -(pu')' + qu$

For the 1d problem (4) with homogeneous boundary conditions a Green's function can be constructed from the fundamental system $\{\varphi_1, \varphi_2\}$.

Recall that $L\varphi_i = 0$. Therefore the Lagrange identity (6) implies that

$$c := (\varphi_1\varphi_2' - \varphi_1'\varphi_2)p \quad \text{is constant.} \tag{20}$$

We assume that this constant is non-zero. Now set

$$G(x,y) := \begin{cases} \frac{1}{c}\varphi_1(x)\varphi_2(y) & \text{if} \quad x \geq y \\ \frac{1}{c}\varphi_2(x)\varphi_1(y) & \text{if} \quad x \leq y. \end{cases} \tag{21}$$

We write G' for the derivative w.r.t. the first variable, which may not exist at $x = y$. Note that $G(\cdot,y)$ and therefore $u =$ (18) satisfy the homogeneous boundary conditions.

Now we verify that G is indeed a Green's function for L by checking (17). Specifically,

$$\int_{y-\varepsilon}^{y+\varepsilon} L[G(\cdot,y)](x)dx \to 1 \quad \text{as} \quad \varepsilon \searrow 0 \qquad \forall y \in I. \tag{22}$$

Since the integrand is zero for $x \neq y$, this implies that it equals $\delta(x-y)$, as in (17).

We assume that φ_i, φ_i' and p are continuous. Fix $y \in I$. Now, for $\varepsilon > 0$ small,

$$\begin{aligned} \text{LHS(22)} &= \int_{y-\varepsilon}^{y+\varepsilon} \{-(pG'(\cdot,y))'(x) + q(x)G(x,y)\}\,dx & \text{(23a)} \\ &= \int_{y+\varepsilon}^{y-\varepsilon} (pG'(\cdot,y))'(x)dx + \mathcal{O}(\varepsilon) & \text{(23b)} \\ &= pG'(x,y)|_{x=y+\varepsilon}^{x=y-\varepsilon} + \mathcal{O}(\varepsilon) & \text{(23c)} \\ &= \frac{1}{c}(p(y-\varepsilon)\varphi_2'(y-\varepsilon)\varphi_1(y) - p(y+\varepsilon)\varphi_1'(y+\varepsilon)\varphi_2(y)) + \mathcal{O}(\varepsilon) & \text{(23d)} \end{aligned}$$

The last expression goes to $\frac{1}{c}p(y)(\varphi_2'\varphi_1 - \varphi_1'\varphi_2)(y)$ as $\varepsilon \searrow 0$, which equals 1 by (20).

Example. TODO: hat/tent

R.A., October 8, 2019