Principles of Mathematical Analysis

1 Measure Theory

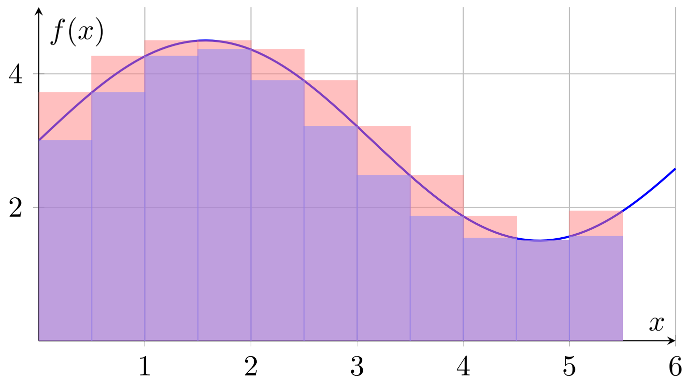
1.1 Riemann Integral

For a bounded function $f : [a, b] \to \mathbb{R}$ and any partition of the interval $[a, b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$, we consider on each subinterval $I_j = [x_{j-1}, x_j], \quad j = 1, \ldots, n$, the quantities:

$$M_j = \sup_{x \in I_j} f(x), \quad m_j = \inf_{x \in I_j} f(x).$$

We also define the upper and lower sums of f with respect to the partition P as:

$$U_f(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \quad L_f(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}).$$



For any two partitions P and Q of $[a, b]$, we have:

$$L_f(P) \leq \text{Area under } f \leq U_f(Q).$$

If P has a value I such that:

$$\sup_P L_f(P) = I = \inf_P U_f(P),$$

then we say that f is Riemann integrable on $[a, b]$ and define the Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x)\, dx = I.$$

Continuous functions on closed intervals are Riemann integrable.

1.2 The Lebesgue Integral

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Lebesgue integrable* on $[a, b]$ if the set of points where f is discontinuous has zero measure.
A set $B \subset \mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, it can be covered by a countable collection of open intervals $\{(a_n, b_n)\}$ such that:

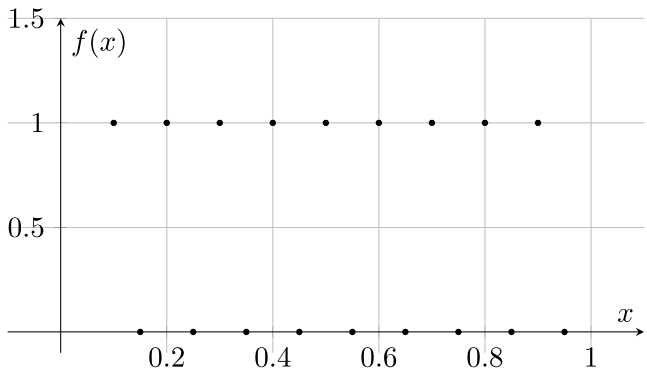
$$B \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

1.2.1 Example: Dirichlet Function

The Dirichlet function:

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

On the interval $[0, 1]$:



We see that f is not Riemann integrable since it is discontinuous everywhere.
But consider,

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$$

and define:

$$f_1(x) = \chi_{\{q_1\}}(x) \to \text{integrable on } [0, 1]$$
$$f_2(x) = \chi_{\{q_1,q_2\}}(x) \to \text{integrable on } [0, 1]$$
$$\vdots$$
$$f_n(x) = \chi_{\{q_1,q_2,\ldots,q_n\}}(x) \to \text{integrable on } [0, 1]$$

Then,

$$\lim_{n\to\infty} f_n(x) = \chi_{\mathbb{Q}}(x).$$

1.2.2 Characteristic Function

For any set $A \subset \mathbb{R}$, the characteristic function $\chi_A : \mathbb{R} \to \{0, 1\}$ is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$\int_0^1 f_1(x)\,dx = 0 = \int_0^1 f_2(x)\,dx = \ldots = \int_0^1 f_n(x)\,dx = 0.$$

Example

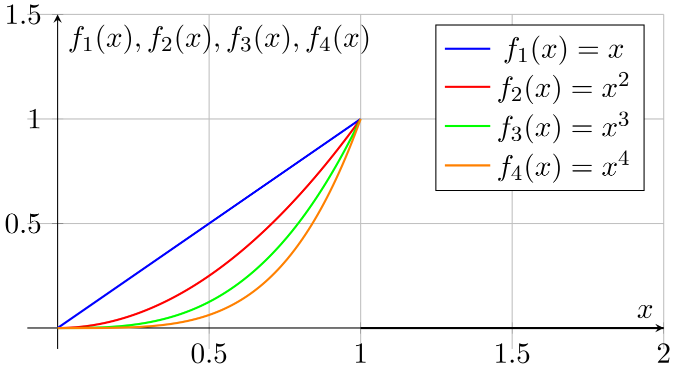
Let

$$f_n(x) = \begin{cases} x^n & \text{if } 0 \leq x < 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \leq 1. \end{cases}$$

so we can see that there is a discontinuity at $x = 1$.



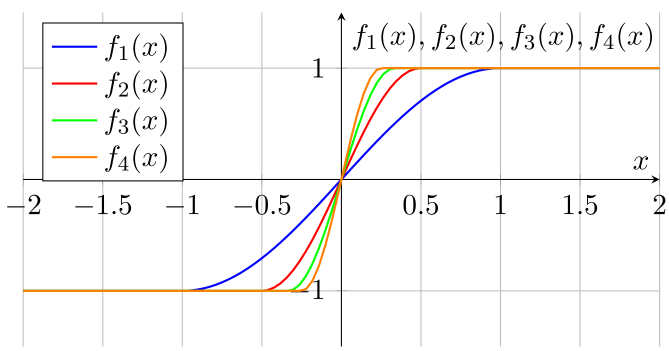
Example

Let

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}, \\ \sin\left(\frac{n\pi x}{2}\right) & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq x. \end{cases}$$

Then, with $f_n(x)$ continuous and differentiable on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Example

The Dirichlet function is not integrable but it is the limit of a sequence of integrable functions, all with integral equal to zero.
We need to define a new kind of convergence.

1.3 Convergences

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges punctually to a function f on $Dom(f)$ if:

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in Dom(f).$$

$$\forall \epsilon > 0, \quad \forall x \in Dom(f), \quad \exists N(\epsilon, x) \in \mathbb{N} : n > N \implies |f_n(x) - f(x)| < \epsilon.$$

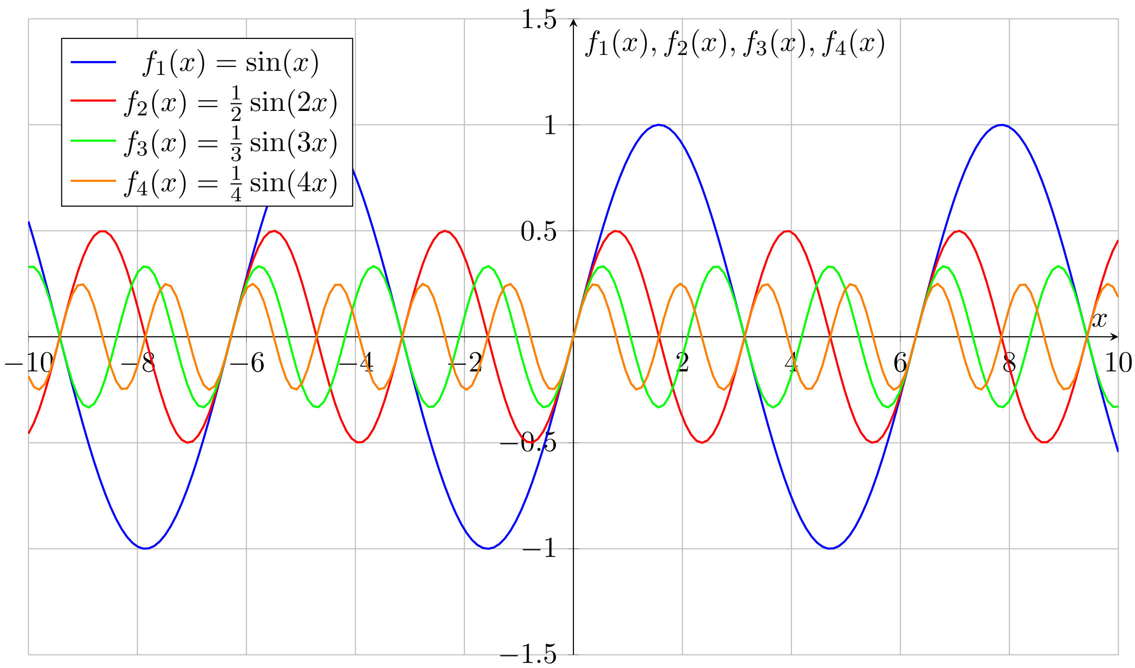
A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to a function f on $Dom(f)$ if:

$$\forall \epsilon, \quad \exists N : n > N \implies |f_n(x) - f(x)| < \epsilon, \quad \forall x \in Dom(f).$$

Example

Let

$$f_n(x) = \frac{1}{n}\sin(nx), \quad x \in \mathbb{R}. \to^{n\to\infty} f(x) = 0.$$



1.3.1 Uniform Convergence

1. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

2. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous in $[a, b]$, then f is continuous on $[a, b]$.

3. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[a, b]$ that converges punctually to some continuous function f on $[a, b]$ and if the sequence of derivatives $\{f'_n\}_{n\in\mathbb{N}}$ converges uniformly to some continuous function g, then f is differentiable on (a, b) and:

$$f'(x) = g(x) = \lim_{n\to\infty} f'_n(x).$$

1.4 Henri Lebesgue (1875-1941)

How can we count money in bills?

1. Add each amount as the bills come in. (Riemann)
2. Make groups by denomination and count each group. (Lebesgue)