Principles of Mathematical Analysis

1 Measure Theory

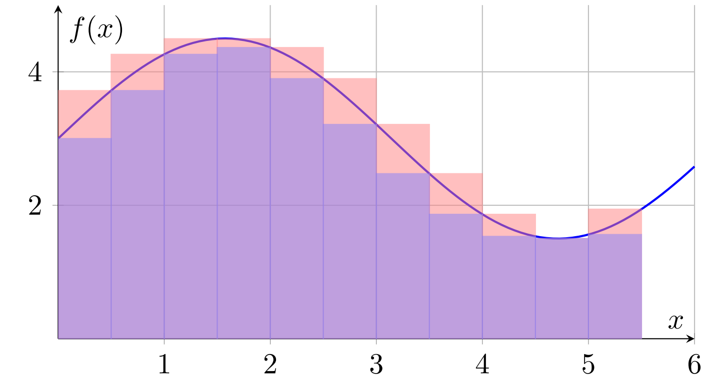
1.1 Riemann Integral

For a bounded function $f : [a, b] \to \mathbb{R}$ and any partition of the interval $[a, b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$, we consider on each subinterval $I_j = [x_{j-1}, x_j], \quad j = 1, \ldots, n$, the quantities:

$$M_j = \sup_{x \in I_j} f(x), \quad m_j = \inf_{x \in I_j} f(x).$$

We also define the upper and lower sums of f with respect to the partition P as:

$$U_f(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad L_f(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}).$$



For any two partitions P and Q of $[a, b]$, we have:

$$L_f(P) \leq \text{Area under } f \leq U_f(Q).$$

If P has a value I such that:

$$\sup_P L_f(P) = I = \inf_P U_f(P),$$

then we say that f is Riemann integrable on $[a, b]$ and define the Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x)\, dx = I.$$

Continuous functions on closed intervals are Riemann integrable.

1.2 The Lebesgue Integral

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Lebesgue integrable* on $[a, b]$ if the set of points where f is discontinuous has zero measure.
A set $B \subset \mathbb{R}$ has *measure zero* if for every $\varepsilon > 0$, it can be covered by a countable collection of open intervals $\{(a_n, b_n)\}$ such that:

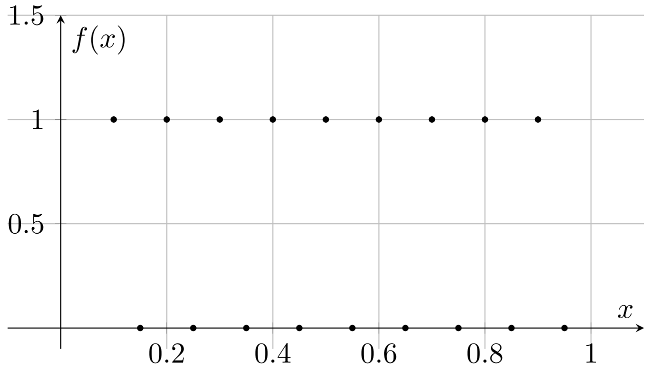
$$B \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

1.2.1 Example: Dirichlet Function

The Dirichlet function:

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

On the interval $[0, 1]$:



We see that f is not Riemann integrable since it is discontinuous everywhere.
But consider,

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$$

and define:

$$f_1(x) = \chi_{\{q_1\}}(x) \to \text{integrable on } [0, 1]$$
$$f_2(x) = \chi_{\{q_1, q_2\}}(x) \to \text{integrable on } [0, 1]$$
$$\vdots$$
$$f_n(x) = \chi_{\{q_1, q_2, \ldots, q_n\}}(x) \to \text{integrable on } [0, 1]$$

Then,

$$\lim_{n \to \infty} f_n(x) = \chi_{\mathbb{Q}}(x).$$

1.2.2 Characteristic Function

For any set $A \subset \mathbb{R}$, the characteristic function $\chi_A : \mathbb{R} \to \{0, 1\}$ is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$\int_0^1 f_1(x)\, dx = 0 = \int_0^1 f_2(x)\, dx = \ldots = \int_0^1 f_n(x)\, dx = 0.$$

Example

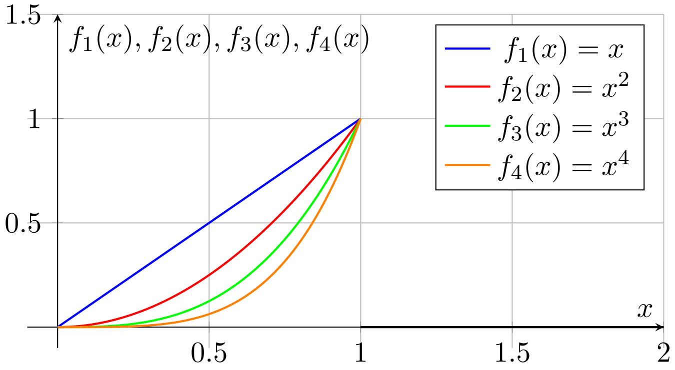
Let

$$f_n(x) = \begin{cases} x^n & \text{if } 0 \leq x < 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \leq 1. \end{cases}$$

so we can see that there is a discontinuity at $x = 1$.



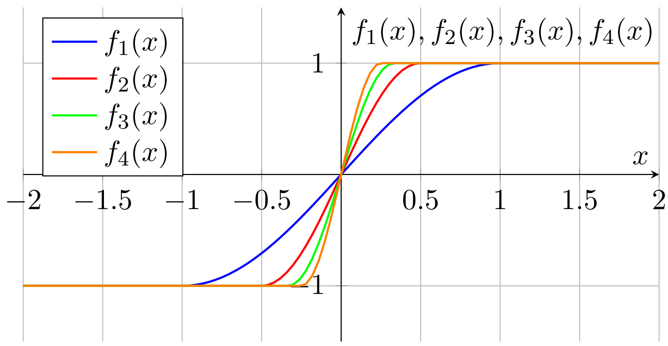
Example

Let

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}, \\ \sin\left(\frac{n\pi x}{2}\right) & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous and differentiable on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Example

The Dirichlet function is not integrable but it is the limit of a sequence of integrable functions, all with integral equal to zero.
We need to define a new kind of convergence.

1.3 Convergences

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges punctually to a function f on $Dom(f)$ if:

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in Dom(f).$$

$$\forall\varepsilon > 0, \quad \forall x \in Dom(f), \quad \exists N(\varepsilon, x) \in \mathbb{N} : n > N \implies |f_n(x) - f(x)| < \varepsilon.$$

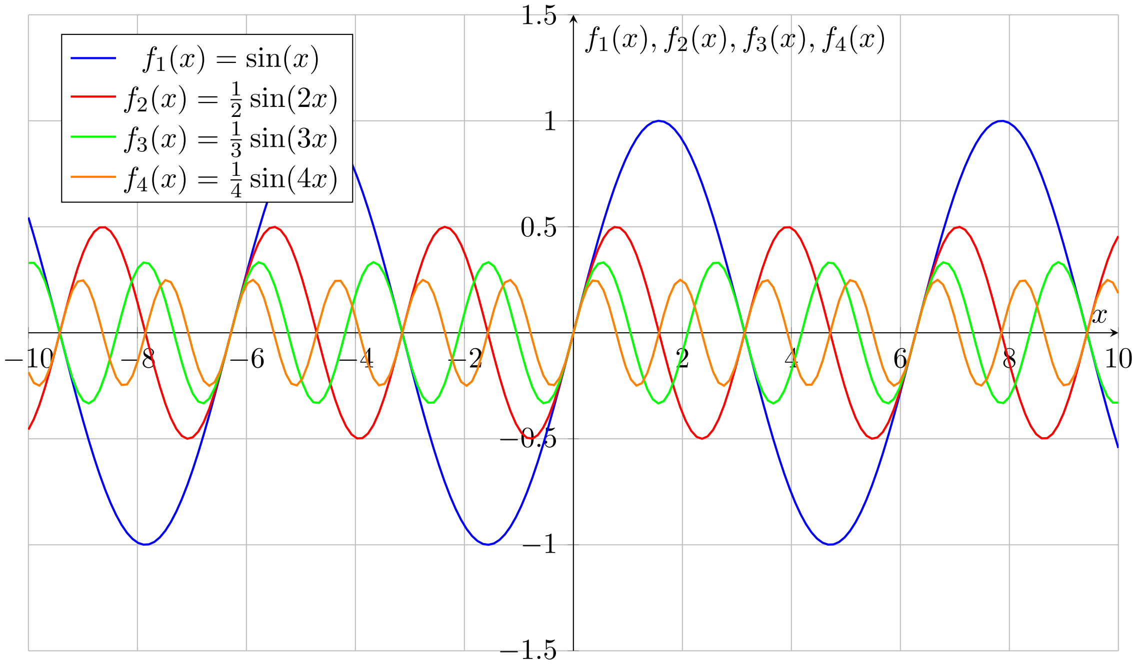
A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to a function f on $Dom(f)$ if:

$$\forall\varepsilon, \quad \exists N : n > N \implies |f_n(x) - f(x)| < \varepsilon, \quad \forall x \in Dom(f).$$

Example

Let

$$f_n(x) = \frac{1}{n}\sin(nx), \quad x \in \mathbb{R}. \to^{n\to\infty} f(x) = 0.$$



1.3.1 Uniform Convergence

1. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

2. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous in $[a, b]$, then f is continuous on $[a, b]$.

3. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[a, b]$ that converges punctually to some continuous function f on $[a, b]$ and if the sequence of derivatives $\{f_n'\}_{n\in\mathbb{N}}$ converges uniformly to some continuous function g, then f is differentiable on (a, b) and:

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x).$$

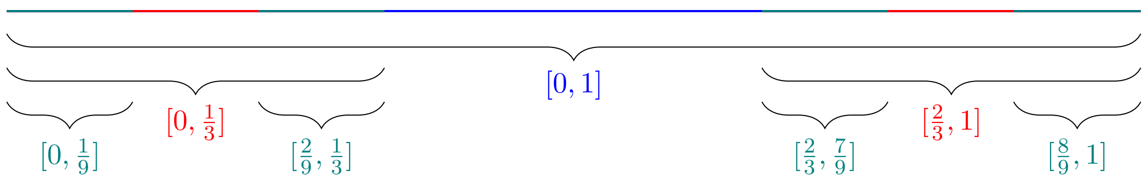
1.3.2 Henri Lebesgue (1875-1941)

How can we count money in bills?

1. Add each amount as the bills come in. (Riemann)
2. Make groups by denomination and count each group. (Lebesgue)

This is the idea behind the Lebesgue integral.

1.4 Cantor Ternary Set



Step 1: $[0,1]$

Step 2: Remove middle third

Step 3: Remove middle thirds of remaining

The Cantor set C is obtained by removing the open middle third of each remaining interval at each step. Then,

$$C = [0,1] \setminus J = \bigcup_{n=1}^{\infty} J_n,$$

where J is the union of all removed intervals and J_n is the set remaining after n steps. For the measure of C:

$$|J| = \sum_{n=1}^{\infty} |J_n| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = 1.$$

Thus, the measure of the Cantor set C is:

$$|C| = |[0,1]| - |J| = 1 - 1 = 0.$$

The Cantor set is not empty; it contains points such as 0, 1, and all endpoints of the removed intervals. It has the following properties:

- It does not contain any intervals.
- It is closed and bounded, hence compact.
- It is a perfect set, which means it is closed and every point is an accumulation point.
- It is uncountable, because there is a bijection between the Cantor set and the interval $[0,1]$ using ternary representation:

  $$\Phi : [0,1] \to C,$$

  where each $x \in C$ is expressed in base 3, and has the form:

  $$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\}.$$

Then, for each $x \in [0,1]$ we define:

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0,1\}.$$

We can then define:

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}.$$

Now, $2b_n \in \{0,2\}$ so $\Phi(x) \in C$. This function is bijective, hence C is uncountable.

2 Measurable Spaces and Topological Spaces

A *Topological Space* $(X, \mathcal{T})$ is a collection $\mathcal{T}$ of subsets of a set X in a topology such that:

- The empty set $\emptyset$ and the whole set X are in $\mathcal{T}$.
- The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
- The intersection of any finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

Example: The Real Line

Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of all open intervals (a,b) where $a < b$ and $a, b \in \mathbb{R}$. Then $(\mathbb{R}, \mathcal{T})$ is a topological space. One can observe that if, for instance, we take the intersection of open intervals like:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\},$$

which is not an open set, hence the requirement for finite intersections.

The sets in a topology $\mathcal{T}$ are called *open sets.* For example, with $X = \bar{\mathbb{R}} = [-\infty, \infty]$, the open sets are all intervals of the form (a,b) where $a < b$. Then, we say that $(\bar{\mathbb{R}}, \mathcal{T})$ is a topological space.

2.1 Metric Spaces

A set X is a *metric space* if there exists a distance function $d : X \times X \to [0, \infty)$, such that for all $x, y, z \in X$:

- $d(x,y) = 0$ if and only if $x = y$ (identity of indiscernibles).
- $d(x,y) = d(y,x)$ (symmetry).
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

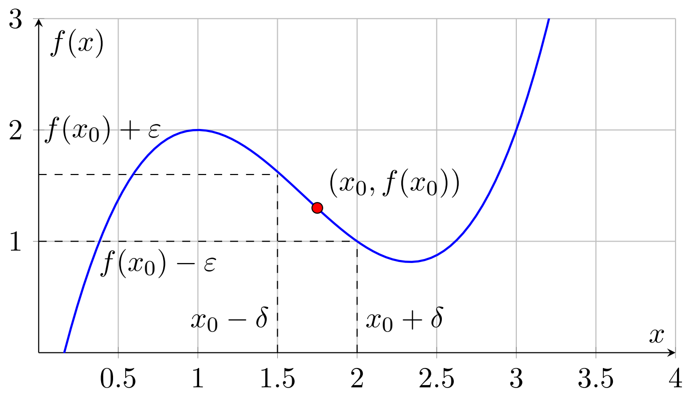
An open ball of center $x \in X$ and radius $r > 0$ is defined as:

$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

2.2 Continuity

A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in [a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in [a, b]$:

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$



2.2.1 Neighborhoods

A *neighborhood* of a set A is any open set that contains A. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a mapping, then f is continuous at a point $x_0 \in X$ if for every neighborhood V of $f(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that:

$$f(U) \subset V.$$

Observation

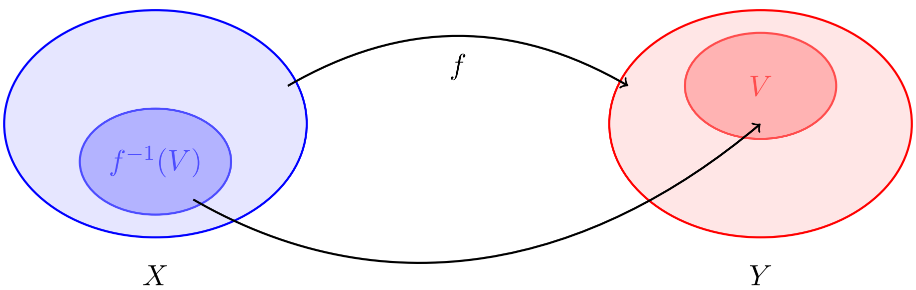
This is equivalent to the ε-δ definition on the $\mathbb{R}^n$ spaces.

2.2.2 Global Continuity

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a mapping, then f is globally continuous if:

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \in \mathcal{T}_X, \quad \forall V \in \mathcal{T}_Y.$$

where $f^{-1}(V)$ is the preimage of V under f.



So, f is continuous if the preimage of every open set in Y is an open set in X.

2.2.3 Proposition

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a mapping, then f is continuous if it is continuous at every point $x \in X$.

2.3 Measurable Spaces

A collection $\mathcal{A}$ of subsets of a space X is a *σ-algebra* if:

1. $\emptyset \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^C \in \mathcal{A}$.
3. If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection with each $A_j \in \mathcal{A}$, then:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$$

The sets of $\mathcal{A}$ are called *measurable sets*. The pair $(X, \mathcal{A})$ is called a *measurable space*. If the third property holds for finite collections, then $\mathcal{A}$ is called an *algebra*.

Example

Is $\mathbb{R}$ with the topology of the usual open sets a σ-algebra? No, because

$$(a, b) \in \mathcal{T} \text{ but } (a, b)^C = (-\infty, a] \cup [b, \infty) \notin \mathcal{T}.$$

Example

The collection $\mathcal{P}(X)$, the power set of X, is a σ-algebra on X.
On X, the collection $\{\emptyset, X\}$ is the smallest σ-algebra.

2.3.1 Properties of measurable spaces

If $(X, \mathcal{A})$ is a measurable space, then:

1. If $\emptyset \in \mathcal{A}$, then $\emptyset^C = X \in \mathcal{A}$.
2. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then:
$$\bigcup_{j=1}^{n} A_j \in \mathcal{A}.$$
3. If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection with each $A_j \in \mathcal{A}$ then, following the second property of σ-algebras:
$$A_j^C \in \mathcal{A}, \quad \forall j \in \mathbb{N}.$$
Then, by the third property of σ-algebras:
$$\bigcup_{j=1}^{\infty} A_j^C \in \mathcal{A}.$$
Finally, by the second property again:
$$\left(\bigcup_{j=1}^{\infty} A_j^C\right)^C = \bigcap_{j=1}^{\infty} A_j \in \mathcal{A}.$$
4. If $A, B \in \mathcal{A}$, then:
$$A \setminus B = A \cap B^C \in \mathcal{A}.$$

2.3.2 Proposition

If $S \subset \mathcal{P}(X)$, then $\sigma(S)$ is called the σ-algebra generated by S:

$$\sigma(S) = \mathcal{A}_S = \bigcap \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra and } S \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}.$$

Example

Let $X = \{1, 2, 3, 4\}$ and $S = \{\{1\}, \{3, 4\}\}$. Then:

$$\sigma(S) = \{\emptyset, X, \{1\}, \{2, 3, 4\}, \{3, 4\}, \{1, 2\}, \{2\}, \{1, 3, 4\}\}.$$

2.3.3 Borel σ-algebra

The Borel σ-algebra on X, denoted by $\mathcal{B}(X)$, is the σ-algebra generated by the open sets of X,

$$\mathcal{B}(X) = \sigma(\mathcal{T}(X)).$$

Its elements are called *Borel sets*.

Example

The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, contains all open intervals, closed intervals, countable sets, and complements of these sets. Examples of these Borel sets include:

$$(a, b), \quad [a, b], \quad (a, b], \quad [a, b), \quad \mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad \{x\}, \quad \mathbb{R}, \quad \emptyset.$$

3 Measurable functions and Integration

A mapping $f : (X, \mathcal{A}) \to (Y, \mathcal{T})$, where $(X, \mathcal{A})$ is a measurable space and $(Y, \mathcal{T})$ is a topological space, is said to be a *measurable function* if the preimage of every open set in Y is a measurable set in X. Formally, for every open set $V \in \mathcal{T}_Y$, we have:

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \in \mathcal{A}.$$

Observation

A mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between two topological spaces is continuous if

$$\forall V \in \mathcal{T}_Y, \quad f^{-1}(V) \in \mathcal{T}_X.$$

Example

If $(X, \mathcal{A})$ is a measurable space and $A \in \mathcal{A}$, then the characteristic function $\chi_A : X \to \{0, 1\}$ defined by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

is a measurable function.

Now, let us consider $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{T})$. For any $V \in \mathcal{T}$, we have:

$$V = (a, b) \quad \text{or} \quad V = (a, b) \cup (c, d) \cup \ldots$$

Then, we can analyze the preimage of V under f:

$$f^{-1}(V) = \begin{cases} A, & \text{if } 1 \in V \\ X \setminus A, & \text{if } 1 \notin V \end{cases}$$

Since both A and $X \setminus A$ are in $\mathcal{A}$, it follows that χ_A is a measurable function.

Example

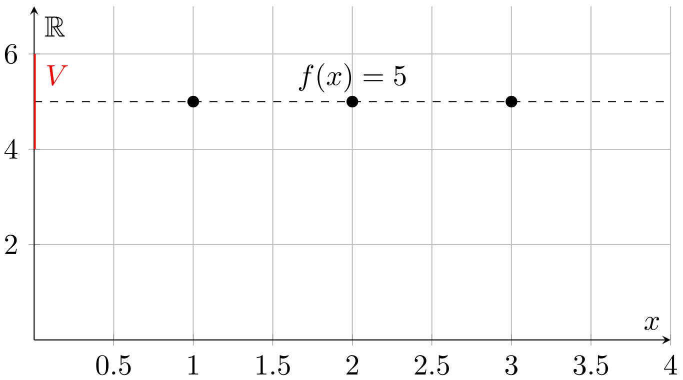
Let $X = \{1, 2, 3\}$ and $\mathcal{A} = \{\emptyset, X\}$. Define $f : X \to \mathbb{R}$ by:

$$f(1) = f(2) = f(3) = 5.$$

Then, for any open set $V \subset \mathbb{R}$:

$$f^{-1}(V) = \begin{cases} X, & \text{if } 5 \in V \\ \emptyset, & \text{if } 5 \notin V \end{cases}$$

Since both X and $\emptyset$ are in $\mathcal{A}$, f is a measurable function.

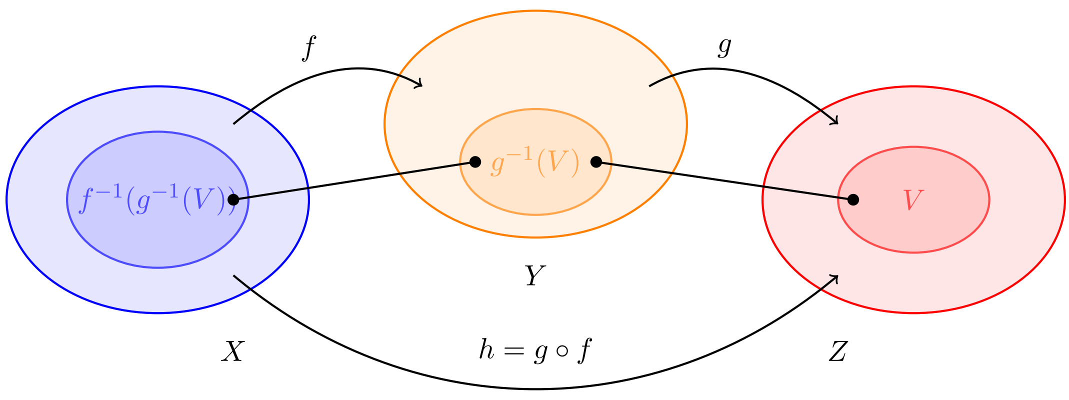


3.1 Composition of Functions and Measurability

Consider two topological spaces $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, and a *continuous* function $g : Y \to Z$:

1. If $(X, \mathcal{T}_X)$ is a *topological space* and $f : X \to Y$ is *continuous*, then the composition $h = g \circ f : X \to Z$ is continuous.

2. If $(X, \mathcal{A})$ is a *measurable space* and $f : X \to Y$ is *measurable*, then the composition $h = g \circ f : X \to Z$ is measurable.



Proof. Consider any open set $V \in \mathcal{T}_Z$. Since g is continuous, the preimage $g^{-1}(V) \in \mathcal{T}_Y$ (it is also an open set, now in $\mathbb{T}_Y$). And then, $f^{-1}(g^{-1}(V)) \in \mathcal{T}_X$ (that is, it is an open set in $\mathbb{T}_X$). Observe that the preimage of $g^{-1}(V)$ under f is:

$$h^{-1}(V) = (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)).$$

Now consider any open set $V \in \mathcal{T}_Z$. Since g is continuous, the preimage $g^{-1}(V) \in \mathcal{T}_Y$. And then, $f^{-1}(g^{-1}(V)) \in \mathcal{A}$ (that is, it is a measurable set in $\mathcal{A}$). $\square$

Example

On $\mathbb{R}$ with $\mathcal{T}$ the topology of the open sets, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the open sets of $\mathbb{R}$. Then $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable space.

3.2 Theorem: Characterizations of Measurable Functions

Given a measurable space $(X, \mathcal{A})$ and $f : X \to \mathbb{R}$, the following statements are equivalent:

1. f is measurable.
2. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) > a\} \in \mathcal{A}$.
3. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) \geq a\} \in \mathcal{A}$.
4. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) < a\} \in \mathcal{A}$.
5. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) \leq a\} \in \mathcal{A}$.
6. For every $a, b \in \mathbb{R}$ with $a < b$, the set $\{x \in X : a < f(x) < b\} \in \mathcal{A}$.
7. The preimage of every open, closed, or Borel set in $\mathbb{R}$ is in $\mathcal{A}$.

3.3 Lemma

Given a measurable function $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{T})$, the family of sets:

$$\mathcal{A}_f = \{B \in \mathbb{R} : f^{-1}(B) \in \mathcal{A}\}$$

is a σ-algebra on $\mathbb{R}$, and it is called the *image σ-algebra*. Then $\mathcal{A}_f$ contains the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ because by definition,

$$(a, \infty) \in \mathcal{A}_f \quad \text{for all } a \in \mathbb{R}.$$

Proof. To show that $\mathcal{A}_f$ is a σ-algebra, we need to verify the three properties:

1. Since $f^{-1}(\emptyset) = \emptyset$ and $\emptyset \in \mathcal{A}$, we have $\emptyset \in \mathcal{A}_f$.
2. If $B \in \mathcal{A}_f$, then $f^{-1}(B) \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, $f^{-1}(B)^C = f^{-1}(B^C) \in \mathcal{A}$. Thus, $B^C \in \mathcal{A}_f$.
3. If $\{B_j\}_{j\in\mathbb{N}}$ is a countable collection with each $B_j \in \mathcal{A}_f$, then $f^{-1}(B_j) \in \mathcal{A}$ for all j. Since $\mathcal{A}$ is a σ-algebra, we have:

$$\bigcup_{j=1}^{\infty} f^{-1}(B_j) = f^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right) \in \mathcal{A}.$$

   Therefore, $\bigcup_{j=1}^{\infty} B_j \in \mathcal{A}_f$.

$\square$

3.4 Measure and Measure Space

A *measure* on a measurable space $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$.
2. If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*.

Observation

Also, there exist negative measures, where $\mu : \mathcal{A} \to [-\infty, \infty]$, and complex measures, where $\mu : \mathcal{A} \to \mathbb{C}$. Furthermore, if $\mu(X) = 1$, then $(X, \mathcal{A}, \mu)$ is called a *probability space*.

Example

Consider the space $X = \{1, 2, 3\}$ and the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$. Define the measure $\mu : \mathcal{A} \to [0, \infty)$ by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Observe that $(X, \mathcal{A}, \mu)$ is a probability space. Also, the measure is countably additive since:

$$\mu(X) = 1 = \mu(\{1, 2\}) + \mu(\{3\}) = 0 + 1 = 1.$$

Observation

On any set X with the σ-algebra $\mathcal{A}$, we can define a measure $\mu : \mathcal{A} \to [0, \infty]$ using a weight function:

$$p : X \to [0, \infty], \quad p(x) \text{ is the weight of } x.$$

If $A \in \mathcal{A}$, then:

$$\mu(A) = \sum_{x \in A} p(x) := \sup_{\{x_1, \ldots, x_n\} \subseteq A} \sum_{j=1}^{n} p(x_j).$$

Example

On $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we can use the weight function $p(x) = 1, \forall x \in \mathbb{N}$. Then, we obtain the *counting measure*:

$$\mu(A) = \sum_{x \in A} 1 = |A|.$$

Example

Now let $p(x) = 1$ for $x = a$, and $p(x) = 0$ for $x \neq a$. Then, we obtain the *Dirac-δ measure* at a:

$$\mu(A) = \sum_{x \in A} p(x) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

3.5 Theorem: Properties of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

1. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ disjoint, then:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Proof. Define $\emptyset = A_{n+1}, A_{n+2}, \ldots$. Then, by the properties of measures:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{n} \mu(A_j) + \sum_{j=n+1}^{\infty} \mu(\emptyset) = \sum_{j=1}^{n} \mu(A_j) + 0 = \sum_{j=1}^{n} \mu(A_j).$$

$\square$

2. If $A, B \in \mathcal{A}$ and $A \subseteq B$, then:
$$\mu(A) \leq \mu(B).$$
And if $\mu(A) < \infty$, then:
$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

Proof. Since $A \subseteq B$, we can write $B = A \cup (B \setminus A)$ with A and $B \setminus A$ disjoint. Then, by the properties of measures:
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$
If $\mu(A) < \infty$, then rearranging gives:
$$\mu(B \setminus A) = \mu(B) - \mu(A).$$
□

3. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of sets in $\mathcal{A}$ (i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$), then:
$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

4. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of increasing sets in $\mathcal{A}$ (i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$), then:
$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n).$$

5. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of decreasing sets in $\mathcal{A}$ (i.e., $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$) and $\mu(A_1) < \infty$, then:
$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n) = \inf_{n\in\mathbb{N}} \mu(A_n).$$

Example

Let $X = \mathbb{N}$ and $A_n = \{n, n+1, n+2, \ldots\}$. Consider the counting measure μ on $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Then:
$$A_1 \supset A_2 \supset A_3 \supset \ldots \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n = \emptyset.$$
Thus:
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(\emptyset) = 0.$$
However, $\mu(A_1) = \infty$, so the condition $\mu(A_1) < \infty$ is necessary.

3.6 Completion of Measure Spaces

A property is said to hold *almost everywhere* (a.e.) if it holds everywhere except on a set of measure zero. A set with measure zero is called a *null set*.

Example

Consider the space $X = \{1, 2, 3\}$ with the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$ and the measure $\mu : \mathcal{A} \to [0, \infty)$ defined by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Then, the set $\{1, 2\}$ is a null set since $\mu(\{1, 2\}) = 0$, and $(X, \mathcal{A}, \mu)$ is a measure space.
Let us define the functions $f, g : X \to \mathbb{R}$ by:

$$f(1) = f(2) = f(3) = 3, \quad g(x) = x.$$

Then, $f(x) = g(x)$ almost everywhere since they differ only on the null set $\{1, 2\}$. However, f is measurable while g is not, because:

$$g^{-1}((2, 4)) = \{3\} \in \mathcal{A},$$

but

$$g^{-1}((0, 2)) = \{1\} \notin \mathcal{A}.$$

A measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if every subset E of a null set N is measurable.

$$\forall N \in \mathcal{A} \text{ with } \mu(N) = 0, \quad \forall E \subseteq N, \quad E \in \mathcal{A}.$$

Example

Consider $X = \mathbb{N}$ with the σ-algebra $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and a counting measure μ. Since the only null set is $\emptyset$, every subset of a null set is measurable. Thus, $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a complete measure space.

Now, consider a Dirac-δ measure μ at $a \in \mathbb{R}$ on $\mathcal{P}(\mathbb{R})$. The Dirac measure is defined by:

$$\mu(E) = \begin{cases} 1 & \text{if } a \in E, \\ 0 & \text{if } a \notin E. \end{cases}$$

In this case, the null set is $\emptyset$, and every subset of $\emptyset$ is measurable. Thus, $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$ is also a complete measure space.

3.7 Theorem: Completion of a Measure Space

Given a measure space $(X, \mathcal{A}, \mu)$, we can construct its completion $(X, \overline{\mathcal{A}}, \overline{\mu})$ as follows:

1. Define $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ as the collection of null sets.
2. Define $\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$ as the collection of sets formed by the union of a measurable set and a null set.
3. Define $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ by:

$$\overline{\mu}(A \cup N) = \mu(A), \quad \text{for } A \in \mathcal{A}, N \in \mathcal{N}.$$

Then, $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space. Furthermore, $\overline{\mathcal{A}}$ is the smallest σ-algebra containing $\mathcal{A}$, and $\overline{\mu}$ is a complete measure extending μ.

Proof. To show that $\overline{\mathcal{A}}$ is a σ-algebra, we need to verify the three properties:

1. Since $\emptyset \in \mathcal{A}$ and $\emptyset \in \mathcal{N}$, we have $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{A}}$.

2. If $B = A \cup N \in \overline{\mathcal{A}}$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$, then:

$$B^C = (A \cup N)^C = A^C \cap N^C = (A^C \cap X) \cup (A^C \cap N^C).$$

Since $A^C \in \mathcal{A}$ and $N^C \in \mathcal{A}$, we have $B^C \in \overline{\mathcal{A}}$.

3. If $\{B_j\}_{j\in\mathbb{N}}$ is a countable collection with each $B_j = A_j \cup N_j \in \overline{\mathcal{A}}$, where $A_j \in \mathcal{A}$ and $N_j \in \mathcal{N}$, then:

$$\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (A_j \cup N_j) = \left(\bigcup_{j=1}^{\infty} A_j\right) \cup \left(\bigcup_{j=1}^{\infty} N_j\right).$$

Since $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ and $\bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$, we have $\bigcup_{j=1}^{\infty} B_j \in \overline{\mathcal{A}}$.

Now we need to check wether $\overline{\mu}$ is well-defined on $\overline{\mathcal{A}}$ and satisfies the properties of a measure:

1. For any $B = A \cup N \in \overline{\mathcal{A}}$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$, we have:

$$\overline{\mu}(B) = \overline{\mu}(A \cup N) = \mu(A) \geq 0.$$

2. If $\{B_j\}_{j\in\mathbb{N}}$ is a countable collection of pairwise disjoint sets in $\overline{\mathcal{A}}$, where $B_j = A_j \cup N_j$ with $A_j \in \mathcal{A}$ and $N_j \in \mathcal{N}$, then:

$$\bigcup_{j=1}^{\infty} B_j = \left(\bigcup_{j=1}^{\infty} A_j\right) \cup \left(\bigcup_{j=1}^{\infty} N_j\right).$$

Since the B_j are pairwise disjoint, the A_j are also pairwise disjoint. Thus, by the properties of measures:

$$\overline{\mu}\left(\bigcup_{j=1}^{\infty} B_j\right) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \overline{\mu}(B_j).$$

□

Example

Consider the space $X = \{1, 2, 3\}$ with the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$ and the measure $\mu : \mathcal{A} \to [0, \infty)$ defined by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Then, the null set is $\mathcal{N} = \{\emptyset, \{1, 2\}\}$. The completion of the measure space is given by:

$$\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, X\} = \mathcal{P}(X).$$

The completed measure $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty)$ is defined by:

$$\overline{\mu}(\emptyset) = \overline{\mu}(\{1\}) = \overline{\mu}(\{2\}) = \overline{\mu}(\{1, 2\}) = 0, \quad \overline{\mu}(\{3\}) = \overline{\mu}(\{1, 3\}) = \overline{\mu}(\{2, 3\}) = \overline{\mu}(X) = 1.$$

3.8 Semi-algebra

A collection $\mathcal{E} \subseteq \mathcal{P}(X)$ is called a *semi-algebra* if:

1. $\emptyset \in \mathcal{E}$.
2. If $A, B \in \mathcal{E}$, then $A \cap B \in \mathcal{E}$.
3. If $A \in \mathcal{E}$, then $A^C = B_1 \cup B_2 \cup \ldots \cup B_n$ where $B_j \in \mathcal{E}$ for $j = 1, 2, \ldots, n$.

Example

On $\mathbb{R}$, the collection of all intervals of the form:

$$(a,b),\quad [a,b),\quad (a,b],\quad [a,b],\quad (-\infty,a),\quad (-\infty,a],\quad (a,\infty),\quad [a,\infty),$$

where $a, b \in \mathbb{R}$, is a semi-algebra.

A set function $\mu : X \to [0,\infty]$ is *σ-finite* if

$$X = \bigcup_{j=1}^{\infty} X_j, \quad X_j \in X, \quad \mu(X_j) < \infty \text{ for all } j.$$

and we say that X is σ-finite with respect to μ.

3.9 Operations with infinity

The following conventions are used when dealing with infinity in measure theory:

- $a + \infty = \infty + a = \infty$ for any $a \in [0,\infty]$.
- $a \cdot \infty = \infty \cdot a = \infty$ for any $a \in (0,\infty]$.
- $0 \cdot \infty = \infty \cdot 0 = 0$.
- Cancellation law: If $a, b \in [0,\infty]$ and $c \in (0,\infty]$, then:

$$a + c = b + c \implies a = b.$$

- If $a, b \in [0,\infty]$ and $c \in (0,\infty)$, then:

$$a \cdot c = b \cdot c \implies a = b.$$

3.10 Caratheodory-Hopf's Theorem

Consider a semi-algebra $\mathcal{E}$ on X and a countably additive function $\mu_0 : \mathcal{E} \to [0,\infty]$. We define for all $A \in \mathcal{P}(X)$:

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : A \subseteq \bigcup_{j=1}^{\infty} E_j, \quad E_j \in \mathcal{E} \right\}.$$