Introduction to Paraxial Fluids of Light - 1

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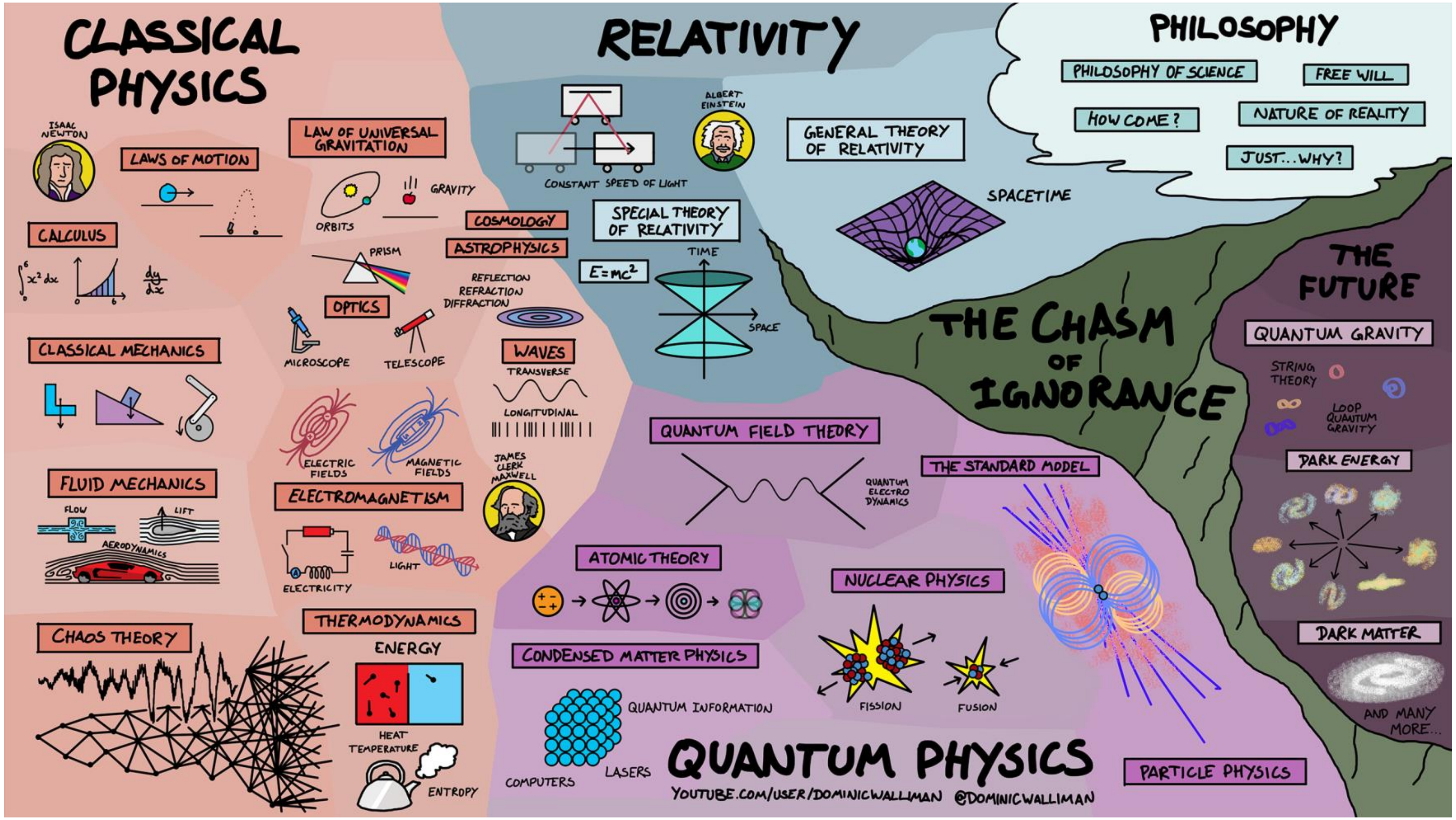
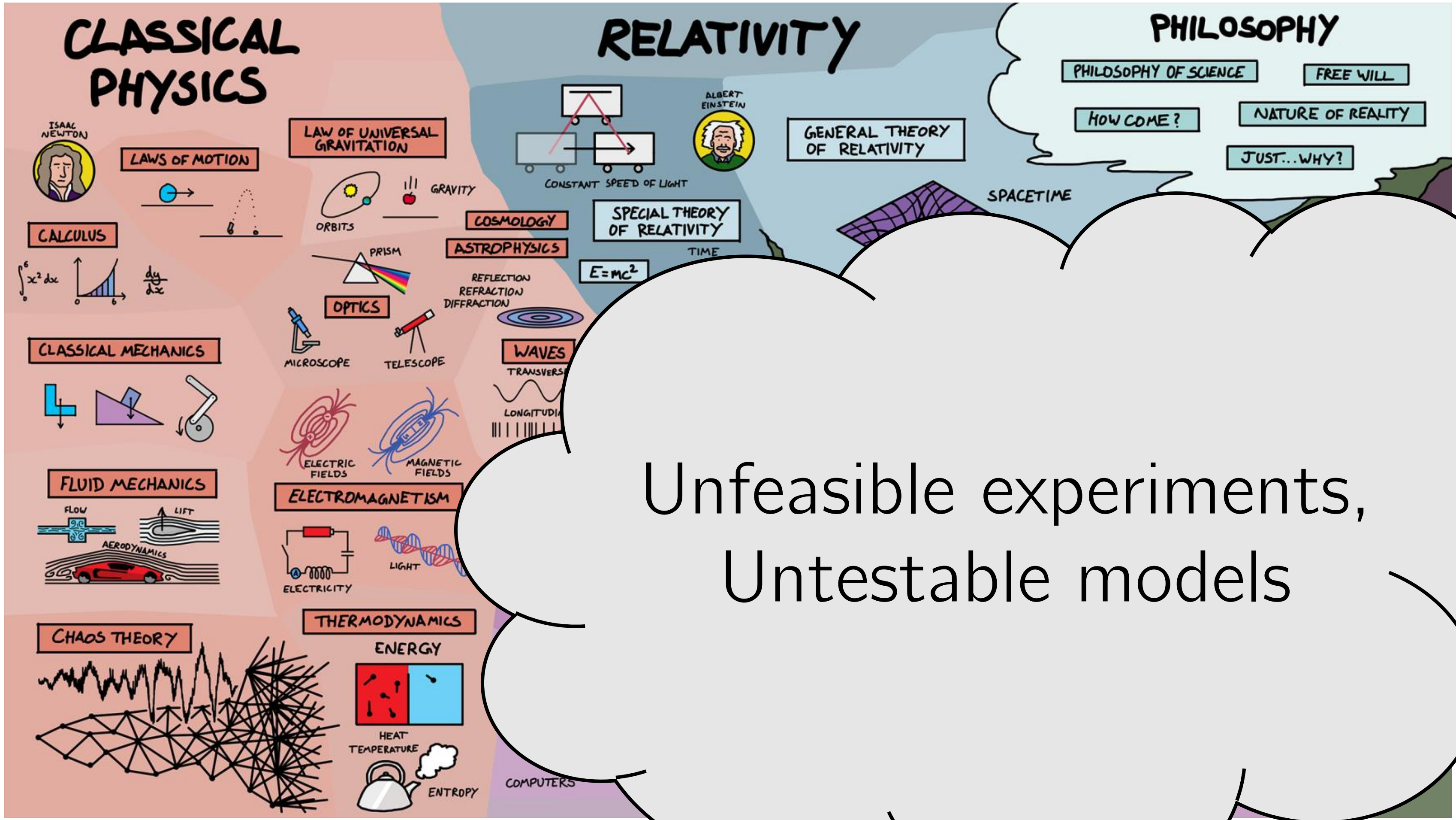
1//Analogue and Quantum Simulations

My Usual Approach

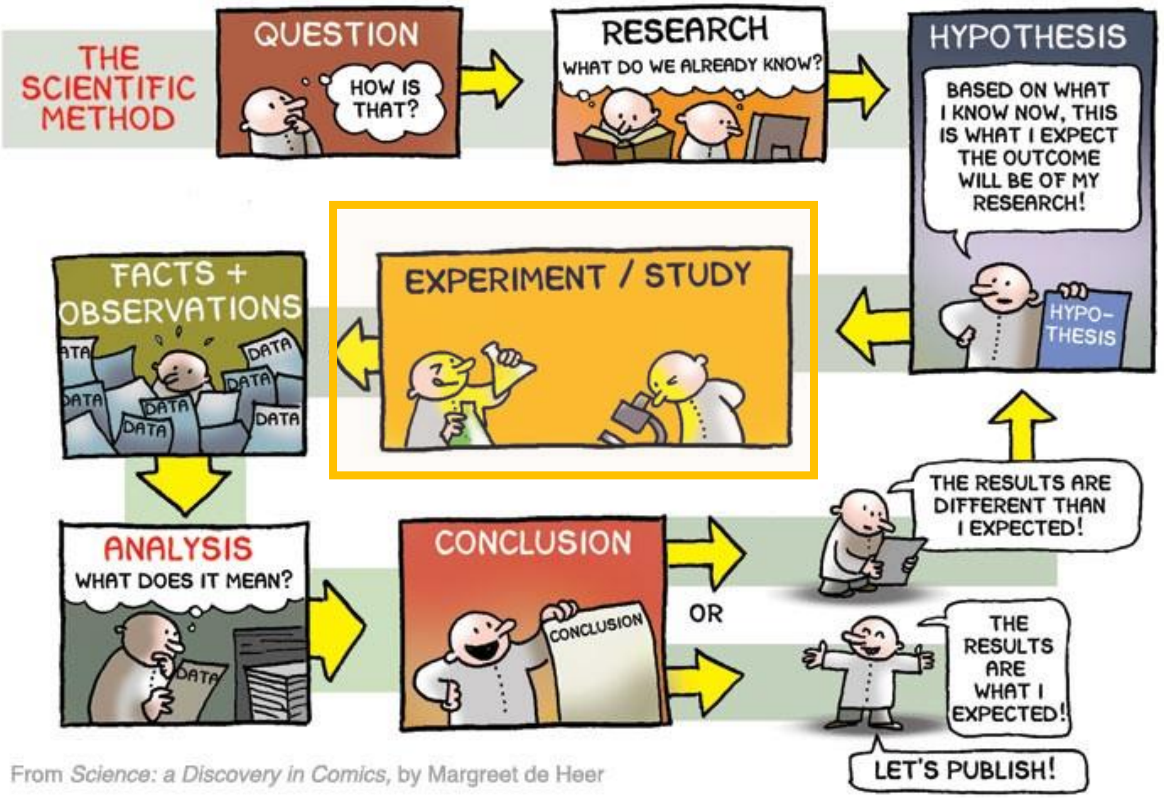


CLASSICAL PHYSICS
ISAAC NEWTON
LAWS OF MOTION
LAW OF UNIVERSAL GRAVITATION
ORBITS
GRAVITY
CALCULUS
COSMOLOGY
ASTROPHYSICS
PRISM
REFLECTION
REFRACTION
DIFFRACTION
OPTICS
MICROSCOPE
TELESCOPE
WAVES
TRANSVERSE
LONGITUDINAL
CLASSICAL MECHANICS
ELECTRIC FIELDS
MAGNETIC FIELDS
JAMES CLERK MAXWELL
FLUID MECHANICS
FLOW
LIFT
AERODYNAMICS
ELECTROMAGNETISM
ELECTRICITY
LIGHT
CHAOS THEORY
THERMODYNAMICS
ENERGY
HEAT
TEMPERATURE
ENTROPY
RELATIVITY
CONSTANT SPEED OF LIGHT
ALBERT EINSTEIN
GENERAL THEORY OF RELATIVITY
SPACETIME
SPECIAL THEORY OF RELATIVITY
E=mc²
TIME
SPACE
PHILOSOPHY
PHILOSOPHY OF SCIENCE
FREE WILL
HOW COME?
NATURE OF REALITY
JUST...WHY?
THE CHASM OF IGNORANCE
THE FUTURE
QUANTUM GRAVITY
STRING THEORY
LOOP QUANTUM GRAVITY
DARK ENERGY
DARK MATTER
AND MANY MORE...
QUANTUM FIELD THEORY
QUANTUM ELECTRO DYNAMICS
THE STANDARD MODEL
ATOMIC THEORY
NUCLEAR PHYSICS
FISSION
FUSION
CONDENSED MATTER PHYSICS
QUANTUM INFORMATION
COMPUTERS
LASERS
QUANTUM PHYSICS
PARTICLE PHYSICS
YOUTUBE.COM/USER/DOMINICWALLIMAN @DOMINICWALLIMAN



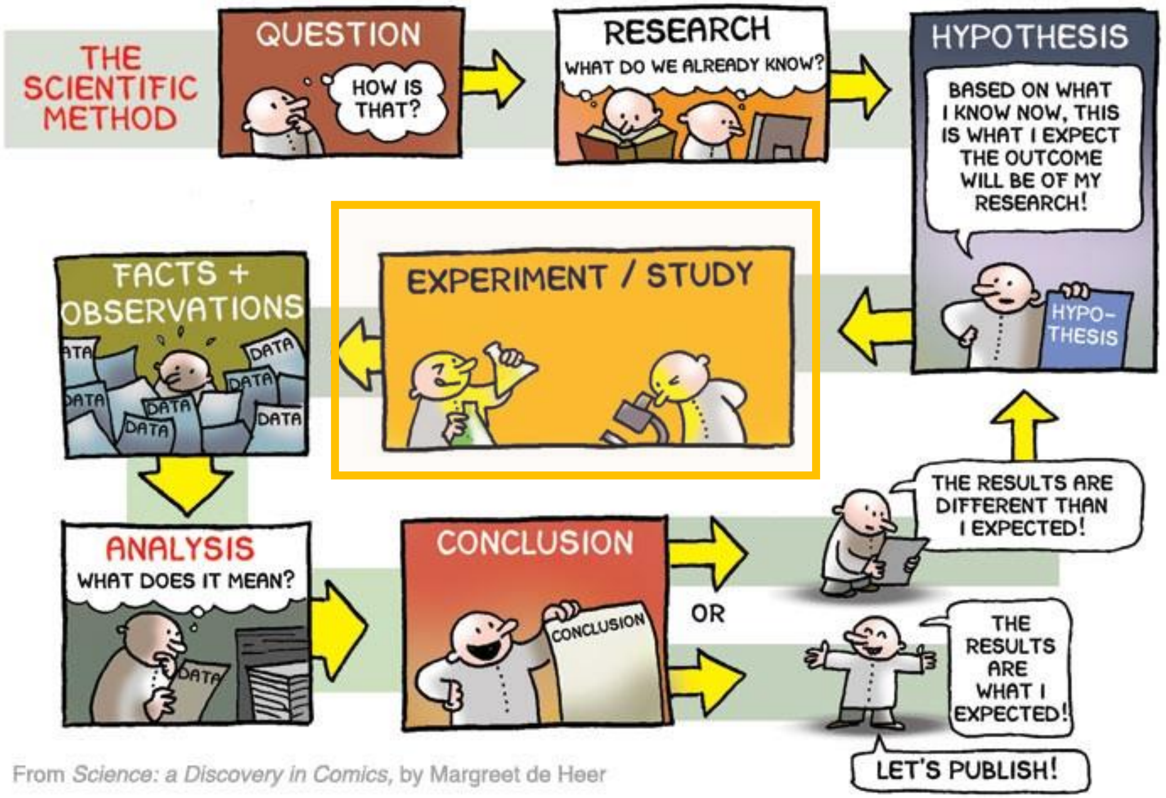
Unfeasible experiments,
Untestable models

Scientific method



From *Science: a Discovery in Comics*, by Margreet de Heer

Scientific method



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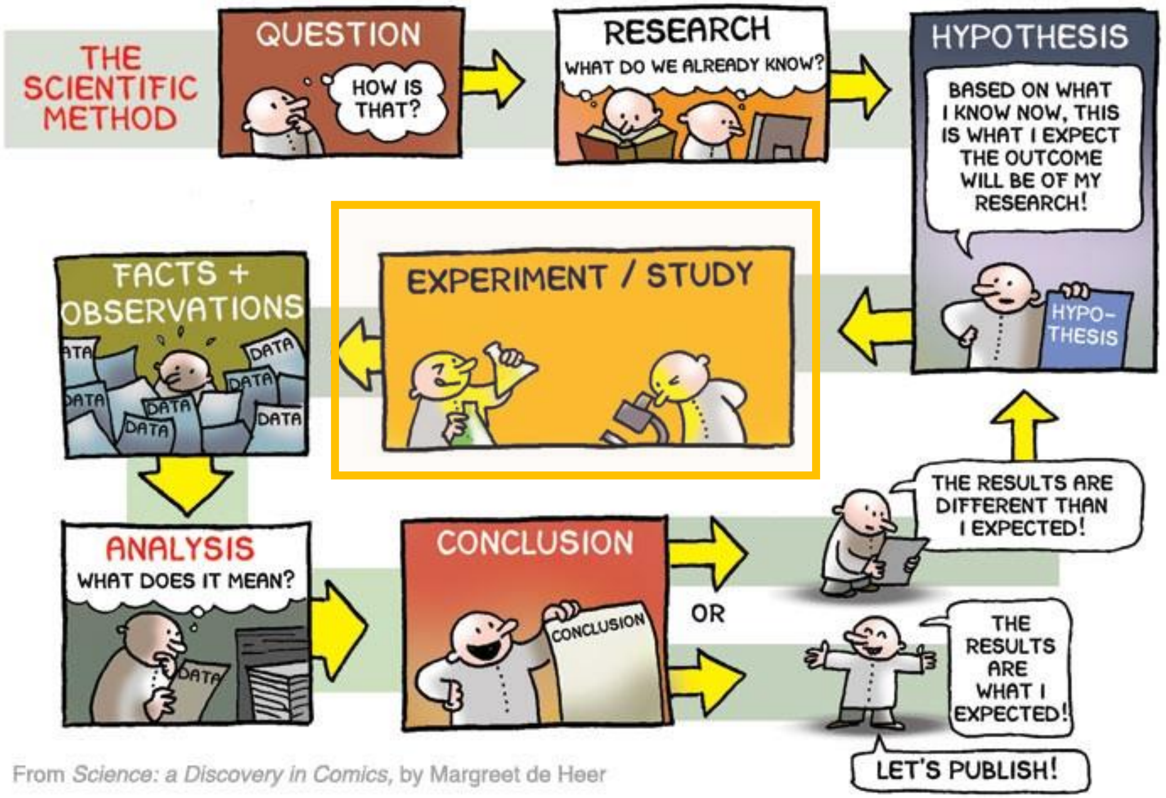
Incompleteness of our research without experiment

Scientific method



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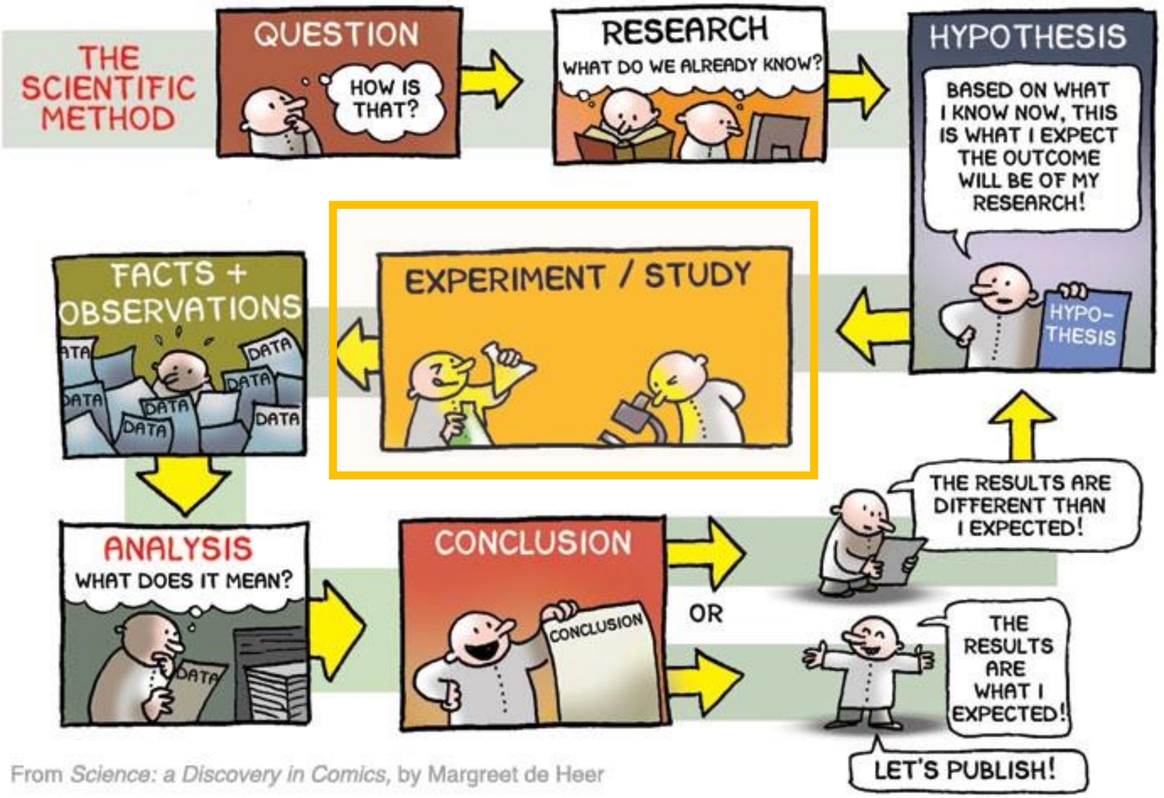
Incompleteness of our research without experiment

Recreate the physics with analogue systems allow to bypass this challenge

Scientific method



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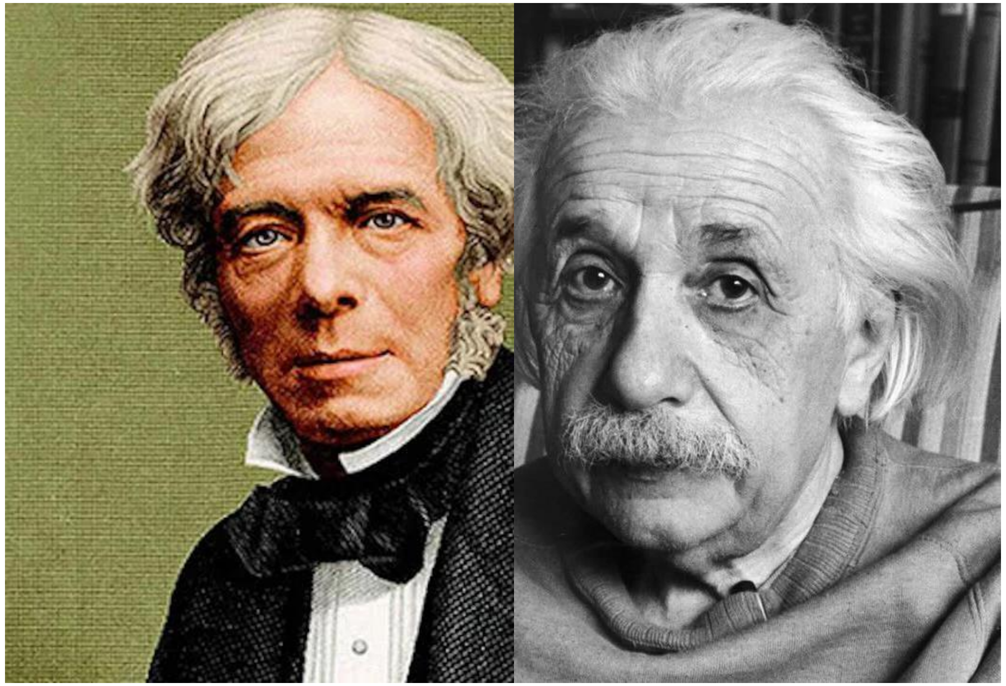
Incompleteness of our research without experiment

Recreate the physics with analogue systems allow to bypass this challenge

Simplistic and Naive

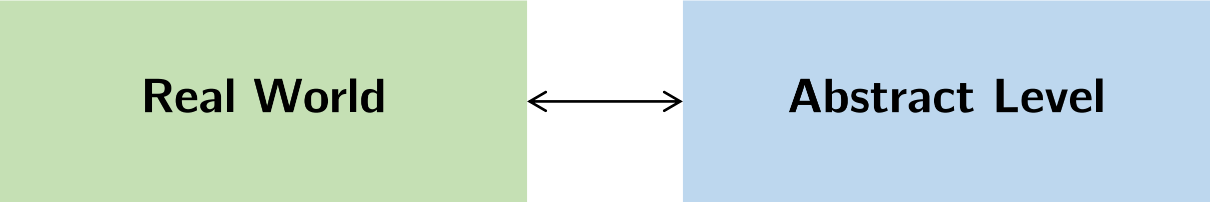


False Dychotomy

experimental vs theoretical physics

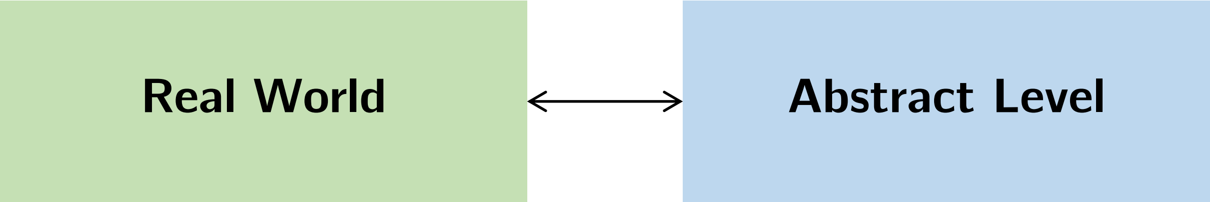
A better perspective

The Role of a Physicist



The Role of a Physicist

Explore how nature works, translating dynamics and phenomena into **mathematical models**



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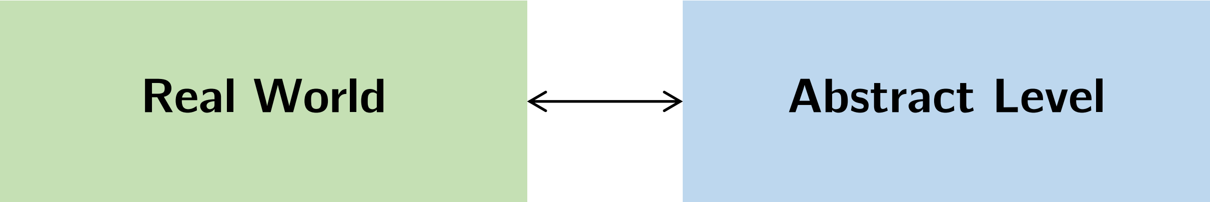
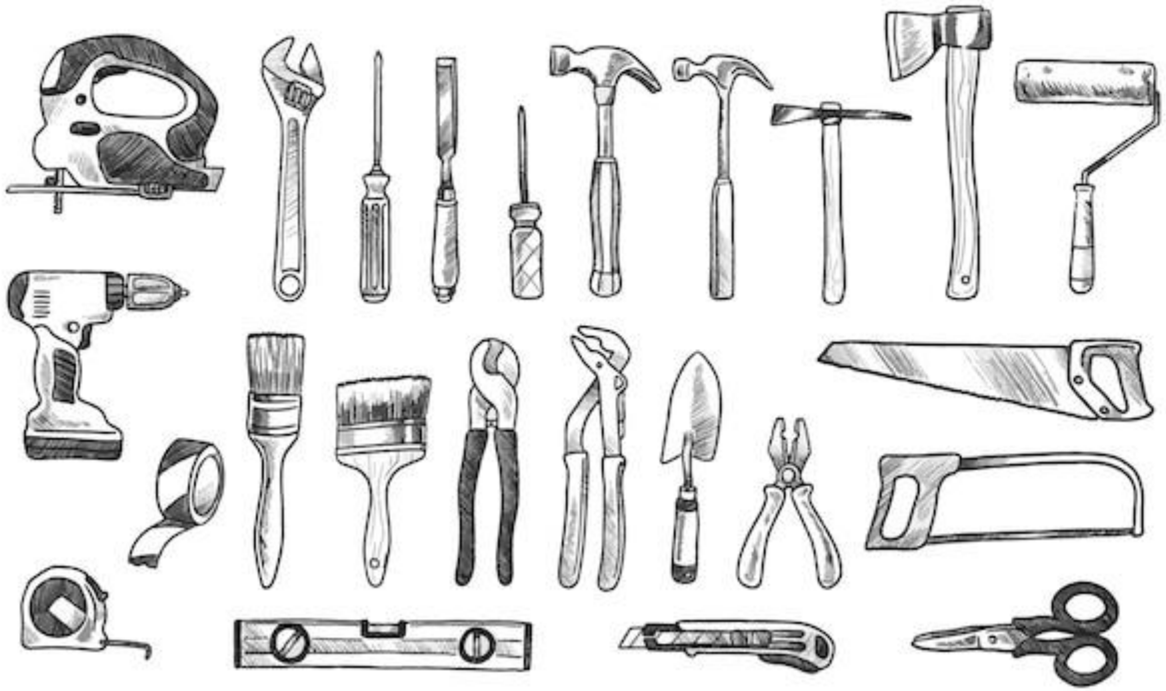


How do we approach a complex problem?



Analogies

The Role of Analogies

A **mental construction** to simplify understanding of complex situations by bridging two well-known concepts

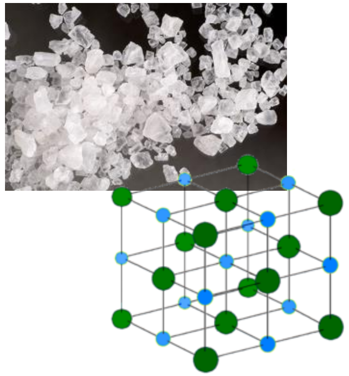
Are to

Are to **?**

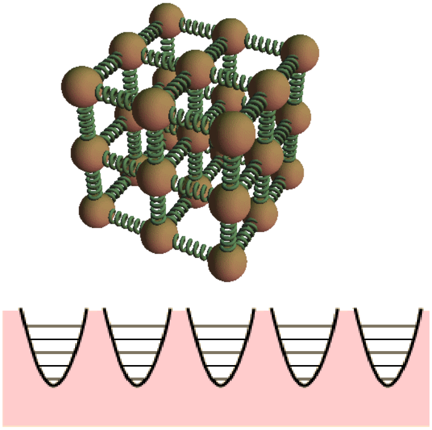
The Role of Analogies

Provide simpler approaches to complex problems

ex. Harmonic Oscillator



Are to

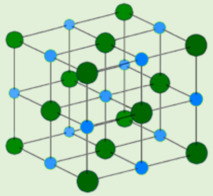


Building Models

Analogies play an important role in building models, bridging **abstract to real-world**

Target System

Real World



Target Model

Abstract Level

$$H_c$$

Building Models

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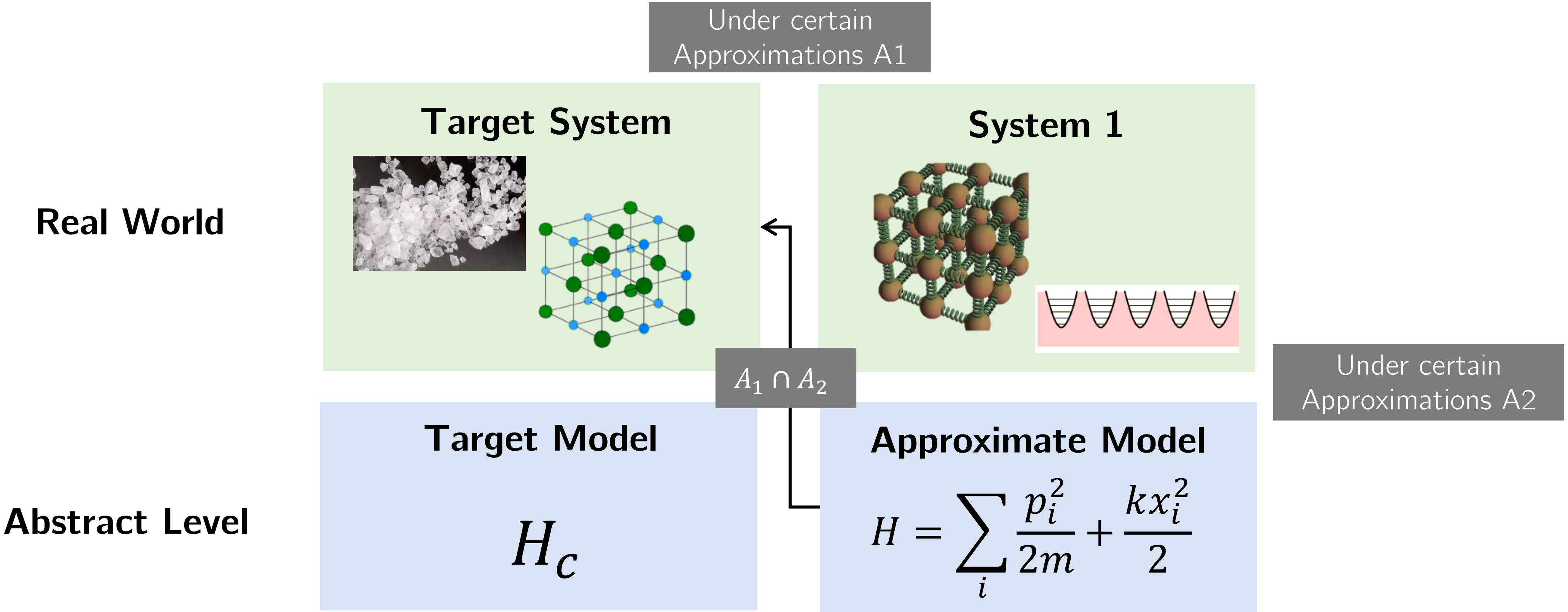

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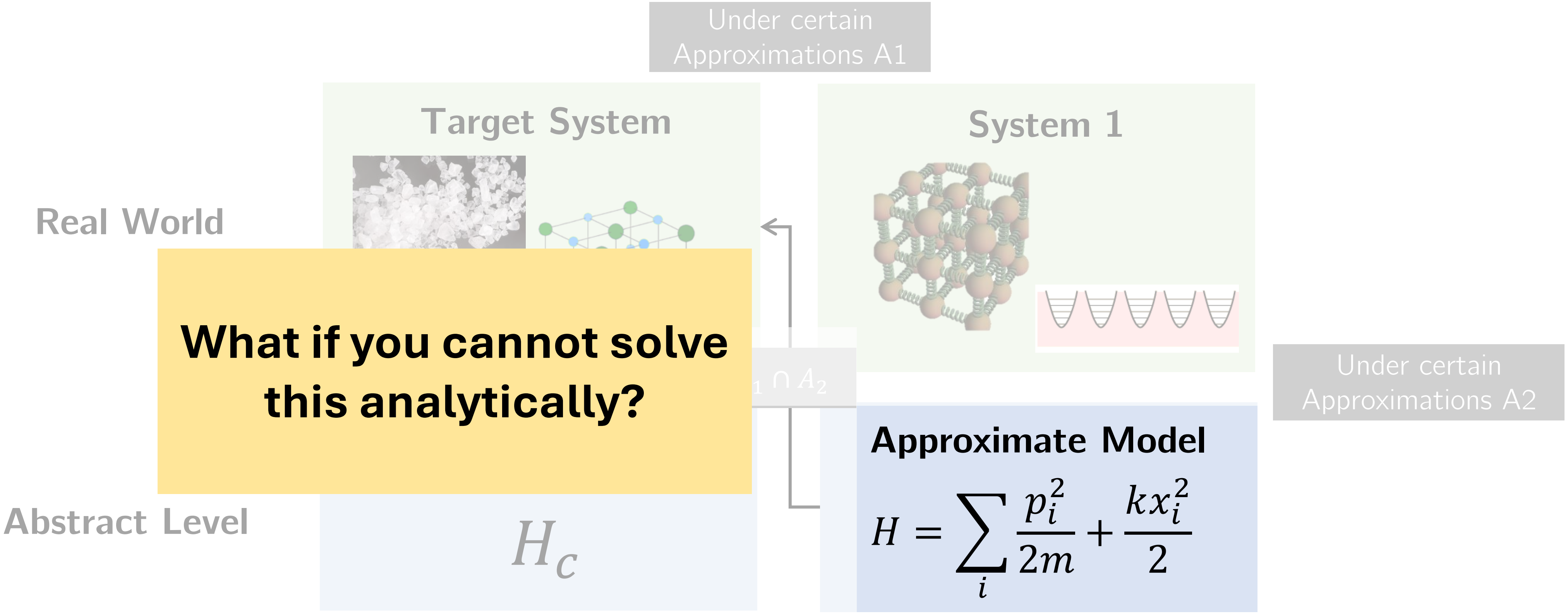
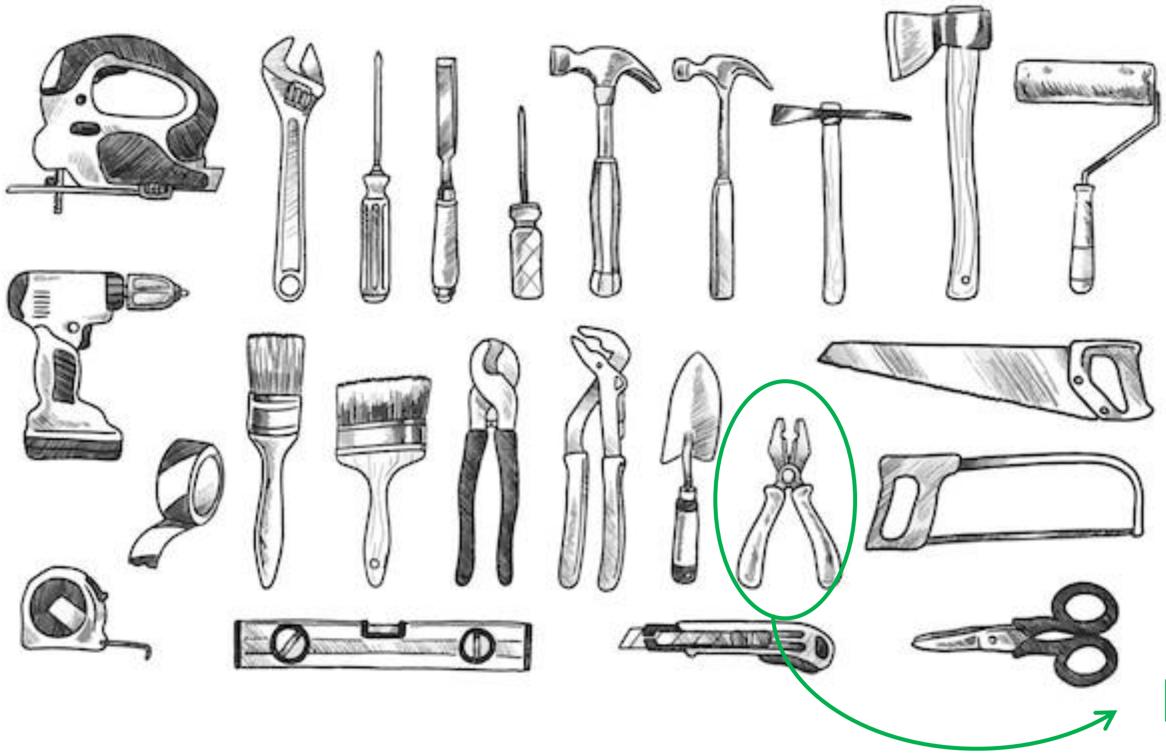
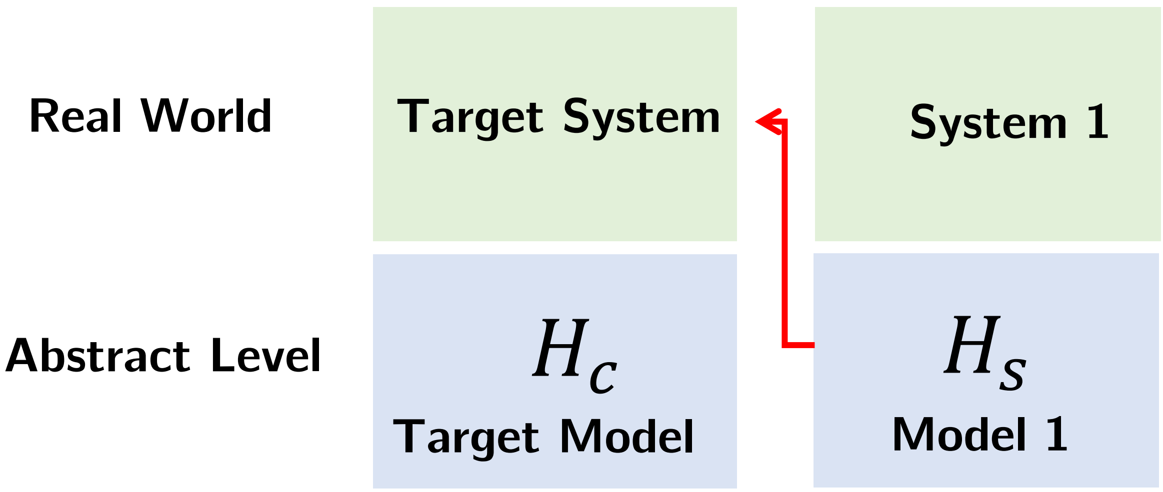


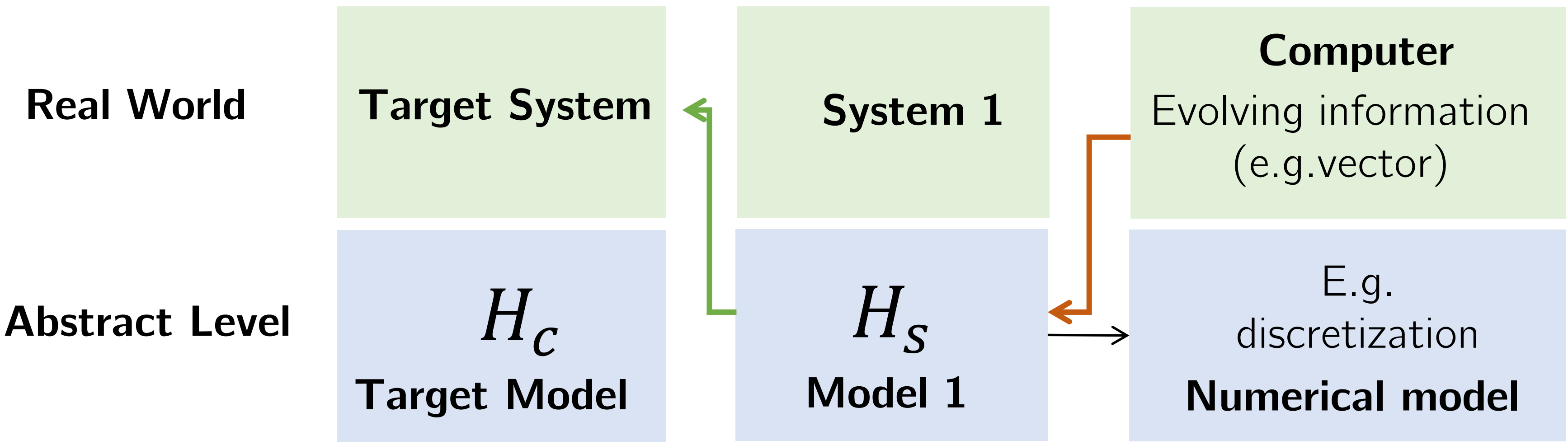


Numerical Simulations

Numerical Simulation



Numerical Simulation



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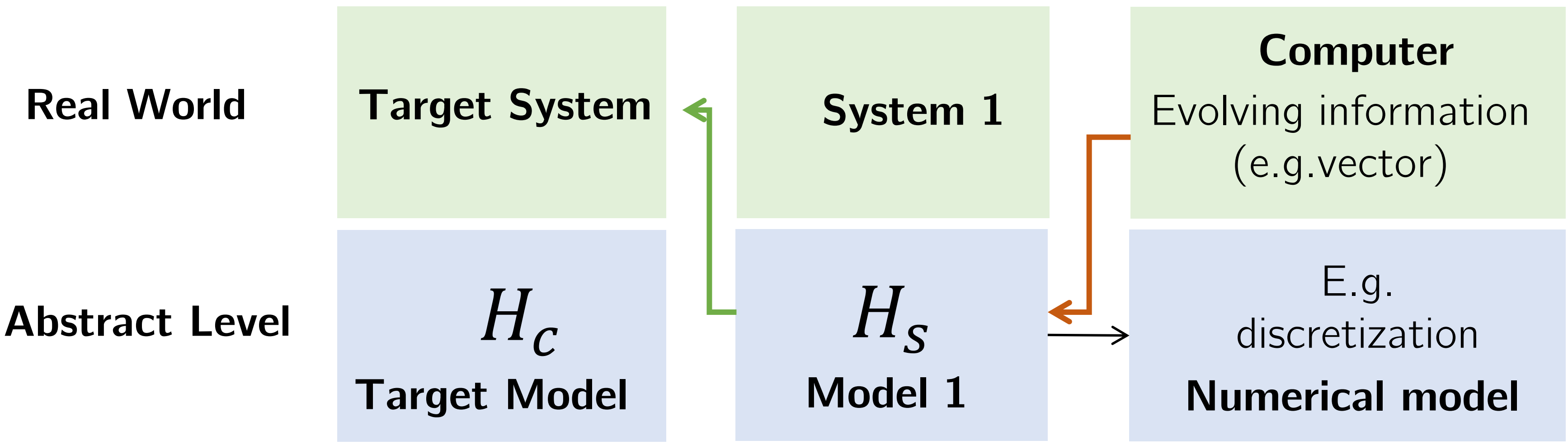
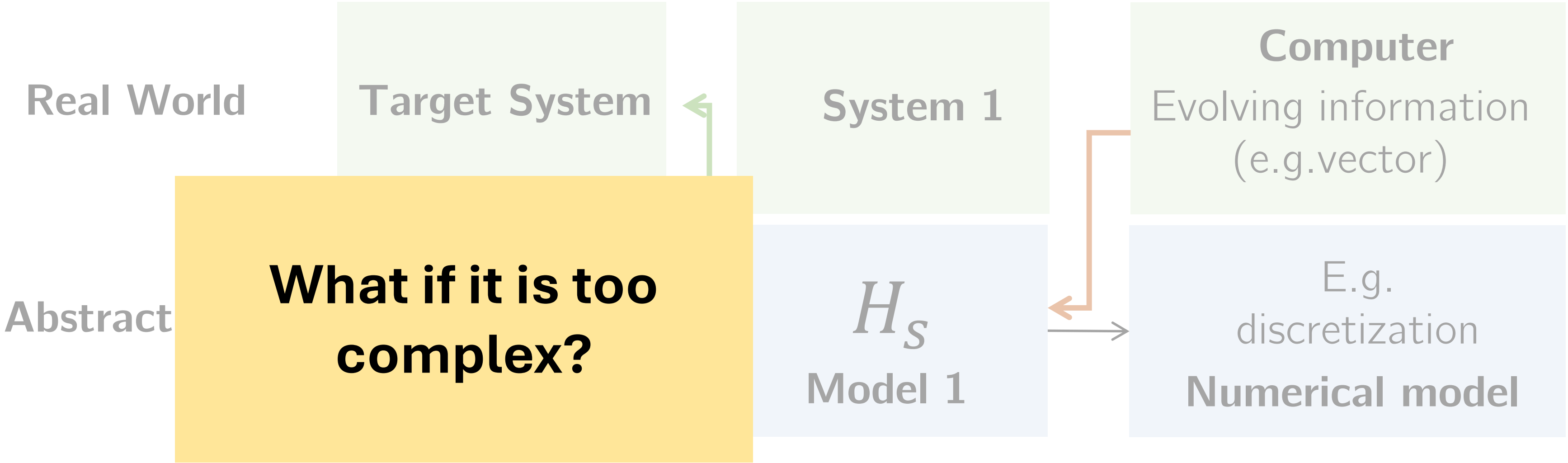


Numerical Simulations

Simulation: utilize a system to provide solutions for a model – from **real-world to abstract level**

Numerical Simulation



Numerical Simulations

Simulation: utilize a system to provide solutions for a model – from **real-world to abstract level**

:(

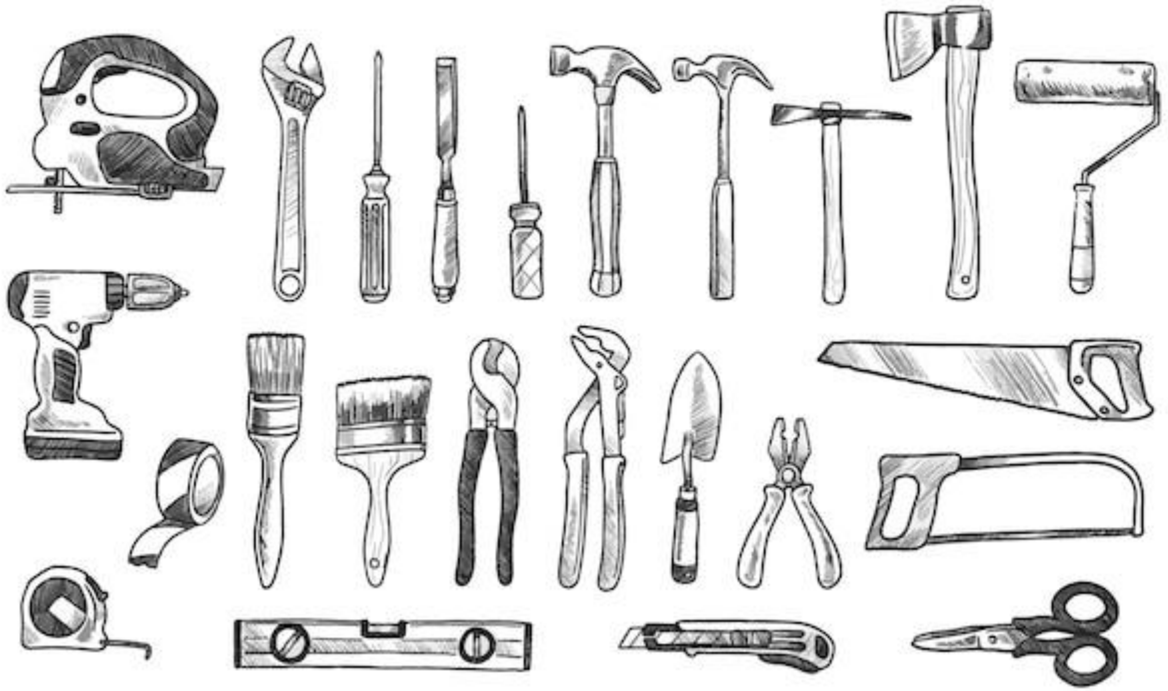
Your PC ran into a problem and needs to restart. We're just collecting some error info, and then we'll restart for you.

20% complete

For more information about this issue and possible fixes, visit https://www.windows.com/stopcode

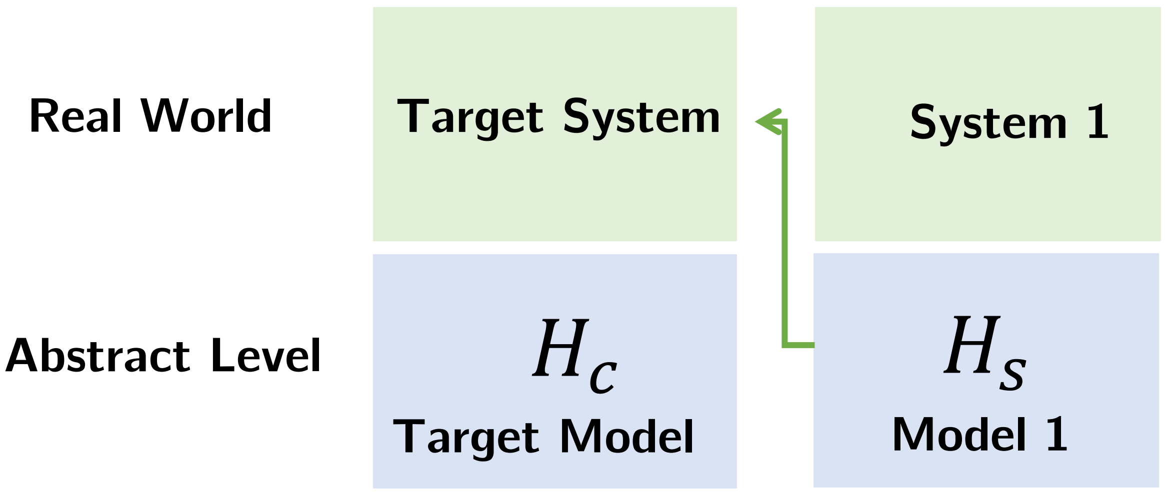
If you call a support person, give them this info:

Stop code: CRITICAL_PROCESS_DIED

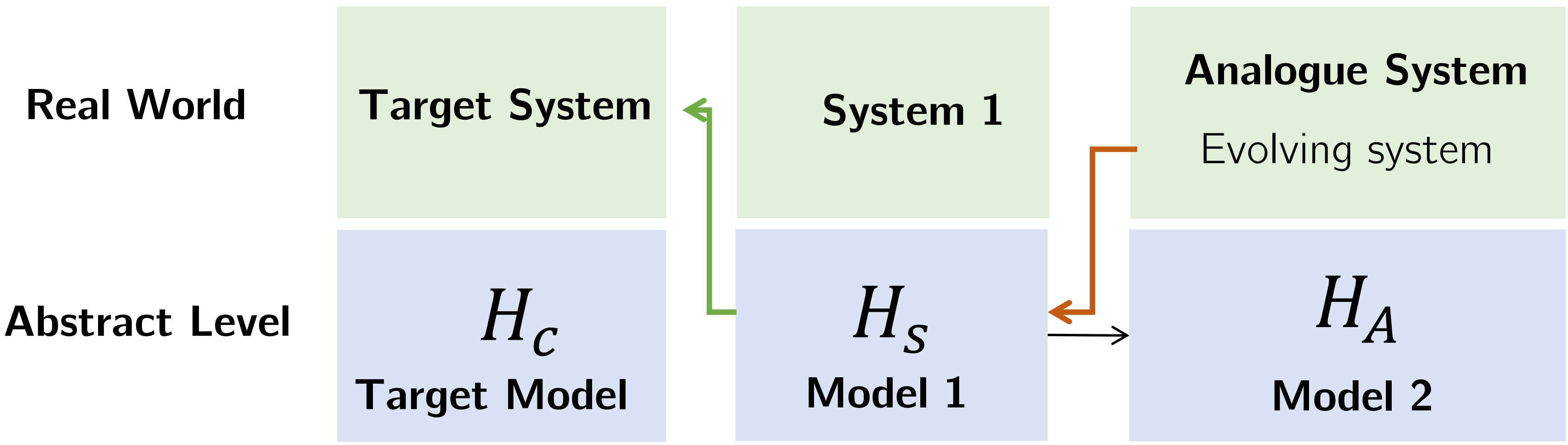


Analogue Physical Simulations

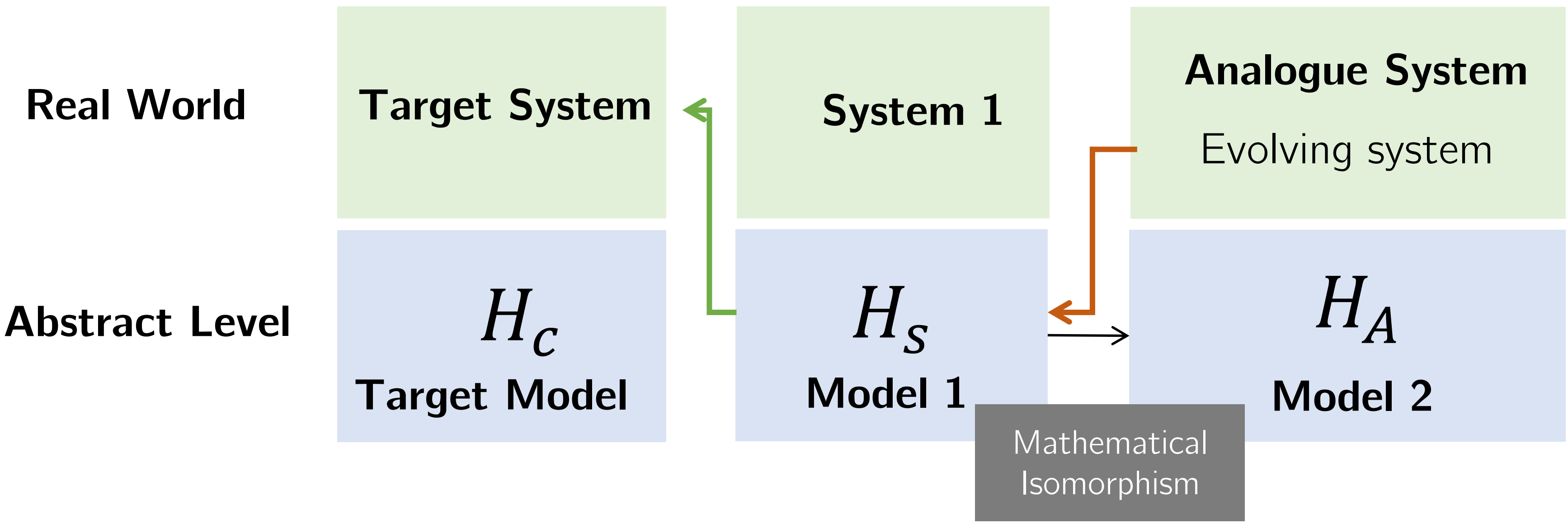
Analogue Physical Simulation



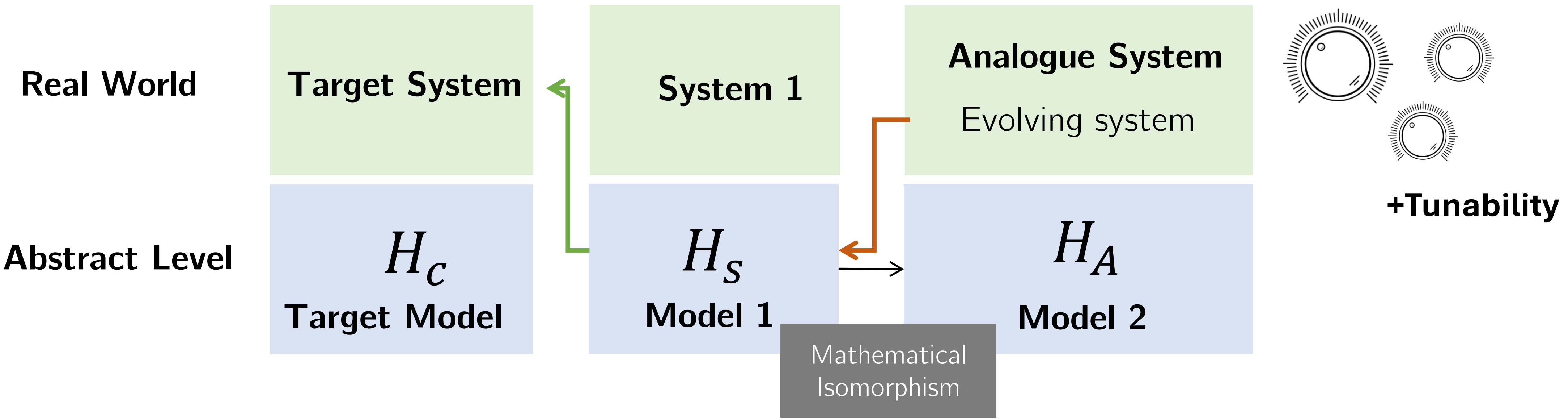
Analogue Physical Simulation



Analogue Physical Simulation

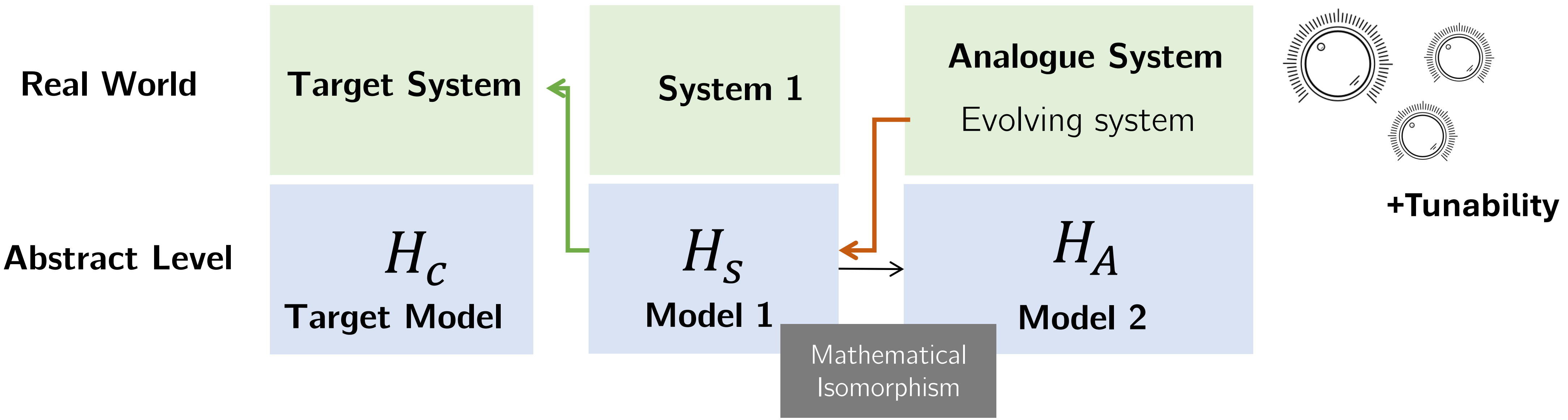


Analogue Physical Simulation



Analogue Systems can also be utilized as Simulators

Analogue Physical Simulation

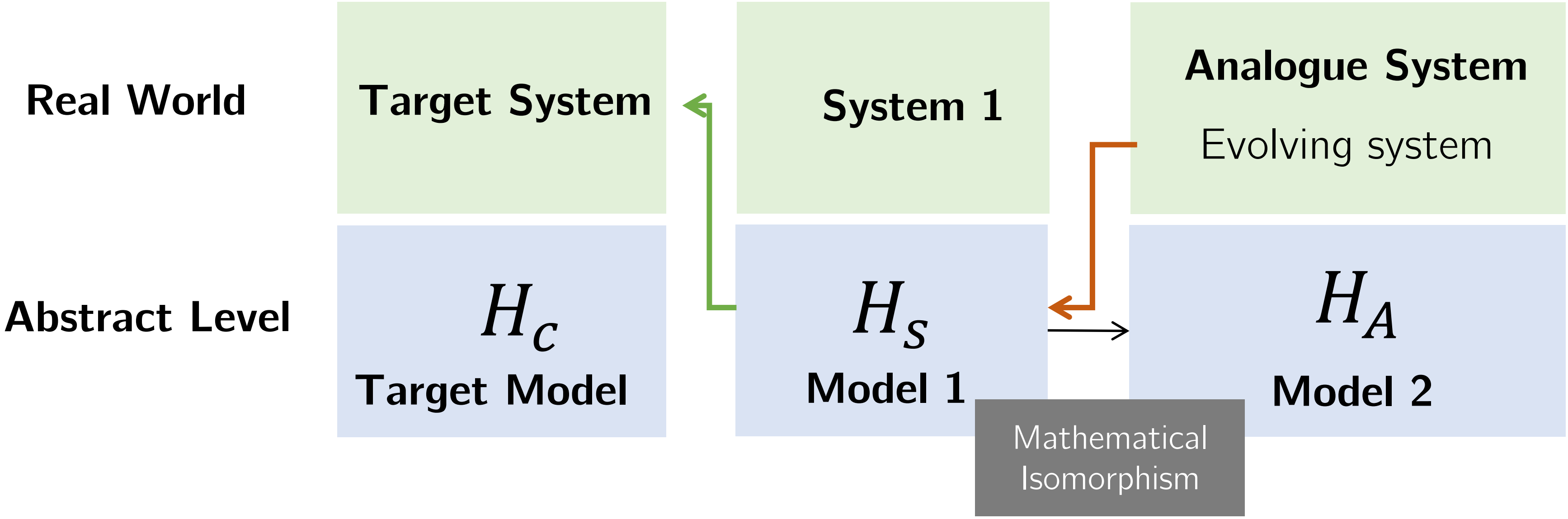


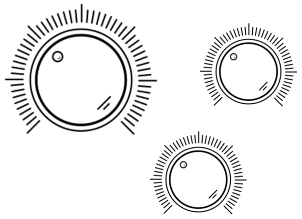
Analogue Systems can also be utilized as Simulators

+May Circumvent issues such as discretization, computer hardware limitation

+May allow faster simulation times for problems of higher complexity

Analogue Physical Simulation





+Tunability

Quantum Simulations:
when the analogue systems is governed by quantum dynamics

Analogue Systems can also be utilized as Simulators

+May Circumvent issues such as discretization, computer hardware limitation

+May allow faster simulation times for problems of higher complexity

Key takeaways

1

Tunable analogue quantum simulators may speed research of complex systems

Macroscopic Quantum Phenomena

Quantum fluid: a generic term for a system that displays quantum dynamics at the **macroscopic** level

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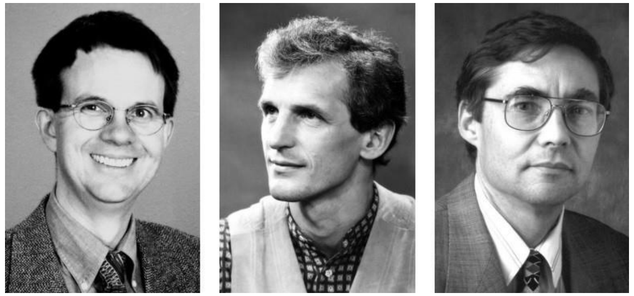
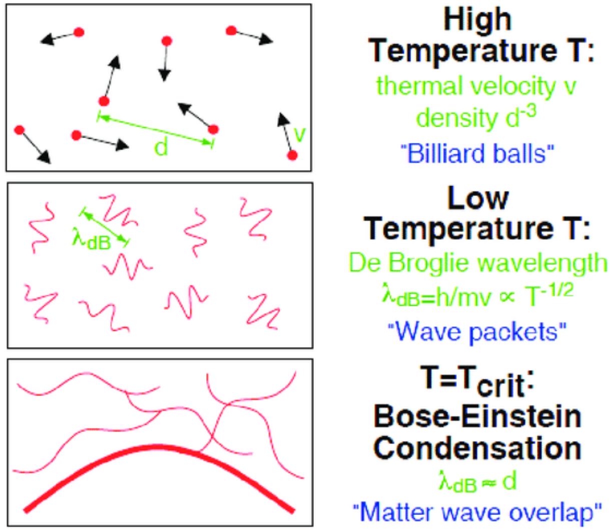
Quantum fluid: a generic term for a system that displays quantum dynamics at the **macroscopic** level

E.g. a Bose-Einstein Condensate (BEC)



Nobel Prize in 2001 (Cornell, Ketterle, Wieman)



Bose - Einstein Condensation

Macroscopic quantum state of matter

$\hat{\psi}(\mathbf{r}, t)$

Quantum wave function

Macroscopic quantum state of matter

$\hat{\psi}(\mathbf{r}, t)$ Quantum wave function

$$\frac{\partial}{\partial t}\hat{\psi}(\mathbf{r}, t) = -\frac{i}{\hbar}\left[\hat{\psi}(\mathbf{r}, t), \hat{H}\right]$$

Heisenberg Equation

Macroscopic quantum state of matter

$$\hat{\psi}(\mathbf{r}, t)$$

Quantum wave function

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Heisenberg Equation

$$\hat{H} = \int \hat{\psi}^{\dagger}(\mathbf{r}, t)\left[\frac{\mathbf{p}^2}{2M} + V_{\mathrm{ext}}(\mathbf{r}, t)\right]\hat{\psi}(\mathbf{r}, t)d\mathbf{r} + \hat{H}_{\mathrm{int}}.$$

Hamiltonian

Macroscopic quantum state of matter

$$\hat{\psi}(\mathbf{r}, t)$$

Quantum wave function

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Hamiltonian

$$\hat{H}_{\text{int}} = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\hat{\psi}^{\dagger}(\mathbf{r}, t)\hat{\psi}^{\dagger}(\mathbf{r}', t)V(\mathbf{r} - \mathbf{r}')\hat{\psi}(\mathbf{r}, t)\hat{\psi}(\mathbf{r}', t).$$

Interaction Hamiltonian

Mean Field Theory

$$\hat{\psi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \delta\hat{\psi}(\mathbf{r}, t), \quad \Phi(\mathbf{r}, t) \equiv \langle\hat{\psi}(\mathbf{r}, t)\rangle,$$

$$n_0(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$$

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$$n_0(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$$

$$\frac{\partial}{\partial t}\Phi(\mathbf{r}, t) = -\frac{i}{\hbar}\left[\frac{\mathbf{p}^2}{2M} + V_{\text{ext}}(\mathbf{r}, t) + \int \Phi^\dagger(\mathbf{r}', t) V(\mathbf{r} - \mathbf{r}') \Phi(\mathbf{r}, t) d\mathbf{r}'\right] \Phi(\mathbf{r}, t).$$

Gross-Pitaevskii equation

$$\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = -\frac{i}{\hbar}\left[\frac{\mathbf{p}^2}{2M} + V_{\text{ext}}(\mathbf{r},t) + \int \Phi^\dagger(\mathbf{r}',t)V(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r},t)d\mathbf{r}'\right]\Phi(\mathbf{r},t).$$

Binary Collisions only $$V(\mathbf{r}-\mathbf{r}') = g\,\delta(\mathbf{r}-\mathbf{r}')\ , \quad g = 4\pi\hbar^2\frac{a}{M}.$$

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = H\Phi(\mathbf{r},t)\ , \quad H \equiv \left[-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}}(\mathbf{r},t) + g\ |\Phi(\mathbf{r},t)|^2\right].$$

Gross-Pitaevskii equation (GPE)

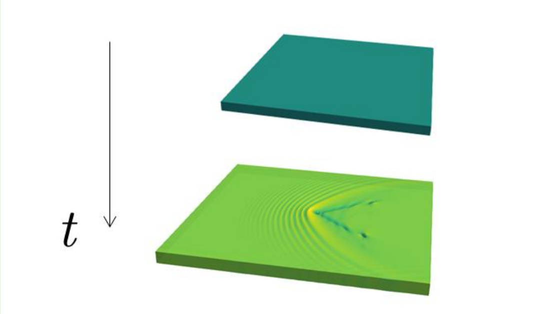
Nonlinear Schrödinger Equation (NLSE)

Analogue Simulation

Real World

System 1

2D Quantum Fluid



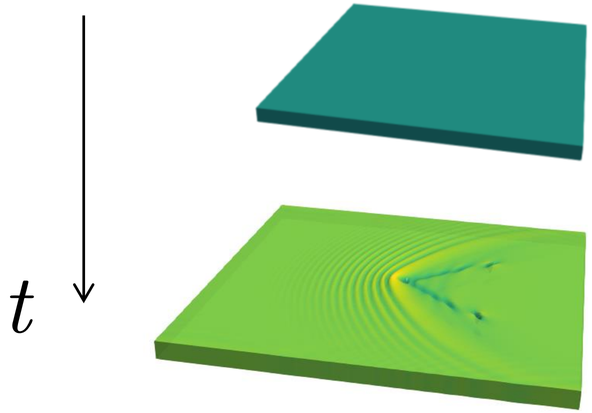
Abstract Level

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\Psi = 0$$

2D Gross-Pittaevskii

An analogue quantum fluid

Bose Einstein Condensate



$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\Psi = 0$$

?

Nonlinear Schrödinger Equation (NLSE) is an interesting nonlinear playground

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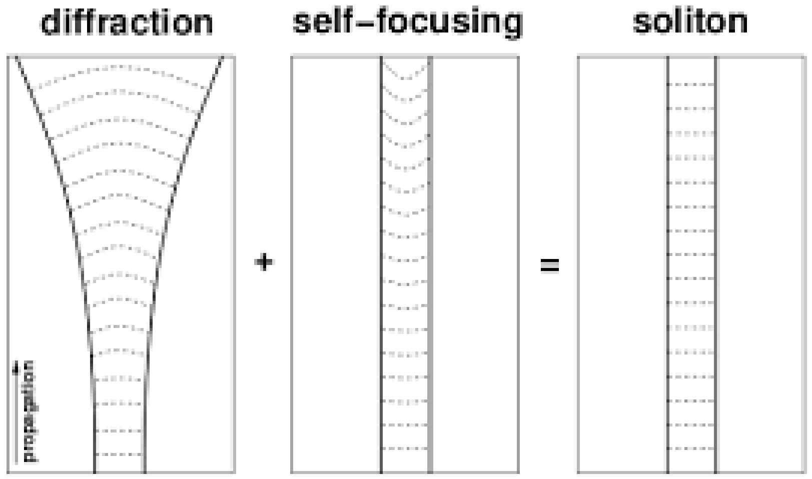
e.g. Solitons – self-localized wave solutions



Nonlinear Schrödinger Equation (NLSE) is an interesting nonlinear playground and universal model in nonlinear physics appearing in:

Quantum Physics;

Fluid Dynamics;

Plasma Physics;

And so on…

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And so on…

And nonlinear optics

Nonlinear optics: an old new world

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It can be argued that NLO have been around since 1931 (two-photon absorption)

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But only after the deployment of the first laser (Maiman 1960) it caught the interest

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Second-harmonic generation (Franken 1961)

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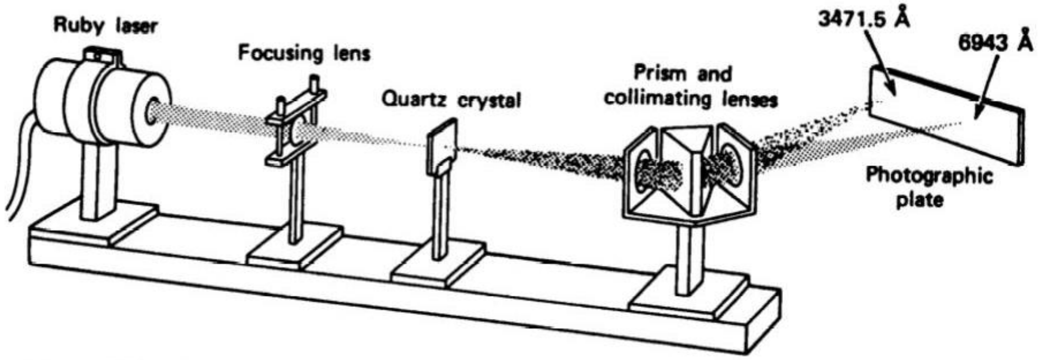

Figure 12.1. Arrangement used in the first experimental demonstration of second-harmonic generation [1]. A ruby-laser beam at $\lambda = 0.694\ \mu m$ is focused on a quartz crystal, causing the generation of a (weak) beam at $\frac{1}{2}\lambda = 0.347\ \mu m$. The two beams are then separated by a prism and detected on a photographic plate.

FIG. 1. A direct reproduction of the first plate in which there was an indication of second harmonic. The wavelength scale is in units of 100 A. The arrow at 3472 A indicates the small but dense image produced by the second harmonic. The image of the primary beam at 6943 A is very large due to halation.

Maxwell Equations

$$
\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \cdot \boldsymbol{D} &= \sigma_{\text{ext}} \\
\nabla \times \boldsymbol{H} &= \boldsymbol{J}_{\text{ext}} + \frac{\partial \boldsymbol{D}}{\partial t} \\
\nabla \cdot \boldsymbol{B} &= 0,
\end{aligned}
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\end{aligned}
$$

(absence of free charges or sources)

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$

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Wave equation

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Polarization

Wave equation

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Polarization

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{r},t) &= \epsilon_0\, \boldsymbol{\chi}\left[\boldsymbol{E}(\boldsymbol{r},t)\right]\boldsymbol{E}(\boldsymbol{r},t)\\ &= \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(1)} \cdot \boldsymbol{E}(\boldsymbol{r},t)}_{\boldsymbol{P}^{(1)}(\boldsymbol{r},t)} + \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(2)} : \boldsymbol{E}(\boldsymbol{r},t)^2}_{\boldsymbol{P}^{(2)}(\boldsymbol{r},t)} + \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(3)} : \boldsymbol{E}(\boldsymbol{r},t)^3}_{\boldsymbol{P}^{(3)}(\boldsymbol{r},t)} + \ldots\end{aligned}$$

Wave equation

Polarization

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{r},t) &= \epsilon_0\, \boldsymbol{\chi}\left[\boldsymbol{E}(\boldsymbol{r},t)\right] \boldsymbol{E}(\boldsymbol{r},t) \\ &= \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(1)} \cdot \boldsymbol{E}(\boldsymbol{r},t)}_{\boldsymbol{P}^{(1)}(\boldsymbol{r},t)} + \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(2)} : \boldsymbol{E}(\boldsymbol{r},t)^2}_{\boldsymbol{P}^{(2)}(\boldsymbol{r},t)} + \underbrace{\epsilon_0\, \boldsymbol{\chi}^{(3)} : \boldsymbol{E}(\boldsymbol{r},t)^3}_{\boldsymbol{P}^{(3)}(\boldsymbol{r},t)} + \ldots \end{aligned}$$

$\chi^{(n)}$ **Susceptibilities**

Polarization

$$P_j^{(n)}(\boldsymbol{r},t) = \epsilon_0 \sum_{i_1 \ldots i_n} \int_{-\infty}^{\infty} \chi^{(n)}_{j\,i_1 \ldots i_n}(\boldsymbol{r}-\boldsymbol{r}_1, \ldots, \boldsymbol{r}-\boldsymbol{r}_n; t-t_1, \ldots, t-t_n) \times$$

$$E_{i_1}(\boldsymbol{r}_1, t_1) \ldots E_{i_n}(\boldsymbol{r}_m, t_n)\, \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_1\, \mathrm{d}t_1 \ldots \mathrm{d}t_n$$

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Tensor that must respect the properties of the media

Polarization

$$P_j^{(n)}(\boldsymbol{r},t) = \epsilon_0 \sum_{i_1 \dots i_n} \int_{-\infty}^{\infty} \chi^{(n)}_{j\, i_1 \dots i_n}(\boldsymbol{r}-\boldsymbol{r}_1,\dots,\boldsymbol{r}-\boldsymbol{r}_n; t-t_1,\dots,t-t_n)\times$$
$$E_{i_1}(\boldsymbol{r}_1,t_1)\dots E_{i_n}(\boldsymbol{r}_m,t_n)\,\mathrm{d}\mathbf{r}_1\dots\mathrm{d}\mathbf{r}_1\,\mathrm{d}t_1\dots\mathrm{d}t_n$$

Tensor that must respect the properties of the media

What does it mean for isotropic media?

Neumann's principle

coordinate transformations (inversion, mirror image and rotation) **T** of field and polarization vectors **E** and **P**

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix} = \begin{pmatrix} T_{x'x} & T_{x'y} & T_{x'z} \\ T_{y'x} & T_{y'y} & T_{y'z} \\ T_{z'x} & T_{z'y} & T_{z'z} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{2.47}$$

$$\mathbf{E}' = \mathbf{T} \cdot \mathbf{E} \tag{2.48}$$

$$\mathbf{E} = \mathbf{T}^{-1} \cdot \mathbf{E}' \tag{2.49}$$

Neumann's principle

employing Einstein's summation convention

$$E_i = (T_{ii'})^T E_{i'} = T_{i'i} E_{i'} \tag{2.51}$$

$$P_i = T_{i'i} P_{i'} \tag{2.52}$$

relations in the two coordinate systems

$$P_i^{(n)} = \varepsilon_0 \chi_{ij\ldots s}^{(n)} E_j \cdots E_s, \tag{2.53}$$

$$P_{i'}^{(n)} = \varepsilon_0 \chi_{i'j'\ldots s'}^{(n)} E_{j'} \cdots E_{s'}, \tag{2.54}$$

Then

$$T_{i'i} P_i^{(n)} = P_{i'}^{(n)} = \varepsilon_0 T_{i'i} \chi_{ij\ldots s}^{(n)} T_{j'j} E_{j'} T_{k'k} E_{k'} \cdots T_{s's} E_{s'} \tag{2.55}$$

$$\chi_{i'j'\ldots s'}^{(n)} = T_{i'i} T_{j'j} \cdots T_{s's} \chi_{ij\ldots s}^{(n)} \tag{2.56}$$

Neumann's principle

Example: let's consider inversion $T_{i'i} = (-1)\delta_{i'i}$

susceptibility tensor of the inverted medium

$$\chi^{(n)}_{i'j'\ldots s'} = (-1)^{n+1}\chi^{(n)}_{ij\ldots s} \tag{2.57}$$

If the medium is invariant under inversion, it follows for *n*=even

$$\chi^{(n)}_{ij\ldots s} = (-1)^{n+1}\chi^{(n)}_{ij\ldots s} = 0 \tag{2.58}$$

i.e., in an inversion symmetric medium, the susceptibility tensors of even orders vanish

Centro-symmetric media

Polarization

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{r},t) &= \epsilon_0\,\boldsymbol{\chi}\left[\boldsymbol{E}(\boldsymbol{r},t)\right]\boldsymbol{E}(\boldsymbol{r},t)\\ &= \underbrace{\epsilon_0\,\boldsymbol{\chi}^{(1)}\cdot\boldsymbol{E}(\boldsymbol{r},t)}_{\boldsymbol{P}^{(1)}(\boldsymbol{r},t)} + \cancel{\underbrace{\epsilon_0\,\boldsymbol{\chi}^{(2)}:\boldsymbol{E}(\boldsymbol{r},t)^2}_{\boldsymbol{P}^{(2)}(\boldsymbol{r},t)}} + \underbrace{\epsilon_0\,\boldsymbol{\chi}^{(3)}:\boldsymbol{E}(\boldsymbol{r},t)^3}_{\boldsymbol{P}^{(3)}(\boldsymbol{r},t)} + \ldots\end{aligned}$$

Polarization

$$\boldsymbol{P}(\boldsymbol{r},t) = \epsilon_0\,\chi\,[\boldsymbol{E}(\boldsymbol{r},t)]\,\boldsymbol{E}(\boldsymbol{r},t)$$
$$= \underbrace{\epsilon_0\,\chi^{(1)}\cdot\boldsymbol{E}(\boldsymbol{r},t)}_{\boldsymbol{P}^{(1)}(\boldsymbol{r},t)} + \cancel{\underbrace{\epsilon_0\,\chi^{(2)}:\boldsymbol{E}(\boldsymbol{r},t)^2}_{\boldsymbol{P}^{(2)}(\boldsymbol{r},t)}} + \underbrace{\epsilon_0\,\chi^{(3)}:\boldsymbol{E}(\boldsymbol{r},t)^3}_{\boldsymbol{P}^{(3)}(\boldsymbol{r},t)} + \ldots$$

Isotropic media – Polarization aligned with electric field

No longer a tensor, a number is sufficient

Polarization

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Isotropic media – Polarization aligned with electric field

No longer a tensor, a number is sufficient

$$P^{(n)}(\boldsymbol{r},t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(n)}(t-t_1,\ldots,t-t_n) \times E(\boldsymbol{r},t_1)\ldots E(\boldsymbol{r},t_n)\,\mathrm{d}t_1\ldots\mathrm{d}t_n.$$

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$$E(\mathbf{r},t) = \frac{1}{2}\left[\mathcal{E}(\mathbf{r})\,e^{i\omega t} + \mathcal{E}^*(\mathbf{r})\,e^{-i\omega t}\right],$$

Third-order polarizations –
four-wave mixing

$\longrightarrow$ polarization oscillating at $+\omega$

$$\binom{3}{2} = 3$$

Wave equation for a cubic nonlinear media

$$\nabla^2 \mathcal{E}(\mathbf{r}) + \frac{\omega^2}{c^2}\left[1 + \chi^{(1)}(\omega)\right]\mathcal{E}(\mathbf{r}) = -\frac{3}{4}\frac{\omega^2}{c^2}\,\chi^{(3)}(\omega)\left|\mathcal{E}(\mathbf{r})\right|^2 \mathcal{E}(\mathbf{r}).$$

Paraxial approximation

$$\mathcal{E}(\mathbf{r}_\perp, z) = \mathcal{E}_0(\mathbf{r}_\perp, z)\, e^{ik(\omega)z}$$

Physical meaning – small angle deviations from the propagation axis z

Paraxial approximation

$$|\nabla_\perp^2 \mathcal{E}_0| / k^2 \sim |\partial_z \mathcal{E}_0| / k \ll 1,$$

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Physical meaning – small angle deviations from the propagation axis z

$$\left|\frac{\partial^2 E}{\partial z^2}\right| \ll \left|2ik\frac{\partial E}{\partial z}\right|$$

Paraxial approximation

At the end we get (adding an absorption alpha)

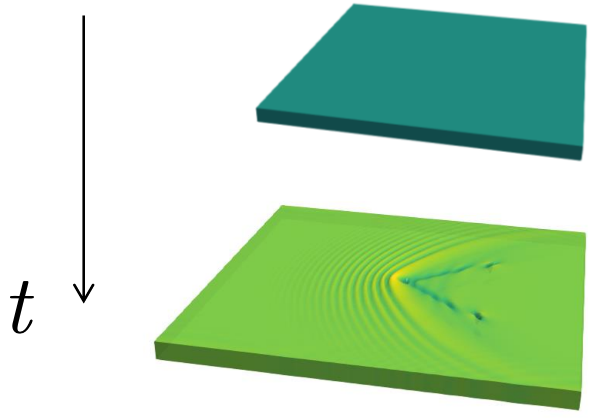
$$i\,\partial_z\,\mathcal{E}_0(\mathbf{r}_\perp, z) = \left[-\frac{1}{2k}\boldsymbol{\nabla}_\perp^2 - \frac{i\alpha}{2} - \frac{3}{8}\frac{k}{n_0^2}\,\chi^{(3)}(\omega)\left|\mathcal{E}_0\left(\mathbf{r}_\perp, z\right)\right|^2\right]\mathcal{E}_0(\mathbf{r}_\perp, z),$$

Addition of a linear refractive index (obstacle-like, delta n)

$$i\,\partial_z\,\mathcal{E}_0(\mathbf{r}_\perp, z) = \left[-\frac{1}{2k}\boldsymbol{\nabla}_\perp^2 - \frac{i\alpha}{2} - k\,\frac{\delta n(\mathbf{r}_\perp, z)}{n_0} - \frac{3}{8}\frac{k}{n_0^2}\,\chi^{(3)}(\omega)\left|\mathcal{E}_0\left(\mathbf{r}_\perp, z\right)\right|^2\right]\mathcal{E}_0(\mathbf{r}_\perp, z). \tag{2.10}$$

An analogue quantum fluid

Bose Einstein Condensate

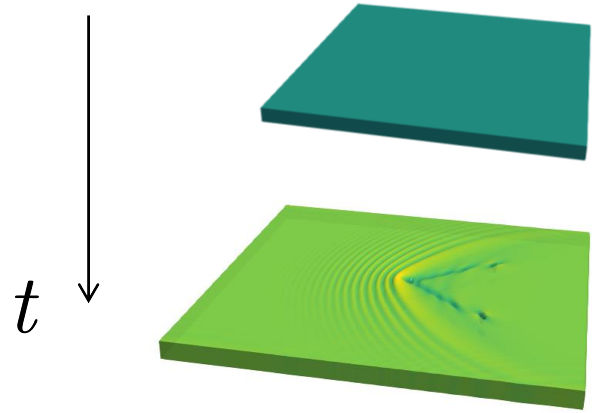


$$i\hbar\partial_t \Psi + \frac{\hbar^2}{2m}\nabla^2\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\Psi = 0$$

?

An analogue quantum fluid

Bose Einstein Condensate



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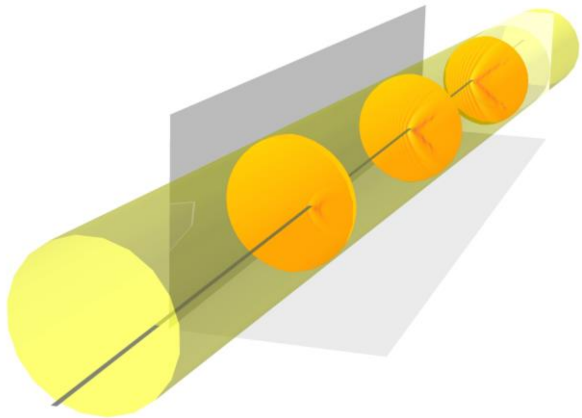
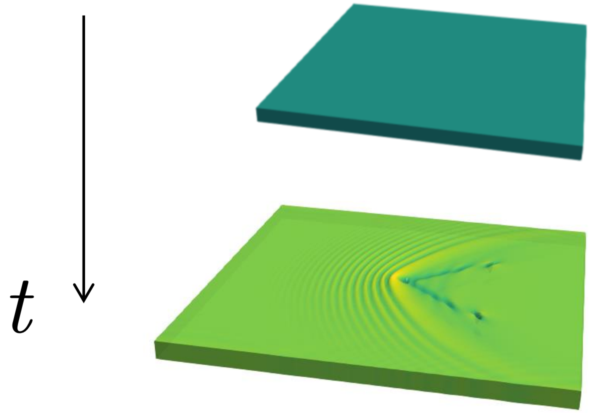
Light in propagating geometries



$$i\partial_z\Omega_p + \tfrac{1}{2}\nabla_\perp^2\Omega_p + V\Omega_p + G\left|\Omega_p\right|^2\Omega_p = 0$$

An analogue quantum fluid

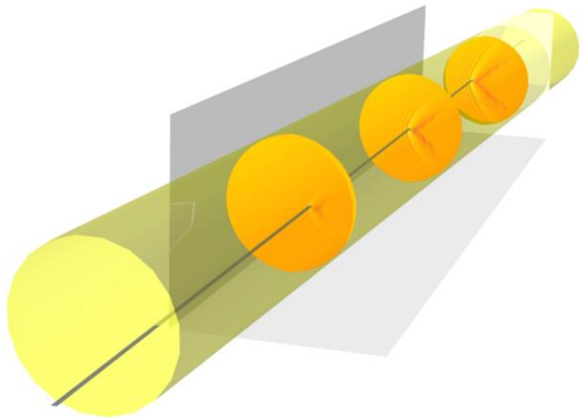
Bose Einstein Condensate



$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\Psi = 0$$

Time t
Density $\left|\Psi\right|^2$
Interaction via collisions g
Atomic mass m
Potential (Confinement, Barrier) V_{ext}

Light in propagating geometries



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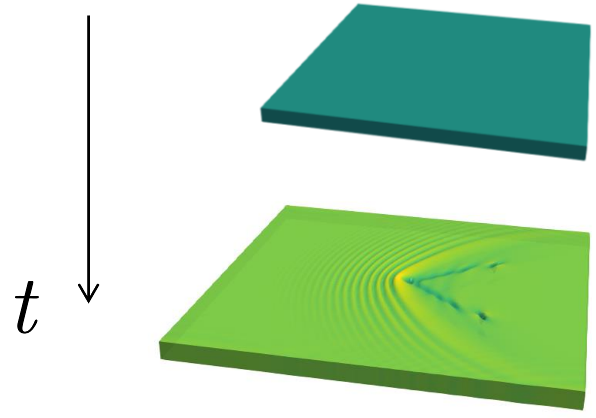
Propagation Distance z
Intensity $\left|\Omega_p\right|^2$
Interaction via nonlinearity G
Effective mass through diffraction k_p
Linear refractive index V

An analogue quantum fluid

Bose Einstein Condensate



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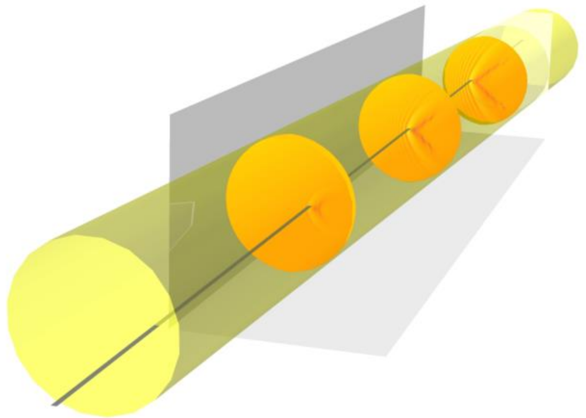

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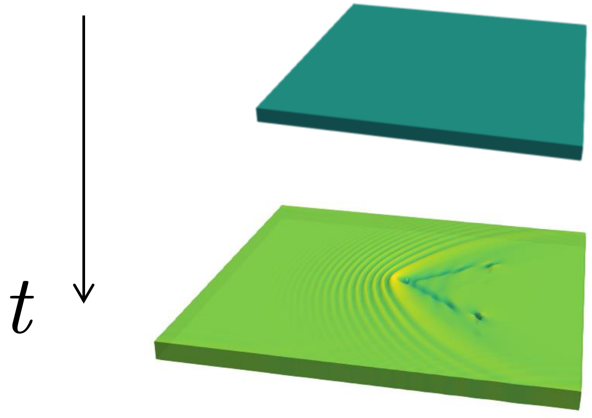

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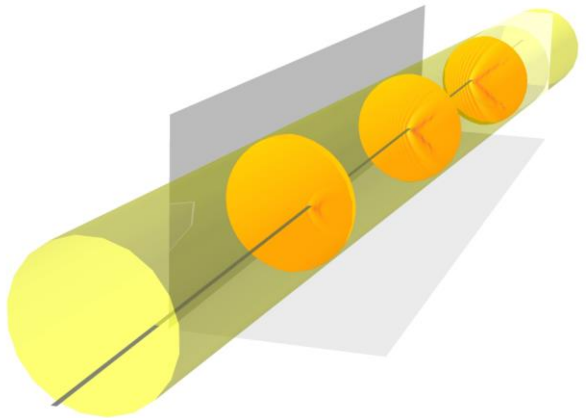

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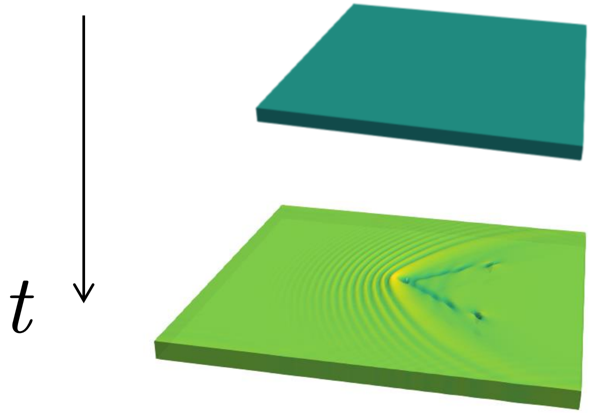

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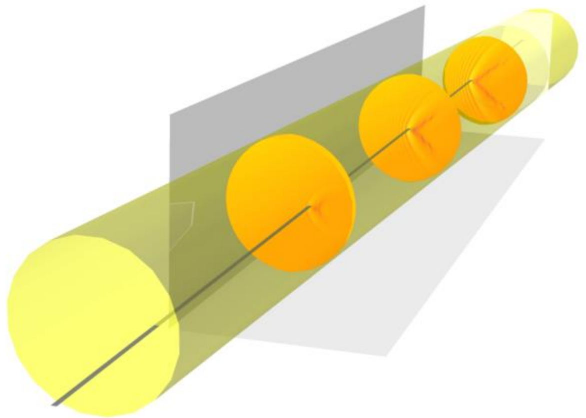

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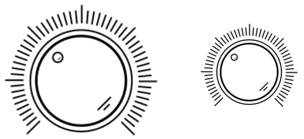
Analogue Quantum Simulation

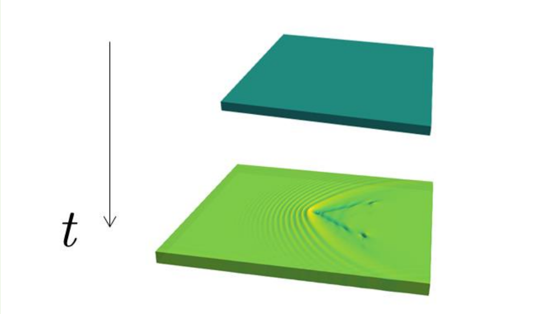


+Tunability

Real World

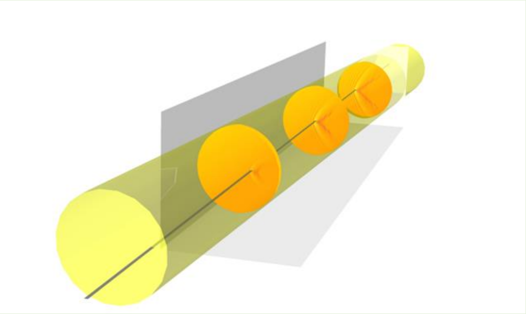
System 1

2D Quantum Fluid



Analogue System

Paraxial Fluid of Light



Abstract Level

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\Psi = 0$$

2D Gross-Pittaevskii

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\nabla_\perp^2 E + \frac{k\chi^{(3)}}{2}|E|^2 E + \frac{k\Delta\chi^{(1)}(\mathbf{r})}{2}E = 0$$

Paraxial Propagation Model

Light Speed Simulation of a 2D Quantum Fluid under GPE
With no discretization

Hydrodynamic interpretation of light

$$i\partial_z E + \frac{1}{2}\nabla_\perp^2 E + V(\boldsymbol{r}_\perp) E + g|E|^2 E = 0$$

Madelung transformation $$E(\boldsymbol{r}_\perp, z) = \sqrt{\rho(\boldsymbol{r}_\perp, z)}e^{i\phi(\boldsymbol{r}_\perp, z)}$$

Hydrodynamic interpretation of light

Set of Hydrodynamic like equations $\boldsymbol{v} = \nabla \phi,$ $V_B \equiv \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}}$

Hydrodynamic interpretation of light

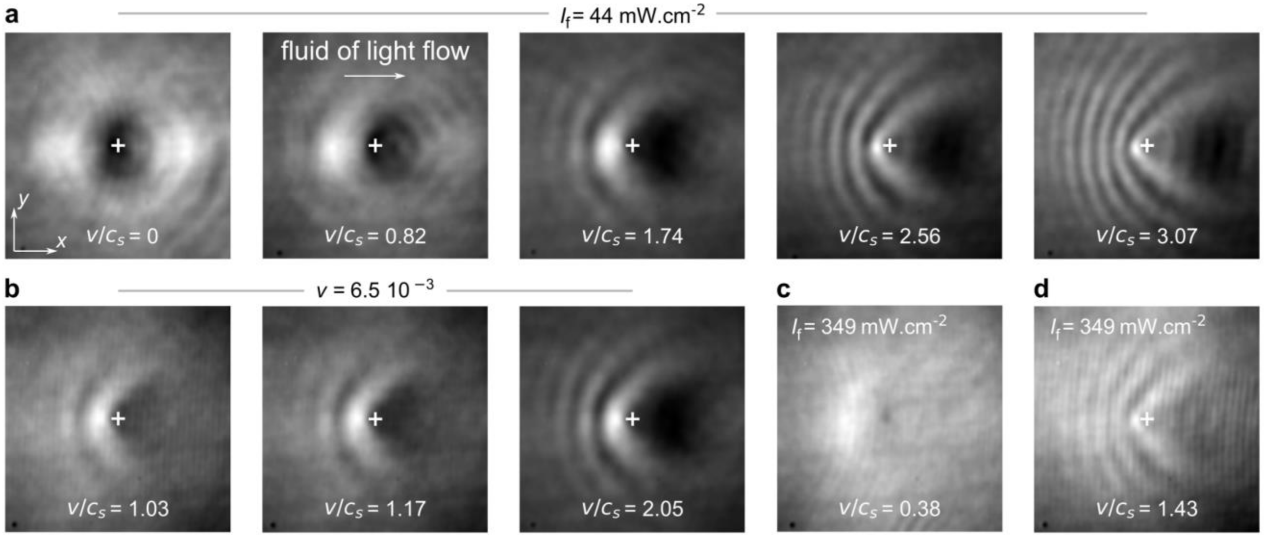
Set of Hydrodynamic like equations $$\boldsymbol{v} = \nabla \phi, \qquad V_B \equiv \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}}$$

$$\partial_z \rho + \nabla (\rho \boldsymbol{v}) = 0$$

$$\partial_z \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = \nabla (V + V_B + g\rho)$$

Hydrodynamic interpretation of light

Light shall behave like a fluid



But we said quantum fluid, where is the quantum part?

$$i\partial_z E + \frac{1}{2}\nabla_\perp^2 E + V\left(r_\perp\right) E + g\left|E\right|^2 E = 0$$

In this self-interaction term, that emulates BECs

Superfluid properties

Small excitations spectra

Bogoliubov dispersion relation

$$\omega_{\mathrm{Bog}}(\boldsymbol{k}_\perp) = \sqrt{k_\perp^2\left(\frac{k_\perp^2}{4} + G|\Omega_p^0|^2\right)}.$$

Superfluid properties

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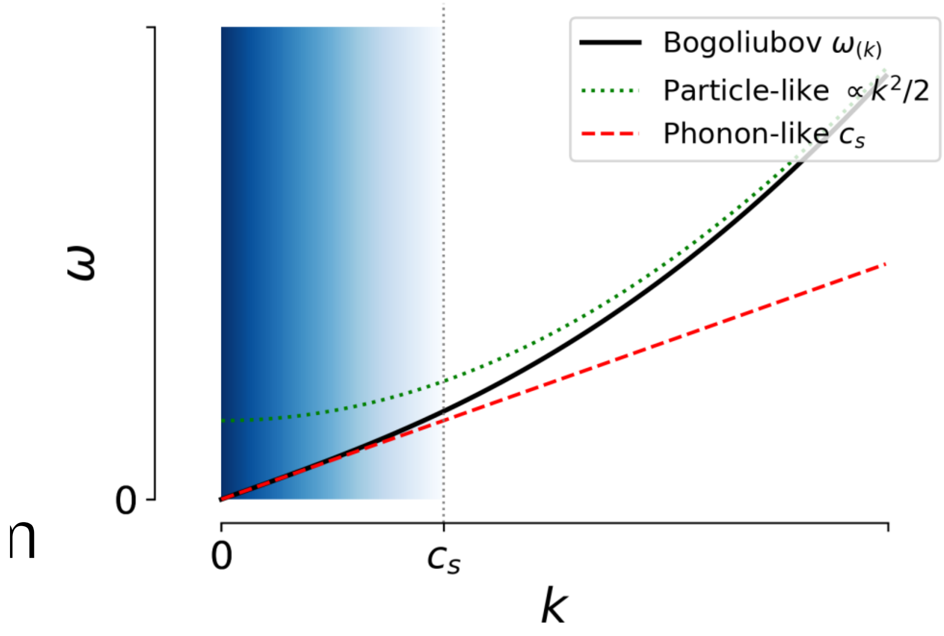

Superfluid properties

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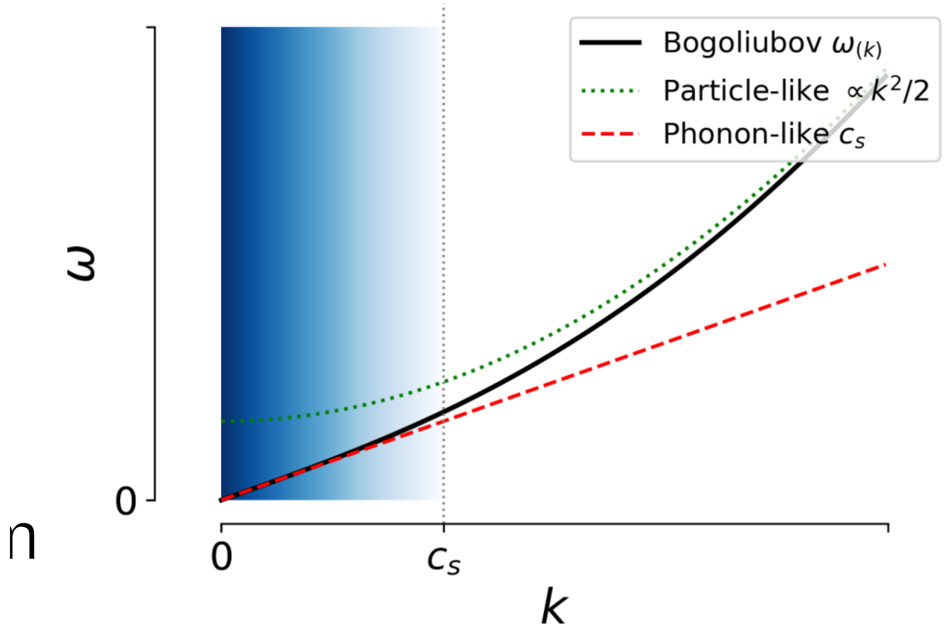
Landau Criteria $v < c_s$



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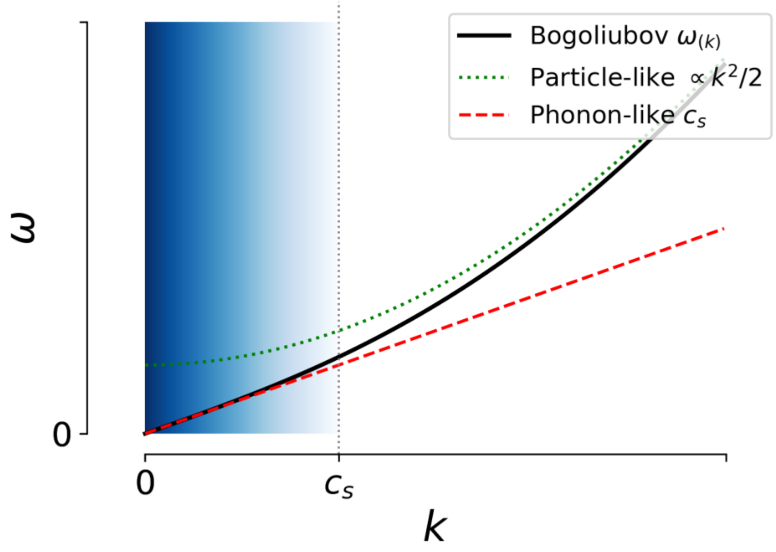

Superfluid properties

absence of drag, loss of momentum