Numerical Linear Algebra

1 Matrices, Vectors, and Norms

1.1 Matrix-Vector Multiplication

Given a matrix $A \in \mathbb{C}^{m \times n}$ and a vector $x \in \mathbb{C}^n$, the matrix-vector product $Ax = b \in \mathbb{C}^m$ is defined as:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \quad \text{for } i = 1, \ldots, m$$

where a_{ij} are the entries of the matrix A.

Observation

The transformation $x \mapsto Ax$ is a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$, i.e., it satisfies:

$$A(x+y) = Ax + Ay, \quad A(\alpha x) = \alpha Ax, \quad \text{for all } x, y \in \mathbb{C}^n, \alpha \in \mathbb{C}$$

1.1.1 A Matrix times a Vector

$$b = Ax = \sum_{j=1}^{n} x_j a_j$$

where a_j is the j-th column of A.

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Example: Vandermonde Matrix

A Vandermonde matrix is defined as:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix} \in \mathbb{C}^{m \times n}$$

Observation

Let fix the sequence $\{x_1, x_2, \ldots, x_m\}$. If p and q are polynomials of degree at most $n-1$ and α is a scalar, then:

1. $(p+q)$ is a polynomial of degree at most $n-1$, and so are $\alpha p, \alpha q$.
2. $(p+q)(x_i) = p(x_i) + q(x_i)$ for $i = 1, \ldots, m$.
3. $(\alpha p)(x_i) = \alpha p(x_i)$ for $i = 1, \ldots, m$.

Observation

Suppose we have a vector $c \in \mathbb{C}^n$ representing the coefficients of a polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$. Then

$$p(x_i) = (Ac)_i = c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{n-1} x_i^{n-1}$$

Any polynomial of degree at most $n-1$ can be represented as

$$p(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

1.2 Matrix-Matrix Multiplication

Given matrices $A \in \mathbb{C}^{l \times m}$ and $C \in \mathbb{C}^{m \times n}$, the matrix-matrix product $B = AC \in \mathbb{C}^{l \times n}$ is defined as:

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}, \quad \text{for } i = 1, \dots, l, j = 1, \dots, n$$

So,

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$b_j = Ac_j = \sum_{k=1}^{m} c_{kj} a_k, \quad j = 1, \dots, n$$

Example: Outer Product

Given two vectors $u \in \mathbb{C}^{m \times 1}$ and $v \in \mathbb{C}^{1 \times n}$, the outer product $uv^T \in \mathbb{C}^{m \times n}$ is defined as:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix}$$

Example

Let $B \in \mathbb{C}^{m \times n}$, let $a_1, a_2, \dots, a_n$ be the columns of $A \in \mathbb{C}^{m \times n}$ and let $R \in \mathbb{C}^{n \times n}$ be an upper triangular matrix with all its superdiagonal entries equal to 1 such that

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Then,

$$b_j = Ar_j = \sum_{k=1}^{j} a_k, \quad j = 1, \dots, n$$

which is known as the indefinite integral operation.

1.3 Range and Null Space

Theorem

The range of a matrix $A \in \mathbb{C}^{m\times n}$ is the space spanned by its columns, i.e. it's column space.

"The set of vectors that can be written as Ax for some $x \in \mathbb{C}^n$."

1.3.1 Null Space and Rank

The null space of a matrix $A \in \mathbb{C}^{m\times n}$ is defined as:

$$\text{span}\{x \in \mathbb{C}^n : Ax = 0\}$$

The rank of a matrix $A \in \mathbb{C}^{m\times n}$ is defined as the dimension of its range:

$$\text{rank}(A) = \dim(\text{range}(A))$$

Theorem

For any matrix $A \in \mathbb{C}^{m\times n}, m \geq n$, A has full rank if and only if it maps no two distinct vectors to the same vector:

$$Ax = Ay \implies x = y, \forall x, y \in \mathbb{C}^n \quad \text{and} \quad Ax \neq Ay \implies x \neq y, \forall x, y \in \mathbb{C}^n$$

1.3.2 Non-singular Matrices

A non-singular matrix, or invertible matrix is a square matrix $A \in \mathbb{C}^{n\times n}$ that has full rank.

1.3.3 Inverse of a Matrix

The matrix $A \in \mathbb{C}^{n\times n}$ is the inverse of Z if and only if:

$$AZ = ZA = I = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

where e_i is the i-th standard basis vector. Furthermore, the inverse of a matrix is unique and

$$\text{rank}(A) = n = \text{rank}(Z)$$

Theorem

For $A \in \mathbb{C}^{n\times n}$, the following statements are equivalent:

1. A has an inverse A^{-1}.
2. A has full rank, i.e. $\text{rank}(A) = n$.
3. The columns of A span $\mathbb{C}^n$, i.e. $\text{range}(A) = \mathbb{C}^n$.
4. The columns of A are linearly independent, i.e. $\text{null}(A) = \{0\}$.
5. 0 is not an eigenvalue of A.
6. 0 is not a singular value of A.
7. $|A| \neq 0$.

1.4 Orthogonal Vectors and Matrices

1.4.1 Complex Conjugate and Conjugate Transpose

Given $z \in \mathbb{C}$, the complex conjugate of z is denoted by $\bar{z}$ or z^*, where if $z = x + iy$, then $\bar{z} = x - iy$.
The conjugate transpose of a matrix $A \in \mathbb{C}^{m\times n}$ is denoted by A^*, where $A^* = \bar{A}^T$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \implies A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{bmatrix}$$

1.4.2 Hermitian Matrices and Skew-Hermitian Matrices

A matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian if $A^* = A$. (Or $A = A^T$ if $A \in \mathbb{R}^{n\times n}$.)
A matrix $A \in \mathbb{C}^{n\times n}$ is called Skew-Hermitian if $A^* = -A$. (Or $A = -A^T$ if $A \in \mathbb{R}^{n\times n}$.)

1.4.3 Inner Product and Norm

Given two vectors $x, y \in \mathbb{C}^n$, the inner product is defined as:

$$\langle x, y\rangle = y^* x = \sum_{i=1}^{n} \bar{y}_i x_i$$

The norm of a vector $x \in \mathbb{C}^n$ is defined as:

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

The modulus of a complex number $z \in \mathbb{C}$ is defined as:

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \quad \text{where } z = x + iy$$

The angle θ between two vectors $x, y \in \mathbb{C}^n$ is defined as:

$$\cos(\theta) = \frac{\langle x, y\rangle}{\|x\|\|y\|}$$

Observation

The inner product is bilinear, which means:

1. $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$
2. $\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$
3. $\langle \alpha x, \beta y\rangle = \alpha\bar{\beta}\,\langle x, y\rangle$ for all $\alpha, \beta \in \mathbb{C}$

For all vectors or matrices A, B of compatible dimensions, we have:

$$(AB)^* = B^* A^*$$

1.4.4 Orthogonal Vectors

Two vectors $x, y \in \mathbb{C}^n$ are orthogonal if $\langle x, y\rangle = 0$.

Observation

If x and y are orthogonal, then they are perpendicular.
With $x = \{x_1, x_2, \ldots, x_n\}$ and $y = \{y_1, y_2, \ldots, y_n\}$, x and y are orthogonal if and only if:

$$\langle x_i, y_j \rangle = 0, \quad \forall i, j = 1, 2, \ldots, n$$

1.4.5 Orthonormal Vectors

A set of vectors $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{C}^n$ is orthonormal if:

$$\langle x_i, x_j \rangle = 0 \text{ for } i \neq j \quad \text{and} \quad \langle x_i, x_i \rangle = \|x_i\|^2 = 1$$

Observation

The vectors in an orthogonal set are linearly independent.

1.4.6 Orthogonal and Unitary Matrices

A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal if its columns form an orthonormal set, i.e. $Q^T Q = QQ^T = I$.
A matrix $U \in \mathbb{C}^{n \times n}$ is unitary if its columns form an orthonormal set, i.e. $U^* U = UU^* = I$.

Observation

If $U \in \mathbb{C}^{n \times n}$ is unitary, then $U^{-1} = U^*$ and

$$\begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

which implies that $\langle u_i, u_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.