

Exercises in Numerical Linear Algebra

Exercise 1.1

Prove the following inequality:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

We start proving that

$$\begin{aligned}\|x\|_2 &\leq \|x\|_1 \\ \|x\|_1^2 &= \left(\sum_{i=1}^n |x_i| \right)^2 = \sum_{i=1}^n |x_i|^2 + \sum_{i \neq j} |x_i| |x_j| \\ \|x\|_2^2 &= \sum_{i=1}^n |x_i|^2\end{aligned}$$

Since $\sum_{i \neq j} |x_i| |x_j| \geq 0$, we have that $\|x\|_1^2 \geq \|x\|_2^2$ and therefore $\|x\|_1 \geq \|x\|_2$.

Now we prove that

$$\|x\|_1 \leq \sqrt{n}\|x\|_2$$

By Cauchy-Schwarz inequality, for a vector $a = (|x_1|, |x_2|, \dots, |x_n|)$ and $b = (1, 1, \dots, 1)$,

$$\|a \cdot b\| \leq \|a\| \|b\|$$

Therefore,

$$\|a \cdot b\| = \sum_{i=1}^n |x_i| = \|x\|_1 \cdot 1 = \|a\|_1 = \|x\|_1$$

$$\|a\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} = \|x\|_2$$

$$\|b\|_2 = \sqrt{\sum_{i=1}^n 1^2} = \sqrt{n}$$

Thus,

$$\|x\|_1 \leq \|x\|_2 \sqrt{n}$$

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Exercise 1.2

Prove the following inequality:

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

We start proving that

$$\|x\|_\infty \leq \|x\|_2$$

Let $\|x\|_\infty = |x_k|$ for some k . Then,

$$\|x\|_2^2 = \sum_{i=1}^n x_i^2 = |x_1|^2 + |x_2|^2 + \dots + |x_k|^2 + \dots + |x_n|^2 \geq |x_k|^2 = \|x\|_\infty^2$$

Taking square roots, we get $\|x\|_2 \geq \|x\|_\infty$.

Now we prove that

$$\|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

Since $\forall i, |x_i| \leq \max_j |x_j| = \|x\|_\infty$, we have

$$\|x\|_i^2 \leq \|x\|_\infty^2$$

Thus,

$$\sqrt{\sum_{i=1}^n |x_i|^2} \leq \sqrt{\sum_{i=1}^n \|x\|_\infty^2} = \sqrt{n\|x\|_\infty^2} = \sqrt{n}\|x\|_\infty$$

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Exercise 1.3

Prove the following inequality:

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

We start proving that

$$\|x\|_\infty \leq \|x\|_1$$

Given that $\|x\|_\infty = |x_k|$ for some k , we have

$$\|x\|_1 = \sum_{i=1}^n |x_i| = |x_1| + |x_2| + \dots + |x_k| + \dots + |x_n| \geq |x_k| = \|x\|_\infty$$

Now we prove that

$$\|x\|_1 \leq n\|x\|_\infty$$

Since $\forall i, |x_i| \leq \max_j |x_j| = \|x\|_\infty$, we have

$$\|x\|_i \leq \|x\|_\infty$$

Thus,

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leq \sum_{i=1}^n \|x\|_\infty = n\|x\|_\infty$$

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Exercise 2.1

Prove the following definition of matrix norm:

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| = \max\{\|col(A)_1\|_1, \|col(A)_2\|_1, \dots, \|col(A)_n\|_1\}$$

With $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, we have

$$\|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1$$

where

$$\|Ax\|_1 = \sum_{i=1}^m |(Ax)_i| = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} x_j \right| \leq \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| |x_j| = \sum_{j=1}^n |x_j| \sum_{i=1}^m |a_{ij}|$$

and we have

$$\sum_{j=1}^n |x_j| \sum_{i=1}^m |a_{ij}| \leq \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \sum_{j=1}^n |x_j| = \mathcal{C} \|x\|_1$$

where $\mathcal{C} = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$.

So now, we need to prove the following:

$$\forall x \in \mathbb{R}^n, \|Ax\|_1 \leq \mathcal{C} \|x\|_1 \quad \text{and} \quad \|Ax\|_1 = \mathcal{C} \|x\|_1, \text{ for some } x \in \mathbb{R}^n.$$

Let us pick k such that

$$c_k = \sum_{i=1}^m |a_{ik}|$$

with $x = e_k$, the standard basis vector with 1 in the k -th position and 0 elsewhere. Then,

$$\sup \|Ax\|_1 = \|Ae_k\|_1 = \|a_k\|_1 = \sum_{i=1}^m |a_{ik}| = c_k$$

and

$$\|x\|_1 = 1$$

Thus,

$$\|Ax\|_1 = c_k \|x\|_1 = \mathcal{C} \|x\|_1$$

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Exercise 2.2

Prove the following definition of matrix norm:

$$\|A\|_2 = \sqrt{\lambda_1}, \text{ where } \lambda_1 \text{ is the largest eigenvalue of } A^T A$$

We have

$$\|Ax\|_2^2 = x^T A^T A x$$

with $x = \alpha_1 z_1, \alpha_2 z_2, \dots, \alpha_n z_n$, where $A^T A z_i = \lambda_i z_i$:

$$x^T A^T A x = \lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2 + \dots + \lambda_n \alpha_n^2 \leq \lambda_1 x^T x = \lambda_1 (\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2)$$

Because of orthonormality, $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = 1$, and we get

$$\|Ax\|_2^2 \leq \lambda_1$$

Now, we need to prove that $\|Ax\|_2^2 = \lambda_1$

$$\|A\|_2^2 = \sup_{\|x\|_2=1} \|Ax\|_2^2 = \lambda_1$$

Exercise 2.3

Prove the following definition of matrix norm:

$$\|A\|_\infty = \max\{\|row(A)_1\|_1, \|row(A)_2\|_1, \dots, \|row(A)_m\|_1\} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

Using the definition of induced norm, we have

$$\|A\|_\infty = \max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} x_j \right| \leq \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}| |x_j|$$

$$\|x\|_\infty = 1 \Rightarrow \|Ax\|_\infty \leq \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{kj}|, \text{ for some } k$$

We define $x \in \mathbb{R}^n$ such that

$$x_j = \begin{cases} \frac{a_{kj}}{|a_{kj}|} & , \text{ if } a_{kj} \neq 0 \\ 0 & , \text{ if } a_{kj} = 0 \end{cases}$$

Then,

$$\|Ax\|_\infty = \max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} x_j \right| = \left| \sum_{j=1}^n a_{kj} x_j \right| = \sum_{j=1}^n |a_{kj}| = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

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