

Exercises in Principles of Mathematical Analysis

Exercise 1.1.1

Prove that \mathbb{Q} has zero measure.

A set $Q \subset \mathbb{R}$ has measure zero if and only if:

$$Q \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{where } A_j \text{ are intervals and } \sum_{j=1}^{\infty} |A_j| < \varepsilon, \forall \varepsilon > 0.$$

Since \mathbb{Q} is countable, we can enumerate its elements as $\{q_1, q_2, q_3, \dots\}$.

We start with:

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1/2}{1 - 1/2} = 1.$$

Then, we can multiply this series by any $\varepsilon > 0$ to get:

$$\sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

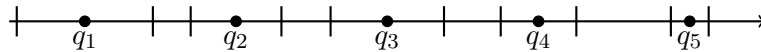
For each rational number q_j , we can construct an interval A_j centered at q_j with length $\frac{\varepsilon}{2^j}$:

$$A_j = \left(q_j - \frac{\varepsilon}{2^{j+1}}, q_j + \frac{\varepsilon}{2^{j+1}} \right).$$

Thus, we have:

$$\mathbb{Q} \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{and} \quad \sum_{j=1}^{\infty} |A_j| = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Since ε can be made arbitrarily small, we conclude that \mathbb{Q} has measure zero.

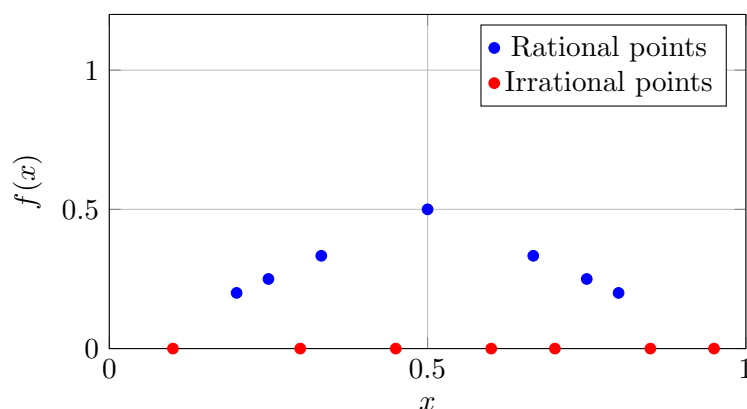


Exercise 1.1.2

For the following function defined on $[0, 1]$:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z}, q \neq 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

1. Show that f is discontinuous only at the rational points.
2. Prove that f is Riemann integrable.



If x is irrational, then

$$\lim_{x \rightarrow x_0} f(x) \stackrel{?}{=} 0 = f(x_0),$$

$\forall \varepsilon > 0$, we want $|f(x) - 0| = |f(x)| < \varepsilon$ if x is close enough to x_0 i.e. $x \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

For example, take $\varepsilon = \frac{1}{3}$, then $|f(x)| < \frac{1}{3}$ if $f(x) = \frac{1}{q} < \frac{1}{3} \Rightarrow q > 3$. So, except for the rationals $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, we have $|f(x)| < \varepsilon = \frac{1}{3}$. We can do this for any $\varepsilon > 0$ by choosing $q > \frac{1}{\varepsilon}$, so

$$\lim_{x \rightarrow x_0} f(x) = 0.$$

So it is continuous at every irrational point. On \mathbb{Q} , the value of $f(x)$ is non-zero, so it is discontinuous at every rational point.

Finally, since the set of discontinuities has measure zero ($|\mathbb{Q}| = 0$), f is Riemann integrable.

Exercise 1.1.4

Consider the sequence of functions given by:

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

Obtain the limit $\lim_{n \rightarrow \infty} f_n(x)$ and study whether the convergence is uniform or not.

$$\sup_{n \in \mathbb{N}} |f_n(x) - f(x)| = 1 \neq 0, \quad \forall x \in \mathbb{R}.$$

So the convergence is not uniform.

Exercise 1.1.5

Prove that the following series converges in $[0, 1]$. Is the convergence uniform?

$$\sum_{n=0}^{\infty} x(1-x)^n = x \sum_{n=0}^{\infty} (1-x)^n = \begin{cases} \frac{x}{1-(1-x)} = \frac{x}{x} = 1 & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

If $|1-x| < 1$, then the series converges. This is true for all $x \in [0, 1]$. The convergence is not uniform since f is not continuous:

$$f_N(x) = \sum_{n=0}^N x(1-x)^n \xrightarrow{N \rightarrow \infty} f(x) = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Exercise 1.1.3

Prove that if a function $f : [a, b] \rightarrow \mathbb{R}$ is monotonous, then it:

1. is bounded

2. is Riemann integrable.

Suppose f is monotonically increasing. Then,

$$f(x) \in [f(a), f(b)], \quad \forall x \in [a, b].$$

So f is bounded.

Now, we build $U_f(P)$ and $L_f(P)$ for a partition $P = \{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \dots, b\}$.

$$U_f(P) = \sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^n f(x_i) \cdot \frac{b-a}{n},$$
$$L_f(P) = \sum_{i=1}^n \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^n f(x_{i-1}) \cdot \frac{b-a}{n}.$$

Then,

$$U_f(P) - L_f(P) = \frac{b-a}{n} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = \frac{b-a}{n} (f(b) - f(a)) \xrightarrow{n \rightarrow \infty} 0.$$

So f is Riemann integrable.

Exercise 1.1.8

Build a sequence of continuous functions on $[0, 1]$ that converges to a continuous function, but in a non-uniform way. Let us define the sequence of functions:

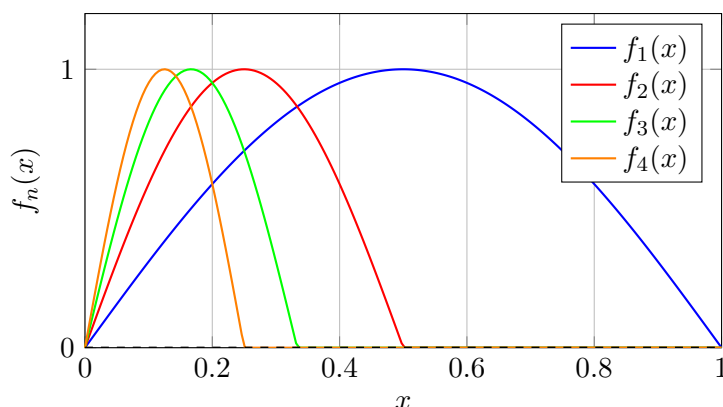
$$f_n(x) = \begin{cases} \sin(n\pi x) & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$$

Each f_n is continuous on $[0, 1]$. Now, we find the limit:

$$\lim_{n \rightarrow \infty} f_n(x) = 0 = f(x), \quad \forall x \in [0, 1].$$

However, the convergence is not uniform. As:

$$\sup_{x \in [0, 1]} |f_n(x) - f(x)| = \sup_{x \in [0, 1]} |f_n(x)| = 1, \quad \forall n \in \mathbb{N}.$$



Exercise 1.1.9

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{n^2 - 1}{(x^2 + 1)(n^2 + 1)} \cdot e^{\frac{-x^4}{n}} dx$$

As $n \rightarrow \infty$,

$$\frac{n^2 - 1}{(x^2 + 1)(n^2 + 1)} \cdot e^{\frac{-x^4}{n}} \rightarrow \frac{1}{x^2 + 1}.$$

So, the limit becomes:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{N \rightarrow \infty} \int_0^N \frac{1}{x^2 + 1} dx + \lim_{M \rightarrow -\infty} \int_M^0 \frac{1}{x^2 + 1} dx = \\ &= [\arctan(x)]_0^N + [\arctan(x)]_M^0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \end{aligned}$$

Exercise 1.2.1

Let $f : X \rightarrow Y$ be a mapping. Given $A \subset Y$, let us define:

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Prove that:

1. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$

If $x \in f^{-1}(Y \setminus A)$, then $f(x) \in Y \setminus A$. So, $f(x) \notin A$, which means $x \notin f^{-1}(A)$. Thus, $x \in X \setminus f^{-1}(A)$.

2. $f^{-1}\left(\bigcup_j A_j\right) = \bigcup_j f^{-1}(A_j).$

If $x \in f^{-1}\left(\bigcup_j A_j\right)$, then $f(x) \in \bigcup_j A_j$. So, there exists some j such that $f(x) \in A_j$. Thus, $x \in f^{-1}(A_j)$ for that j , which means $x \in \bigcup_j f^{-1}(A_j)$.

Exercise 1.2.2

Let $f : X \rightarrow Y$ be a mapping between two topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$. **Prove that f is continuous if and only if f is continuous at every $x \in X$.**

For a topological space (X, \mathcal{T}_X) , a function $f : X \rightarrow Y$ is continuous if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V) \in \mathcal{T}_X$.

Now, if f is continuous at every $x \in X$, then for every open set $V \in \mathcal{T}_Y$, we have $f^{-1}(V) \in \mathcal{T}_X$. Conversely, if f is continuous, then for every $x \in X$, and for every open set $V \in \mathcal{T}_Y$ containing $f(x)$, we have $f^{-1}(V) \in \mathcal{T}_X$. Thus, f is continuous at every $x \in X$.

Exercise 1.2.3

Show that if $X = \{1, 2, 3\}$, then $\mathcal{F} := \{\emptyset, \{2, 3\}, X\}$ is not a σ -algebra.

To show that \mathcal{F} is not a σ -algebra, we need to verify the three properties of a σ -algebra:

1. **Contains the empty set:** $\emptyset \in \mathcal{F}$.
2. **Closed under complementation:** The complement of $\{2, 3\}$ in X is $\{1\}$, which is **not** in \mathcal{F} . Since \mathcal{F} is not closed under complementation, it is not a σ -algebra.

Exercise 1.2.4

Let \mathcal{S} be a family of subsets of X , $\mathcal{S} \subseteq \mathcal{P}(X)$. **Prove that**

$$\mathcal{A}_{\mathcal{S}} := \bigcap \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra, } \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}$$

is the smallest σ -algebra containing \mathcal{S} .

To prove that $\mathcal{A}_{\mathcal{S}}$ is the smallest σ -algebra containing \mathcal{S} , we need to show the following:

1. $\mathcal{A}_{\mathcal{S}}$ is a σ -algebra.

We know that $\mathcal{A}_{\mathcal{S}}$ contains the empty set since every σ -algebra contains the empty set. It is closed under complementation and countable unions because these properties hold for each σ -algebra in the intersection.

2. $\mathcal{S} \subseteq \mathcal{A}_{\mathcal{S}}$.

By definition, $\mathcal{A}_{\mathcal{S}}$ is the intersection of all σ -algebras containing \mathcal{S} , so \mathcal{S} is contained in $\mathcal{A}_{\mathcal{S}}$.

3. If \mathcal{B} is any σ -algebra containing \mathcal{S} , then $\mathcal{A}_{\mathcal{S}} \subseteq \mathcal{B}$.

Since $\mathcal{A}_{\mathcal{S}}$ is the intersection of all σ -algebras containing \mathcal{S} , it must be contained in any such σ -algebra \mathcal{B} .

Thus, $\mathcal{A}_{\mathcal{S}}$ is the smallest σ -algebra containing \mathcal{S} .

Exercise 1.2.5

Let $X = \{a, b, c, d\}$. Construct the σ -algebra generated by $E_1 = \{a\}$ and by $E_2 = \{a, b\}$. The σ -algebra generated by $E_1 = \{a\}$ is:

$$\mathcal{A}_{E_1} = \{\emptyset, \{a\}, \{b, c, d\}, X\}.$$

The σ -algebra generated by $E_2 = \{a, b\}$ is:

$$\mathcal{A}_{E_2} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}.$$

Exercise 1.2.12

Let $u, v : X \rightarrow \mathbb{R}$ be measurable functions and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Prove that

1. $u + v$, uv and $|u|^\alpha$ are measurable.

$$u + v = \phi(u, v)$$

$$uv = \phi(u, v)$$

$$|u|^\alpha = \phi(u)$$

Every ϕ is continuous, and the composition of measurable functions is measurable. Therefore, $u + v$, uv , and $|u|^\alpha$ are measurable.

Exercise 1.2.13

Let (X, \mathcal{A}) be a measurable space and let $f : X \rightarrow \mathbb{R}$ be a function. Prove that the following assertions are equivalent:

1. $\{x \in X : f(x) < \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.
2. $f^{-1}(B) \in \mathcal{A}$ for every Borel set B . Borel sets on $(\mathbb{R}, \mathcal{T})$ are σ -algebras generated by \mathcal{T}

We start by proving that (2) implies (1). The open sets $V \in \mathcal{T}$ are all in $\mathcal{B}(\mathbb{R})$, so they are all borelians. In (1), we have

$$\{x \in X : f(x) < \alpha\} \stackrel{?}{\in} \mathcal{A}$$

which is equivalent to

$$f^{-1}((\alpha, \infty)), \quad (\alpha, \infty) \in \mathcal{B}(\mathbb{R}).$$

Thus,

$$f^{-1}((\alpha, \infty)) \in \mathcal{A} \text{ by (2).}$$

Now, we prove that (1) implies (2). In \mathcal{A} , since it is a sigma algebra, it must contain $\mathcal{B}(\mathbb{R})$, that is the smallest σ -algebra generated by the open sets.

Exercise 1.2.15

Prove that if f is a real function on a measurable space X such that $\{x \in X : f(x) \geq r\}$ is measurable for every rational r , then f is measurable.

We have

$$\forall \alpha \in \mathbb{R}, \quad \exists \{r_n\} \subset \mathbb{Q} : r_n \nearrow \alpha.$$

and $\{r_n\}$ is increasing (since $r_n \nearrow \alpha$).

Note: $r_n \nearrow \alpha$ means that r_n is an increasing sequence that converges to α .

We want to prove that

$$f^{-1}((\alpha, \infty)) = \{x \in X : f(x) > \alpha\} \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}.$$

Consider the increasing sequence $\{r_n\}$ such that

$$\lim_{n \rightarrow \infty} r_n = \alpha.$$

With this we express

$$(\alpha, \infty) = \bigcap_{n=1}^{\infty} [r_n, \infty)$$

and then

$$\{x \in X : f(x) \geq r\} = f^{-1}([r, \infty)) \in \mathcal{A}, \quad \forall r \in \mathbb{Q}.$$

Thus,

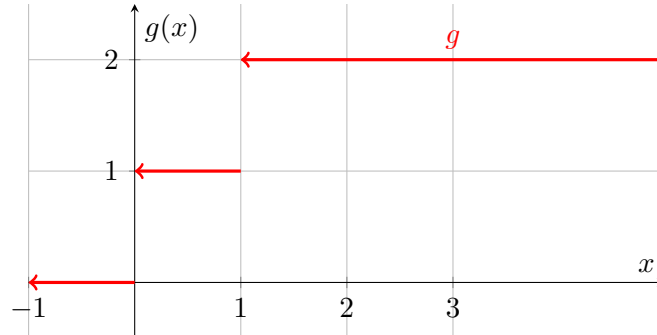
$$f^{-1}((\alpha, \infty)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [r_n, \infty)\right) = \bigcap_{n=1}^{\infty} f^{-1}([r_n, \infty)) \in \mathcal{A}.$$

Exercise 1.2.16

Let \mathcal{M} be the σ -algebra in \mathbb{R} given by $\mathcal{M} = \{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$. Let g be the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0], \\ 1 & \text{if } x \in (0, 1], \\ 2 & \text{if } x \in (1, \infty). \end{cases}$$

Is g measurable? How are the measurable functions $f : (\mathbb{R}, \mathcal{M}) \rightarrow \mathbb{R}$, with the usual topology of the open sets?



$$\forall V \in \mathcal{T}, \quad g^{-1}(V) \in \mathcal{M}.$$

With $V = (-1, 3)$, we have $g^{-1}((-1, 3)) = \mathbb{R} \in \mathcal{M}$.

With $W = (0, 1)$, we have $g^{-1}((0, 1)) = \emptyset \in \mathcal{M}$.

With $H = (-1, 2)$, we have $g^{-1}((-1, 2)) = (-\infty, 1] \notin \mathcal{M}$. So g is not measurable.

Exercise 1.2.18

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\bar{\mathbb{R}} = [-\infty, \infty]$. Prove that

- (a) $\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$.
- (b) $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$.
- (c) If $a_n \leq b_n$ for all n , then $\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n$.
- (d) Show with an example that strict inequality can hold in part b).

Definition: If $\{a_n\}$ is a sequence in $\bar{\mathbb{R}}$, then the sequence:

$$b_k := \sup\{a_k, a_{k+1}, \dots\}$$

is decreasing (non-increasing), that is $b_k \geq b_{k+1}$ for all k . It is bounded from below by $-\infty$, so it converges to its infimum. The limit is called the *limit superior* of the sequence $\{a_n\}$ and is denoted by:

$$\limsup_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} (\sup\{a_k, a_{k+1}, \dots\})$$

This limit always exists in $\bar{\mathbb{R}}$. Similarly, consider the sequence:

$$c_k := \inf\{a_k, a_{k+1}, \dots\}$$

which is increasing (non-decreasing), that is $c_k \leq c_{k+1}$ for all k . It is bounded from above by ∞ , so it converges to its supremum. The limit is called the *limit inferior* of the sequence $\{a_n\}$ and is denoted by:

$$\liminf_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} (\inf\{a_k, a_{k+1}, \dots\}).$$

Example

Let $a_n = (-1)^n \arctan(n)$, then

$$\limsup_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} (\sup\{a_k, a_{k+1}, \dots\}) = \lim_{k \rightarrow \infty} (\sup(\arctan(k))) = \frac{\pi}{2},$$

$$\liminf_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} (\inf\{a_k, a_{k+1}, \dots\}) = \lim_{k \rightarrow \infty} (\inf(-\arctan(k))) = -\frac{\pi}{2}.$$

So the sequence does not converge, since $\limsup_{n \rightarrow \infty} a_n \neq \liminf_{n \rightarrow \infty} a_n$, it does not have a limit.

$\limsup_{n \rightarrow \infty} \{a_n\}$ is the largest value for which there is a subsequence converging to it, and $\liminf_{n \rightarrow \infty} \{a_n\}$ is the smallest value for which there is a subsequence converging to it.

Exercise 1.3.3

Let (X, \mathcal{A}) be a measurable space and let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a countably additive function on the σ -algebra \mathcal{A} .

- (a) Show that if μ satisfies that $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\emptyset) = 0$ (and therefore μ is a measure).
- (b) Find an example for which $\mu(\emptyset) \neq 0$ (and therefore the countably additivity property does not imply that μ is a measure).

For μ to be countably additive, so it must satisfy:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j),$$

for any countable collection of disjoint sets $\{A_j\} \subseteq \mathcal{A}$.

Suppose there exists some $A \in \mathcal{A}$ such that $\mu(A) < \infty$. We can express A as the union of two disjoint sets:

$$A = A \cup \emptyset.$$

By countable additivity, we have:

$$\mu(A) = \mu(A) + \mu(\emptyset).$$

Rearranging this gives:

$$\mu(\emptyset) = \mu(A) - \mu(A) = 0.$$

Thus, if μ is countably additive and there exists some $A \in \mathcal{A}$ with $\mu(A) < \infty$, then $\mu(\emptyset) = 0$. An example where $\mu(\emptyset) \neq 0$ is the function $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ defined by:

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

This function is not a measure because it does not satisfy $\mu(\emptyset) = 0$.