

## 1.1

Consider  $C^1(\mathbb{R})$  solutions of the differential equation  $\dot{x}(t) = -x(t)$ .

- Sketch all solutions in the  $x$ - $t$  plane.
- What do they have in common?
- For  $T > 0$ , how do  $x(0) \mapsto x(T)$  and  $x(T) \mapsto x(0)$  differ qualitatively?
- Parameterize the family of solution curves implicitly à la  $F(x, t, \lambda) = 0$ .
- Indicate the “Lie group orbits”  $\lambda \mapsto F(x, t, \lambda)$  for various fixed  $(x, t)$ .
- Find and solve a differential equation for the curve in the  $x$ - $t$  plane that goes through  $(c, 0)$  and is perpendicular to the solution curves.

## 1.2

A lake with constant volume  $V$  [m<sup>3</sup>] is fed by a river with flow rate  $r$  [m<sup>3</sup>/s]. A factory upstream begins releasing a pollutant into the river at a rate  $p$  [m<sup>3</sup>/s]. Propose and solve a differential equation for the concentration  $c(t)$  [m<sup>3</sup>/m<sup>3</sup>] of the pollutant in the lake.

## 1.3

Excerpt from R. FISHER, “The genetical theory of natural selection”, 1930, p. 42:

An increase in numbers of any organism will impair its environment in a manner analogous to, and probably more definitely than, an increase in the numbers or efficiency of its competitors. [...] The situation is represented by the differential equation

$$\frac{dM}{dt} + \frac{M}{C} = W - D \quad (1)$$

in which  $M$  is the mean of the Malthusian parameter,  $C$  is a constant expressing the relation between fitness and population increase [...],  $W$  is the rate of actual increase in fitness determined by natural selection, and  $D$  is the rate of loss due to the deterioration of the environment. If  $C$ ,  $W$  and  $D$  are constant the equation has the solution

$$M = \frac{W - D}{C} + Ae^{-t/C} \quad (2)$$

in which  $A$  is an arbitrary constant, dependent on the initial conditions.

Questions:

- What will the value of  $M$  approach to?
- What is the fate of the species in this model?

## 1.4

Find, if possible, a differential equation “ $\dot{x}(t) = \dots$ ” with a nontrivial solution in  $C^1([0, \infty))$  for which  $x(t) = 0$  for all  $t$  sufficiently large — i.e.  $x$  converges in finite time.

## 1.5

Let  $\gamma$ ,  $g$  and  $a \geq 0$  be continuous real-valued functions on  $[0, T]$ .

- Sketch the function  $s \mapsto A(s) := \exp(\int_s^T a(\tau) d\tau)$ .

Suppose

$$\gamma(t) \leq g(t) + \int_0^t \gamma(s) a(s) ds \quad \forall t \in [0, T]. \quad (3)$$

Prove

- the Gronwall–Bellman inequality:

$$\gamma(T) \leq g(T) + \int_0^T g(s) a(s) A(s) ds. \quad (4)$$

- assuming that  $g$  is continuously differentiable, the corollary:

$$\gamma(T) \leq g(0) A(0) + \int_0^T g'(s) A(s) ds. \quad (5)$$

It may be helpful to start with a differential inequality for  $x(t) := \int_0^t \gamma(s) a(s) ds$ .

## 1.6

Fix  $T > 0$ , suppose  $f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous in the second variable. Assume there exists a solution to the differential equation  $\dot{x}(t) = f(t, x(t))$  on the interval  $(0, T)$  for any initial value near some fixed  $x_0$ .

- Show that  $x(0) \mapsto x(T)$  is Lipschitz continuous. Is your statement sharp?
- Do we need to *assume* existence?
- Suppose  $f$  depends on a parameter  $p$ . Fix  $x_p(0)$ . What is the derivative of the solution  $x_p(T)$  w.r.t.  $p$ ? Propose conditions on  $f$  for the derivative to exist.

## 1.7

Develop a solution formula to the initial value problem

$$\dot{x}(t) = g(t)h(x(t)), \quad x(t_0) = x_0, \quad (6)$$

in terms of the given functions  $g$  and  $h$ . Apply to

- $\dot{x}(t) = t x(t)$ ,
- $\dot{x}(t) = t/x(t)$ ,
- $\dot{x}(t) = x^2(t)$ ,
- $\dot{x}(t) = \sqrt{x(t)}$ ,
- $\dot{x}(t) = (1 + t^2)(1 + x^2(t))$ ,
- $\dot{x}(t) = x(t) \log x(t)$ .

What is the maximal temporal interval of existence in each case?

## 1.8

The following are simple (and simplistic) models for the volume of a growing tumor:

- $\dot{V} = aV$  (Exponential)
- $\dot{V} = aV^b$  (Mendelsohn)
- $\dot{V} = aV(1 - \frac{1}{b}V)$  (Logistic)
- $\dot{V} = aV(b + V)^{-1}$  (Linear)
- $\dot{V} = aV(b + V)^{-1/3}$  (Surface)
- $\dot{V} = aV \log \frac{b}{c+V}$  (Gompertz)
- $\dot{V} = aV^{2/3} - bV$  (Bertalanffy)

Herein,  $a$ ,  $b$  and  $c$  are positive constants. For each model, answer the following questions:

1. What is the maximum size of the tumor?
2. What is the condition on the constants for positive growth?
3. What is then the doubling time?
4. What a possible motivation for the model?

Now add a term “ $-\gamma V$ ”.

5. Interpret this term.
6. What is now the maximum size of the tumor?
7. What is the minimum concentration  $\gamma$  for complete cure?

Using the experimental data from

<https://www.nature.com/articles/s41598-019-39109-1/figures/4>

select a suitable model for Fig. 4a and 4b and estimate the parameters.

## 1.9

Give the maximal interval of existence  $(0, t^*)$  for the initial value problem  $\dot{x}(t) = x^2(t)$  with  $x(0) := x_0 > 0$ .

- Why is  $t^*$  called the “blow up time”?
- Can you continue the solution past the blow up time?

Now consider  $\dot{z}(t) = z^2(t)$ ,  $t > 0$ , for a complex-valued function  $z$  with the initial value  $z(0) = x_0 + i\epsilon$ , where  $x_0 > 0$  and  $\epsilon \neq 0$  are real. Where needed, assume  $x_0 = 1$ .

- What happens to the blow up time?
- Decomposing  $z = x + iy$ , write down the differential equations for  $x$  and  $y$ .
- Argue that  $x^2(t) + (y(t) - R_0)^2 = R_0^2$  for a certain real constant  $R_0$  and comment.
- Sketch the evolution in  $(t, x, y)$ -space for a small  $\epsilon > 0$ . What happens for  $\epsilon \searrow 0$ ?

## 1.10

Develop a solution formula to the initial value problem

$$\dot{x}(t) = a(t)x(t) + b(t), \quad x(t_0) = x_0, \quad (7)$$

in terms of the given functions  $a$  and  $b$ .

- Suppose  $a$  and  $b$  are independent of  $t$ . Investigate the limits  $a \rightarrow 0$  and  $b \rightarrow 0$ .
- Apply your formula to  $t \dot{x}(t) = 2x(t) + t$ . Don't forget to discuss  $t = 0$ .
- Check your formula on examples of your choice.
- Is this a good model for global seafood stock?
- Via a suitable transformation, solve  $\dot{y}(t) = y(t) - y^\alpha(t)$ , where  $\alpha \in \mathbb{R}$ . What is remarkable about this “Bernoulli differential equation”?
- Solve the “logistic differential equation”  $\dot{y}(t) = y(t)(1 - y(t))$  with  $0 \leq y(0) \leq 1$ . Explain this model in terms of “growth” and “competition”.
- Propose a better model for global seafood stock.

### 1.11

Derive the Tsiolkovsky equation

$$\Delta v = v_{\text{exhaust}} \log \frac{m_{\text{before}}}{m_{\text{after}}} \quad (8)$$

that gives the velocity acquired by a propulsion rocket (no external forces).

### 1.12

Propose a nontrivial system of differential equations “ $\dot{x}(t) = \dots$ ,  $\dot{y}(t) = \dots$ ” together with an initial condition at  $t = 0$  for which the function

$$t \mapsto v(t) := x^2(t) + y^2(t) \quad (9)$$

satisfies, for all  $t > 0$ ,

- $\dot{v}(t) = -2v(t)$ .
- $\dot{v}(t) = 0$ .
- $\dot{v}(t) = 2x(t) + 2y(t)$ .

In each case, comment graphically on the possible trajectories  $t \mapsto (x(t), y(t))$  in  $\mathbb{R}^2$ .

### 1.13

On the “spatial” interval  $I := (0, 1)$  define the functions  $\varphi_k := \sqrt{2} \sin(\pi k \cdot)$ . Check whether the set  $\Phi := \{\varphi_k : k = 1, 2, 3, \dots\}$  is

- orthonormal in the Lebesgue space  $L_2(I)$ ,
- a basis thereof.

Now consider the heat equation

$$\partial_t u(t, x) - \partial_{xx} u(t, x) = f(t, x), \quad t > 0, \quad x \in I, \quad (10)$$

with the initial condition  $u(0, \cdot) = g$  and the boundary conditions  $u|_{\partial I} = 0$ .

- What is the physical motivation for this PDE?
- Solve the PDE formally by expansion in a suitable spatial basis.
- What qualitative properties of  $u$  can you infer?
- Can you propose a suitable functional framework?
- Does the solution  $u$  depend continuously on the data  $f$  and  $g$ ?

## 1.14

Let  $A \in \mathbb{R}^{N \times N}$  be a symmetric positive definite matrix. Let  $J := (0, T)$  be a nontrivial interval. Let  $g \in \mathbb{R}^N$  and  $f: J \rightarrow \mathbb{R}^N$  continuous. Consider the bilinear form

$$B(u, v) := \int_J \{ \langle Au, v \rangle + \langle A^{-1}\dot{u}, \dot{v} \rangle \} dt + \langle u(T), v(T) \rangle \quad (11)$$

and the functional

$$F(v) := \int_J \langle f, v + A^{-1}\dot{v} \rangle dt + \langle g, v(0) \rangle, \quad (12)$$

defined for arbitrary continuously differentiable  $u, v: J \rightarrow \mathbb{R}^N$ .

- What can you say about the functional  $\mathcal{J}(u) := \frac{1}{2}B(u, u) - F(u)$ ?
- Does  $\mathcal{J}$  have a minimum?
- What is the variational first order condition for the minimum of  $\mathcal{J}$ ?
- What is the differential equation satisfied by the minimum?
- What are practical implications?

## 1.15

Consider the first order system of differential equations for  $N$  functions  $\mathbf{x}_i: \mathbb{R} \rightarrow \mathbb{R}^d$ ,

$$\dot{\mathbf{x}}_i = \frac{1}{N} \sum_{j \neq i} \phi(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{x}_j - \mathbf{x}_i), \quad i = 1, \dots, N, \quad (13)$$

where  $\phi$  is the “influence function”, say  $\phi(r) := (1 + r)^{-s}$  for some  $s > 0$ .

- Explain qualitatively the behavior of the model.
- What could this be a model for?
- Find an invariant.
- What do you expect to happen in the long run?
- Can you verify your hypothesis?

Consider the modified system

$$\dot{\mathbf{x}}_i = \frac{1}{N} \sum_{j \neq i} \frac{\phi_{ij}}{\sum_k \phi_{ik}} (\mathbf{x}_j - \mathbf{x}_i), \quad (14)$$

where  $\phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|)$ .

- What could be the motivation for this modification?
- How does this system differ from the previous one in an essential way?

## 1.16

These days, papers are published with a short abstract or even with a few “highlights” as bullet points. Write the “highlights” emphasizing the role of differential equations for

R. M. SOLOW, *A contribution to the theory of economic growth*, The quarterly journal of economics, 70 (1956), pp. 65–94.

according to Elsevier’s guidelines:

<https://www.elsevier.com/authors/journal-authors/highlights>

## 1.17

The following is from the IMO 2011, a problem by Geoffrey Smith (UK):

Let  $S$  be a set of  $N \geq 2$  points in the plane (no three points are collinear). A *windmill* is a process that starts with a line  $\ell_0$  through a single point  $P_0 \in S$ . The line rotates clockwise about the pivot  $P_0$  until it meets some other point  $P_1 \in S$ . This point takes over as the new pivot, and so on.

Describe this windmill as a differential equation.

Optionally, solve the problem itself:

Show that there is a choice of  $P_0$  and  $\ell_0$  for which the resulting windmill meets each point of  $S$  infinitely many times.

## 1.18

The Navier–Stokes equations in cylindrical coordinates for a fluid flowing through a round pipe reduce (under reasonable assumptions of steady, symmetric, laminar and fully developed flow) to

$$\frac{1}{r} \frac{d}{dr} \left( r \mu \frac{du}{dr} \right) = \frac{dp}{dz}, \quad (15)$$

where  $u$  is the axial velocity along the pipe (independent of  $z$ ) and  $p$  is the pressure (independent of  $r$ ), as functions of the radial coordinate  $r$  and the axial coordinate  $z$ . We use SI units; the constant  $\mu$  is the dynamic viscosity [ $\text{Pa} \times \text{s}$ ] of the fluid that expresses its resistance to flow. The “no-slip” boundary conditions dictate that the velocity must be zero at the wall.

- Why are both sides equal to some constant, and what is the physical meaning of this constant?
- The pressure drop across the pipe of length  $L$  is  $\Delta P < 0$ . What is the flow rate [ $\text{m}^3/\text{s}$ ] of the fluid through the pipe?
- Estimate the flow rate at the water tap and thus the pressure at the source.

### 1.19

For each natural  $N$ , let  $(X_t^{(N)})_{t \geq 0}$  be a family of random variables. Define a second family by  $Y_t^{(N)} := X_t^{(N)}/N$ . Set  $y(t) := \lim_{N \rightarrow \infty} \mathbb{E}[Y_t^{(N)}]$ , assuming the limit exists. Derive a differential equation for  $y$  where the process  $X_t^{(N)}$  is

- the Poisson process with rate  $\lambda > 0$ ;
- the queue with Poisson arrivals at rate  $N$  and exponential service times of mean 1.

### 1.20

Let  $J$  be the  $n \times n$  matrix

$$J := \lambda I + N \quad \text{where} \quad \lambda \in \mathbb{C} \quad \text{and} \quad N := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}. \quad (16)$$

Derive an expression for  $\exp(tJ)$ ,  $t \in \mathbb{R}$ .