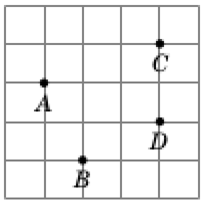
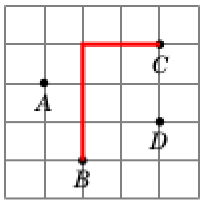
Rotating coordinates

Some problems are easier to solve if Manhattan distances are used instead of Euclidean distances. As an example, consider a problem where we are given n points in the two-dimensional plane and our task is to calculate the maximum Manhattan distance between any two points.

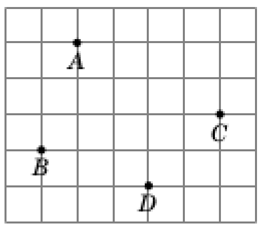
For example, consider the following set of points:



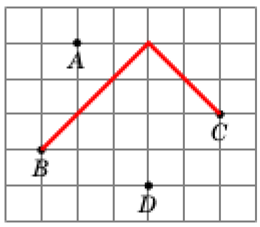
The maximum Manhattan distance is 5 between points B and C:



A useful technique related to Manhattan distances is to rotate all coordinates 45 degrees so that a point (x, y) becomes $(x+y, y-x)$. For example, after rotating the above points, the result is:



And the maximum distance is as follows:



Consider two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ whose rotated coordinates are $p'_1 = (x'_1, y'_1)$ and $p'_2 = (x'_2, y'_2)$. Now there are two ways to express the Manhattan distance between p_1 and p_2:

$$|x_1 - x_2| + |y_1 - y_2| = \max(|x'_1 - x'_2|, |y'_1 - y'_2|)$$

For example, if $p_1 = (1,0)$ and $p_2 = (3,3)$, the rotated coordinates are $p'_1 = (1,-1)$ and $p'_2 = (6,0)$ and the Manhattan distance is

$$|1-3| + |0-3| = \max(|1-6|, |-1-0|) = 5.$$

The rotated coordinates provide a simple way to operate with Manhattan distances, because we can consider x and y coordinates separately. To maximize the Manhattan distance between two points, we should find two points whose rotated coordinates maximize the value of

$$\max(|x'_1 - x'_2|, |y'_1 - y'_2|).$$

This is easy, because either the horizontal or vertical difference of the rotated coordinates has to be maximum.