Exercises for the course

Machine Learning 1

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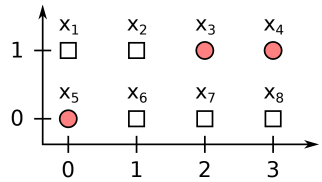
Exercise Sheet 12

Exercise 1: Boosted Classifiers (25 + 25 P)

We consider a two-dimensional dataset $x_1, \dots, x_8 \in \mathbb{R}^2$ with binary labels $y_1, \dots, y_8 \in \{-1, 1\}$.



Red circles denote the first class ($y_i = +1$) and white squares denote the second class ($y_i = -1$). We decide to classify this data with a boosted classifier and use the nearest mean classifier as a weak classifier. The boosted classifier is given by

$$f(x) = \mathrm{sign}\Big(\alpha_0 + \sum_{t=1}^{T} \alpha_t h_t(x)\Big)$$

where $\alpha_0, \dots, \alpha_T \in \mathbb{R}$ are the boosting coefficients. The tth nearest mean classifier is given by

$$h_t(x) = \begin{cases} +1 & \|x - \mu_t^+\| < \|x - \mu_t^-\| \\ -1 & \text{else} \end{cases} \qquad \text{with} \quad \mu_t^+ = \frac{\sum_{i:y_i=+1} p_i^{(t)} x_i}{\sum_{i:y_i=+1} p_i^{(t)}} \quad \text{and} \quad \mu_t^- = \frac{\sum_{i:y_i=-1} p_i^{(t)} x_i}{\sum_{i:y_i=-1} p_i^{(t)}}.$$

where $p_1^{(t)}, \dots, p_N^{(t)}$ are the data weighting terms for this classifier.

(a) *Draw* at hand a possible boosted classifier that classifies the dataset above, i.e. *draw* the decision boundary of the weak classifiers $h_t(x)$ and of the final boosted classifier $f(x)$. We use the convention $\mathrm{sign}(0) = 0$.

(b) *Write* the weighting terms $p_i^{(t)}$ and the coefficients $\alpha_0, \dots, \alpha_T$ associated to the classifiers you have drawn.

(Note: In this exercise, the boosted classifier does not *need to derive from a particular algorithm. Instead, the number of weak classifiers, the coefficients and the weighting terms can be picked at hand with the sole constraint that the final classifier implements the desired decision boundary.)*

Exercise 2: AdaBoost as an Optimization Problem (25 + 25 P)

Consider AdaBoost for binary classification applied to some dataset $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$. The algorithm starts with uniform weighting ($\forall_{i=1}^N : \ p_i^{(1)} = 1/N$) and performs the following iteration:

for $t = 1 \dots T$:

Step 1:	$\mathcal{D}, p^{(t)} \mapsto h_t$	(learn tth weak classifier using weighting $p^{(t)}$)
Step 2:	$\epsilon_t = \mathbb{E}_{p^{(t)}}[1_{(h_t(x) \neq y)}]$	(compute the weighted error of the classifier)
Step 3:	$\alpha_t = \frac{1}{2} \log\Big(\frac{1-\epsilon_t}{\epsilon_t}\Big)$	(set its contribution to the boosted classifier)
Step 4:	$\forall_{i=1}^N : \ p_i^{(t+1)} = Z_t^{-1} p_i^{(t)} \exp(-\alpha_t y_i h_t(x_i))$	(set a new weighting for the data)

The term $\mathbb{E}_{p^{(t)}}[\cdot]$ denotes the expectation under the data weighting $p^{(t)}$, and Z_t is a normalization term. An interesting property of AdaBoost is that it can be shown to minimize some objective function

$$\mathcal{G}(\boldsymbol{\alpha}) = \sum_{i=1}^{N} \exp(-y_i f_{\boldsymbol{\alpha},t}(x_i))$$

where $f_{\boldsymbol{\alpha},t}(x) = \sum_{\tau=1}^{t} \alpha_\tau h_\tau(x)$ is the output score of the boosted classifier after t iterations.

(a) *Show* that the objective can be rewritten as $\mathcal{G}(\boldsymbol{\alpha}) = N \cdot \Big(\prod_{\tau=1}^{t-1} Z_\tau\Big) \cdot \sum_{i=1}^{N} p_i^{(t)} \exp(-y_i \alpha_t h_t(x_i))$.

(b) *Show* that Step 3 of the AdaBoost procedure above is equivalent to computing $\alpha_t = \arg\min_{\alpha_t} \mathcal{G}(\boldsymbol{\alpha})$.