Principles of Mathematical Analysis

1 Measure Theory

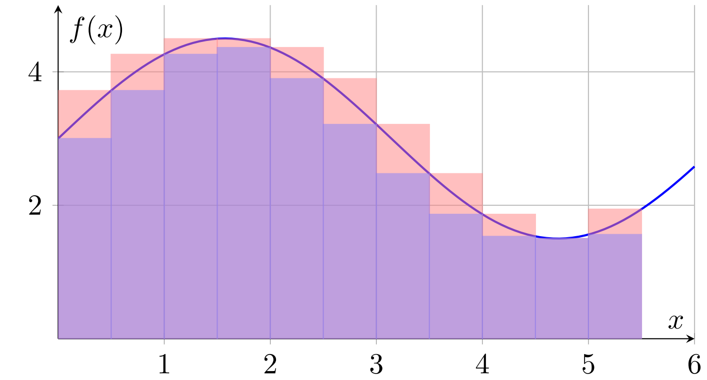
1.1 Riemann Integral

For a bounded function $f : [a, b] \to \mathbb{R}$ and any partition of the interval $[a, b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$, we consider on each subinterval $I_j = [x_{j-1}, x_j]$, $\quad j = 1, \ldots, n$, the quantities:

$$M_j = \sup_{x \in I_j} f(x), \quad m_j = \inf_{x \in I_j} f(x).$$

We also define the upper and lower sums of f with respect to the partition P as:

$$U_f(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad L_f(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}).$$



For any two partitions P and Q of $[a, b]$, we have:

$$L_f(P) \leq \text{Area under } f \leq U_f(Q).$$

If P has a value I such that:

$$\sup_P L_f(P) = I = \inf_P U_f(P),$$

then we say that f is Riemann integrable on $[a, b]$ and define the Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x)\, dx = I.$$

Continuous functions on closed intervals are Riemann integrable.

1.2 The Lebesgue Integral

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Lebesgue integrable* on $[a, b]$ if the set of points where f is discontinuous has zero measure.
A set $B \subset \mathbb{R}$ has *measure zero* if for every $\varepsilon > 0$, it can be covered by a countable collection of open intervals $\{(a_n, b_n)\}$ such that:

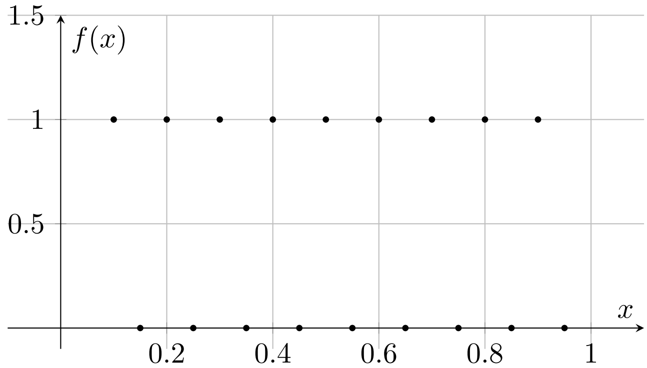
$$B \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

1.2.1 Example: Dirichlet Function

The Dirichlet function:

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

On the interval $[0, 1]$:



We see that f is not Riemann integrable since it is discontinuous everywhere.
But consider,

$$\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$$

and define:

$$f_1(x) = \chi_{\{q_1\}}(x) \to \text{integrable on } [0, 1]$$
$$f_2(x) = \chi_{\{q_1, q_2\}}(x) \to \text{integrable on } [0, 1]$$
$$\vdots$$
$$f_n(x) = \chi_{\{q_1, q_2, \dots, q_n\}}(x) \to \text{integrable on } [0, 1]$$

Then,

$$\lim_{n \to \infty} f_n(x) = \chi_{\mathbb{Q}}(x).$$

1.2.2 Characteristic Function

For any set $A \subset \mathbb{R}$, the characteristic function $\chi_A : \mathbb{R} \to \{0, 1\}$ is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$\int_0^1 f_1(x)\, dx = 0 = \int_0^1 f_2(x)\, dx = \dots = \int_0^1 f_n(x)\, dx = 0.$$

Example

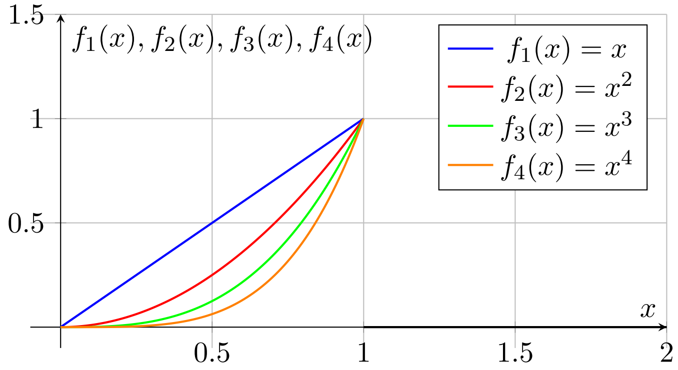
Let

$$f_n(x) = \begin{cases} x^n & \text{if } 0 \leq x < 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \leq 1. \end{cases}$$

so we can see that there is a discontinuity at $x = 1$.



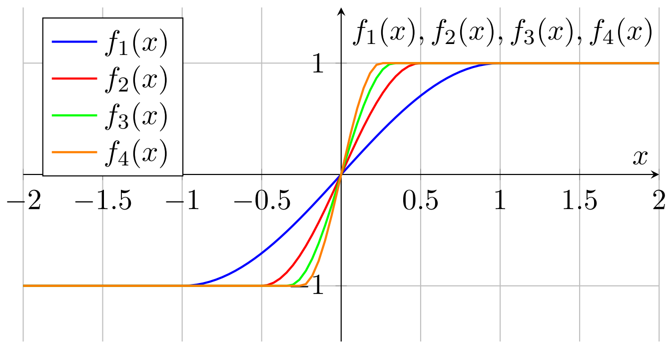
Example

Let

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}, \\ \sin\left(\frac{n\pi x}{2}\right) & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous and differentiable on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Example

The Dirichlet function is not integrable but it is the limit of a sequence of integrable functions, all with integral equal to zero.
We need to define a new kind of convergence.

1.3 Convergences

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges punctually to a function f on $Dom(f)$ if:

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in Dom(f).$$

$$\forall \varepsilon > 0, \quad \forall x \in Dom(f), \quad \exists N(\varepsilon, x) \in \mathbb{N} : n > N \implies |f_n(x) - f(x)| < \varepsilon.$$

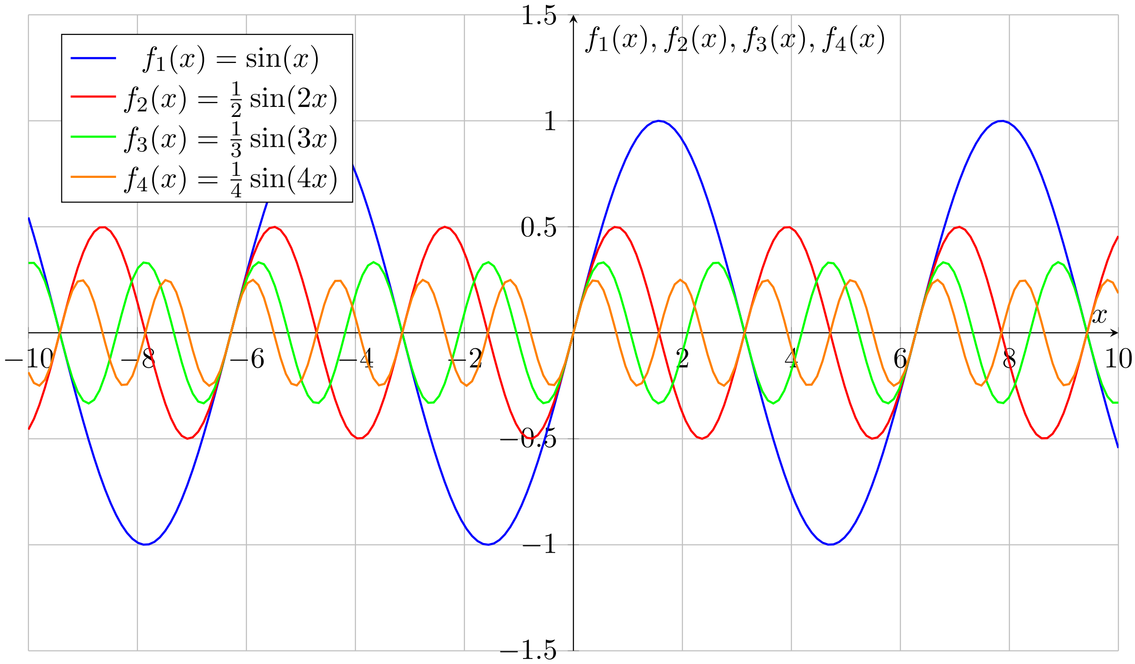
A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to a function f on $Dom(f)$ if:

$$\forall \varepsilon, \quad \exists N : n > N \implies |f_n(x) - f(x)| < \varepsilon, \quad \forall x \in Dom(f).$$

Example

Let

$$f_n(x) = \frac{1}{n}\sin(nx), \quad x \in \mathbb{R}. \to^{n\to\infty} f(x) = 0.$$



1.3.1 Uniform Convergence

1. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

2. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous in $[a, b]$, then f is continuous on $[a, b]$.

3. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[a, b]$ that converges punctually to some continuous function f on $[a, b]$ and if the sequence of derivatives $\{f'_n\}_{n\in\mathbb{N}}$ converges uniformly to some continuous function g, then f is differentiable on (a, b) and:

$$f'(x) = g(x) = \lim_{n\to\infty} f'_n(x).$$

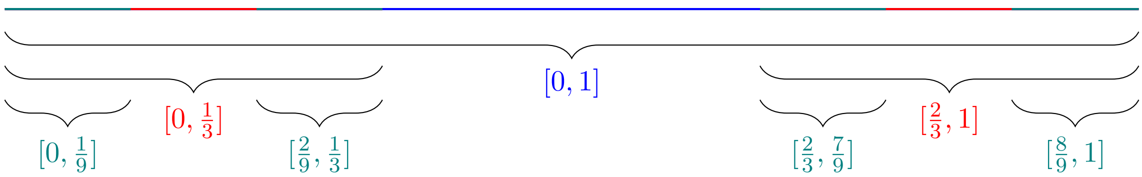
1.3.2 Henri Lebesgue (1875-1941)

How can we count money in bills?

1. Add each amount as the bills come in. (Riemann)
2. Make groups by denomination and count each group. (Lebesgue)

This is the idea behind the Lebesgue integral.

1.4 Cantor Ternary Set



Step 1: $[0,1]$

Step 2: Remove middle third

Step 3: Remove middle thirds of remaining

The Cantor set C is obtained by removing the open middle third of each remaining interval at each step. Then,

$$C = [0,1] \setminus J = \bigcup_{n=1}^{\infty} J_n,$$

where J is the union of all removed intervals and J_n is the set remaining after n steps. For the measure of C:

$$|J| = \sum_{n=1}^{\infty} |J_n| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = 1.$$

Thus, the measure of the Cantor set C is:

$$|C| = |[0,1]| - |J| = 1 - 1 = 0.$$

The Cantor set is not empty; it contains points such as 0, 1, and all endpoints of the removed intervals. It has the following properties:

- It does not contain any intervals.
- It is closed and bounded, hence compact.
- It is a perfect set, which means it is closed and every point is an accumulation point.
- It is uncountable, because there is a bijection between the Cantor set and the interval $[0,1]$ using ternary representation:

  $$\Phi : [0,1] \to C,$$

  where each $x \in C$ is expressed in base 3, and has the form:

  $$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\}.$$

Then, for each $x \in [0,1]$ we define:

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0,1\}.$$

We can then define:

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}.$$

Now, $2b_n \in \{0,2\}$ so $\Phi(x) \in C$. This function is bijective, hence C is uncountable.

2 Measurable Spaces and Topological Spaces

A *Topological Space* $(X, \mathcal{T})$ is a collection $\mathcal{T}$ of subsets of a set X in a topology such that:

- The empty set $\emptyset$ and the whole set X are in $\mathcal{T}$.
- The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
- The intersection of any finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

Example: The Real Line

Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of all open intervals (a,b) where $a < b$ and $a, b \in \mathbb{R}$. Then $(\mathbb{R}, \mathcal{T})$ is a topological space. One can observe that if, for instance, we take the intersection of open intervals like:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is not an open set, hence the requirement for finite intersections.

The sets in a topology $\mathcal{T}$ are called *open sets.* For example, with $X = \bar{\mathbb{R}} = [-\infty, \infty]$, the open sets are all intervals of the form (a,b) where $a < b$. Then, we say that $(\bar{\mathbb{R}}, \mathcal{T})$ is a topological space.

2.1 Metric Spaces

A set X is a *metric space* if there exists a distance function $d : X \times X \to [0, \infty)$, such that for all $x, y, z \in X$:

- $d(x,y) = 0$ if and only if $x = y$ (identity of indiscernibles).
- $d(x,y) = d(y,x)$ (symmetry).
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

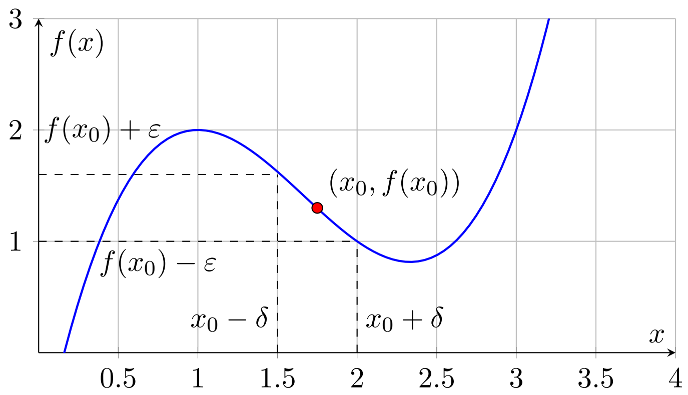
An open ball of center $x \in X$ and radius $r > 0$ is defined as:

$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

2.2 Continuity

A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in [a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in [a, b]$:

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$



2.2.1 Neighborhoods

A *neighborhood* of a set A is any open set that contains A. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a mapping, then f is continuous at a point $x_0 \in X$ if for every neighborhood V of $f(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that:

$$f(U) \subset V.$$

Observation

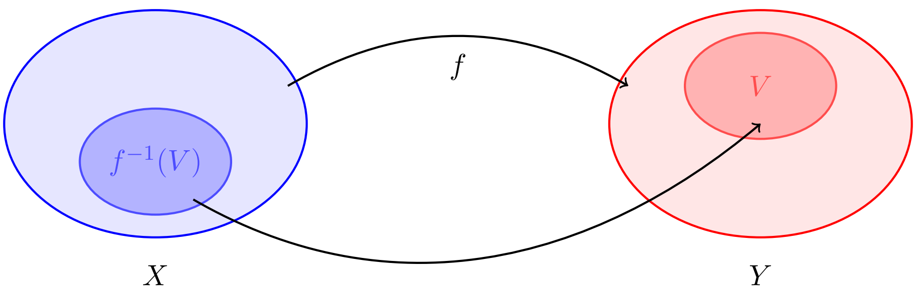
This is equivalent to the ε-δ definition on the $\mathbb{R}^n$ spaces.

2.2.2 Global Continuity

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a mapping, then f is globally continuous if:

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \in \mathcal{T}_X, \quad \forall V \in \mathcal{T}_Y.$$

where $f^{-1}(V)$ is the preimage of V under f.



So, f is continuous if the preimage of every open set in Y is an open set in X.

2.2.3 Proposition

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a mapping, then f is continuous if it is continuous at every point $x \in X$.

2.3 Measurable Spaces

A collection $\mathcal{A}$ of subsets of a space X is a *σ-algebra* if:

1. $\emptyset \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^C \in \mathcal{A}$.
3. If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection with each $A_j \in \mathcal{A}$, then:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$$

The sets of $\mathcal{A}$ are called *measurable sets.* The pair $(X, \mathcal{A})$ is called a *measurable space.* If the third property holds for finite collections, then $\mathcal{A}$ is called an *algebra.*

Example

Is $\mathbb{R}$ with the topology of the usual open sets a σ-algebra? No, because

$$(a, b) \in \mathcal{T} \text{ but } (a, b)^C = (-\infty, a] \cup [b, \infty) \notin \mathcal{T}.$$

Example

The collection $\mathcal{P}(X)$, the power set of X, is a σ-algebra on X.
On X, the collection $\{\emptyset, X\}$ is the smallest σ-algebra.

2.3.1 Properties of measurable spaces

If $(X, \mathcal{A})$ is a measurable space, then:

1. If $\emptyset \in \mathcal{A}$, then $\emptyset^C = X \in \mathcal{A}$.
2. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then:
$$\bigcup_{j=1}^{n} A_j \in \mathcal{A}.$$
3. If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection with each $A_j \in \mathcal{A}$ then, following the second property of σ-algebras:
$$A_j^C \in \mathcal{A}, \quad \forall j \in \mathbb{N}.$$
Then, by the third property of σ-algebras:
$$\bigcup_{j=1}^{\infty} A_j^C \in \mathcal{A}.$$
Finally, by the second property again:
$$\left(\bigcup_{j=1}^{\infty} A_j^C\right)^C = \bigcap_{j=1}^{\infty} A_j \in \mathcal{A}.$$
4. If $A, B \in \mathcal{A}$, then:
$$A \setminus B = A \cap B^C \in \mathcal{A}.$$

2.3.2 Proposition

If $S \subset \mathcal{P}(X)$, then $\sigma(S)$ is called the σ-algebra generated by S:

$$\sigma(S) = \mathcal{A}_S = \bigcap\{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra and } S \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}.$$

Example

Let $X = \{1, 2, 3, 4\}$ and $S = \{\{1\}, \{3, 4\}\}$. Then:

$$\sigma(S) = \{\emptyset, X, \{1\}, \{2, 3, 4\}, \{3, 4\}, \{1, 2\}, \{2\}, \{1, 3, 4\}\}.$$

2.3.3 Borel σ-algebra

The Borel σ-algebra on X, denoted by $\mathcal{B}(X)$, is the σ-algebra generated by the open sets of X,

$$\mathcal{B}(X) = \sigma(\mathcal{T}(X)).$$

Its elements are called *Borel sets*.

Example

The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, contains all open intervals, closed intervals, countable sets, and complements of these sets. Examples of these Borel sets include:

$$(a, b), \quad [a, b], \quad (a, b], \quad [a, b), \quad \mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad \{x\}, \quad \mathbb{R}, \quad \emptyset.$$

3 Measurable functions and Integration

A mapping $f : (X, \mathcal{A}) \to (Y, \mathcal{T})$, where $(X, \mathcal{A})$ is a measurable space and $(Y, \mathcal{T})$ is a topological space, is said to be a *measurable function* if the preimage of every open set in Y is a measurable set in X. Formally, for every open set $V \in \mathcal{T}_Y$, we have:

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \in \mathcal{A}.$$

Observation

A mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between two topological spaces is continuous if

$$\forall V \in \mathcal{T}_Y, \quad f^{-1}(V) \in \mathcal{T}_X.$$

Example

If $(X, \mathcal{A})$ is a measurable space and $A \in \mathcal{A}$, then the characteristic function $\chi_A : X \to \{0, 1\}$ defined by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

is a measurable function.

Now, let us consider $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{T})$. For any $V \in \mathcal{T}$, we have:

$$V = (a, b) \quad \text{or} \quad V = (a, b) \cup (c, d) \cup \dots$$

Then, we can analyze the preimage of V under f:

$$f^{-1}(V) = \begin{cases} A, & \text{if } 1 \in V \\ X \setminus A, & \text{if } 1 \notin V \end{cases}$$

Since both A and $X \setminus A$ are in $\mathcal{A}$, it follows that χ_A is a measurable function.

Example

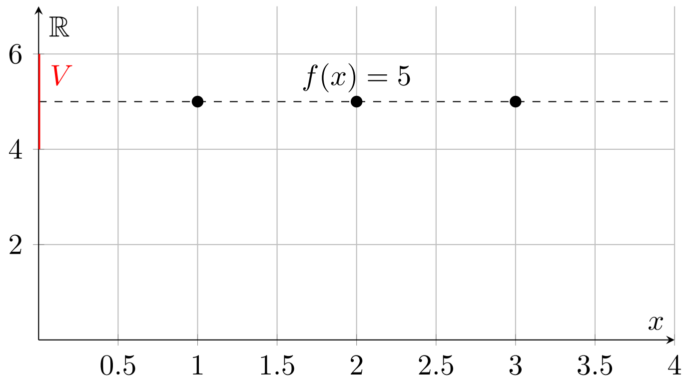
Let $X = \{1, 2, 3\}$ and $\mathcal{A} = \{\emptyset, X\}$. Define $f : X \to \mathbb{R}$ by:

$$f(1) = f(2) = f(3) = 5.$$

Then, for any open set $V \subset \mathbb{R}$:

$$f^{-1}(V) = \begin{cases} X, & \text{if } 5 \in V \\ \emptyset, & \text{if } 5 \notin V \end{cases}$$

Since both X and $\emptyset$ are in $\mathcal{A}$, f is a measurable function.

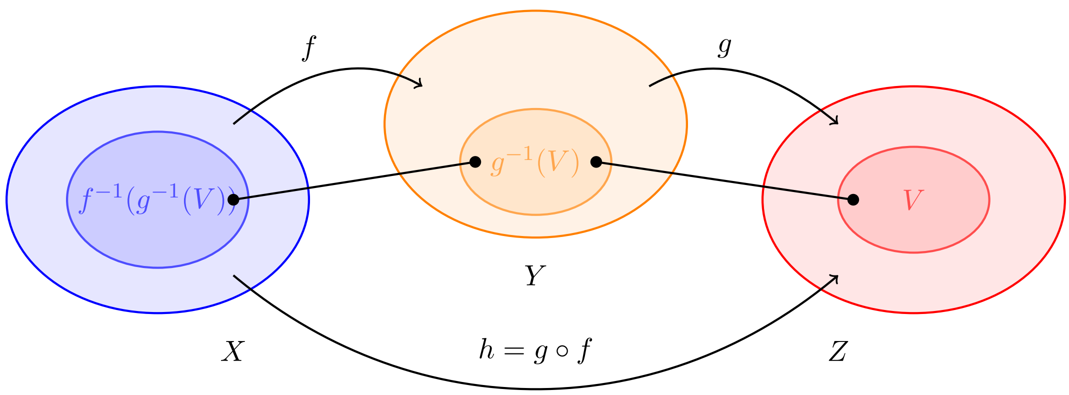


3.1 Composition of Functions and Measurability

Consider two topological spaces $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, and a *continuous* function $g : Y \to Z$:

1. If $(X, \mathcal{T}_X)$ is a *topological space* and $f : X \to Y$ is *continuous*, then the composition $h = g \circ f : X \to Z$ is continuous.

2. If $(X, \mathcal{A})$ is a *measurable space* and $f : X \to Y$ is *measurable*, then the composition $h = g \circ f : X \to Z$ is measurable.



Proof. Consider any open set $V \in \mathcal{T}_Z$. Since g is continuous, the preimage $g^{-1}(V) \in \mathcal{T}_Y$ (it is also an open set, now in $\mathbb{T}_Y$). And then, $f^{-1}(g^{-1}(V)) \in \mathcal{T}_X$ (that is, it is an open set in $\mathbb{T}_X$). Observe that the preimage of $g^{-1}(V)$ under f is:

$$h^{-1}(V) = (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)).$$

Now consider any open set $V \in \mathcal{T}_Z$. Since g is continuous, the preimage $g^{-1}(V) \in \mathcal{T}_Y$. And then, $f^{-1}(g^{-1}(V)) \in \mathcal{A}$ (that is, it is a measurable set in $\mathcal{A}$). $\square$

Example

On $\mathbb{R}$ with $\mathcal{T}$ the topology of the open sets, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the open sets of $\mathbb{R}$. Then $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable space.

3.2 Theorem: Characterizations of Measurable Functions

Given a measurable space $(X, \mathcal{A})$ and $f : X \to \mathbb{R}$, the following statements are equivalent:

1. f is measurable.
2. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) > a\} \in \mathcal{A}$.
3. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) \geq a\} \in \mathcal{A}$.
4. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) < a\} \in \mathcal{A}$.
5. For every $a \in \mathbb{R}$, the set $\{x \in X : f(x) \leq a\} \in \mathcal{A}$.
6. For every $a, b \in \mathbb{R}$ with $a < b$, the set $\{x \in X : a < f(x) < b\} \in \mathcal{A}$.
7. The preimage of every open, closed, or Borel set in $\mathbb{R}$ is in $\mathcal{A}$.

3.3 Lemma

Given a measurable function $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{T})$, the family of sets:

$$\mathcal{A}_f = \{B \in \mathbb{R} : f^{-1}(B) \in \mathcal{A}\}$$

is a σ-algebra on $\mathbb{R}$, and it is called the *image σ-algebra*. Then $\mathcal{A}_f$ contains the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ because by definition,

$$(a, \infty) \in \mathcal{A}_f \quad \text{for all } a \in \mathbb{R}.$$

Proof. To show that $\mathcal{A}_f$ is a σ-algebra, we need to verify the three properties:

1. Since $f^{-1}(\emptyset) = \emptyset$ and $\emptyset \in \mathcal{A}$, we have $\emptyset \in \mathcal{A}_f$.
2. If $B \in \mathcal{A}_f$, then $f^{-1}(B) \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, $f^{-1}(B)^C = f^{-1}(B^C) \in \mathcal{A}$. Thus, $B^C \in \mathcal{A}_f$.
3. If $\{B_j\}_{j \in \mathbb{N}}$ is a countable collection with each $B_j \in \mathcal{A}_f$, then $f^{-1}(B_j) \in \mathcal{A}$ for all j. Since $\mathcal{A}$ is a σ-algebra, we have:

$$\bigcup_{j=1}^{\infty} f^{-1}(B_j) = f^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right) \in \mathcal{A}.$$

   Therefore, $\bigcup_{j=1}^{\infty} B_j \in \mathcal{A}_f$.

□

3.4 Measure and Measure Space

A *measure* on a measurable space $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$.
2. If $\{A_j\}_{j \in \mathbb{N}}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*.

Observation

Also, there exist negative measures, where $\mu : \mathcal{A} \to [-\infty, \infty]$, and complex measures, where $\mu : \mathcal{A} \to \mathbb{C}$. Furthermore, if $\mu(X) = 1$, then $(X, \mathcal{A}, \mu)$ is called a *probability space*.

Example

Consider the space $X = \{1, 2, 3\}$ and the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$. Define the measure $\mu : \mathcal{A} \to [0, \infty)$ by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Observe that $(X, \mathcal{A}, \mu)$ is a probability space. Also, the measure is countably additive since:

$$\mu(X) = 1 = \mu(\{1, 2\}) + \mu(\{3\}) = 0 + 1 = 1.$$

Observation

On any set X with the σ-algebra $\mathcal{A}$, we can define a measure $\mu : \mathcal{A} \to [0, \infty]$ using a weight function:

$$p : X \to [0, \infty], \quad p(x) \text{ is the weight of } x.$$

If $A \in \mathcal{A}$, then:

$$\mu(A) = \sum_{x \in A} p(x) := \sup_{\{x_1, \ldots, x_n\} \subseteq A} \sum_{j=1}^{n} p(x_j).$$

Example

On $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we can use the weight function $p(x) = 1, \forall x \in \mathbb{N}$. Then, we obtain the *counting measure*:

$$\mu(A) = \sum_{x \in A} 1 = |A|.$$

Example

Now let $p(x) = 1$ for $x = a$, and $p(x) = 0$ for $x \neq a$. Then, we obtain the *Dirac-δ measure* at a:

$$\mu(A) = \sum_{x \in A} p(x) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

3.5 Theorem: Properties of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

1. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ disjoint, then:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Proof. Define $\emptyset = A_{n+1}, A_{n+2}, \ldots$. Then, by the properties of measures:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{n} \mu(A_j) + \sum_{j=n+1}^{\infty} \mu(\emptyset) = \sum_{j=1}^{n} \mu(A_j) + 0 = \sum_{j=1}^{n} \mu(A_j).$$

□

2. If $A, B \in \mathcal{A}$ and $A \subseteq B$, then:
$$\mu(A) \leq \mu(B).$$
And if $\mu(A) < \infty$, then:
$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

Proof. Since $A \subseteq B$, we can write $B = A \cup (B \setminus A)$ with A and $B \setminus A$ disjoint. Then, by the properties of measures:
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$
If $\mu(A) < \infty$, then rearranging gives:
$$\mu(B \setminus A) = \mu(B) - \mu(A).$$
□

3. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of sets in $\mathcal{A}$ (i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$), then:
$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

4. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of increasing sets in $\mathcal{A}$ (i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$), then:
$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n).$$

5. If $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of decreasing sets in $\mathcal{A}$ (i.e., $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$) and $\mu(A_1) < \infty$, then:
$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n) = \inf_{n\in\mathbb{N}} \mu(A_n).$$

Example

Let $X = \mathbb{N}$ and $A_n = \{n, n+1, n+2, \ldots\}$. Consider the counting measure μ on $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Then:
$$A_1 \supset A_2 \supset A_3 \supset \ldots \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n = \emptyset.$$
Thus:
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(\emptyset) = 0.$$
However, $\mu(A_1) = \infty$, so the condition $\mu(A_1) < \infty$ is necessary.

3.6 Completion of Measure Spaces

A property is said to hold *almost everywhere* (a.e.) if it holds everywhere except on a set of measure zero. A set with measure zero is called a *null set*.

Example

Consider the space $X = \{1, 2, 3\}$ with the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$ and the measure $\mu : \mathcal{A} \to [0, \infty)$ defined by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Then, the set $\{1, 2\}$ is a null set since $\mu(\{1, 2\}) = 0$, and $(X, \mathcal{A}, \mu)$ is a measure space.
Let us define the functions $f, g : X \to \mathbb{R}$ by:

$$f(1) = f(2) = f(3) = 3, \quad g(x) = x.$$

Then, $f(x) = g(x)$ almost everywhere since they differ only on the null set $\{1, 2\}$. However, f is measurable while g is not, because:

$$g^{-1}((2, 4)) = \{3\} \in \mathcal{A},$$

but

$$g^{-1}((0, 2)) = \{1\} \notin \mathcal{A}.$$

A measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if every subset E of a null set N is measurable.

$$\forall N \in \mathcal{A} \text{ with } \mu(N) = 0, \quad \forall E \subseteq N, \quad E \in \mathcal{A}.$$

Example

Consider $X = \mathbb{N}$ with the σ-algebra $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and a counting measure μ. Since the only null set is $\emptyset$, every subset of a null set is measurable. Thus, $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a complete measure space.

Now, consider a Dirac-δ measure μ at $a \in \mathbb{R}$ on $\mathcal{P}(\mathbb{R})$. The Dirac measure is defined by:

$$\mu(E) = \begin{cases} 1 & \text{if } a \in E, \\ 0 & \text{if } a \notin E. \end{cases}$$

In this case, the null set is $\emptyset$, and every subset of $\emptyset$ is measurable. Thus, $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$ is also a complete measure space.

3.7 Theorem: Completion of a Measure Space

Given a measure space $(X, \mathcal{A}, \mu)$, we can construct its completion $(X, \overline{\mathcal{A}}, \overline{\mu})$ as follows:

1. Define $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ as the collection of null sets.
2. Define $\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$ as the collection of sets formed by the union of a measurable set and a null set.
3. Define $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ by:

$$\overline{\mu}(A \cup N) = \mu(A), \quad \text{for } A \in \mathcal{A}, N \in \mathcal{N}.$$

Then, $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space. Furthermore, $\overline{\mathcal{A}}$ is the smallest σ-algebra containing $\mathcal{A}$, and $\overline{\mu}$ is a complete measure extending μ.

Proof. To show that $\overline{\mathcal{A}}$ is a σ-algebra, we need to verify the three properties:

1. Since $\emptyset \in \mathcal{A}$ and $\emptyset \in \mathcal{N}$, we have $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{A}}$.

2. If $B = A \cup N \in \overline{\mathcal{A}}$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$, then:

$$B^C = (A \cup N)^C = A^C \cap N^C = (A^C \cap X) \cup (A^C \cap N^C).$$

Since $A^C \in \mathcal{A}$ and $N^C \in \mathcal{A}$, we have $B^C \in \overline{\mathcal{A}}$.

3. If $\{B_j\}_{j \in \mathbb{N}}$ is a countable collection with each $B_j = A_j \cup N_j \in \overline{\mathcal{A}}$, where $A_j \in \mathcal{A}$ and $N_j \in \mathcal{N}$, then:

$$\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (A_j \cup N_j) = \left(\bigcup_{j=1}^{\infty} A_j \right) \cup \left(\bigcup_{j=1}^{\infty} N_j \right).$$

Since $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ and $\bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$, we have $\bigcup_{j=1}^{\infty} B_j \in \overline{\mathcal{A}}$.

Now we need to check wether $\overline{\mu}$ is well-defined on $\overline{\mathcal{A}}$ and satisfies the properties of a measure:

1. For any $B = A \cup N \in \overline{\mathcal{A}}$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$, we have:

$$\overline{\mu}(B) = \overline{\mu}(A \cup N) = \mu(A) \geq 0.$$

2. If $\{B_j\}_{j \in \mathbb{N}}$ is a countable collection of pairwise disjoint sets in $\overline{\mathcal{A}}$, where $B_j = A_j \cup N_j$ with $A_j \in \mathcal{A}$ and $N_j \in \mathcal{N}$, then:

$$\bigcup_{j=1}^{\infty} B_j = \left(\bigcup_{j=1}^{\infty} A_j \right) \cup \left(\bigcup_{j=1}^{\infty} N_j \right).$$

Since the B_j are pairwise disjoint, the A_j are also pairwise disjoint. Thus, by the properties of measures:

$$\overline{\mu} \left(\bigcup_{j=1}^{\infty} B_j \right) = \mu \left(\bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \overline{\mu}(B_j).$$

□

Example

Consider the space $X = \{1, 2, 3\}$ with the σ-algebra $\mathcal{A} = \{\emptyset, X, \{1, 2\}, \{3\}\}$ and the measure $\mu : \mathcal{A} \to [0, \infty)$ defined by:

$$\mu(\emptyset) = \mu(\{1, 2\}) = 0, \quad \mu(X) = \mu(\{3\}) = 1.$$

Then, the null set is $\mathcal{N} = \{\emptyset, \{1, 2\}\}$. The completion of the measure space is given by:

$$\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, X\} = \mathcal{P}(X).$$

The completed measure $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty)$ is defined by:

$$\overline{\mu}(\emptyset) = \overline{\mu}(\{1\}) = \overline{\mu}(\{2\}) = \overline{\mu}(\{1, 2\}) = 0, \quad \overline{\mu}(\{3\}) = \overline{\mu}(\{1, 3\}) = \overline{\mu}(\{2, 3\}) = \overline{\mu}(X) = 1.$$

3.8 Semi-algebra

A collection $\mathcal{E} \subseteq \mathcal{P}(X)$ is called a *semi-algebra* if:

1. $\emptyset \in \mathcal{E}$.

2. If $A, B \in \mathcal{E}$, then $A \cap B \in \mathcal{E}$.

3. If $A \in \mathcal{E}$, then $A^C = B_1 \cup B_2 \cup \ldots \cup B_n$ where $B_j \in \mathcal{E}$ for $j = 1, 2, \ldots, n$.

Example

On $\mathbb{R}$, the collection of all intervals of the form:

$$(a,b), \quad [a,b), \quad (a,b], \quad [a,b], \quad (-\infty,a), \quad (-\infty,a], \quad (a,\infty), \quad [a,\infty),$$

where $a,b \in \mathbb{R}$, is a semi-algebra.

A set function $\mu : X \to [0,\infty]$ is *σ-finite* if

$$X = \bigcup_{j=1}^{\infty} X_j, \quad X_j \in X, \quad \mu(X_j) < \infty \text{ for all } j.$$

and we say that X is σ-finite with respect to μ.

3.9 Operations with infinity

The following conventions are used when dealing with infinity in measure theory:

- $a + \infty = \infty + a = \infty$ for any $a \in [0,\infty]$.
- $a \cdot \infty = \infty \cdot a = \infty$ for any $a \in (0,\infty]$.
- $0 \cdot \infty = \infty \cdot 0 = 0$.
- Cancellation law: If $a,b \in [0,\infty]$ and $c \in (0,\infty]$, then:

$$a + c = b + c \implies a = b.$$

- If $a,b \in [0,\infty]$ and $c \in (0,\infty)$, then:

$$a \cdot c = b \cdot c \implies a = b.$$

3.10 Outer Measure

An *outer measure* on a set X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that:

1. $\mu^*(\emptyset) = 0$.
2. If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.
3. If $\{A_j\}_{j\in\mathbb{N}} \subseteq \mathcal{P}(X)$, then:

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Example

Consider the set $X = \{1,2,3\}$ and define the function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by:

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

Then, μ^* is an outer measure on X since it satisfies the three properties:

1. $\mu^*(\emptyset) = 0$.

2. If $A \subseteq B$, then:

$$\mu^*(A) \leq \mu^*(B)$$

holds trivially since both sides are either 0 or 1.

3. For any collection $\{A_j\}_{j\in\mathbb{N}}$, we have:

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$$

since the left side is either 0 or 1, and the right side is at least 0.

Remark

Given an outer measure μ^* on a set X, a set $A \subseteq X$ is said to be μ^**-measurable* if:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \text{for all } E \subseteq X.$$

3.11 Caratheodory-Hopf's Theorem

Consider $\mathcal{M} = \{M \subseteq X : M \text{ is } \mu^*\text{-measurable}\}$. Then:

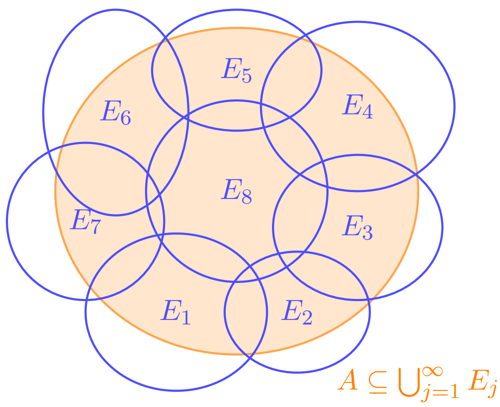
1. $\mathcal{M}$ is a σ-algebra.
2. The restriction $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure on $\mathcal{M}$.

To define an outer measure, we can start with a semi-algebra. Consider a semi-algebra $\mathcal{E} \subseteq \mathcal{P}(X)$ and a countably additive function $\mu_0 : \mathcal{E} \to [0, \infty]$. Then, we can define an outer measure μ^* for all $A \subseteq \mathcal{P}(X)$:

$$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(E_j) : A \subseteq \bigcup_{j=1}^{\infty} E_j, E_j \in \mathcal{E}\right\}.$$

and define $\mathcal{M} = \{M \subseteq X : M \text{ is } \mu^*\text{-measurable}\}$. Then:

1. μ^* is an outer measure and $\mu^*|_{\mathcal{M}} = \mu$ is a complete measure on $\mathcal{M}$, where μ extends μ_0.
2. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\sigma(\mathcal{E})$.



3.12 Semi-open Intervals and Elementary Measure

A *semi-open interval* in $\mathbb{R}^n$ is a set of the form:

$$I = I_1 \times I_2 \times \ldots \times I_n,$$

where each I_j is a semi-open interval in $\mathbb{R}$. The collection $\mathcal{E}$ of all semi-open intervals in $\mathbb{R}^n$ forms a semi-algebra. Furthermore, $\sigma(\mathcal{E})$ is the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.

The *elementary measure* μ_0 of a semi-open interval $I = I_1 \times I_2 \times \ldots \times I_n$ is defined as:

$$\mu_0(I) = \prod_{j=1}^{n} (b_j - a_j),$$

where $I_j = [a_j, b_j)$ for $j = 1, 2, \ldots, n$. If any I_j is of the form $(-\infty, b_j)$ or $[a_j, \infty)$ (i.e., unbounded), we set $\mu_0(I) = \infty$. Also, $\mu_0(I_j = \emptyset) = 0$. This elementary measure μ_0 is countably additive on the semi-algebra $\mathcal{E}$. It is σ-finite since:

$$X = \bigcup_{j=1}^{\infty} X_n \quad \text{such that} \quad \mu_0(X_n) < \infty.$$

3.13 Theorem: Lebesgue Measure

There exists a unique measure space $(\mathbb{R}^n, \mathcal{M}, m)$ such that

$$\mathcal{M} = \overline{\mathcal{B}(\mathbb{R}^n)} \quad \text{and} \quad m|_{\mathcal{E}} = \mu_0.$$

In particular,

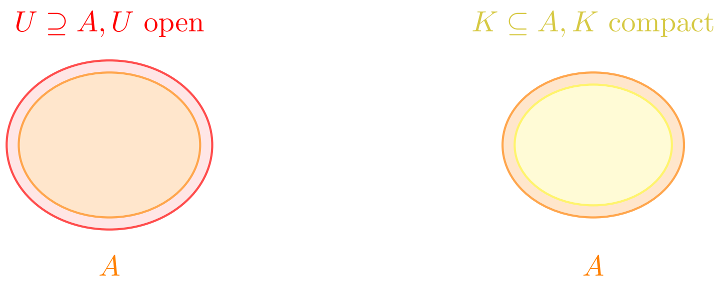
1. $\forall M \in \mathcal{M}, m = B \cup N$ where $B \in \mathcal{B}(\mathbb{R}^n)$ and N is a null set, i.e., $m(N) = 0$.
2. $\forall N \in \mathcal{M}$ with $m(N) = 0$, there exists $B \in \mathcal{B}(\mathbb{R}^n)$ such that $N \subseteq B$ and $m(B) = 0$.

This unique measure m is called the *Lebesgue measure* on $\mathbb{R}^n$. The Lebesgue measure fulfills the following properties:

1. Define $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$ as the collection of null sets. Then, if $\{N_k\}_{k=1}^{\infty} \subseteq \mathcal{N}$ is a sequence of null sets, we have:
$$\bigcup_{k=1}^{\infty} N_k \in \mathcal{N}.$$
2. If $a \in \mathbb{R}^n$ is a point, then $\{a\} \in \mathcal{N}$ and $m(\{a\}) = 0$.
3. If $A \subseteq \mathbb{R}^n$ is countable, then $A \in \mathcal{N}$ and $m(A) = 0$.
4. There exist non-countable sets in $\mathcal{N}$. For example, the Cantor ternary set $\mathcal{C} \subseteq [0, 1]$ is uncountable and $m(\mathcal{C}) = 0$.
5. If $H \subseteq \mathbb{R}^n$ is a shifted $(n-1)$-dimensional hyperplane, then $H \in \mathcal{N}$ and $m(H) = 0$.
6. The Borelian σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is strictly contained in $\mathcal{M}$, i.e., $\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{M} \subsetneq \mathcal{P}(\mathbb{R}^n)$.
7. If $A \subset \mathbb{R}^n$ is an open set, then $A \in \mathcal{M}$ and $m(A) > 0$.
8. If $K \subset \mathbb{R}^n$ is a compact set (i.e., closed and bounded), then $K \in \mathcal{M}$ and $m(K) < \infty$.

9. The Lebesgue measure m is *regular*, i.e., for every $A \in \mathcal{M}$:

$$m(A) = \inf\{m(U) : U \supseteq A, U \text{ open}\} = \sup\{m(K) : K \subseteq A, K \text{ compact}\}.$$



3.13.1 Theorem: Heine-Borel

On $\mathbb{R}^n$, a set K is compact if and only if it is closed and bounded. In general, on a topological space, a set is compact if for any $\{B_j\}_{j\in A}$ such that:

$$K \subset \bigcup_{j\in A} B_j,$$

there exists a finite subcover $\{B_{j_1}, B_{j_2}, \dots, B_{j_n}\}$ such that:

$$K \subset \bigcup_{i=1}^{n} B_{j_i}.$$

A measure μ on a topological space with its Borel σ-algebra is called a *Radon measure* if it is finite on compact sets and outer regular on Borel sets.

Observation

The Lebesgue measure m on $\mathbb{R}^n$ is a Radon measure. What are the other Radon measures on $\mathbb{R}^n$?

3.13.2 Theorem: Characterization of Radon Measures on $\mathbb{R}^n$

The Lebesgue measure m is the unique translation-invariant Radon measure on $\mathbb{R}^n$ (up to a multiplicative constant).

1. $(\mathbb{R}^n, \mathcal{M}, m)$ is translation-invariant, i.e., for any $A \in \mathcal{M}$ and any $x \in \mathbb{R}^n$:

$$m(A + x) = m(A),$$

   where $A + x = \{a + x : a \in A\}$.

2. If $\mu : \mathcal{M} \to [0, \infty]$ is another translation-invariant Radon measure on $\mathbb{R}^n$, then there exists a constant $k > 0$ such that:

$$\mu(A) = k \cdot m(A) \quad \text{for all } A \in \mathcal{M}.$$

3.14 Lebesgue-Stieltjes Measure

Observe that a measure μ on $\mathcal{B}(\mathbb{R})$ satisfies:

$$\mu((-\infty, t]) \quad t \in \mathbb{R} \quad \text{is an increasing function and}$$

$$\mu((a, b]) = \mu((-\infty, b] \setminus (-\infty, a]) = \mu((-\infty, b]) - \mu((-\infty, a]) \quad \text{for } a < b.$$

We define $g(t) = \mu((-\infty, t])$ for $t \in \mathbb{R}$. Then g is an increasing function and we have the following theorem:

3.14.1 Theorem: Lebesgue-Stieltjes Measure

Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then, there exists a unique Radon measure μ_g on $\mathcal{B}(\mathbb{R})$ such that

$$\mu_g((a, b]) = g(b^+) - g(a^+) \quad \text{for all } a < b.$$

where:

$$g(t^+) = \lim_{x \to t^+} g(x) \quad \text{and} \quad g(t^-) = \lim_{x \to t^-} g(x).$$

The measure μ_g is called the *Lebesgue-Stieltjes measure* with *distribution function* g.

Observation

If we consider g a right-continuous increasing function, then:

$$g(t^+) = \lim_{x \to t^+} g(x) = g(t).$$

Thus, in this case:

$$\mu_g((a, b]) = g(b) - g(a) \quad \text{for all } a < b.$$

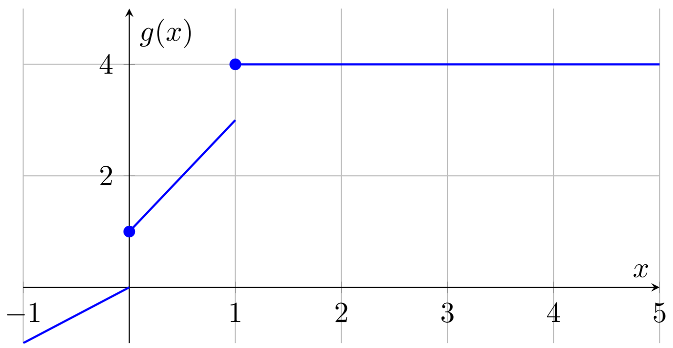
Example

Consider the function

$$g(x) = \begin{cases} x & \text{if } x < 0, \\ 2x + 1 & \text{if } 0 \le x < 1, \\ 4 & \text{if } 1 \le x \end{cases}$$

Then, g is an increasing right-continuous function. Then, it defines a Lebesgue-Stieltjes measure μ_g. For example:

$$\mu_g((0, 2]) = g(2^+) - g(0^+) = 4 - 1 = 3.$$



Observation

If μ is a Radon measure on $\mathcal{B}(\mathbb{R})$, then there exists a unique increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $\mu = \mu_g$.

3.14.2 Properties of Lebesgue-Stieltjes Measure

Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function and μ_g be the associated Lebesgue-Stieltjes measure. Then:

a) $\mu_g(\{x\}) = g(x^+) - g(x^-)$ for all $x \in \mathbb{R}$.

b) g is continuous at x if and only if $\mu_g(\{x\}) = 0$.

c) $\mu_g([a,b]) = g(b^+) - g(a^-)$ for all $a < b$.

d) $\mu_g((a,b)) = g(b^-) - g(a^+)$ for all $a < b$.

e) $\mu_g([a,b)) = g(b^-) - g(a^-)$ for all $a < b$.

f) If $I \subset \mathbb{R}$ is an interval, then $\mu_g(I) = 0$ if and only if g is constant on I.

Example

Consider the function in the previous example:

$$g(x) = \begin{cases} x & \text{if } x < 0, \\ 2x+1 & \text{if } 0 \leq x < 1, \\ 4 & \text{if } 1 \leq x \end{cases}$$

Then, the associated Lebesgue-Stieltjes measure μ_g satisfies:

- $\mu_g(\{0\}) = g(0^+) - g(0^-) = 1 - 0 = 1$.
- $\mu_g(\{1\}) = g(1^+) - g(1^-) = 4 - 3 = 1$.
- $\mu_g((0,1)) = g(1^-) - g(0^+) = 3 - 1 = 2$.
- $\mu_g([0,1]) = g(1^+) - g(0^-) = 4 - 0 = 4$.

4 Integration

A *simple function* on a measure space $(X, \mathcal{A}, \mu)$ is a *measurable* function whose change consists of a finite number of values. Then, it is of the form:

$$s(x) = \sum_{j=1}^{n} c_j \chi_{A_j}(x),$$

where $c_j \in [0, \infty)$, $A_j \in \mathcal{A}$ for $j = 1, 2, \ldots, n$, and χ_{A_j} is the characteristic function of A_j.

Example

$$s(x) = \chi_{[a,b]}(x) \text{ is simple, and so is } s(x) = \chi_{\mathbb{Q}}(x).$$

4.1 Theorem: Approximation of measurable functions

For any measurable function $f : X \to [0, \infty]$, there exists a sequence of simple functions $\{s_n\}_{n=1}^{\infty}$ such that:

1. $0 \leq s_n(x) \leq s_{n+1}(x) \leq f(x)$ for all $x \in X$.
2. $\lim_{n\to\infty} s_n(x) = f(x)$ for all $x \in X$.

Proof. For each $n \in \mathbb{N}$, define the simple function $s_n : X \to [0, \infty)$ by:

$$s_n(x) = \sum_{j=0}^{n2^n-1} \frac{j}{2^n} \chi_{A_{j,n}}(x) + n\chi_{A_{n,n}}(x),$$

where:

$$A_{j,n} = \left\{ x \in X : \frac{j}{2^n} \leq f(x) < \frac{j+1}{2^n} \right\} \text{ for } j = 0, 1, \ldots, n2^n - 1,$$

and

$$A_{n,n} = \{x \in X : f(x) \geq n\}.$$

Then, it is easy to verify that the sequence $\{s_n\}_{n=1}^{\infty}$ satisfies the required properties. $\square$

4.2 Integral of a simple function

Let $s : X \to [0, \infty)$ be a simple function of the form:

$$s(x) = \sum_{j=1}^{n} c_j \chi_{A_j}(x),$$

where $c_j \in [0, \infty)$, $A_j \in \mathcal{A}$ for $j = 1, 2, \ldots, n$. Then, the integral of s with respect to the measure μ is defined as:

$$\int_X s \, d\mu = \sum_{j=1}^{n} c_j \mu(A_j).$$

The itegral of a positive measurable function $f : X \to [0, \infty]$ is defined as:

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : 0 \leq s \leq f, s \text{ is simple} \right\}.$$

This integral is called the *Lebesgue integral* of f with respect to the measure μ.

Example

Consider the Dirichlet function of $\mathbb{Q}$, defined as:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, we can compute the Lebesgue integral of f with respect to the Lebesgue measure m:

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}} \, dm = m(\mathbb{Q}) \cdot 1 + m(\mathbb{I}) \cdot 0 = 0.$$

4.3 Properties of the Lebesgue Integral

Let $f, g : X \to [0, \infty]$ be measurable functions and $A, B, E \in \mathcal{A}$. Then:

1. $$\int_E f \, d\mu = \int_X f \chi_E \, d\mu.$$

   Proof. With f measurable, $\exists \{s_n\}_{n=1}^{\infty}$ simple functions such that $s_n \nearrow f$. Consider $f(x)\chi_E(x)$, then $\{r_n(x) = s_n(x)\chi_E(x)\}_{n=1}^{\infty}$ is a sequence of simple functions such that $r_n \nearrow f\chi_E$. Thus:

   $$\int_X f\chi_E \, d\mu = \lim_{n\to\infty} \int_X r_n \, d\mu = \lim_{n\to\infty} \int_X s_n \chi_E \, d\mu = \lim_{n\to\infty} \int_E s_n \, d\mu = \int_E f \, d\mu.$$

   □

2. $$\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu.$$

   Proof. Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences of simple functions such that $s_n \nearrow f$ and $t_n \nearrow g$. Then, $s_n + t_n \nearrow f + g$. Thus:

   $$\int_E (f + g) \, d\mu = \lim_{n\to\infty} \int_E (s_n + t_n) \, d\mu = \lim_{n\to\infty} \left(\int_E s_n \, d\mu + \int_E t_n \, d\mu \right) = \int_E f \, d\mu + \int_E g \, d\mu.$$

   □

3. $$\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu \quad \text{for any } \lambda \in \mathbb{R}.$$

4. If $f(x) \leq g(x)$ for all $x \in E$, then:
$$\int_E f \, d\mu \leq \int_E g \, d\mu.$$

5. If $A \subseteq B$, then:
$$\int_A f \, d\mu \leq \int_B f \, d\mu.$$

6. If $f = 0$ almost everywhere on E, that is, $\mu(\{x \in E : f(x) \neq 0\}) = 0$, then:
$$\int_E f \, d\mu = 0.$$

7. If $\mu(E) = 0$, then:
$$\int_E f \, d\mu = 0.$$

8. If $A \cap B = \emptyset$, then:
$$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu.$$

9. If $f = g$ almost everywhere on X, then:
$$\int_X f \, d\mu = \int_X g \, d\mu.$$

4.4 General Functions

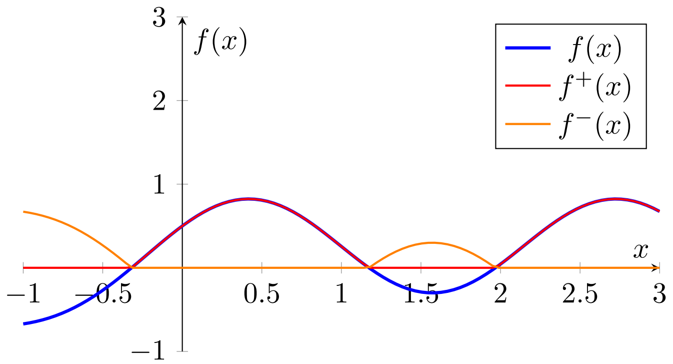
Given a function $f : X \to [-\infty, \infty]$, we can write it as:

$$f = f^+ - f^-,$$

where:

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\}.$$

Both f^+ and f^- are non-negative measurable functions.



Then, we define the integral of f as:

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

provided that at least one of the integrals on the right-hand side is finite.

4.5 Lebesgue Space

On a measure space $(X, \mathcal{A}, \mu)$, the *Lebesgue space* $L^1(X, \mu)$ is defined as:

$$L^1(X,\mu) = \left\{ f : X \to \mathbb{R} \text{ measurable} : \int_X |f| \, d\mu < \infty \right\}.$$

The elements of $L^1(X, \mu)$ are equivalence classes, where two functions f and g are considered equivalent if they are equal almost everywhere, i.e., $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$. These functions are called *Lebesgue-integrable functions*.

Example

With the Lebesgue measure m on $\mathbb{R}$, the function $f(x) = \dfrac{1}{x^2}$ is in $L^1([1, \infty], m)$, but not in $L^1([0, 1], m)$.

Example

On $\mathbb{N}$, consider the counting measure μ defined by $\mu(A) = |A|$ for any $A \subseteq \mathbb{N}$. Then, $L^1(\mathbb{N}, \mu)$ is the space formed by functions $f : \mathbb{N} \to \mathbb{R}$ such that:

$$\sum_{n=1}^{\infty} |f(n)| < \infty.$$

So f is such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent (i.e., convergent regardless of the order of its terms).

Example

Now, consider again on $\mathbb{N}$, the counting measure μ. Let $f(x) = \dfrac{(-1)^n}{2^n}$. Then, $f \in L^1(\mathbb{N}, \mu)$ since:

$$\int_{\mathbb{N}} |f| \, d\mu = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

4.6 Corollary

$L^1(\mu)$ is a vector space, that is if $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{R}$, then:

$$\alpha f + \beta g \in L^1(\mu), \quad \text{and} \quad \int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

Remark

If $f \in L^1(\mu)$, then:

$$|f| \in L^1(\mu) \text{ and } \left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$$

.

4.7 Theorem: Monotone Convergence

Consider a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$ such that:

1. $0 \leq f_n(x) \leq f_{n+1}(x) \leq \infty$ for all $x \in X$ and $n \in \mathbb{N}$.
2. $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$.

Then:

$$\int_X f(x)\, d\mu = \int_X \lim_{n\to\infty} f_n(x)\, d\mu = \lim_{n\to\infty} \int_X f_n(x)\, d\mu.$$

This integral may be infinite.

Example

On $X = [0, \infty]$, we define

$$f_n(x) = \begin{cases} 1 - x^n & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x \end{cases}$$

for $n \in \mathbb{N}$. Then,

$$0 \le f_n(x) \le f_{n+1}(x) \le 1 \text{ for all } x \in [0, \infty]$$

and

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x \end{cases} = f(x).$$

Thus, by the Monotone Convergence Theorem:

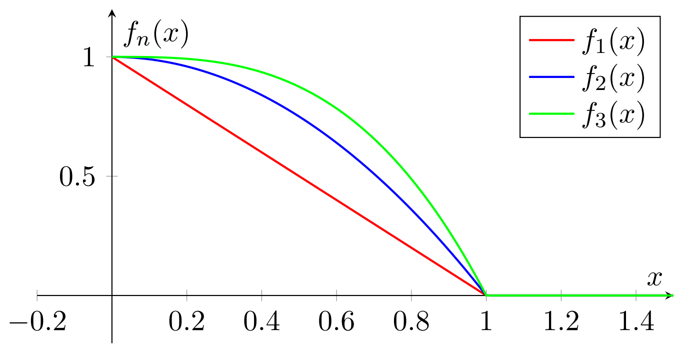
$$\int_0^\infty f(x)\, dm = \int_0^\infty \lim_{n\to\infty} f_n(x)\, dm = \lim_{n\to\infty} \int_0^\infty f_n(x)\, dm.$$

Now, we compute:

$$\int_0^\infty f_n(x)\, dm = \int_0^1 (1 - x^n)\, dm = \left[x - \frac{x^{n+1}}{n+1}\right]_0^1 = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Thus:

$$\int_0^\infty f(x)\, dm = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$



Example

Now consider the sequence of functions

$$f_n(x) = \begin{cases} 2^n & \text{if } 0 \le x < \frac{1}{2^n}, \\ 0 & \text{if } \frac{1}{2^n} \le x \end{cases}$$

for $n \in \mathbb{N}$. Then,

$$0 \le f_n(x) \le f_{n+1}(x) \le \infty \text{ for all } x \in [0, \infty]$$

and

$$\lim_{n\to\infty} f_n(x) = 0 \text{ for all } x \in [0, \infty].$$

Thus, by the Monotone Convergence Theorem:

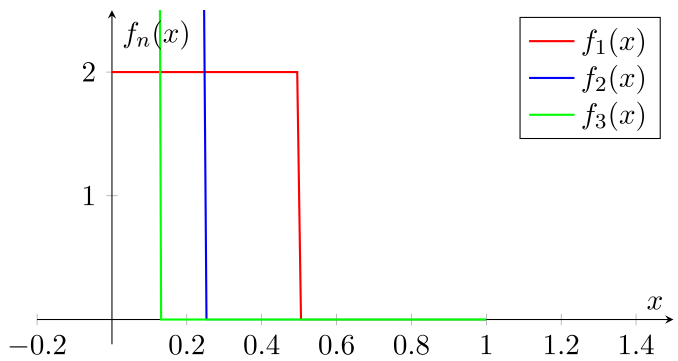
$$\int_0^\infty f(x)\,dm = \int_0^\infty \lim_{n\to\infty} f_n(x)\,dm = \lim_{n\to\infty} \int_0^\infty f_n(x)\,dm.$$

Now, we compute:

$$\int_0^\infty f_n(x)\,dm = \int_0^{\frac{1}{2^n}} 2^n\,dm = 2^n \cdot \frac{1}{2^n} = 1.$$

Thus:

$$\int_0^\infty f(x)\,dm = \lim_{n\to\infty} 1 = 1.$$



4.7.1 Proof of the Monotone Convergence Theorem

We know that:

$$\left\{ \int_X f_n(x)\,d\mu \right\}_{n=1}^\infty$$

is an increasing sequence, since $f_n(x) \le f_{n+1}(x)$ for all $x \in X$. Thus, the limit:

$$\lim_{n\to\infty} \int_X f_n(x)\,d\mu$$

exists (possibly infinite). Also, since $f_n(x) \le f_{n+1}(x)$ for all $x \in X$, we have:

$$\int_X f_n(x)\,d\mu \le \int_X f_{n+1}(x)\,d\mu \quad \text{for all } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \int_X f_n(x)\,d\mu \le \int_X f(x)\,d\mu.$$

To prove the reverse inequality, consider an increasing sequence of simple functions approximating each f_n:

$$\begin{gathered} r_{1,1}, r_{1,2}, \ldots, r_{1,k_1} \nearrow f_1, \\ r_{2,1}, r_{2,2}, \ldots, r_{2,k_2} \nearrow f_2, \\ \vdots \\ r_{n,1}, r_{n,2}, \ldots, r_{n,k_n} \nearrow f_n. \end{gathered}$$

Define a new sequence of simple functions $\{s_m\}_{m=1}^\infty$ as follows:

$$s_m = \max\{r_{n,k} : n, k \le m\}.$$

Then, $s_m \nearrow f$ as $m \to \infty$. Thus:

$$\int_X f(x)\,d\mu = \lim_{m\to\infty} \int_X s_m(x)\,d\mu \le \lim_{n\to\infty} \int_X f_n(x)\,d\mu. \quad \square$$

4.8 Corollary

For a sequence of positive measurable functions $\{f_n\}_{n=1}^{\infty}$:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in X,$$

we have:

$$\int_X f(x)\, d\mu = \sum_{n=1}^{\infty} \int_X f_n(x)\, d\mu.$$

Proof. Define the sequence of functions:

$$s_k(x) = \sum_{n=1}^{k} f_n(x) \quad \text{for } k \in \mathbb{N}.$$

Then, $s_k(x) \nearrow f(x)$ as $k \to \infty$. Thus, by the Monotone Convergence Theorem:

$$\int_X f(x)\, d\mu = \lim_{k\to\infty} \int_X s_k(x)\, d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n(x)\, d\mu = \sum_{n=1}^{\infty} \int_X f_n(x)\, d\mu. \quad \square$$

$\square$

4.9 Fatou's Lemma

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on X. Then:

$$\int_X \liminf_{n\to\infty} f_n(x)\, d\mu \le \liminf_{n\to\infty} \int_X f_n(x)\, d\mu.$$

Proof. Define the sequence of functions:

$$g_n(x) = \inf_{k\ge n} f_k(x) \quad \text{for } n \in \mathbb{N}.$$

Then, $g_n(x) \nearrow \liminf_{n\to\infty} f_n(x)$ as $n \to \infty$. Thus, by the Monotone Convergence Theorem:

$$\int_X \liminf_{n\to\infty} f_n(x)\, d\mu = \lim_{n\to\infty} \int_X g_n(x)\, d\mu.$$

Also, since $g_n(x) \le f_n(x)$ for all $x \in X$ and $n \in \mathbb{N}$, we have:

$$\int_X g_n(x)\, d\mu \le \int_X f_n(x)\, d\mu \quad \text{for all } n \in \mathbb{N}.$$

Thus:

$$\lim_{n\to\infty} \int_X g_n(x)\, d\mu \le \liminf_{n\to\infty} \int_X f_n(x)\, d\mu. \quad \square$$

$\square$

Example

Consider the sequence of functions:

$$f_n(x) = \begin{cases} \chi_{[0,1]}(x) & \text{if } n \text{ is odd,} \\ 1 - \chi_{[0,1]}(x) & \text{if } n \text{ is even.} \end{cases}$$

Then, for all $x \in \mathbb{R}$:

$$\liminf_{n\to\infty} f_n(x) = 0.$$

Thus:

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)\, dm = \int_{\mathbb{R}} 0\, dm = 0.$$

On the other hand:

$$\int_{\mathbb{R}} f_n(x)\, dm = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \infty & \text{if } n \text{ is even.} \end{cases}$$

Thus:

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dm = 0.$$

Therefore, Fatou's Lemma holds:

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n(x)\, dm = 0 \leq 0 = \liminf_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dm.$$

4.10 Theorem: Dominated Convergence

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on X such that:

1. $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.
2. There exists a function $g \in L^1(X, \mu)$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$ and $n \in \mathbb{N}$.

Then:

$$\int_X f(x)\, d\mu = \lim_{n\to\infty} \int_X f_n(x)\, d\mu, \quad f \in L^1(X, \mu) \quad \text{and} \quad \lim_{n\to\infty} \int_X |f_n(x) - f(x)|\, d\mu = 0.$$

Proof. Each $f_n \in L^1(\mu)$, and so is f, since:

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq g(x) \quad \text{for all } x \in X.$$

Also, since:

$$|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq g(x) + g(x) = 2g(x) \quad \text{for all } x \in X,$$

we have:

$$2g(x) - |f_n(x) - f(x)| \geq 0 \quad \text{for all } x \in X.$$

Thus, by Fatou's Lemma:

$$\int_X \liminf_{n\to\infty} \left(2g(x) - |f_n(x) - f(x)|\right)\, d\mu \leq \liminf_{n\to\infty} \int_X \left(2g(x) - |f_n(x) - f(x)|\right)\, d\mu.$$

Now, since:

$$\liminf_{n\to\infty} \left(2g(x) - |f_n(x) - f(x)|\right) = \int_X 2g(x) - \limsup_{n\to\infty} |f_n(x) - f(x)| = 2g(x) - 0 = 2g(x),$$

we have:

$$\int_X 2g(x)\,d\mu \leq \liminf_{n\to\infty}\left(2\int_X g(x)\,d\mu - \int_X |f_n(x)-f(x)|\,d\mu\right).$$

Thus:

$$\limsup_{n\to\infty}\int_X |f_n(x)-f(x)|\,d\mu \leq 0.$$

Since the integral is non-negative, we conclude that:

$$\lim_{n\to\infty}\int_X |f_n(x)-f(x)|\,d\mu = 0.$$

Finally, by the triangle inequality:

$$\left|\int_X f_n(x)\,d\mu - \int_X f(x)\,d\mu\right| \leq \int_X |f_n(x)-f(x)|\,d\mu,$$

we have:

$$\lim_{n\to\infty}\left|\int_X f_n(x)\,d\mu - \int_X f(x)\,d\mu\right| = 0.$$

□

Example

Let us consider:

$$\lim_{n\to\infty}\int_0^1 \frac{\log(n+x)}{n}\cdot\sin(x)\,dx.$$

We have:

$$|f_n(x)| = \left|\frac{\log(n+x)}{n}\cdot\sin(x)\right| \leq \left|\frac{\log(n+x)}{n}\right| \leq 1 \in L^1([0,1]).$$

So with $g(x)=1$, we can apply the Dominated Convergence Theorem:

$$\lim_{n\to\infty}\int_0^1 \frac{\log(n+x)}{n}\cdot\sin(x)\,dx = \int_0^1 \lim_{n\to\infty}\frac{\log(n+x)}{n}\cdot\sin(x)\,dx = \int_0^1 0\cdot\sin(x)\,dx = 0.$$

4.11 Corollary: Uniform Convergence

Let $X, \mathcal{A}, \mu$ be a finite measure space, that is, $\mu(X)<\infty$. If a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on X, then $f\in L^1(X,\mu)$ and:

$$\lim_{n\to\infty}\int_X f_n(x)\,d\mu = \int_X f(x)\,d\mu.$$

Proof. The limit of $f(x)$ satisfies:

$$\inf_n |f_n(x)| \leq |f(x)| \leq \sup_n |f_n(x)| \quad \text{for all } x\in X.$$

Since $f_n\in L^1(\mu)$, f is also integrable. Besides, by the uniform convergence:

$$\exists N\in\mathbb{N} : |f_n(x)-f(x)|<\epsilon \text{ for all } n\geq N \text{ and } x\in X.$$

Also:

$$|f_n(x)| \leq |f(x)|+\epsilon \implies \int_X |f_n(x)|\,d\mu \leq \int_X |f(x)|\,d\mu + \epsilon\mu(X) < \infty.$$

Thus, by the Dominated Convergence Theorem:

$$\lim_{n\to\infty}\int_X f_n(x)\,d\mu = \int_X f(x)\,d\mu.$$

□

4.12 Corollary

Let $X, \mathcal{A}, \mu$ be a measure space and $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that:

$$\sum_{n=1}^{\infty} \int_X |f_n(x)| \, d\mu < \infty.$$

Then $f \in L^1(X, \mu)$,

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges to } f : X \to \mathbb{R},$$

and

$$\int_X f(x) \, d\mu = \sum_{n=1}^{\infty} \int_X f_n(x) \, d\mu.$$

Proof. Define:

$$g_N(x) = \sum_{n=1}^{N} f_n(x)$$

Then:

$$\int_X |g_N(x)| \, d\mu \leq \sum_{n=1}^{N} \int_X |f_n(x)| \, d\mu \leq \sum_{n=1}^{\infty} \int_X |f_n(x)| \, d\mu < \infty.$$

Thus, $g_N \in L^1(X, \mu)$ for all $N \in \mathbb{N}$. Now,

$$\sum_{n=1}^{\infty} \int_X f_n(x) \, d\mu = \int_X \sum_{n=1}^{\infty} f_n(x) \, d\mu,$$

and finally:

$$\lim_{N \to \infty} \sum_{n=1}^{N} \int_X f_n(x) \, d\mu = \int_X \lim_{N \to \infty} g_N(x) \, dx.$$

□