Quantum Sensing with Non-classical Light States

Week

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Quantum engineering is a multidisciplinary approach to quantum physics, focused on exploiting **quantum phenomenology to develop a novel technology**, usually referred to as **quantum technology**.

In short, to understand the concept you can establish a parallel with the history of Electricity: in the early eighteenth century, Physicists like Volta, Ampére, Ohm, and Maxwell established the foundations of electricity; later in the nineteenth century led by the likes of Bell, Tesla, Edison and Hertz give birth to electrical engineering by bringing electricity laws to the real world in the form of technology. As a result, today most of the electronic technology used by humanity is based on the fundamental principles established by Faraday and his contemporaries.

Quantum Engineering aims the same: to bring the principles and laws of quantum physics developed in the twentieth century by scientists like Planck, Einstein, Bohr, and Schrödinger, who started the so-called **first quantum revolution**, into real-world applications and sparked a **second quantum revolution**.

Indeed, the first quantum revolution, rooted in the early 20th century, brought a foundational understanding of quantum mechanics, elucidating the wave-particle duality of light and matter, the quantization of energy levels, and the principle of superposition. This leads to the development of groundbreaking technologies such as semiconductors, lasers, and magnetic resonance imaging (MRI).

In a distinct spirit, the second quantum revolution aims to leverage the strange innerworkings of quantum mechanics to deploy new technology. Unlike the first, which primarily involved passive applications of quantum principles[1] - **the second quantum revolution actively exploits phenomena such as superposition, entanglement, and quantum coherence to develop radically new technologies**[2]. Essentially, Quantum Technology is often divided into four research topics:

- **Quantum computation** - exploits new units of information (qubits) that theoretically can offer massive parallel computation power;

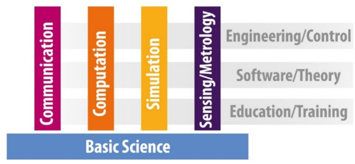

Figure 1. Quantum technology pathways according to European union's Quantum Flagship.

[1] *Work and they are quantum*

[2] *Work because they are quantum*

- **Quantum communication** – uses quantum properties for new strategies of cryptography and secure data transmission;
- **Quantum sensors and metrology** - explore quantum properties for enhanced and novel types of sensitivity;
- **Quantum simulation** – exploits analogue systems following similar physical and mathematical laws to simulate the behavior of complex quantum systems.

Among these, **quantum sensing** stands out for its potential to revolutionize how we measure and interact with the world. Quantum sensors harness quantum mechanical effects to achieve unprecedented sensitivity and precision, far surpassing the capabilities of classical sensors. This quantum advantage enables the detection of minute variations in physical quantities such as time, magnetic fields, acceleration, and gravitational waves. This sets the stage for major impact across various domains as diverse as fundamental physics (e.g. cosmology), healthcare (e.g. quantum-based brain activity detectors), or oil prospection (e.g. gravimetric sensors).

In this chapter, we briefly grasp this fascinating topic, approaching it first from the optical interferometric-based sensors perspective. In particular, we will demonstrate how both **entangled states** and **photon correlation**. Finally, we also address the broader implications of quantum sensing to metrology, where quantum physics is already playing a critical role in redefining the International System of Units (SI), providing more stable, accurate, and reliable standards for measurements of time, length, and other fundamental quantities. But before entering in detail in quantum sensing methodologies, we should first make a small note on quantum light emitters and in particular, Single Photon Sources.

1 Generating Quantum States of Light and Single Photon Sources

As said before, when we talk about quantum sensing we are talking about enabling using non-classical properties of light. This means that we need to depart from using lasers and utilize non-classical states of light instead, such as the Fock states.

In this line, single photon sources play a crucial role in quantum technologies, allowing the creation of Fock states with unitary occupancy $|1\rangle$. Depending on the final application there are many ways of generating single photons:

- **Quantum Dots:** semiconductor nanocrystals that act as artificial atoms and emit single photons when excited. Depending on the size and composition, they may offer high tunable emission wavelengths. They can also work as sensors by direct incorporation in various substrates.

- **Nitrogen-Vacancy Centers in Diamond:** Defects in diamond where a nitrogen atom replaces a carbon atom adjacent to a vacancy. When optically excited, they may emit single photons at room temperature. Again they may be used as sensors as they feature sensitivity to magnetic and electric fields, as well as temperature variations. Similarly, there are other solid-state emitters such as SiV and GeV centers that work similarly.

- **Nonlinear Crystals:** Nonlinear crystals, such as beta barium borate (BBO) or potassium titanyl phosphate (KTP), can be used to generate single photons through spontaneous parametric down-conversion (SPDC). In this process, a high-energy pump photon is converted into a pair of lower-energy photons. this is particularly advantageous in two distinct ways: first, when detecting one of the photons you know that in the other path, it will be the other photon - the heralded photon, allowing you to precisely create and measure experimental outcomes; secondly, depending on the generation process (Type I - same polarization, spatial entanglement; Type II - orthogonal polarizations, spatial and polarization entanglement, also referred to as *hyperentanglement*) the photons are usually entangled, allowing to assess novel quantum-based phenomenologies.

So in practice, you can generate single photons in multiple ways, but experimentally, if you need to detect with high precision the event, you end up needing a nonlinear crystal and heralded scheme photon generation.

2 Sensing with Entangled States: NOON states in Interferometers

Entanglement is a fundamental phenomenon in quantum mechanics where the quantum states of two or more particles become intertwined such that the state of one particle cannot be described independently of the state of the others, even when the particles are separated by large distances. This non-local correlation enables quantum systems to exhibit properties that are impossible in classical systems.

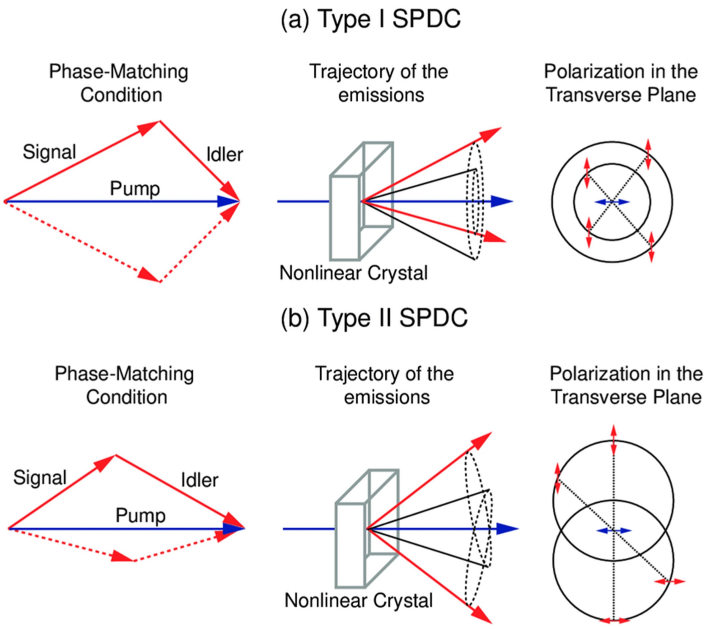

Figure 2. Comparison of the two types of parametric down-conversion. a)In Type I photons from pump beam are converted into signal and idler photons with identical polarization and forming a set of concentric cones. b)In Type II the signal and idler have orthogonal polarizations and are separated into two cones due to birefrigence effects.

Entanglement can be quantified by the reduced density matrix, obtained from the density matrix

$$\rho = \sum_j p_j |\psi\rangle\langle\psi| \tag{2.1}$$

as the trace over a given subspace A,

$$\rho_A = tr_A(\rho) \tag{2.2}$$

. A pure state $|\psi\rangle$ is entangled if the reduced density matrix ρ_A (or ρ_B) is not a pure state, i.e., it has a mixed state form.

Definition. Lemmas on Separability and Entanglement: (formal proofs can consult the paper by Chong, Keiter, and Stolze, *Multiparticle entanglement and ranks of density matrices*).

- **Lemma 1**: A state is pure if and only if the rank of its density matrix ρ is equal to 1, i.e. $rank(\rho) = 1$.
- **Lemma 2**: A pure state is entangled if and only if the rank of at least one of its reduced density matrices is greater than 1.
- **Lemma 3**: Given a pure state ρ if its particles are separated into

two parts U and V, the $rank(\rho_U) = 1$ holds if and only if these two parts are separable, that is, $\rho = \rho_U \otimes \rho_V$

Note that the above conditions only hold for pure states and not for a mixed state, which requires different and somewhat more complex procedures, which, for simplicity, we left out of the scope of this unit.

2.1 NOON states

NOON states, a type of maximally entangled quantum state, provide significant advantages in interferometric measurements by enhancing phase sensitivity. This allows measurements to surpass the standard quantum limit and approach the Heisenberg limit. A NOON state with N photons is represented as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + |0, N\rangle),$$

where $|N, 0\rangle$ indicates N photons in one mode and 0 in the other, and $|0, N\rangle$ vice versa.

Proof of Maximal Entanglement in NOON States: A NOON state is given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + |0, N\rangle).$$

The density matrix for this state is:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|N, 0\rangle + |0, N\rangle)(\langle N, 0| + \langle 0, N|).$$

To find the reduced density matrix for one mode, trace out the other mode. For mode A:

$$\rho_A = \mathrm{Tr}_B(\rho) = \frac{1}{2}(|N\rangle\langle N| + |0\rangle\langle 0|).$$

Similarly, for mode B:

$$\rho_B = \mathrm{Tr}_A(\rho) = \frac{1}{2}(|N\rangle\langle N| + |0\rangle\langle 0|).$$

Since ρ_A and ρ_B are maximally mixed, this indicates that the NOON state is maximally entangled. The rank of the reduced density matrices equals the rank of the total density matrix, satisfying Lemma 3 and confirming maximal entanglement.

In an interferometer in the configuration of Figure 3, the phase difference

ϕ of a NOON state will be given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + e^{iN\phi}|0,N\rangle).$$

Indeed this is easily obtained from the fact that the new operator $\hat{a}_2^{\dagger'} = e^{-i\phi}\hat{a}_2^{\dagger}$, meaning that

$$|\psi\rangle = \frac{1}{\sqrt{2N!}}((\hat{a}_1^{\dagger'})^N|0,0\rangle + (e^{i\phi}\hat{a}_2^{\dagger'})^N|0,0\rangle)$$

and ultimately defines the phase shift operator as $\hat{P}_s(\phi) = e^{i\phi\hat{n}}$. At this point it is interesting to stop for a moment and appreciate what is happening: each photon is shifting a phase ϕ which accumulates in the final state. **This is contrary to what happens for the coherent state and encloses the quantum advantage of these states, as the phase shift of a NOON state increases N-fold**.

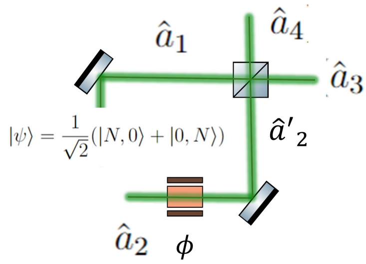

Figure 3. A Mach-Zehnder with a NOON state.

Exercise 1. Using the definition of a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.3}$$

demonstrate that the phase shift operator gives $\hat{P}_s(\phi)|\alpha\rangle = e^{i\phi}|\alpha\rangle$, which is the expected result from the classical picture.

2.2 Enhanced Phase Sensitivity

It is easy to demonstrate that at the output ports 3 and 4, the average number at each port is just $N/2$, except for the case of $N = 1$, where we would recover the classical Mach-Zehnder formula. This means that we cannot interrogate this interferometer in the same manner. So why would we bother to use these states?

Instead of looking at the intensity measured at the output ports, we will look at value of the parity operator at a particular output port (e.g. 3) [3]

$$\hat{\Pi}_3 = (-1)^{\hat{n}_3} = exp(i\pi\hat{n}_3).$$

Written in the output ports, using the transformations

[3] Which experimentally can be done using a Photon number resolved detector.

(2.4)

we have that our state is given by

$$
\begin{aligned}
|\psi_{out}\rangle &= \frac{1}{\sqrt{2}^{N+1}\sqrt{N!}} \sum_k \binom{N}{k} \left((\hat{a}_3^\dagger)^k (i\hat{a}_4^\dagger)^{N-k} + (i\hat{a}_3^\dagger)^k (\hat{a}_4^\dagger)^{N-k} e^{i\phi N}\right)|0,0\rangle \\
&= \frac{1}{\sqrt{2}^{N+1}} \sum_k \sqrt{\binom{N}{k}} (i^{N-k} + i^k e^{i\phi N})|k, N-k\rangle
\end{aligned}
\tag{2.5}
$$

where we use the fact that $(\hat{a}_3^\dagger)^k (\hat{a}_4^\dagger)^{N-k}|0,0\rangle = \sqrt{k!(N-k)!}|k, N-k\rangle$. Applying the parity operator and taking the mean and keeping only the non-zero terms

$$
\begin{aligned}
\langle\psi_{out}|\hat{\Pi}_3|\psi_{out}\rangle &= \frac{1}{2^{N+1}} \sum_k \binom{N}{k} (-1)^k ((-i)^{N-k} + (-i)^k e^{-i\phi N})(i^{N-k} + i^k e^{i\phi N}) \\
&= \frac{1}{2^{N+1}} \sum_k \binom{N}{k} ((-1)^k 2 + (-i)^N e^{i\phi N} + i^N e^{-i\phi N}) \\
&= \frac{1}{2^{N+1}} \sum_k \binom{N}{k} ((-1)^k 2 + e^{i(\phi-\pi/2)N} + e^{-i(\phi-\pi/2)N}) \\
&= \frac{1}{2^{N+1}} \sum_k \binom{N}{k} ((-1)^k 2 + 2cos((\phi-\pi/2)N))
\end{aligned}
\tag{2.6}
$$

Using the binomial sum properties

$$
\sum_k \binom{N}{k} (-1)^k = 0, \sum_k \binom{N}{k} = 2^N \tag{2.7}
$$

it is possible to obtain

$$
\Pi_3 = cos(N\theta)
$$

which depends not only on $\theta = \phi - \pi/2$ but also on the number of input photons N. By using the fact that the parity operator either outputs 1 or -1, the variance for the parity operator is simply given by $Var(P) = \sqrt{1 - \langle\hat{\Pi}_3\rangle^2} = |sin(N\theta)|$

For small phase shifts, the sensitivity is determined by the uncertainty in the measurement of ϕ, which is given by:

$$
u(\phi) = \frac{\sqrt{Var(P)}}{\left|\frac{dP}{d\phi}\right|} = \frac{|sin(N\theta)|}{|Nsin(N\theta)|} = \frac{1}{N}.
$$

Compared to the classical shot noise uncertainty that scales as $1/\sqrt{N}$, NOON states can thus offer enhanced phase resolution and sensitivity, demonstrating the Heisenberg limit[4] and its **quantum advantage**.

In terms of applications, NOON states offer tremendous potential for super-resolution and super-sensitivity measurements, with applications in **gravitational wave detection** and **bio and chemical sensing**, as well as **quantum litography**[5]. Yet, there are several challenges to their widespread implementation including:

- **Generation and detection:** Creating and detecting NOON states with high photon numbers and maintaining coherence over large distances or timescales is technically demanding.
- **Loss and Decoherence:** NOON states are highly susceptible to losses and decoherence, which can degrade their performance.

[4] The Heisenberg limit can be obtained from the Quantum Fisher information and states that for a system of N entangled particles or photons the precision of a variable ϕ measurement is $\phi \geq 1/N$

[5] beating the classical diffraction limit

3 Quantum Interference

Quantum interference is one of the hallmarks of quantum mechanics that occurs when the probability amplitudes of different paths combine, leading to constructive or destructive interference patterns. As in classical optical waves, this interference underpins many quantum technologies from quantum sensing to quantum computing, and allows unique properties of quantum states that classical physics cannot replicate and that often may be exploited to get some quantum advantages.

3.1 Hong-Ou-Mandel Effect

First demonstrated by Chung Ki Hong, Zhe-Yu Ou, and Leonard Mandel in 1987, the Hong-Ou-Mandel (HOM) effect is an interesting and counterintuitive display of quantum interference that besides its unique physical behavior, is now being actively explored in the context of quantum sensing and quantum technology.

In very generic terms, HOM interference occurs when two indistinguishable photons enter a beamsplitter from different input ports. Considering a 50:50 beamsplitter, the input state of the system can be described as $|1\rangle_1|1\rangle_2$ where each photon is in a different path. Usually, we look at the beam splitter from the perspective of the input modes. Yet, to analyze this situation is convenient to look at how the input operators are expressed in terms of the output ones. Using the matrix formalism and

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{3.1}$$

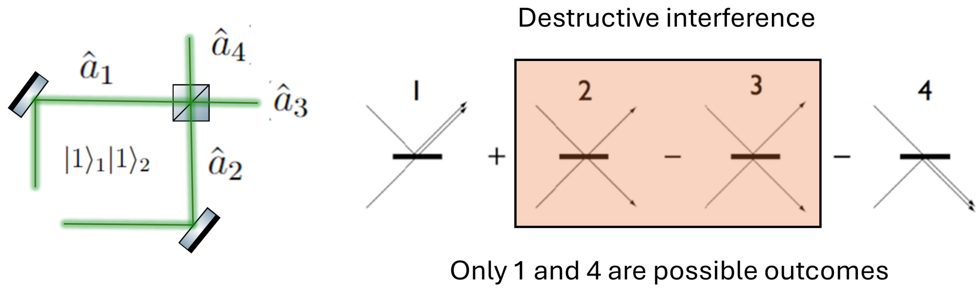

Figure 4. Hong-Ou-Mandel Interference: due to interference of quantum paths for indistinguishable photons, the output of this interferometer is always two photons exiting one of the ports.

it is trivial to obtain

$$\begin{aligned}\hat{a}_1 &= \frac{1}{\sqrt{2}}(\hat{a}_3 - i\hat{a}_4)\\ \hat{a}_2 &= \frac{1}{\sqrt{2}}(\hat{a}_4 - i\hat{a}_3).\end{aligned} \tag{3.2}$$

Taking the input state $|1\rangle_1|1\rangle_2 = \hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle_1|0\rangle_2$ and applying the transformations we obtain

$$\hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle_1|0\rangle_2 = \frac{i}{\sqrt{2}}\left(|2\rangle_3|0\rangle_4 + |0\rangle_3|2\rangle_4\right) \tag{3.3}$$

Looking at the results, it means that when the two photons are indistinguishable and arrive simultaneously at the beamsplitter, **they bunch together and exit from the same output port.** The output state is thus always two photons exiting one of the ports, either from one or the other side[6].

[6] however the average number of photons will be equal

3.2 Second-Order Correlation Function

To quantify the degree of HOM interference we shall introduce the second-order correlation function. The second-order coherence function $g^{(2)}(\tau)$ is a critical parameter in understanding the statistical properties of light and its potential quantum advantages.

The classical second-order correlation function may be defined as:

$$\begin{aligned}g^{(2)}(\tau) &= \frac{\langle E_1^*(t)E_2^*(t+\tau)E_2(t+\tau)E_1(t)\rangle}{\langle E_1^*(t)E_1(t)\rangle\langle E_2^*(t+\tau)E_2(t+\tau)\rangle}\\ &= \frac{\langle I_1(t)I_2(t+\tau)\rangle}{\langle I_1(t)\rangle\langle I_2(t+\tau)\rangle}\end{aligned} \tag{3.4}$$

where I_i is the field at the detector i. This can be generalized to the quantum formalism as

$$g^{(2)}(\tau) = \frac{\langle \hat{E}_1^{(-)}(t)\hat{E}_2^{(-)}(t+\tau)\hat{E}_2^{(+)}(t+\tau)\hat{E}_1^{(+)}(t)\rangle}{\langle \hat{E}_1^{(-)}(t)\hat{E}_1^{(+)}(t)\rangle\langle \hat{E}_2^{(-)}(t+\tau)\hat{E}_2^{(+)}(t+\tau)\rangle}$$

where $\hat{E}_i^{(+)}(t)$ and $\hat{E}_i^{(-)}(t)$ are the positive and negative frequency parts of the electric field operator at detector i.

For the HOM interference, we will be interested in computing the second-order correlation at the detectors as

$$g^{(2)}(0) = \frac{\langle \hat{a}_3^\dagger(t)\hat{a}_4^\dagger(t+\tau)\hat{a}_4(t+\tau)\hat{a}_3(t)\rangle}{\langle \hat{a}_3^\dagger(t)\hat{a}_3(t)\rangle\langle \hat{a}_4^\dagger(t+\tau)\hat{a}_4(t+\tau)\rangle}$$

which at zero time delay ($\tau = 0$) simplifies to:

$$g^{(2)}(0) = \frac{\langle \hat{a}_3^\dagger\hat{a}_4^\dagger\hat{a}_4\hat{a}_3\rangle}{\langle \hat{a}_3^\dagger\hat{a}_3\rangle\langle \hat{a}_4^\dagger\hat{a}_4\rangle}$$

After some cumbersome operations and proper use of the commutators, it is possible to show that for the initial state $|1\rangle_1|1\rangle_2$ we will have $g^{(2)}(0) = 0$ which is the characteristic signature of the photon anti-bunching.

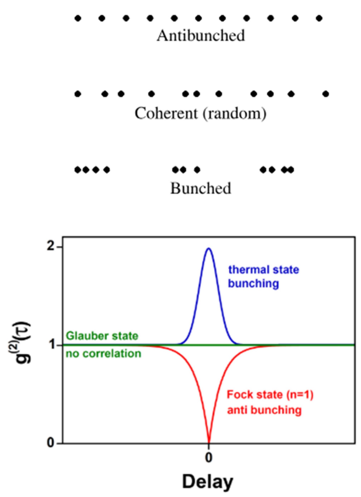

Figure 5. Comparison of different types of $g^{(2)}$ depending on the light characteristics.

Bunching and Anti-bunching: by computing the second-order correlation of the output ports of our interferometer for distinct types of light we may obtain very distinct behaviors:

- **Anti-Bunching of non-classical states:** Quantum Light Sources feature $g^{(2)}(\tau) < 1$ (anti-bunching) indicate non-classical behavior and correlations;
- **Bunching of thermal light:** featuring gaussian intensity fluctuations feature $g^{(2)}(\tau) > 1$, also indicating a correlation but associated to noisy fluctuations rather than the intrinsic nature of light;
- **Coherent light:** coherent light features $g^{(2)}(\tau) = 1$ meaning no correlations.

3.3 Monitoring the Hong-Ou-Mandel Shift for Quantum Sensing

To compute $g^{(2)}(\tau)$ based on the temporal shift τ, we need to consider a temporal shape for the wavepacket of the photon. This is not a trivial task and it will depend greatly on the type of process originating the photons[7].

[7] e.g. if its pulsed or not, the spectral bandwidth, etc.

In general terms, the Hong-Ou-Mandel dip can be experimentally (see

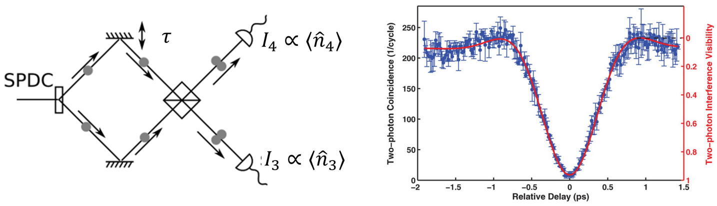

Figure 7. Experimental configuration to obtain the Hong-Ou-Mandel dip (with typical experimental results on the right) in terms of a temporal delay. The addition of a temporal delay affects the indistinguishably of the photons and thus departs from the zero second-order correlation.

Figure 7) approximated by

$$g^{(2)}(\tau) = 1 - e^{-\frac{\tau^2}{2\sigma^2}}$$

where σ is usually associated with the coherence time of the photons, with typical times around the picosecond scale. Note that in the HOM case, the $g^{(2)}$ function is also associated with the **coincidences**, i.e. the probability of detecting events at the same time in both detectors located on the output ports.

It is possible to demonstrate that HOM may feature some quantum advantage if[8]. In experimental world, and by monitoring this dip, one can deploy a quantum sensing strategy that harnesses this quantum advantage. Indeed, depending on the properties of the dip one can enjoy enhanced sensitivity and high spatial and temporal resolutions, supporting applications such as:

[8] following the argument discussed here

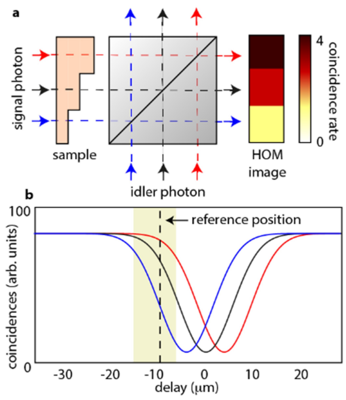

Figure 6. Concept of the Hong-Ou-Mandel Microscopy.

- **Quantum Microscopy and Imaging:** HOM imaging can be used to achieve super-resolution microscopy, now paving for the observation of biological samples and nanostructures with unprecedented detail.
- **Material Characterization:** The sensitivity of HOM imaging makes it ideal for characterizing materials at the microscopic level, detecting defects, and analyzing surface properties.
- **Quantum Sensing:** HOM interferometry may also be used to assess optical properties, such as refractive index variations and path length differences.

Of course, in spite of the theoretical potential, any quantum technology only makes sense if you enable some **quantum advantage**. In this case, the quantum advantage comes from the utilization of quantum interference and entanglement, which enhance resolution, sensitivity, and noise reduction beyond classical limits, opening up new possibilities in imaging and metrology.

4 Quantum for metrology and the Redefinition of SI Units

Quantum metrology leverages the principles of quantum mechanics to improve measurement precision and accuracy beyond classical limits. This field has played a crucial role in the recent redefinition of several SI units, ensuring they are based on fundamental constants rather than physical artifacts.

The International System of Units (SI) is the modern form of the metric system and the most widely used system of measurement globally. Formally published in 1960, SI provides a standardized framework for measuring physical quantities, ensuring consistency and accuracy in science, industry, and commerce. It consists of seven base units: the meter (length), kilogram (mass), second (time), ampere (electric current), kelvin (temperature), mole (amount of substance), and candela (luminous intensity).

Since its establishment in 1960, there was a push to standardization in terms of immutable physical phenomena: At this time the metre was redefined: the definition was changed from the prototype of the metre to a certain number of wavelengths of a spectral line of a krypton-86 radiation, making it derivable from universal natural phenomena, immutable in every reference frame. But other units such as the kilogram remained as physical prototypes.

This picture changed recently in 2019, with the SI undergoing a significant redefinition again, with four of the seven base units—kilogram, ampere, kelvin, and mole—were redefined based on fundamental constants of nature. This change ensures greater stability and universality of measurements across all scientific disciplines.

Kilogram: Previously defined by the International Prototype of the Kilogram (a platinum-iridium alloy cylinder), the kilogram is now defined using the Planck constant h as

$$1kg = \frac{h}{6.62607015 \times 10^{-34} m^2 s^{-1}}. \tag{4.1}$$

This equivalence can be experimentally established using a Kibble balance (see Figure 8). In essence, you just need to measure the Planck constant in every single point in the universe to define the kilogram.

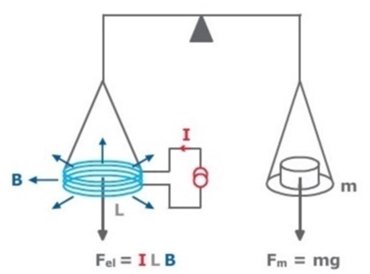

Figure 8. Conceptual illustration of the interworkings of the Kibble's Balance, which establishes a direct link between mass and Planck constant.

The redefinition of SI units ensures that measurements are stable and reproducible across different times and locations, underpinning advancements in science and technology. This change reflects a shift from artifact-based standards to those based on immutable constants of nature, essentially driven by Quantum Metrology. By grounding the SI units in the unchanging constants of quantum mechanics, we ensure the highest possible precision and reliability.