Exercises in Numerical Linear Algebra

Exercise 1.1

Prove the following inequality:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

We start proving that

$$\|x\|_2 \leq \|x\|_1$$

$$\|x\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2 = \sum_{i=1}^{n} |x_i|^2 + \sum_{i \neq j} |x_i||x_j|$$

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$$

Since $\sum_{i \neq j} |x_i||x_j| \geq 0$, we have that $\|x\|_1^2 \geq \|x\|_2^2$ and therefore $\|x\|_1 \geq \|x\|_2$.

Now we prove that

$$\|x\|_1 \leq \sqrt{n}\|x\|_2$$

By Cauchy-Schwarz inequality, for a vector $a = (|x_1|, |x_2|, \ldots, |x_n|)$ and $b = (1, 1, \ldots, 1)$,

$$\|a \cdot b\| \leq \|a\|\|b\|$$

Therefore,

$$|a \cdot b| = \sum_{i=1}^{n} |x_i| = \|x\|_1 \cdot 1 = \|a\|_1 = \|x\|_1$$

$$\|a\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2$$

$$\|b\|_2 = \sqrt{\sum_{i=1}^{n} 1^2} = \sqrt{n}$$

Thus,

$$\|x\|_1 \leq \|x\|_2\sqrt{n}$$

$\blacksquare$

Exercise 1.2

Prove the following inequality:

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

We start proving that

$$\|x\|_\infty \leq \|x\|_2$$

Let $\|x\|_\infty = |x_k|$ for some k. Then,

$$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2 = |x_1|^2 + |x_2|^2 + \ldots + |x_k|^2 + \ldots + |x_n|^2 \geq |x_k|^2 = \|x\|_\infty^2$$

Taking square roots, we get $\|x\|_2 \geq \|x\|_\infty$.

Now we prove that

$$\|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

Since $\forall i, |x_i| \leq \max_j |x_j| = \|x\|_\infty$, we have

$$\|x\|_i^2 \leq \|x\|_\infty^2$$

Thus,

$$\sqrt{\sum_{i=1}^{n} |x_i|^2} \leq \sqrt{\sum_{i=1}^{n} \|x\|_\infty^2} = \sqrt{n\|x\|_\infty^2} = \sqrt{n}\|x\|_\infty$$

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Exercise 1.3

Prove the following inequality:

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

We start proving that

$$\|x\|_\infty \leq \|x\|_1$$

Given that $\|x\|_\infty = |x_k|$ for some k, we have

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \ldots + |x_k| + \ldots + |x_n| \geq |x_k| = \|x\|_\infty$$

Now we prove that

$$\|x\|_1 \leq n\|x\|_\infty$$

Since $\forall i, |x_i| \leq \max_j |x_j| = \|x\|_\infty$, we have

$$\|x\|_i \leq \|x\|_\infty$$

Thus,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \|x\|_\infty = n\|x\|_\infty$$

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Exercise 2.1

Prove the following definition of matrix norm:

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| = \max\{\|col(A)_1\|_1, \|col(A)_2\|_1, \ldots, \|col(A)_n\|_1\}$$

With $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n$, we have

$$\|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1$$

where

$$\|Ax\|_1 = \sum_{i=1}^{m} |(Ax)_i| = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} |x_j| \sum_{i=1}^{m} |a_{ij}|$$

and we have

$$\sum_{j=1}^{n} |x_j| \sum_{i=1}^{m} |a_{ij}| \leq \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \sum_{j=1}^{n} |x_j| = \mathcal{C}\|x\|_1$$

where $\mathcal{C} = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$.
So now, we need to prove the following:

$$\forall x \in \mathbb{R}^n,\ \|Ax\|_1 \leq \mathcal{C}\|x\|_1 \quad \text{and} \quad \|Ax\|_1 = \mathcal{C}\|x\|_1, \text{ for some } x \in \mathbb{R}^n.$$

Let us pick k such that

$$c_k = \sum_{i=1}^{m} |a_{ik}|$$

with $x = e_k$, the standard basis vector with 1 in the k-th position and 0 elsewhere. Then,

$$\sup \|Ax\|_1 = \|Ae_k\|_1 = \|a_k\|_1 = \sum_{i=1}^{m} |a_{ik}| = c_k$$

and

$$\|x\|_1 = 1$$

Thus,

$$\|Ax\|_1 = c_k\|x\|_1 = \mathcal{C}\|x\|_1$$

$\blacksquare$

Exercise 2.2

Prove the following definition of matrix norm:

$$\|A\|_2 = \sqrt{\lambda_1}, \text{ where } \lambda_1 \text{ is the largest eigenvalue of } A^TA$$

We have

$$\|Ax\|_2^2 = x^TA^TAx$$

with $x = \alpha_1 z_1, \alpha_2 z_2, \ldots, \alpha_n z_n$, where $A^TAz_i = \lambda_i z_i$:

$$x^TA^TAx = \lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 + \ldots + \lambda_n\alpha_n^2 \leq \lambda_1 x^Tx = \lambda_1\left(\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2\right)$$

Because of orthonormality, $\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = 1$, and we get

$$\|Ax\|_2^2 \leq \lambda_1$$

Now, we need to prove that $\|Ax\|_2^2 = \lambda_1$

$$\|A\|_2^2 = \sup_{\|x\|_2=1} \|Ax\|_2^2 = \lambda_1$$

Exercise 2.3

Prove the following definition of matrix norm:

$$\|A\|_\infty = \max\{\|row(A)_1\|_1, \|row(A)_2\|_1, \dots, \|row(A)_m\|_1\} = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

Using the definition of induced norm, we have

$$\|A\|_\infty = \max_{1\le i\le m} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}||x_j|$$

$$\|x\|_\infty = 1 \Rightarrow \|Ax\|_\infty \le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{kj}|, \text{ for some } k$$

We define $x \in \mathbb{R}^n$ such that

$$x_j = \begin{cases} \dfrac{a_{kj}}{|a_{kj}|} & , \text{ if } a_{kj} \ne 0 \\ 0 & , \text{ if } a_{kj} = 0 \end{cases}$$

Then,

$$\|Ax\|_\infty = \max_{1\le i\le m} \left| \sum_{j=1}^{n} a_{ij}x_j \right| = \left| \sum_{j=1}^{n} a_{kj}x_j \right| = \sum_{j=1}^{n} |a_{kj}| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

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