1.1

Consider $C^1(\mathbb{R})$ solutions of the differential equation $\dot{x}(t) = -x(t)$.

- Sketch all solutions in the x-t plane.
- What do they have in common?
- For $T > 0$, how do $x(0) \mapsto x(T)$ and $x(T) \mapsto x(0)$ differ qualitatively?
- Parameterize the family of solution curves implicitly à la $F(x, t, \lambda) = 0$.
- Indicate the "Lie group orbits" $\lambda \mapsto F(x, t, \lambda)$ for various fixed (x, t).
- Find and solve a differential equation for the curve in the x-t plane that goes through $(c, 0)$ and is perpendicular to the solution curves.

1.2

A lake with constant volume V [m^3] is fed by a river with flow rate r [m^3/s]. A factory upstream begins releasing a pollutant into the river at a rate p [m^3/s]. Propose and solve a differential equation for the concentration $c(t)$ [m^3/m^3] of the pollutant in the lake.

1.3

Excerpt from R. Fisher, "The genetical theory of natural selection", 1930, p. 42:

> An increase in numbers of any organism will impair its environment in a manner analogous to, and probably more definitely than, an increase in the numbers or efficiency of its competitors. [..] The situation is represented by the differential equation
>
> $$\frac{dM}{dt} + \frac{M}{C} = W - D \tag{1}$$
>
> in which M is the mean of the Malthusian parameter, C is a constant expressing the relation between fitness and population increase [..], W is the rate of actual increase in fitness determined by natural selection, and D is the rate of loss due to the deterioration of the environment. If C, W and D are constant the equation has the solution
>
> $$M = \frac{W - D}{C} + Ae^{-t/C} \tag{2}$$
>
> in which A is an arbitrary constant, dependent on the initial conditions.

Questions:

- What will the value of M approach to?
- What is the fate of the species in this model?

1.4

Find, if possible, a differential equation "$\dot{x}(t) = \ldots$" with a nontrivial solution in $C^1([0,\infty))$ for which $x(t) = 0$ for all t sufficiently large — i.e. x converges in finite time.

1.5

Let γ, g and $a \geq 0$ be continuous real-valued functions on $[0,T]$.

- Sketch the function $s \mapsto A(s) := \exp(\int_s^T a(\tau)d\tau)$.

Suppose

$$\gamma(t) \leq g(t) + \int_0^t \gamma(s)a(s)ds \quad \forall t \in [0,T]. \tag{3}$$

Prove

- the Gronwall–Bellman inequality:

$$\gamma(T) \leq g(T) + \int_0^T g(s)a(s)A(s)ds. \tag{4}$$

- assuming that g is continuously differentiable, the corollary:

$$\gamma(T) \leq g(0)A(0) + \int_0^T g'(s)A(s)ds. \tag{5}$$

It may be helpful to start with a differential inequality for $x(t) := \int_0^t \gamma(s)a(s)ds$.

1.6

Fix $T > 0$, suppose $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the second variable. Assume there exists a solution to the differential equation $\dot{x}(t) = f(t, x(t))$ on the interval $(0,T)$ for any initial value near some fixed x_0.

- Show that $x(0) \mapsto x(T)$ is Lipschitz continuous. Is your statement sharp?
- Do we need to *assume* existence?
- Suppose f depends on a parameter p. Fix $x_p(0)$. What is the derivative of the solution $x_p(T)$ w.r.t. p? Propose conditions on f for the derivative to exist.

1.7

Develop a solution formula to the initial value problem

$$\dot{x}(t) = g(t)h(x(t)), \qquad x(t_0) = x_0, \tag{6}$$

in terms of the given functions g and h. Apply to

- $\dot{x}(t) = t\,x(t)$,
- $\dot{x}(t) = t/x(t)$,
- $\dot{x}(t) = x^2(t)$,
- $\dot{x}(t) = \sqrt{x(t)}$,
- $\dot{x}(t) = (1+t^2)(1+x^2(t))$,
- $\dot{x}(t) = x(t)\log x(t)$.

What is the maximal temporal interval of existence in each case?

1.8

The following are simple (and simplistic) models for the volume of a growing tumor:

- $\dot{V} = aV$ (Exponential)
- $\dot{V} = aV^b$ (Mendelsohn)
- $\dot{V} = aV(1 - \frac{1}{b}V)$ (Logistic)
- $\dot{V} = aV\,(b+V)^{-1}$ (Linear)
- $\dot{V} = aV\,(b+V)^{-1/3}$ (Surface)
- $\dot{V} = aV\log\frac{b}{c+V}$ (Gompertz)
- $\dot{V} = aV^{2/3} - bV$ (Bertalanffy)

Herein, a, b and c are positive constants. For each model, answer the following questions:

1. What is the maximum size of the tumor?
2. What is the condition on the constants for positive growth?
3. What is then the doubling time?
4. What a possible motivation for the model?

Now add a term "$-\gamma V$".

5. Interpret this term.
6. What is now the maximum size of the tumor?
7. What is the minimum concentration γ for complete cure?

Using the experimental data from

`https://www.nature.com/articles/s41598-019-39109-1/figures/4`

select a suitable model for Fig. 4a and 4b and estimate the parameters.

1.9

Give the maximal interval of existence $(0, t^\star)$ for the initial value problem $\dot{x}(t) = x^2(t)$ with $x(0) := x_0 > 0$.

- Why is $t^\star$ called the "blow up time"?
- Can you continue the solution past the blow up time?

Now consider $\dot{z}(t) = z^2(t)$, $t > 0$, for a complex-valued function z with the intial value $z(0) = x_0 + i\epsilon$, where $x_0 > 0$ and $\epsilon \neq 0$ are real. Where needed, assume $x_0 = 1$.

- What happens to the blow up time?
- Decomposing $z = x + iy$, write down the differential equations for x and y.
- Argue that $x^2(t) + (y(t) - R_0)^2 = R_0^2$ for a certain real constant R_0 and comment.
- Sketch the evolution in (t, x, y)-space for a small $\epsilon > 0$. What happens for $\epsilon \searrow 0$?

1.10

Develop a solution formula to the initial value problem

$$\dot{x}(t) = a(t)x(t) + b(t), \qquad x(t_0) = x_0, \tag{7}$$

in terms of the given functions a and b.

- Suppose a and b are indenpendent of t. Investigate the limits $a \to 0$ and $b \to 0$.
- Apply your formula to $t\,\dot{x}(t) = 2x(t) + t$. Don't forget to discuss $t = 0$.
- Check your formula on examples of your choice.
- Is this a good model for global seafood stock?
- Via a suitable transformation, solve $\dot{y}(t) = y(t) - y^\alpha(t)$, where $\alpha \in \mathbb{R}$. What is remarkable about this "Bernoulli differential equation"?
- Solve the "logistic differential equation" $\dot{y}(t) = y(t)(1 - y(t))$ with $0 \leq y(0) \leq 1$. Explain this model in terms of "growth" and "competition".
- Propose a better model for global seafood stock.

1.11

Derive the Tsiolkovsky equation

$$\Delta v = v_{\text{exhaust}} \log \frac{m_{\text{before}}}{m_{\text{after}}} \tag{8}$$

that gives the velocity acquired by a propulsion rocket (no external forces).

1.12

Propose a nontrivial system of differential equations "$\dot{x}(t) = \ldots$, $\dot{y}(t) = \ldots$" together with an initial condition at $t = 0$ for which the function

$$t \mapsto v(t) := x^2(t) + y^2(t) \tag{9}$$

satisfies, for all $t > 0$,

- $\dot{v}(t) = -2v(t)$.
- $\dot{v}(t) = 0$.
- $\dot{v}(t) = 2x(t) + 2y(t)$.

In each case, comment graphically on the possible trajectories $t \mapsto (x(t), y(t))$ in $\mathbb{R}^2$.

1.13

On the "spatial" interval $I := (0, 1)$ define the functions $\varphi_k := \sqrt{2}\sin(\pi k\,\cdot\,)$. Check whether the set $\Phi := \{\varphi_k : k = 1, 2, 3, \ldots\}$ is

- orthonormal in the Lebesgue space $L_2(I)$,
- a basis thereof.

Now consider the heat equation

$$\partial_t u(t, x) - \partial_{xx} u(t, x) = f(t, x), \qquad t > 0, \quad x \in I, \tag{10}$$

with the initial condition $u(0, \cdot\,) = g$ and the boundary conditions $u|_{\partial I} = 0$.

- What is the physical motivation for this PDE?
- Solve the PDE formally by expansion in a suitable spatial basis.
- What qualitative properties of u can you infer?
- Can you propose a suitable functional framework?
- Does the solution u depend continuously on the data f and g?

1.14

Let $A \in \mathbb{R}^{N\times N}$ be a symmetric positive definite matrix. Let $J := (0,T)$ be a nontrivial interval. Let $g \in \mathbb{R}^N$ and $f\colon J \to \mathbb{R}^N$ continuous. Consider the bilinear form

$$B(u,v) := \int_J \{\langle Au, v\rangle + \langle A^{-1}\dot{u}, \dot{v}\rangle\} dt + \langle u(T), v(T)\rangle \tag{11}$$

and the functional

$$F(v) := \int_J \langle f, v + A^{-1}\dot{v}\rangle dt + \langle g, v(0)\rangle, \tag{12}$$

defined for arbitrary continuously differentiable $u, v\colon J \to \mathbb{R}^N$.

- What can you say about the functional $\mathcal{J}(u) := \frac{1}{2}B(u,u) - F(u)$?
- Does $\mathcal{J}$ have a minimum?
- What is the variational first order condition for the minimum of $\mathcal{J}$?
- What is the differential equation satisfied by the minimum?
- What are practical implications?

1.15

Consider the first order system of differential equations for N functions $\mathbf{x}_i\colon \mathbb{R} \to \mathbb{R}^d$,

$$\dot{\mathbf{x}}_i = \frac{1}{N}\sum_{j\neq i} \phi(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{x}_j - \mathbf{x}_i), \quad i = 1,\dots,N, \tag{13}$$

where ϕ is the “influence function”, say $\phi(r) := (1+r)^{-s}$ for some $s > 0$.

- Explain qualitatively the behavior of the model.
- What could this be a model for?
- Find an invariant.
- What do you expect to happen in the long run?
- Can you verify your hypothesis?

Consider the modified system

$$\dot{\mathbf{x}}_i = \frac{1}{N}\sum_{j\neq i} \frac{\phi_{ij}}{\sum_k \phi_{ik}}(\mathbf{x}_j - \mathbf{x}_i), \tag{14}$$

where $\phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|)$.

- What could be the motivation for this modification?
- How does this system differ from the previous one in an essential way?

1.16

These days, papers are published with a short abstract or even with a few “highlights” as bullet points. Write the “highlights” emphasizing the role of differential equations for

> R. M. Solow, *A contribution to the theory of economic growth*, The quarterly journal of economics, 70 (1956), pp. 65–94.

according to Elsevier’s guidelines:

> `https://www.elsevier.com/authors/journal-authors/highlights`

1.17

The following is from the IMO 2011, a problem by Geoffrey Smith (UK):

> Let S be a set of $N \geq 2$ points in the plane (no three points are collinear). A *windmill* is a process that starts with a line ℓ_0 through a single point $P_0 \in S$. The line rotates clockwise about the pivot P_0 until it meets some other point $P_1 \in S$. This point takes over as the new pivot, and so on.

Describe this windmill as a differential equation.

Optionally, solve the problem itself:

> Show that there is a choice of P_0 and ℓ_0 for which the resulting windmill meets each point of S infinitely many times.

1.18

The Navier–Stokes equations in cylindrical coordinates for a fluid flowing through a round pipe reduce (under reasonable assumptions of steady, symmetric, laminar and fully developed flow) to

$$\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{du}{dr}\right) = \frac{dp}{dz}, \tag{15}$$

where u is the axial velocity along the pipe (independent of z) and p is the pressure (independent of r), as functions of the radial coordinate r and the axial coordinate z. We use SI units; the constant μ is the dynamic viscosity [Pa $\times$ s] of the fluid that expresses its resistance to flow. The “no-slip” boundary conditions dictate that the velocity must be zero at the wall.

- Why are both sides equal to some constant, and what is the physical meaning of this constant?
- The pressure drop across the pipe of length L is $\Delta P < 0$. What is the flow rate [m^3/s] of the fluid through the pipe?
- Estimate the flow rate at the water tap and thus the pressure at the source.

1.19

For each natural N, let $(X_t^{(N)})_{t\geq 0}$ be a family of random variables. Define a second family by $Y_t^{(N)} := X_t^{(N)}/N$. Set $y(t) := \lim_{N\to\infty} \mathbb{E}[Y_t^{(N)}]$, assuming the limit exists. Derive a differential equation for y where the process $X_t^{(N)}$ is

- the Poisson process with rate $\lambda > 0$;
- the queue with Poisson arrivals at rate N and exponential service times of mean 1.

1.20

Let J be the $n \times n$ matrix

$$J := \lambda I + N \quad \text{where} \quad \lambda \in \mathbb{C} \quad \text{and} \quad N := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}. \tag{16}$$

Derive an expression for $\exp(tJ)$, $t \in \mathbb{R}$.