Exercises for the course

# Machine Learning 1

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## Exercise Sheet 6

A kernel function $k\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ must satisfy the *Mercer's condition*, which verifies that for any sequence of data points $x_1, \ldots, x_n \in \mathbb{R}^d$ and coefficients $c_1, \ldots, c_n \in \mathbb{R}$ the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \, c_j \, k(x_i, x_j) \geq 0$$

is satisfied. If it is the case, the kernel is called a *Mercer kernel*.

Conversely, the *representer theorem* states that if $k$ is a Mercer kernel on $\mathbb{R}^d$, then there exists a Hilbert space (i.e., a finite or infinite dimensional $\mathbb{R}$-vector space with norm and scalar product) $\mathcal{F}$, the so-called feature space, and a continuous map $\varphi : \mathbb{R}^d \to \mathcal{F}$, such that

$$k(x, x') = \langle \, \varphi(x) \, , \, \varphi(x') \, \rangle_{\mathcal{F}} \quad \text{for all } x, x' \in \mathbb{R}^d.$$

### Exercise 1: Mercer Kernels (3 × 20 P)

(a) *Show* that the following are Mercer kernels.

i. $k(x, x') = \langle x, x' \rangle$

ii. $k(x, x') = f(x) \cdot f(x')$ where $f\colon \mathbb{R}^d \to \mathbb{R}$ is an arbitrary continuous function

(b) Let $k_1, k_2$ be two Mercer kernels, for which we assume the existence of a finite-dimensional feature map associated to them. *Show* that the following are again Mercer kernels.

i. $k(x, x') = k_1(x, x') + k_2(x, x')$

ii. $k(x, x') = k_1(x, x') \cdot k_2(x, x')$

(c) *Show* using the results above that the polynomial kernel of degree $d$, where $k(x, x') = (\langle x, x' \rangle + \vartheta)^d$ and $\vartheta \in \mathbb{R}^+$, is a Mercer kernel.

### Exercise 2: The Feature Map (4 × 10 P)

Consider the homogenous polynomial kernel $k$ of degree 2 which is $k : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, where

$$k(x, y) = \langle x, y \rangle^2 = \Big( \sum_{i=1}^{2} x_i \, y_i \Big)^2.$$

(a) *Show* that $\mathcal{F} = \mathbb{R}^3$ and $\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ \sqrt{2}\, x_1 \, x_2 \\ x_2^2 \end{pmatrix}$ are possible choices for feature space and feature map.

(b) Consider the unit circle $C = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \; ; \; 0 \leq \theta < 2\pi \right\}$. *Show* that the image $\varphi(C)$ lies on a plane $H$ in $\mathbb{R}^3$.

(c) Consider the plane $A = \left\{ \begin{pmatrix} t \\ s \end{pmatrix} \; ; \; t, s \in \mathbb{R} \right\}$. *Find* a point $P$ in $\mathcal{F}$ which is not contained in $\varphi(A)$.

(d) *Find* a feature map associated to the kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with $k(x, y) = \langle x, y \rangle^2 = \Big( \sum_{i=1}^{d} x_i \, y_i \Big)^2$.