Boundary value problems (DRAFT October 7, 2019)

Introduction

Let $I = (a, b)$ be a non-trivial interval. An important class of differential equations (historically, theoretically, didactically and practically) are of the form

$$-(pu')' + qu = f \qquad \text{on} \qquad (a, b) \tag{1}$$

where $u\colon (a, b) \to \mathbb{R}$ is the unknown function, while p, q and g are known, sufficiently smooth functions. Typically,

$$p \geq 0 \text{ is non-negative.} \tag{2}$$

Example. Consider $u'' + a_1 x' + a_0 x = g$. Set $p(s) := \exp(\int_a^s a_1(\tau) d\tau)$ and $q := -pa_0$.

To streamline the notation we introduce the differential operator L,

$$Lu := -(pu')' + qu. \tag{3}$$

Thus, Eqn. 1 becomes

$$Lu = g. \tag{4}$$

The equation (1) is linear in u and involves two derivatives. From the ODE theory we expect that the homogeneous equation

$$L\varphi = 0 \tag{5}$$

has two (linearly independent) solutions φ_1 and φ_2, forming a so-called fundamental system. If, moreover, φ_0 is a particular solution to the original non-homogeneous equation (4) then so is any combination, for any $c_1, c_2 \in \mathbb{R}$,

$$L(\varphi_0 + c_1\varphi_1 + c_2\varphi_2) = L\varphi_0 + c_1 L\varphi_1 + c_2 L\varphi_2 = g. \tag{6}$$

The solution space is two-dimensional, parameterized by the two constants $c_1, c_2 \geq 0$. To fully specify the solution, two additional constraints are needed.

The operator L is self-adjoint (almost)

The particular form of L is significant for the following reason, called "Lagrange identity" in a more general context.

Observation. For any two smooth functions u and v,

$$uLv - vLu = -u(pv')' + v(pu')' = (p(u'v - uv'))'. \tag{7}$$

The significance of the observation is seen by defining the bilinear form

$$B(u,v) := \int_I (Lu)(x)v(x)dx. \tag{8}$$

Flipping L onto v using (7), we have the identity

$$B(u,v) = B(v,u) + \int_I (p(u'v - uv'))'dx, \tag{9}$$

which says that B is almost symmetric but the last anti-symmetric term. However, if we restrict u and v to functions that vanish on the boundary of I, the bilinear form is *symmetric.* This unleashes operator-theoretic tools which form the basis of the *finite element method* for boundary value problems like (4).

Boundary conditions I

In *initial value problems*, we specify two conditions on one end of the interval, such as $u(a) := u_0$ and $u'(a) := u_1$. Such equations typically model oscillatory evolution over time.

In *boundary value problems* we specify conditions at both ends of the interval, called boundary conditions. For example,

$$\text{homogeneous Dirichlet condition at } a: \qquad D_a u := u(a) \overset{!}{=} 0 \tag{10}$$

$$\text{homogeneous Neumann condition at } b: \qquad N_b u := u'(b) \overset{!}{=} 0. \tag{11}$$

More generally, so-called Robin boundary conditions can be imposed:

$$H_a u := \alpha_0 u(a) + \alpha_1 u'(a) \overset{!}{=} \eta_a \tag{12a}$$

$$H_b u := \beta_0 u(b) + \beta_1 u'(b) \overset{!}{=} \eta_b \tag{12b}$$

where α_i, β_i and η_x are constants. Note that H_a and H_b are *linear operators* that take a function and return a real number (if defined). In more general boundary value problems posed on d-dimensional domains, boundary operators return functions on the $(d-1)$-dimensional boundary of the domain; these functions are called *traces* of u.

For the candidate solution in the form $u = \varphi_0 + c_1\varphi_1 + c_2\varphi_2$ we can write the set of boundary conditions in matrix-vector form,

$$\begin{pmatrix} H_a\varphi_1 & H_a\varphi_2 \\ H_b\varphi_1 & H_b\varphi_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \eta_a \\ \eta_b \end{pmatrix} - \begin{pmatrix} H_a\varphi_0 \\ H_b\varphi_0 \end{pmatrix}, \tag{13}$$

providing conditions on the unknonwn coefficients c_1 and c_2. If the determinant of the matrix is non-zero, these coefficients are uniquely determined (but otherwise there may be no solution, one solution or infinitely many solutions).

Example. Take $p := 1$ and $q := -(\pi k)^2$, where $k \in \mathbb{N}$, and $g = 0$, on the interval $(0, 1)$ subject to the homogeneous Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 0$. There is no need for the particular solution, in other words $\varphi_0 = 0$. Determine c_1 and c_2.

Boundary conditions II

In practice, that is in algorithms and software packages for boundary value problems, there are other ways to impose the (linear) boundary conditions (12). A relatively easy way is to decompose $u = \tilde{u} + u_0$ where u_0 is *any* function that satisfies the boundary conditions (12). Then, setting $\tilde{f} := f - Lu_0$, the function $\tilde{u}$ satisfies the non-homogeneous boundary value problem

$$L\tilde{u} = \tilde{f} \tag{14}$$

with homogeneous boundary conditions $H.\tilde{u} = 0$.

Example. On $I = (-1, 1)$, take

$$Lu := -u'' + u = 0 \quad \text{with} \quad u(\pm 1) = \pm 1. \tag{15}$$

Then $u_0(x) := x$ satisfies the boundary conditions. Therefore, if $\tilde{u}$ solves

$$(L\tilde{u})(x) \stackrel{!}{=} -(Lu_0)(x) = x \quad \forall x \in (-1, 1), \quad \text{with} \quad \tilde{u}(\pm 1) = 0, \tag{16}$$

then $u := \tilde{u} + u_0$ solves the original problem.

In this way, we do not need the fundamental system $\{\varphi_1, \varphi_2\}$, which is in general hard to find anyway. In the following we therefore focus on homogeneous boundary conditions.

Green's function

We consider boundary value problems $Lu = f$ on an interval I with homogeneous boundary conditions. The Green's function is largely a theoretical device for representing solutions that is based on the following idea (that also works in higher dimensions).

Let δ denote the Dirac functional at 0 that formally verifies

$$\int_I f(y)\delta(y - x)dy = f(x) \tag{17}$$

for all smooth functions f.

Example. Let H be the Heaviside function with $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$. Show that

$$\int_{\mathbb{R}} f(y)H'(y - x)dy = f(x) \quad \forall f \in C^1(\mathbb{R}). \tag{18}$$

The Green's function to the operator L is a scalar-valued function G of two variables such that

$$L[G(\cdot, y)](x) = \delta(y - x) \quad \forall x \in I, \tag{19}$$

where we agree that L acts on the first variable of G. Now set

$$u(x) := \int_I G(x,y)f(y)dy. \tag{20}$$

Then, bravely exchanging differentiation in x and integration in y,

$$(Lu)(x) \overset{(20)}{=} \int_I L[G(\cdot,y)](x)f(y)dy \overset{(19)}{=} \int_I \delta(y-x)f(y)dy \overset{(17)}{=} f(x) \quad \forall x \in I. \tag{21}$$

Remark. If $x \mapsto G(x,y)$ satisfies the homogeneous boundary conditions then so does (20).

Green's function for (4)

For the one-dimensional problem (4) with homogeneous boundary conditions a Green's function can be constructed from the fundamental system $\{\varphi_1, \varphi_2\}$. Recall that this means in particular that $L\varphi_i = 0$.

First note that the Lagrange identity (7) implies that the quantity

$$c := p(\varphi_1\varphi_2' - \varphi_1'\varphi_2) \tag{22}$$

is a constant. We assume that this constant in non-zero. To simplify the notation we rescale φ_i so that $c = 1$.

Now set

$$G(x,y) := \begin{cases} \varphi_1(x)\varphi_2(y) & \text{if} \quad x \geq y \\ \varphi_1(y)\varphi_2(x) & \text{if} \quad y \geq x. \end{cases} \tag{23}$$

Remark. We can write G using the Heaviside function as

$$G(x,y) = \varphi_1(x)\varphi_2(y)H(x-y) + \varphi_1(y)\varphi_2(x)H(y-x). \tag{24}$$

Remark. The function u defined by (20) satisfies the *homogeneous* boundary conditions because $G(\cdot,y)$ does (for any $y \in I$).

Remark. The function $x \mapsto G(x,y)$ is continuous but likely not differentiable at $x = y$.

Now we verify that G is indeed a Green's function for L by checking (19). Specifically, we check that

$$\int_{y-\epsilon}^{y+\epsilon} L[G(\cdot,y)](x)dx = 1 \tag{25}$$

for any $y \in I$ as $\epsilon \searrow 0$. Since $L[G(\cdot,y)](x) = 0$ whenever $x \neq y$, this implies $L[G(\cdot,y)](x) = \delta(x-y)$, which is equivalent to (19).

Indeed, using the Lagrange identity (7) with $u = G(\cdot, y)$ and $v = 1$,

$$
\begin{aligned}
\mathrm{LHS}(25) &= -\int_{y-\epsilon}^{y+\epsilon} (pG'(\cdot, y))'(x)dx && (26)\\
&= -pG'(x,y)\big|_{x=y-\epsilon}^{x=y+\epsilon} && (27)\\
&= -(p(y+\epsilon)\varphi_1'(y+\epsilon)\varphi_2(y) - p(y-\epsilon)\varphi_2'(y-\epsilon)\varphi_1(y)) && (28)\\
&\to p(\varphi_1\varphi_2' - \varphi_1'\varphi_2)(y) \quad \text{as} \quad \epsilon \searrow 0 && (29)\\
&= 1. && (30)
\end{aligned}
$$

Remark. We assumed that p and φ_i' are continuous.

R.A., October 7, 2019