MAP-fis - Programa Doutoral em Física

Quantum fluids of light for analogue computing

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1 Introduction

In Physics, we often tend to base the development of new theories by establishing analogies with older models. The harmonic oscillator is perhaps the best-known example, as a simple model widely used as a first approach to tackle many problems. More elaborate analogies are found in the history of Physics, for example, Maxwell derived his equations by establishing an analogy between the electromagnetic fields and spinning vortices in aether [1], or Bekenstein that discussed the thermal properties of black holes with an analogy with the laws of thermodynamics [2]. Another type of analogies is also encountered, where the primary goal was not to develop new theories, but rather how to establish a correspondence between the dynamics and phenomena of different systems that share a similar description. For example, Lifshitz [3] tried to encode gravitational fields in dielectric media or Walther Gordon [4] who tried the opposite, encode dielectric mediums in gravitational fields.

In recent years, this idea of different systems being able to be described by a similar model catch the attention of a scientific community who start to search for known physical systems with a similar mathematical description. In particular, those capable of being implemented in the laboratory under controlled conditions, as a way to mimic other systems or phenomena that otherwise would be hard or even impossible to study. The main goal is not to completely recreate the original system, but rather to emulate certain phenomena that are crucial for their better understating. For example, one does not try to emulate a complete black hole in the laboratory. Instead, by controlling the intensity and phase profiles of a light beam propagating in a nonlinear material, it is possible to mimic de space-time curvature near a black hole, see Figure 1, and thus opening the possibility to experimentally study how the perturbations behave in this zones. Furthermore, with analogous horizons it is possible to have access not only to what happens with the perturbation outside of the horizon, but also to what happens to the perturbations that fall to the inside. A measurement that, even with the real system, would be impossible to obtain. Certainly, these types of experiments will not provide definitive validations for the underlying theories and models, but from them, it is possible to measure properties of the system that, in some situations, are hidden in the real systems, gain new insights about the underlying models and advance physics through interdisciplinary research. Analogue computing (also known as physical analogues or physical simulators) summarizes the effort of this line of investigation and many analogues models, ranging from gravity to superfluidity, have been theoretical and experimentally studied in recent years.

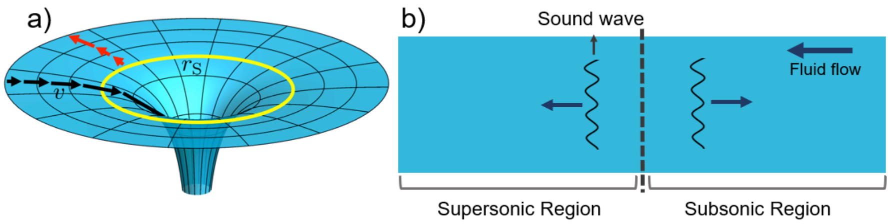

Figure 1: Analogy between an astrophysical and an analogous event horizon with a moving fluid. a) Represents the space-time geometry around a black hole, where the event horizon is represented by the yellow circle with the Schwarzschild radius r_S. b) An analogue of an event horizon is created by establishing a zone where the fluid flow passes from subsonic to supersonic velocities when compared with the sound speed of the medium. Sound wave perturbations in the supersonic region will be trapped, while the ones in the subsonic region can escape, and the gray dashed line works as an event horizon. Image a) was taken from [5].

The most important challenge in developing and test analogue models resides in finding materials with tunable properties that are capable of emulating different systems and explore a wide variety of behaviors and regimes. Several systems have been proposed for these types of experiments, such as Bose-Einstein condensates, Cold-atoms, polaritonic systems, and nonlinear optical media. Among these, optical media stand as the most relevant alternative, since they combine the experimental accessibility with tunable mechanisms and with the capacity of emulating a wide range of systems. In particular, quantum fluids of light, where light propagating in a nonlinear material is interpreted as a fluid, became a popular choice for producing analogue models, since these systems are easy to produce in the laboratory, when compared with Bose-Einstein condensates or Cold-atoms, for example, and offer a wide variety of mechanisms to easily tailor the experiment to better emulate the desired system. Furthermore, some nonlinear materials have nonlocal nonlinearities, where the effect extends beyond the beam waist, which opens the opportunity to study a new class of analogue models.

Considering now this essay, the goal is to present the analogue computing research field, and then focus on a particular mathematical model for developing analogue models and explore the physical systems capable of emulating it in the laboratory. In more detail, we begin by looking at the history and development of analogue models, in particular for gravity analogues, and then we focus the discussion on the quantum fluids of light, not only as an analogue system by itself but also as a support medium for other systems. Following this, we present the generalized nonlinear Schrödinger system as the base mathematical model to develop these systems and explore the numerical methods and computational techniques to study it. Then, we discuss the materials that allow emulating the model in the laboratory with a focus on the nonlinear optical media. Finally, we present the work planned for the Ph.D.

2 Analogue Gravity

The modern days of physical analogues, and in particular of analogue gravity, began with the work of Unruh [6], where he demonstrated the analogy between a curved-space time and sound waves flowing in a supercritical fluid flow. This initial theoretical work inspired new developments, for example to create analogues of the Hawking radiation [7–9], artificial black holes [10], explore the similarities between particle physics, cosmology, and condensed matter physics [11] and to create other analogue gravity phenomenology [12, 13]. Experimental works with artificial horizons began to appear some years later with different support systems, ranging from surface waves in a water tank [14], light propagation in optical fibers [15], bulk crystals [16] to Bose-Einstein condensates [17]. In addition, experimental work with white hole horizons and measurements of analogue Hawking radiation were also performed with surface waves in moving water [18] and the results agreed with the theory. More recently, analogues of Hawking radiation were measured with light pulses in optical fibers [19, 20], and Bose-Einstein condensates [21]. For an overview of this line of research and future perspectives see work [22] and references therein.

In parallel, two-dimensional analogue black holes began to be explored, which allows the introduction of the phenomenology associated with rotation in the system. Due to the rotation, scattered waves from the analogue horizon can extract energy from them and be amplified in the process. This effect constitutes an analogue of superradiance [23], and it was experimentally observed with waves propagating on the surface of the water with an analogue of the black-hole created with a draining vortex [23], with light propagating in thermo-optical media, where the horizon was created by manipulating the phase profile of the optical beam [24], and in defocusing Kerr systems [25]. Currently, the goal is to use vortex pairs to extract energy from the black hole similar to the Penrose effect, the particle equivalent of superradiance. Theoretical work has been done with Bose-Einstein condensates [26, 27] and with nonlinear optical media [28].

Gravity analogues are extensible to other systems and many models have been developed through the years, for boson stars [29], scalar dark matter models [30,31], gravitational effects [32] and others [12,13]. Some work has also been done with cosmological phenomena. For example, for the Casimir effect [33–35], to study the cosmological redshift and Hubble friction with rapid expanding Bose-Einstein condensates [36] or to study the false vacuum decay with coupled Bose-Einstein condensates [37, 38] or ultracold atoms [39]. With stimulated Sakharov oscillations in superfluidic optical systems it is possible to have an analogue of fluctuations imprinted in the primordial universe [40]. The field of gravity analogues is in constant development, where their paradigm in the last years changed from theory-driven to experimental-driven line o research. For more details and more references, see [41].

In past years, a new platform for emulating gravitational system appeared: quantum fluids of light, where light propagating on a nonlinear material is interpreted as fluid, and sonic horizons where produced in polaritons in microcavities [42]. Quantum fluids of light can be created with other systems, for example by simply propagating light in nonlinear media. The interest in these setups is that they are simple to implement and manipulate in the laboratory, and offer a wide variety of mechanisms that allow better control of the emulation. Indeed, many of the recent analogue gravity models are based on quantum fluids of light [24, 29, 32], however, their applicability extends for other analogue systems, as we shall see in the next section.

3 Quantum fluids of light

Quantum fluids of light is another branch of the analogue models that explores the possibility of creating fluids with light and the capacity of these to present superfluidic effects. For light to behave as a fluid a nonlinear medium is required to mediate the photon-photon interactions. Moreover, it is also necessary that the photons have an effective mass and this can be achieved by two different configurations: by confining the photons in microcavity geometries with a quantum well, where from the coupling between the quantum-well excitations with the cavity photons results a bosonic quasi-particle with low mass known as polaritons [43, 44]; or by propagating light in bulk nonlinear materials, where the effective mass appears from the diffraction in the transverse plane [45, 46]. In both situations the physical system can be modeled by the nonlinear Schrödinger equation and through the Madelung transformation [47] this model is transformed in a set of hydrodynamic equations, from which the fluid interpretation naturally appears.

Considering the microcavity geometries, different materials may be used as nonlinear medium, for example atomic gases [48], Josephson junctions [49] or semi-conducting solid-state media [50] and many optical analogues have been developed in this type of systems, ranging from superfluidic effects, such as as superfluid flow [51] and Mach-Cherenkov cones [51], quantized vortices [52], oblique dark solitons [53] or quantum turbulence [53], to gravity analogues, such as analogue horizons [42, 54] with rotating geometries [54], Hawking radiation [55], as well as thermalization [56] and photon condensation [23]. However, this type of configuration presents some disadvantages, since experiments often require temperatures around few kelvins [44, 51–53], in spite room temperature experiments have recently been demonstrated to create a polariton superfluid [57], and these systems are intrinsically dissipative and difficult to scale to larger systems [43].

When compared with the microcavity geometries, bulk nonlinear materials stand as a suitable alternative for creating fluids of light and implementing analogue models, since they are much easier to implement in the laboratory. However, they have an important difference: while with microcavity geometries the evolution of time is associated with the natural evolution of the system, in bulk nonlinear systems the time is mapped onto the direction of propagation and different slices along this axis correspond to different moments in time, see Figure 2. Thus, these systems are limited to 2D+1 configurations, which is enough for most of the analogue models developed so far. With short pulses, it is possible to have 3D+1 configurations, but the mathematical model is more complex and the analogies may not be valid.

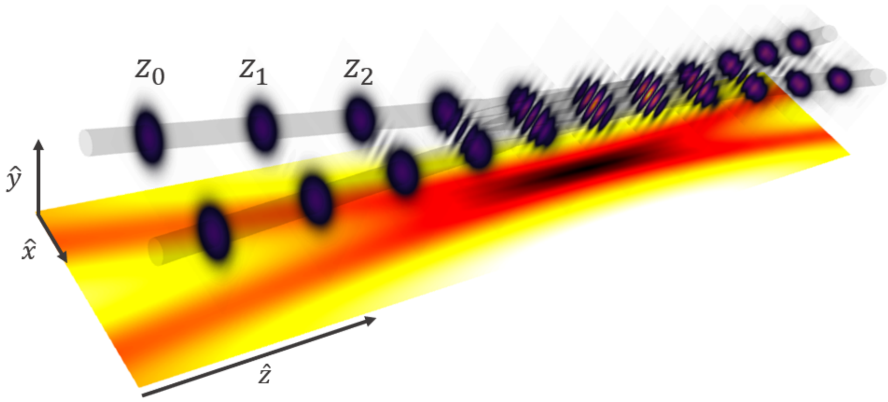

Figure 2: Interaction between two Gaussian distributions propagating in a thermo-optical medium that emulates the interaction between two galactic cores in a scalar field dark-matter scenario [30]. The original system is mapped onto a two-dimensional configuration, where the z coordinate has the role of time. Thus, different slices z_i represent different instants of time in the evolution.

Normally, with local nonlinearities, where the effect is restricted to the beam waist, the materials used are Kerr ($\chi^{(3)}$) or higher-order nonlinear materials [58]. Many optical analogues have been developed with these systems, ranging from superfluid behavior [59] to dispersive superfluid-like shock waves [60], and also including drag force cancellation [59, 61] and analogues of the Hawking radiation [62, 63]. Recently, it has been theoretical and experimentally explored the Berezinskii-Kosterlitz-Thouless transition (a topological phase transition) in photon fluids, with photorefractive crystals (Kerr medium) [64,65]. Although this transition was also predicted for microcavity geometries, for this configuration there are some difficulties in defining the temperature analogue of the phase transition [65]. These works open the possibility of studying phase transitions of condensed matter with nonlinear optics [64]. Quantum depletion and entanglement of phonons in Sakharov oscillations were recently suggested in the context of light superfluids [40]. Thermalization processes have also been explored with bulk nonlinear media [66]. Furthermore, these systems have also been considered for reservoir computing with soliton chains as physical implementations of neuromorphic hardware for regression and classification tasks [67, 68].

Another branch of bulk nonlinear materials is composed of those with nonlocal nonlinearities, where the effect extends beyond the beam waist. Usually, bulk systems with this type of response are the thermo-optical media [29, 30], nematic liquid crystals [69, 70], or atomic vapors [71]. The nonlocal character of these systems offers the opportunity to develop new types of optical analogues that require long-range interactions. With this class of materials, it is possible to produce analogue system that range from superfluidity [70, 72, 73], to gravity analogues of dark-matter [30, 31], boson stars [29], gravitational effects [32], among others [74, 75].

Most of the optical systems and optical analogues discussed so far have a common characteristic: all of them, under the appropriate approximations, can be described by a nonlinear Schrödinger equation with local and or nonlocal nonlinearities. Indeed, this mathematical model has vast applicability in physics and other fields, and thus it is a natural choice as a base model to develop analogue models. In the next section, we will explore this model in more detail.

4 The General Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation, with local an nonlocal nonlinearities, has a plethora of different applications in physics. Indeed, this mathematical model is able to describe not only a large class of different physical phenomena, such as gravitational effects [32], some dark matter models [30, 31], boson stars [29], superfluidity [59, 70, 72, 73] and others [74], but also systems capable of being implemented in the laboratory, such as local [58] and nonlocal [29, 30, 69, 70] optical media, polaritonic systems [43] or Bose-Einstein condensates [21, 26, 27]. The large applicability of this mathematical model makes this system a good candidate for developing optical analogues.

The nonlinear Schrödinger equation, in the most general form, is written as

$$i\frac{\partial\psi}{\partial\tilde{t}} = a\nabla^2\psi + b\phi\psi + F\left(|\psi|^2\right)\psi, \tag{1}$$

$$\nabla^2\phi + c\phi = d|\psi|^2, \tag{2}$$

where $\psi = \psi\left(\vec{r}, \tilde{t}\right)$ is the wave function that represents the physical quantity that is evolving, $\phi = \phi\left(\vec{r}\right)$ represents the nonlocal potential and a to e are constants to be chosen accordingly to the system. Furthermore, the term F represents local nonlinear terms and has the form

$$F\left(|\psi|^2\right) = \sum_n \alpha_n |\psi|^{2n}, \tag{3}$$

and the amplitudes α_n can be constant or have a spatial profile. The parameter $\tilde{t}$ can have different physical meanings accordingly to the system that is describing. For example, in the context of Bose-Einstein condensates or polaritonic systems, $\tilde{t}$ has dimensions of time, while when describing a continuous beam of light propagating on nonlinear optical media it has dimensions of space, see Figure 2.

Equations 1 and 2 can also describe a fluid by applying the Madelung transformation [47]. This transformation consists in rewriting the wave field as $\psi = \sqrt{\rho}e^{iS}$, where $\rho = |\psi|^2$ and $\vec{v} = \nabla S$ represent the density and velocity of the fluid, respectively. Introducing this transformation into the model and after some manipulation, the system can be written as

$$\frac{\partial\rho}{\partial\tilde{t}} - 2a\nabla\cdot\left(\rho\vec{v}\right) = 0 \tag{4}$$

$$\frac{\partial\vec{v}}{\partial\tilde{t}} - 2a\left(\vec{v}\cdot\nabla\right)\vec{v} = -\nabla\left[a\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + b\phi + F\left(\rho\right)\right] \tag{5}$$

$$\nabla^2\phi + c\phi = d\rho, \tag{6}$$

where these equations correspond, respectively, to the Euler equations, and to an equation of state for ϕ. The first term in the right-hand side of the second Euler equation (5) is commonly known as the Bohm potential or quantum pressure. This transformation shows how this model can be used to emulate fluid phenomena, which is of great interest in the development of analogue models, in particular for those based on superfluid flow. Furthermore, an equation of motion for the phase perturbations δS is obtained by linearizing equations 4 and 5, and it corresponds to a Klein-Gordon equation for a minimally coupled massless scalar field with a metric that can be tailored by controlling the density and phase profiles [24]. It is thus possible to emulate different space-time metrics, for example, those that describe the surroundings of black holes, and explore how small perturbations behave in these situations.

Working with equations 1 and 2 is no easy task. In particular, in the context of analogue models, predicting the dynamics of the systems is very complex and requires powerful techniques. In the next section, we will explore how numerical methods and high-performance computing can help in the development and study of new analogue models.

5 Numerical Methods

The general model presented in equations 1 and 2 is most of the time non-integrable and analytical treatment is difficult. Only for specific situations, analytical solutions are known. For example, for a one-dimensional system with $\phi = 0$ and $F = \alpha_1|\psi|^2$, the model supports a wide variety of solutions, known as solitons [76]. In the context of nematic liquid crystals, where $\phi \neq 0$ and $F = 0$, solutions are also known for one and two dimensions [77]. Nevertheless, knowing these solutions is not enough, and numerical methods are required. Due to the wide applicability of this model, many numerical models have been developed and study over the years [78], and within all the possibilities, the Split-Step Fourier Method stands as the most indicated numerical scheme for this model. This method consists of dividing the linear and nonlinear parts of the left-hand side of equation 1 and integrate them separately. However, since we are working with operators, the exponential laws to separate both parts are not applicable and the Strang-Splitting method has to be used [79]. The linear part, corresponding to the kinetic term $a\nabla^2\psi$, is thus solved in the reciprocal space since the nabla operator is diagonal in this space and given by $\mathcal{F}(\nabla^2) = -k^2$. The remaining terms, corresponding to the nonlinear parts, are integrated in the direct space. Figure 3 show a schematically representation of SSFM and for more details about the numerical scheme and their implementation see [80] and references therein.

The Split-Step Fourier Method can also be used as a tool for searching for stationary solutions, by a technique known as the Imaginary-time propagation method, which con-

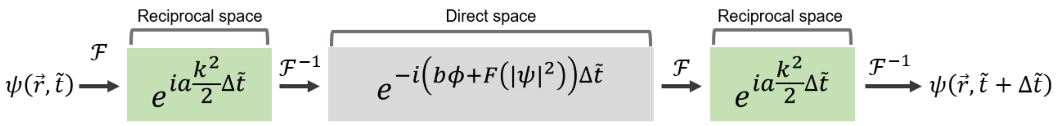

Figure 3: Schematic representation of an integration step with the Split-Step Fourier Method. The method consists in transforming the ψ field between the direct and reciprocal spaces and multiplying it by the corresponding evolution operators. The $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier and inverse Fourier transforms, respectively. In this scheme, the symmetric splitting is used to reduce the error, where the kinetic operator is performed before and after the nonlinear term.

sists in making a Wick rotation on the parameter $\tilde{t}' \rightarrow -i\tilde{t}$. This transformation transforms the equation 1 in a diffusion-like equation with emission and absorption terms, due to the potentials. By propagating an initial random guess, the system will converge to the ground state when $\tilde{t}' \rightarrow \infty$ [81, 82]. This technique is mostly used for finding the ground state, in particular when an analytical treatment is difficult, however excited states may also be found with additional conditions [82].

The implementation of the Split-Step Fourier Method to tackle a wide variety of systems with great precision with reasonable amounts of time requires large computation power which is normally obtained with supercomputers. However, this method is expensive and not available to everyone. At our research group, we use a different paradigm based on General-purpose computing on graphics processing units (GPGPUs or simply GPUs). GPUs are highly parallelizable devices that are very fast at performing simple arithmetic operations, that offer a computation power that can outperform the commonly used CPUs paradigm and are inexpensive when compared with a supercomputer setup. Indeed, we already developed some solvers based on this technology, in particular for the nonlinear Schrödinger equation, and have successfully applied them in different works [70, 80, 83, 84]. More recently, our research group started the development of high-performance solvers for distributed memory architectures based on Message Passing Interface (MPI) and OpenMP (Open Multi-Processing). These tools are fundamental for developing and explore analogue models that due to its complexity, are usually associated with an higher computational burden.

The nonlinear Schrödinger equation is indeed the most indicated model to explore and develop analogue models since it describes a wide variety of physical phenomena. But the effort is only worthy if there are physical systems that can be implemented in the laboratory and that are described by the same, or at least similar, mathematical model.

This topic will be explored in the next section.

6 Optical Media

The success in developing optical analogues relies on finding physical systems that are governed by the same model that we want to emulate and that are easy to implement in the laboratory under controlled conditions. Furthermore, these systems should also have some types of mechanisms that allows us to control the properties of the material. This is an important feature since not only allows us to adjust the material to better emulate the original system, but also to perform emulations under different conditions. Since the beginning of the development of analogue models, a wide variety of materials were proposed as a support media to realize the analogue experiments. Currently, there are a wide variety of options, ranging from Bose-Einstein condensates [21, 26, 27], to polaritonic systems [43], ultra-cold atoms [38, 39], and light propagating on nonlinear optical media [29,30,58,69,70]. Among these possibilities, the nonlinear optical media stand as an interesting candidate for implementing these type of experiments since they are easy to implement and offer interesting mechanisms to control their properties. Indeed, in the last years, many experiments have been focused on these systems, and in this section we will explore the available materials and what kind of nonlinearities they can produce to emulate the generalized nonlinear Schrödinger equation.

6.1 Nonlinear Local Media

Light propagating on a nonlinear material is fully described by the Maxwell equations, however, in most of the situations, such detailed and complex descriptions is unnecessary. To overcome this issue, in many situations it is possible to introduce two approximations and simplify the model: (1) the slowly varying approximations (SVEA), it is used for monochromatic beams with envelopes that vary very slowly when compared with the frequency and wavenumber of the carrier; (2) the paraxial approximation, where the envelope is assumed to be constant over time. Under this approximations, the Maxwell equations are reduced to nonlinear Schrödinger equation

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\nabla_{\perp}E + k_0\Delta n E = 0, \tag{7}$$

where E is the envelope, z is the propagation direction and $\nabla_{\perp}$ is taken along the dimensions perpendicular to z. The Δn represents the nonlinear potentials and these can take different forms according to the material that is being considered. In particular, the nonlinearities can be of different types

- **Kerr Nonlinearity** - $\Delta n = \alpha|E|^2$ - This nonlinearity occurs in crystals with $\chi^{(3)}$ susceptibility [46]. This nonlinearity can be focusing ($\alpha > 0$) or defocussing ($\alpha < 0$).
- **Higher-order Nonlinearities** - $\Delta n = \sum_i \alpha_i |E|^{2i}$ - This is identical to the Kerr nonlinearity, however now higher orders of the susceptibility $\chi^{(i)}$ must be considered [58].
- **Saturation Nonlinearity** - $\Delta n \propto 1 - \frac{1}{1+|E|^2/|E_{sat}|^2}$ - This type of nonlinearity saturates for fields amplitudes above $|E_{sat}|^2$ and it is present in photo-refractive materials [85] or semiconductor doped fibers [86].
- **Linear latice** - $\Delta n = F(|E|^2) + V(\vec{r})$ - This nonlinearity may correspond to one of the previously, with the addition of a confinement or perturbative potential [46].
- **Nonlinear latices** - $\Delta n = R(\vec{r}) \sum_i |E|^{2i}$ - This nonlinearity corresponds to systems with inhomogeneous susceptibilities [86].

There are other types of nonlinearities, which for the present context they are not very important, however a detailed description of all of them can be found in [86] and in the references therein.

With only local nonlinearities, these optical systems can emulate equation 1 with a wide variety of nonlinear terms, which allowed the development of optical analogues that range from superfluidity effects [46, 59] to gravity phenomena [25, 87, 88].

6.2 Nonlinear Nonlocal Media

The field of analogue models expanded with the introduction of support mediums with nonlocal nonlinearities. These types of nonlinearities brought to these experiments the possibility of having long-range interactions, which are important when describing gravitational phenomena, for example. Different materials have this nonlocal character, ranging from thermo-optical materials [72] to nematic liquid crystals [69], and including Bose-Einstein condensates [89], quantum gases [90, 91] and photorefractive materials [92]. Among these, we will focus on the thermo-optical materials, and on the nematic liquid crystals. These last ones are very interesting for optical analogues, since they offer external mechanisms that allow us to easily control their properties and consequently fine-tune the emulations.

In the nonlocal media, the envelope is described by an equation identical to 7, but now the nonlinear potential Δn is more complex. The nonlocal character of media is usually calculated in two different ways, that in some cases are equivalent. In the first approach, the Δn is calculated through a Poisson-like equation

$$\nabla_{\perp}^{2}\Delta n + c\Delta n = d|E|^{2}, \tag{8}$$

where c and d are constants. For some materials, the c parameter may not exist, which means that the nonlocal term has an infinite range, however, there are others where $c \neq 0$. The presence of this term limits the range of interaction of the nonlocal potential and can be interpreted as a Debye-like screening effect that determines the characteristic nonlocal length around the source field E. In the second approach, the Δn is given by a convolution

$$\Delta n \propto \int G\left(\vec{r} - \vec{r'}\right)|\psi\left(\vec{r'}\right)|d^{3}\vec{r'}, \tag{9}$$

where the kernel G represents the material being considered, which corresponds to the Green's function of equation 8. This formulation is useful, since certain kernels may be approximated with more friendly functions that facilitate the theoretical calculations.

Thermo-optical materials [93] are known for having a refractive index that depends on the temperature accordingly to

$$\Delta n = \beta\Delta T, \tag{10}$$

where β is the thermo-optical coefficient and ΔT is the temperature variation relative to some reference value. In the presence of a heat source, which in the current case is $\alpha|E|^{2}$, where α is the linear absorption coefficient, the temperature distribution is governed by the heat equation

$$\rho c_{P}\frac{\partial\Delta T}{\partial t} - \kappa\nabla^{2}\Delta T = \alpha|E|^{2}, \tag{11}$$

where ρ is the density of the medium, c_P is the specific heat capacity and κ is the thermal conductivity. The heat equation 11 can be simplified by assuming that the diffusing along the direction of propagation is negligible when compared with the diffusing along the transverse dimensions [94], and that the temperature distribution is stationary. Under these conditions, the refractive index is given by

$$\nabla_{\perp}^{2}\Delta n = -\frac{\alpha\beta}{\kappa}|E|^{2}. \tag{12}$$

Thus, in thermo-optical materials, the beam heats the medium, and since the temperature distribution is governed by the heat equation, the nonlinear response is nonlocal. Equation 12 assumes that the medium is infinite in the transverse dimensions ($c = 0$), however this can be corrected by considering a distributed loss model [29]. This model considers $c = -1/\sigma^2$, where $\sigma \approx D/2$ measures the nonlocal length of the system, and D is the diameter of the medium [29]. Notice that this model only affects the nonlinear potential, the envelope equation still assumes that the medium is infinite.

Thermo-optical materials offer mechanism that allow to adjust the model parameters. For example, it is possible to modify the value of α by adding dopants or changing the density [95, 96]. Furthermore, by controlling the temperature of the material [97], changing the angle of incidence of the beam on the material [97] or the intensity profile [73], or even by changing the geometry of the cell that holds the thermo-optical medium [98], can also be used to tune the properties of the medium. A drawback of thermo-optical systems is that their nonlinear effect emerges from the observation of the beam by the medium, and this naturally leads to losses. By combining equations 7 and 12, we have a medium that is capable of emulating the desired model. Furthermore, it also possible to have local nonlinearities with whit these systems [30, 31], which extends their capacity of emulating the generalized nonlinear Schrödinger model. Indeed, they have been used as a platform to emulate dark-matter models [30,31], Boson-stars [29] or superfluidity phenomena [72, 73], to name a few.

Nematic liquid crystals (NLCs) are another nonlocal medium capable of emulating the general Schrödinger model and that can be used for developing optical analogues. NLCs are composed of elongated organic molecules that randomly flow in a liquid and they tend to align along a common direction, which is known as molecular director $\hat{n}$ that makes an angle θ with the direction of propagation [69], see Figure 4. This natural alignment makes the NLCs uniaxial positive with the optical axis parallel to the molecular director. The relative alignment between the beam propagation direction and the molecular director, will dictate the refractive index perceived by the beam and the direction of $\hat{n}$ can be modified by different mechanisms, represented in Figure 4, that can be used in simultaneous:

A) by applying a low-frequency electric field it is possible to tune the angle θ for the desired value;

B) by choosing how the molecules are anchored to the boundaries of the cell that contains the NLCs;

C) by propagating a light beam inside the nematic liquid crystal cell.

Mechanisms A and B are used to impose a reference orientation to the molecular director, which corresponds to an angle θ_0, where B is somehow limited on the angles θ_0 that can originate, A can be used to fine-tune this value. The light beam propagating in the cell will generate dipoles along the organic molecules, assuming that the beam is polarized in the same plane as the molecular director, and will force them to rotate, mechanism C. If the intensity of the beam is high enough to overcome the inertia of the molecules to rotate, the Freédericksz threshold [69], this will induce a rotation on the molecular director of θ'.

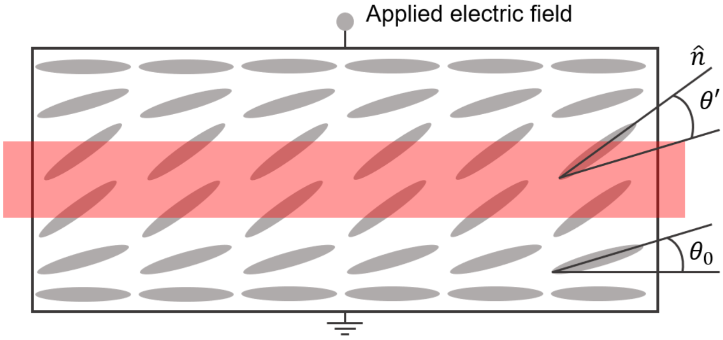

Figure 4: Schematic representation of a nematic liquid crystal cell. The organic molecules are anchored to the boundaries of the cell and an applied electric field is used to establish a base orientation of θ_0 on the molecular director. The propagation of a light beam inside the cell represented a red, induces an extra rotation of θ' in the molecular director.

The value of θ_0 will only be important in the definition of the system parameters, as we will see, and the nonlinear potential will be due to the θ', since $\Delta n \propto s\theta'$, where the NLC is self-focusing for $s = 1$ self-defocusing for $s = -1$. Since the molecules are elongated and can interact with their neighbors, it is easy to understand that this change is nonlocal. It is possible to show that the distribution of θ' is given by

$$\nu\nabla^2\theta' - 2q\theta' = -2|E|^2, \tag{13}$$

where parameter q is related to the pre-tilt of the molecular director (θ_0) and ν is the normalized elastic coefficient that measures the nonlocal character of the medium. In particular ν small ($\nu \to 0$) indicates a local medium, while a larger ν corresponds to a highly nonlocal NLC [99, 100]. Although not explicitly indicated, the parameters ν and q depend on the value of θ_0, and it is here that resides the true value of nematic liquid crystals as a support medium for optical analogues, since by applying an external electric field is it possible to adjust the system parameters as desired [99, 101]. This allows, for example, with the same experimental setup to test a certain model under different regimes. Furthermore, the nonlinear potential is polarization-dependent and this can also be used as a tunning mechanism [102, 103]. The anchoring conditions also offer some tunability, although this mechanism is more restricted and can only be changed during the fabrication of the NLCs cells.

Combining equations 7 and 13 results in another medium capable way of emulating the general nonlinear Schrödinger equation, with the advantage that with external controls it is possible to adjust the model as desired. By exploring some thermal properties of nematic liquid crystals, is possible to have a system with competing nonlinearities governed by two Poisson-like equations [104, 105], and this configuration offers the possibility to explore self-organization of light [71], for example.

Thus, Nematic liquid crystals stand as suitable candidates for developing optical analogues, and there is almost no work regarding this material as a support medium. Our group, to our best knowledge, was the first to use NLCs in the context of optical analogues, in particular, to study superfluidity phenomena [70]. However, there is much work to be done with NLCs, not only from a theoretical perspective, but it is also necessary to experimentally validate them as suitable media for anlogue systems. We pretend to follow this research direction during the Ph.D., and how we intend to do it will be detailed in the next section.

7 Thesis Proposal

The research program of this Ph.D. aims to explore experimentally accessible systems that share identical or similar models with phenomena that are hard or even impossible to recreate and characterize in the laboratory under controlled conditions supported by theoretical and numerical studies. In particular, we will identify and model theoretically new media capable of supporting these so-called optical analogues, and then produce experiments to validate them. The primary objectives of this doctoral research program can be organized into:

1) Explore and identify different optical systems, their models, and how they can be used to implement optical analogues and study fundamental phenomena. It corresponds to a review of the state-of-the-art of optical media capable of supporting optical analogues and their properties. Ultimately, the idea is to combine and improve existing models, while identifying interesting phenomena and applications

for analogue models.

2) Identify base equations that describe the systems explored in the objective 1, as well as physical mechanisms not contemplated in these equations that are important in the description of the system, and identifying how to model them numerically. This objective aims the implementation of numerical solvers based on GPGPU supercomputing (with the Arryfire framework) that will later serve as a numerical testbed to study the optical systems and design numerical analogues. The main solver to develop is around the multidimensional solver of the generalized Schrödinger (including nonlinear, dissipative, and nonlocal effects) with the possibility of coupling different Schrödinger fields and adding a variety of source terms. In addition to addressing wider classes of models and including multiple field coupling, the main challenge is to balance model complexity/completeness with available simulation resources. Other solvers may also be necessary.

3) Use the previous models and numerical tools explored in objectives 1 and 2 to investigate optical analogues and study fundamental phenomena in new optical systems. The methodology is to employ numerical simulations as a testbed to investigate phenomena identified in objective 1, explore different parameter domains and physical processes, to test the solvers, and identify new implementations for optical analogues. The main challenge will be establishing the relation between the original and the analogue optical system and producing a numerical model of the analogue both representative and resource-effective.

4) Apply the developed numerical tools to propose experimental setups for optical analogues and establish these systems as test-beds for quantum-optical simulators. Also, we will implement simulations of experimental setups for optical analogues (hence the importance of developing high dimensionality solvers) not only to optimize experimental designs but also to identify the relevant parameters and explain experimental results obtained in objective 5.

5) Implement laboratory demonstrations at Glasgow, integrated into the Extreme-Light research group, of some of the systems explored in the previous objectives. The methodology is to learn the experimental procedures for these types of experiments and test some selected optical analogues based on the results of the simulations of objective 4, providing their validation as analogue simulators. Besides the technical difficulties of experimental work, the main challenge will be to relate the results with those of the simulations and ultimately with the phenomenology

of the original system.

Besides the core objectives of the program, we considered two complementary sub-objectives whose realization cannot be guaranteed given the high level of uncertainty involved and are not a requirement for the success of the research project, but which can augment its outreach:

A) Use the access to the IBM Quantum Computer (QC) (via QuantaLab as part of the IBM-Portugal program) to verify the possibility of transferring the algorithms developed initially for GPGPUs to a QC, exploring that both the QC and the GPGPU at a fundamental level perform matrix operations. This must be considered as highly exploratory/speculative since quantum computing and quantum algorithms are still relatively new and there are many obstacles to overcome.

B) Implementation of a pilot optical analogue laboratory, mainly to study these optical analogues in Nematic Liquid Crystal, at INESC-TEC based on the experimental skills acquired at Glasgow.

Objectives 2, 3 and 4 will be supported by the computational resources available at INESC-TEC/CAP (specifically, a high-performance computing GPGPU system) within the quantum simulation research group [70, 80, 83, 106–108]. Objective 5 will be performed at the University of Glasgow in Extreme-Light research group and it will be supported by the experimental infrastructures and vast experience in optical analogues [24, 29, 35, 72, 73].

7.1 State of Development

The research plan presented in the previous section constitutes a continuation of the work initiated during my Masters's thesis, where we explored the possibility of using Nematic liquid Crystals in the context of optical analogues, more precisely for producing superfluids of light [70]. Following this work, most of objective 1 has already been completed and is being complemented with the new works that are being published, and we are continuing to explore nematic liquid crystals in the context of optical analogues. In particular, we are currently exploring two different analogue systems that overlap with each other: superfluid effects, and gravity analogues. More precisely, with the former we are exploring the drag-force suppression, vortex nucleation, and the formation of light structures in NLCs with competing nonlinearities. With the gravity analogues, we are studying the possibility of NLCs to emulate curved space-times. Some theoretical and numerical experiments have already been conducted for both systems, which constitutes some development towards objective 3. Furthermore, we foresee other systems that we aim to study within the context of NLCs, such as quantum turbulence with two fluids model [109] and as a support medium for reservoir computing [67, 68].

Regarding objective 2, an initial version of the general nonlinear Schrödinger equation solver based on GPGPU supercomputing has already been produced in our group [80]. Furthermore, we have also developed solvers for other systems, for example for the Maxwell-Bloch model [106], and the complete Maxwell equations with matter interactions [107, 108]. More recently, we began the development of a high-performance solver for the Maxwell-Bloch model for distributed memory systems based in MPI and OpenMP, which may also be useful for the numerical studies. These previous works reduce the risk associated with this objective by guaranteeing that at least a simpler version of the codes already exist and can support the following objectives. Moreover, they allowed us to gain experience with the underlying technologies of GPGPU and High-performance supercomputing, which now facilitates the development of new solvers. The main challenge now is to optimize the solvers and couple them with other models.

In parallel, we are exploring the possibility of reusing the solvers for other problems in physics. In particular, we have a collaboration with a team of the Centro de Física da Universidade do Porto, where we are exploring the utilization of the generalized Schrödinger formalism to test the impact of new extensions of the Theory of General Relativity in the dynamics of systems, in particular those based on theories with non-minimal coupling between curvature and matter [84]. We already adapted the solvers for this scenario, and we are using the imaginary-time propagation method to find stationary solutions and study their properties.

8 Conclusion

The field of Analogue computing brings the possibility of experimental test certain physical phenomena under controlled conditions that otherwise would be hard or even impossible to study. This idea assents on the universal character found in Nature, and in particular in Physics, where it is possible to find a wide variety of systems that, under appropriate approximations, are described by a similar mathematical model. Among these different systems, some of them are capable of being implemented in the laboratory, which, by proper mapping, permits to emulate in the laboratory the remaining ones. The success of this field relies on finding materials capable of mimicking the problem of interest and with tunable mechanisms that allow testing the emulation in different conditions. This topic was discussed in this essay.

We began by discussing the development of this field as well the current state of research, with special focus on gravity analogues and on quantum fluids of light in bulk nonlinear materials, where the latter works not only as an analogue system by itself but also as a support for other analogue models. Next, we presented the generalized nonlinear Schrödinger equation as the base mathematical model to explore and implement analogue models. Furthermore, we also presented the fluid interpretation that naturally arises from the Schrödinger equation through the Madelung transformation and how this can be used to emulate curved space-times. Since the generalized nonlinear Schrödinger equations are hard to work with, we presented the Split-step Fourier method as the indicated numerical method to tackle this model and briefly discussed the GPGPU paradigm on which the solvers are implemented. Following this discussion, we focused our attention on the nonlinear optical materials capable of emulating the generalized nonlinear Schrödinger model. In particular, we focused on two different classes of materials: the local and nonlocal mediums. Regarding the latter, we focused the analysis on thermo-optical systems, commonly used in the context of analogue models, and on nematic liquid crystals, which are another medium capable of supporting optical analogues and that are almost unexplored in this context. In parallel, we explored the tunable mechanisms offered of both systems. Finally, we outlined the research objectives of the Ph.D. and the current state of development of the ones already initiated.

Analogue computing is still a relatively new field with interesting opportunities for expansion, in particular for systems based on the quantum fluids of light. Furthermore, the utilization of nematic liquid crystals as a support medium for these systems is a direction almost unexplored, which, in combination with the tunable opportunities offered by them, constitutes a line of research with great potential. Thus, we pretend to follow this line of investigation which is supported by the work developed and experience gained so far.

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