Exercises in Principles of Mathematical Analysis

Exercise 1.1.1

Prove that $\mathbb{Q}$ has zero measure.
A set $Q \subset \mathbb{R}$ has measure zero if and only if:

$$Q \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{where } A_j \text{ are intervals and } \sum_{j=1}^{\infty} |A_j| < \varepsilon, \forall \epsilon > 0.$$

Since $\mathbb{Q}$ is countable, we can enumerate its elements as $\{q_1, q_2, q_3, \ldots\}$.
We start with:

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1/2}{1 - 1/2} = 1.$$

Then, we can multiply this series by any $\epsilon > 0$ to get:

$$\sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \varepsilon.$$

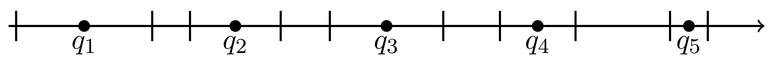
For each rational number q_j, we can construct an interval A_j centered at q_j with length $\dfrac{\varepsilon}{2^j}$:

$$A_j = \left(q_j - \frac{\varepsilon}{2^{j+1}}, q_j + \frac{\varepsilon}{2^{j+1}}\right).$$

Thus, we have:

$$\mathbb{Q} \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{and } \sum_{j=1}^{\infty} |A_j| = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Since ε can be made arbitrarily small, we conclude that $\mathbb{Q}$ has measure zero.

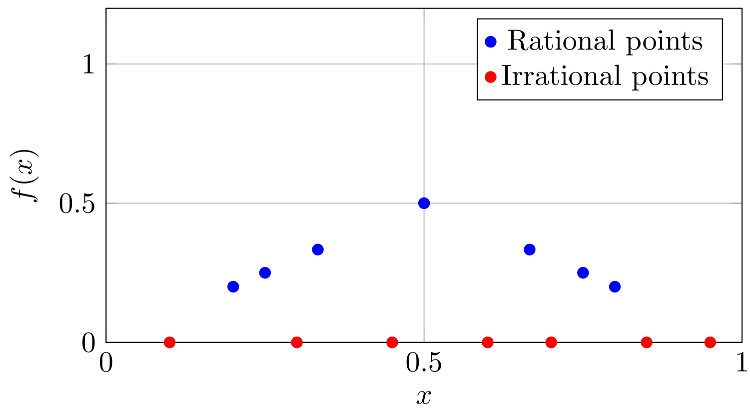


Exercise 1.1.2

For the following function defined on $[0,1]$:

$$f(x) = \begin{cases} \frac{1}{q} & \textbf{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z}, q \neq 0 \\ 0 & \textbf{if } x \notin \mathbb{Q} \end{cases}$$

1. **Show that f is discontinuous only at the rational points.**
2. **Prove that f is Riemann integrable.**



If x is irrational, then

$$\lim_{x \to x_0} f(x) \stackrel{?}{=} 0 = f(x_0),$$

$\forall \varepsilon > 0$, we want $|f(x) - 0| = |f(x)| < \varepsilon$ if x is close enough to x_0 i.e. $x \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.
For example, take $\varepsilon = \frac{1}{3}$, then $|f(x)| < \frac{1}{3}$ if $f(x) = \frac{1}{q} < \frac{1}{3} \Rightarrow q > 3$. So, except for the rationals $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, we have $|f(x)| < \varepsilon = \frac{1}{3}$. We can do this for any $\varepsilon > 0$ by choosing $q > \frac{1}{\varepsilon}$, so

$$\lim_{x \to x_0} f(x) = 0.$$

So it is continuous at every irrational point. On $\mathbb{Q}$, the value of $f(x)$ is non-zero, so it is discontinuous at every rational point.
Finally, since the set of discontinuities has measure zero ($|\mathbb{Q}| = 0$), f is Riemann integrable.

Exercise 1.1.4

Consider the sequence of functions given by:

$$f_n(x) = \begin{cases} 1 & \textbf{if } |x| \leq n \\ 0 & \textbf{if } |x| > n \end{cases}$$

Obtain the limit $\lim_{n\to\infty} f_n(x)$ and study whether the convergence is uniform or not.

$$\sup_{n\in\mathbb{N}} |f_n(x) - f(x)| = 1 \neq 0, \quad \forall x \in \mathbb{R}.$$

So the convergence is not uniform.

Exercise 1.1.5

Prove that the following series converges in $[0,1]$. Is the convergence uniform?

$$\sum_{n=0}^{\infty} x(1-x)^n = x\sum_{n=0}^{\infty}(1-x)^n = \begin{cases} \dfrac{x}{1-(1-x)} = \dfrac{x}{x} = 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

If $|1-x| < 1$, then the series converges. This is true for all $x \in [0,1]$. The convergence is not uniform since f is not continuous:

$$f_N(x) = \sum_{n=0}^{N} x(1-x)^n \overset{N\to\infty}{\longrightarrow} f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Exercise 1.1.3

Prove that if a function $f : [a,b] \to \mathbb{R}$ is monotonus, then it:

1. **is bounded**
2. **is Riemann integrable.**

Suppose f is monotonically increasing. Then,

$$f(x) \in [f(a), f(b)], \quad \forall x \in [a,b].$$

So f is bounded.

Now, we build $U_f(P)$ and $L_f(P)$ for a partition $P = \{a, a + \dfrac{b-a}{n}, a + 2\dfrac{b-a}{n}, \ldots, b\}$.

$$U_f(P) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n},$$

$$L_f(P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n}.$$

Then,

$$U_f(P) - L_f(P) = \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = \frac{b-a}{n}(f(b) - f(a)) \overset{n\to\infty}{\longrightarrow} 0.$$

So f is Riemann integrable.

Exercise 1.1.8

Build a sequence of continuous functions on $[0,1]$ that converges to a continuous function, but in a non-uniform way. Let us define the sequence of functions:

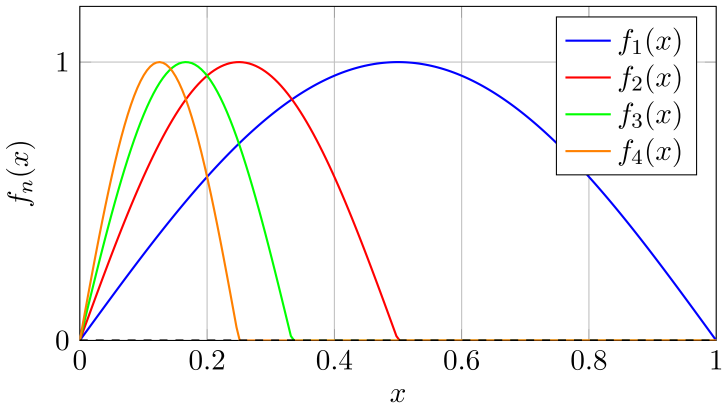
$$f_n(x) = \begin{cases} \sin(n\pi x) & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$$

Each f_n is continuous on $[0,1]$. Now, we find the limit:

$$\lim_{n\to\infty} f_n(x) = 0 = f(x), \quad \forall x \in [0,1].$$

However, the convergence is not uniform. As:

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |f_n(x)| = 1, \quad \forall n \in \mathbb{N}.$$



Exercise 1.1.9

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n^2-1}{(x^2+1)(n^2+1)} \cdot e^{\frac{-x^4}{n}} \, dx$$

As $n \to \infty$,

$$\frac{n^2-1}{(x^2+1)(n^2+1)} \cdot e^{\frac{-x^4}{n}} \to \frac{1}{x^2+1}.$$

So, the limit becomes:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx = \lim_{N\to\infty} \int_0^N \frac{1}{x^2+1} \, dx + \lim_{M\to-\infty} \int_M^0 \frac{1}{x^2+1} \, dx =$$

$$= [\arctan(x)]_0^N + [\arctan(x)]_M^0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Exercise 1.2.1

Let $f : X \to Y$ be a mapping. Given $A \subset Y$, let us define:

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Prove that:

1. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

   If $x \in f^{-1}(Y \setminus A)$, then $f(x) \in Y \setminus A$. So, $f(x) \notin A$, which means $x \notin f^{-1}(A)$. Thus, $x \in X \setminus f^{-1}(A)$.

2. $f^{-1}\left(\bigcup_j A_j\right) = \bigcup_j f^{-1}(A_j)$.

   If $x \in f^{-1}\left(\bigcup_j A_j\right)$, then $f(x) \in \bigcup_j A_j$. So, there exists some j such that $f(x) \in A_j$. Thus, $x \in f^{-1}(A_j)$ for that j, which means $x \in \bigcup_j f^{-1}(A_j)$.

Exercise 1.2.2

Let $f : X \to Y$ be a mapping between two topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$. Prove that f is continuous if and only if f is continuous at every $x \in X$.
For a topological space $(X, \mathcal{T}_X)$, a function $f : X \to Y$ is continuous if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V) \in \mathcal{T}_X$.
Now, if f is continuous at every $x \in X$, then for every open set $V \in \mathcal{T}_Y$, we have $f^{-1}(V) \in \mathcal{T}_X$.
Conversely, if f is continuous, then for every $x \in X$, and for every open set $V \in \mathcal{T}_Y$ containing $f(x)$, we have $f^{-1}(V) \in \mathcal{T}_X$. Thus, f is continuous at every $x \in X$.

Exercise 1.2.3

Show that if $X = \{1, 2, 3\}$, then $\mathcal{F} := \{\emptyset, \{2, 3\}, X\}$ is not a σ-algebra.
To show that $\mathcal{F}$ is not a σ-algebra, we need to verify the three properties of a σ-algebra:
1. **Contains the empty set**: $\emptyset \in \mathcal{F}$.
2. **Closed under complementation**: The complement of $\{2, 3\}$ in X is $\{1\}$, which **is not in** $\mathcal{F}$. Since $\mathcal{F}$ is not closed under complementation, it is not a σ-algebra.

Exercise 1.2.4

Let $\mathcal{S}$ be a family of subsets of X, $S \subseteq \mathcal{P}(X)$. Prove that

$$\mathcal{A}_{\mathcal{S}} := \bigcap \{\mathcal{A} : \mathcal{A} \textbf{ is a } \sigma\textbf{-algebra}, \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}$$

is the smallest σ-algebra containing $\mathcal{S}$.
To prove that $\mathcal{A}_{\mathcal{S}}$ is the smallest σ-algebra containing $\mathcal{S}$, we need to show the following:

1. $\mathcal{A}_{\mathcal{S}}$ is a σ-algebra.
   We know that $\mathcal{A}_{\mathcal{S}}$ contains the empty set since every σ-algebra contains the empty set. It is closed under complementation and countable unions because these properties hold for each σ-algebra in the intersection.

2. $\mathcal{S} \subseteq \mathcal{A}_{\mathcal{S}}$.
   By definition, $\mathcal{A}_{\mathcal{S}}$ is the intersection of all σ-algebras containing $\mathcal{S}$, so $\mathcal{S}$ is contained in $\mathcal{A}_{\mathcal{S}}$.

3. If $\mathcal{B}$ is any σ-algebra containing $\mathcal{S}$, then $\mathcal{A}_\mathcal{S} \subseteq \mathcal{B}$.
   Since $\mathcal{A}_\mathcal{S}$ is the intersection of all σ-algebras containing $\mathcal{S}$, it must be contained in any such σ-algebra $\mathcal{B}$.

Thus, $\mathcal{A}_\mathcal{S}$ is the smallest σ-algebra containing $\mathcal{S}$.

Exercise 1.2.5

Let $X = \{a, b, c, d\}$. Construct the σ-algebra generated by $E_1 = \{a\}$ and by $E_2 = \{a, b\}$. The σ-algebra generated by $E_1 = \{a\}$ is:

$$\mathcal{A}_{E_1} = \{\emptyset, \{a\}, \{b, c, d\}, X\}.$$

The σ-algebra generated by $E_2 = \{a, b\}$ is:

$$\mathcal{A}_{E_2} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}.$$

Exercise 1.2.12

Let $u, v : X \to \mathbb{R}$ be measurable functions and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

1. $u + v$, uv and $|u|^\alpha$ are measurable.

$$u + v = \phi(u, v)$$

$$uv = \phi(u, v)$$

$$|u|^\alpha = \phi(u)$$

   Every ϕ is continuous, and the composition of measurable functions is measurable. Therefore, $u + v$, uv, and $|u|^\alpha$ are measurable.

Exercise 1.2.13

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be a function. Prove that the following assertions are equivalent:

1. $\{x \in X : f(x) < \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.
2. $f^{-1}(B) \in \mathcal{A}$ for every Borel set B. Borel sets on $(\mathbb{R}, \mathcal{T})$ are σ-algebras generated by $\mathcal{T}$

We start by proving that (2) implies (1). The open sets $V \in \mathcal{T}$ are all in $\mathcal{B}(\mathbb{R})$, so they are all borelians. In (1), we have

$$\{x \in X : f(x) < \alpha\} \overset{?}{\in} \mathcal{A}$$

which is equivalent to

$$f^{-1}((\alpha, \infty)), \quad (\alpha, \infty) \in B(\mathbb{R}).$$

Thus,

$$f^{-1}((\alpha, \infty)) \in \mathcal{A} \text{ by (2).}$$

Now, we prove that (1) implies (2). In $\mathcal{A}$, since it is a sigma algebra, it must contain $B(\mathcal{R})$, that is the smallest σ-algebra generated by the open sets.

Exercise 1.2.15

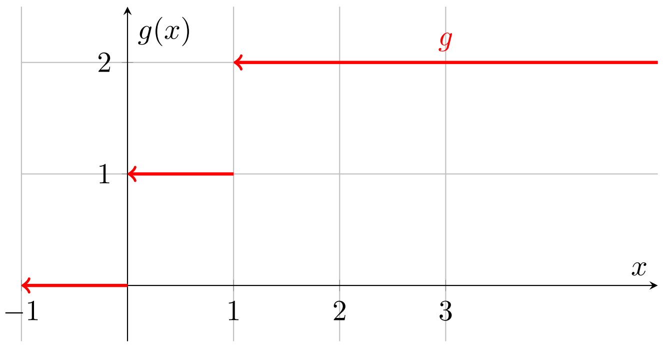
Prove that if f is a real function on a measurable space X such that $\{x \in X : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

We have
$$\forall \alpha \in \mathbb{R}, \quad \exists \{r_n\} \subset \mathbb{Q} : r_n \nearrow \alpha.$$
and $\{r_n\}$ is increasing (since $r_n \nearrow \alpha$).
Note: $r_n \nearrow \alpha$ means that r_n is an increasing sequence that converges to α.

We want to prove that
$$f^{-1}((\alpha, \infty)) = \{x \in X : f(x) > \alpha\} \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}.$$
Consider the increasing sequence $\{r_n\}$ such that
$$\lim_{n\to\infty} r_n = \alpha.$$
With this we express
$$(\alpha, \infty) = \bigcap_{n=1}^{\infty} [r_n, \infty)$$
and then
$$\{x \in X : f(x) \geq r\} = f^{-1}((r, \infty)) \in \mathcal{A}, \quad \forall r \in \mathbb{Q}.$$
Thus,
$$f^{-1}((\alpha, \infty)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [r_n, \infty)\right) = \bigcap_{n=1}^{\infty} f^{-1}([r_n, \infty)) \in \mathcal{A}.$$

Exercise 1.2.16

Let $\mathcal{M}$ be the σ-algebra in $\mathbb{R}$ given by $\mathcal{M} = \{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$. Let g be the function $g : \mathbb{R} \to \mathbb{R}$ defined by
$$g(x) = \begin{cases} 0 & \textbf{if } x \in (-\infty, 0], \\ 1 & \textbf{if } x \in (0, 1], \\ 2 & \textbf{if } x \in (1, \infty). \end{cases}$$
Is g measurable? How are the measurable functions $f : (\mathbb{R}, \mathcal{M}) \to \mathbb{R}$, with the usual topology of the open sets?



$$\forall V \in \mathcal{T}, \quad g^{-1}(V) \in \mathcal{M}.$$
With $V = (-1, 3)$, we have $g^{-1}((-1, 3)) = \mathbb{R} \in \mathcal{M}$.
With $W = (0, 1)$, we have $g^{-1}((0, 1)) = \emptyset \in \mathcal{M}$.
With $H = (-1, 2)$, we have $g^{-1}((-1, 2)) = (-\infty, 1] \notin \mathcal{M}$. So g is not measurable.

Exercise 1.2.18

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\bar{\mathbb{R}} = [-\infty, \infty]$. Prove that

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(c) If $a_n \leq b_n$ for all n, then $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.

(d) Show with an example that strict inequality can hold in part b).

Definition: If $\{a_n\}$ is a sequence in $\bar{\mathbb{R}}$, then the sequence:

$$b_k := \sup\{a_k, a_{k+1}, \ldots\}$$

is decreasing (non-increasing), that is $b_k \geq b_{k+1}$ for all k. It is bounded from below by $-\infty$, so it converges to its infimum. The limit is called the *limit superior* of the sequence $\{a_n\}$ and is denoted by:

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} (\sup\{a_k, a_{k+1}, \ldots\})$$

This limit always exists in $\bar{\mathbb{R}}$. Similarly, consider the sequence:

$$c_k := \inf\{a_k, a_{k+1}, \ldots\}$$

which is increasing (non-decreasing), that is $c_k \leq c_{k+1}$ for all k. It is bounded from above by ∞, so it converges to its supremum. The limit is called the *limit inferior* of the sequence $\{a_n\}$ and is denoted by:

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} (\inf\{a_k, a_{k+1}, \ldots\}).$$

Example

Let $a_n = (-1)^n \arctan(n)$, then

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} (\sup\{a_k, a_{k+1}, \ldots\}) = \lim_{k\to\infty} (\sup(\arctan(k))) = \frac{\pi}{2},$$

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} (\inf\{a_k, a_{k+1}, \ldots\}) = \lim_{k\to\infty} (\inf(-\arctan(k))) = -\frac{\pi}{2}.$$

So the sequence does not converge, since $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$, it does not have a limit.

$\limsup_{n\to\infty}\{a_n\}$ is the largest value for which there is a subsequence converging to it, and $\liminf_{n\to\infty}\{a_n\}$ is the smallest value for which there is a subsequence converging to it.

Exercise 1.3.3

Let (X, A) be a measurable space and let $\mu : A \to [0, \infty]$ be a countably additive function on the σ-algebra $\mathcal{A}$.

(a) Show that if μ satisfies that $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\emptyset) = 0$ (and therefore μ is a measure).

(b) Find an example for which $\mu(\emptyset) \neq 0$ (and therefore the countably additivity property does not imply that μ is a measure).

For μ to be countably additive, so it must satisfy:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j),$$

for any countable collection of disjoint sets $\{A_j\} \subseteq \mathcal{A}$.
Suppose there exists some $A \in \mathcal{A}$ such that $\mu(A) < \infty$. We can express A as the union of two disjoint sets:

$$A = A \cup \emptyset.$$

By countable additivity, we have:

$$\mu(A) = \mu(A) + \mu(\emptyset).$$

Rearranging this gives:

$$\mu(\emptyset) = \mu(A) - \mu(A) = 0.$$

Thus, if μ is countably additive and there exists some $A \in \mathcal{A}$ with $\mu(A) < \infty$, then $\mu(\emptyset) = 0$.
An example where $\mu(\emptyset) \neq 0$ is the function $\mu : \mathcal{P}(X) \to [0, \infty]$ defined by:

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

This function is not a measure because it does not satisfy $\mu(\emptyset) = 0$.

Exercise 2.2.1

Let $f_n : [0, 1] \to [0, 1]$ be continuous functions such that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in [0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

Try to find $F(x) \in L^1([0, 1])$ such that $f_n(x) \leq F(x)$, for example $F(x) = 1$. Then, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx = \int_0^1 0\, dx = 0.$$

1 Exercise 2.2.4

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be an integrable function. Prove Markov's inequality:

$$\mu(\{x \in X : |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f(x)|\, d\mu, \quad \forall \varepsilon > 0.$$

By the properties of the Lebesgue integral, we have:

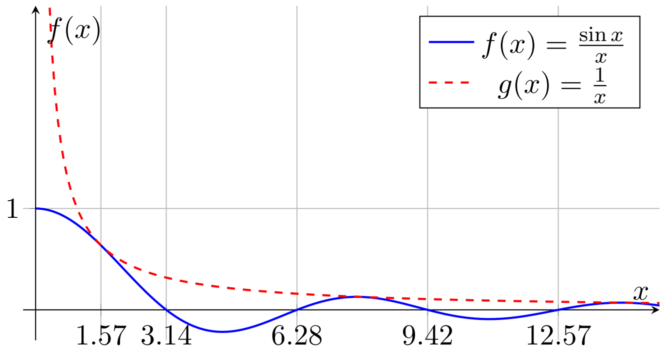
$$\mu(A = \{x \in X : |f(x)| \geq \varepsilon\}) = \int_X \chi_A(x)\, d\mu = \int_A 1\, d\mu \leq \int_A \frac{|f(x)|}{\varepsilon}\, d\mu \leq \frac{1}{\varepsilon} \int_X |f(x)|\, d\mu.$$

2 Exercise 2.2.5

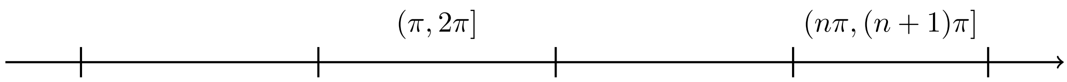
Consider the function

$$f(x) = \frac{\sin x}{x}, \qquad x \in (0, \infty).$$

Prove that $f \notin L^1((0,\infty))$
Hint: divide the interval $(0,\infty)$ into subintervals of the form $[n\pi, (n+1)\pi]$, for $n = 1, 2, \ldots$



We divide the interval $(0,\infty)$ into subintervals of the form $[n\pi, (n+1)\pi]$, for $n = 1, 2, \ldots$:



We have:

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx =$$

$$= \int_0^\pi \sin x\, dx \cdot \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{(n+1)} = [\cos x]_0^\pi \cdot \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} = \frac{2}{\pi} \cdot \infty = \infty.$$

Thus, $f \notin L^1((0,\infty))$.

Exercise 2.2.6

Consider the function

$$f(x) = \frac{1 - \cos x}{x(1 + x^2)}, \qquad x \in (0, \infty).$$

Prove that $f \in L^1((0,\infty))$**.**
We have:

$$\int_0^\infty \left|\frac{1-\cos x}{x(1+x^2)}\right| dx = \int_0^\infty \frac{1-\cos x}{x(1+x^2)}\, dx$$

Using the Taylor expansion of $\cos x$ around 0:

$$\cos x = 1 - \frac{x^2}{2} + \cdots \implies 1 - \cos x = \frac{x^2}{2} + \varepsilon \quad \text{on some } (0, \delta),$$

Then in $(0,\delta)$:

$$\left|\frac{1-\cos x}{x^2/2}\right| < 1 + \varepsilon$$

And,

$$|f(x)| < \frac{(1+\varepsilon)x^2/2}{x(1+x^2)} = \frac{(1+\varepsilon)x}{2(1+x^2)} \in L^1((0,\delta)).$$

On (δ, ∞), we have:

$$|f(x)| \leq \frac{2}{x(1+x^2)} \leq \frac{2}{x^3} \in L^1((\delta, \infty)).$$

Thus, $f \in L^1((0, \infty))$.