Principles of Mathematical Analysis

1 Measure Theory

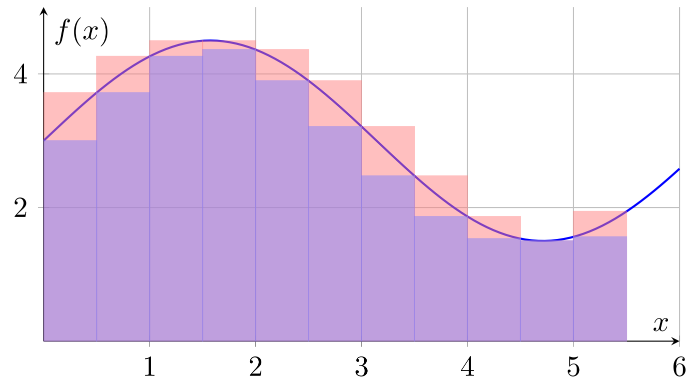
1.1 Riemann Integral

For a bounded function $f : [a,b] \to \mathbb{R}$ and any partition of the interval $[a,b]$, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$, we consider on each subinterval $I_j = [x_{j-1}, x_j], \quad j = 1, \ldots, n$, the quantities:

$$M_j = \sup_{x \in I_j} f(x), \quad m_j = \inf_{x \in I_j} f(x).$$

We also define the upper and lower sums of f with respect to the partition P as:

$$U_f(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}), \quad L_f(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$



For any two partitions P and Q of $[a,b]$, we have:

$$L_f(P) \leq \text{Area under } f \leq U_f(Q).$$

If P has a value I such that:

$$\sup_P L_f(P) = I = \inf_P U_f(P),$$

then we say that f is Riemann integrable on $[a,b]$ and define the Riemann integral of f over $[a,b]$ as:

$$\int_a^b f(x)\, dx = I.$$

Continuous functions on closed intervals are Riemann integrable.

1.2 The Lebesgue Integral

A bounded function $f : [a,b] \to \mathbb{R}$ is said to be *Lebesgue integrable* on $[a,b]$ if the set of points where f is discontinuous has zero measure.
A set $B \subset \mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, it can be covered by a countable collection of open intervals $\{(a_n, b_n)\}$ such that:

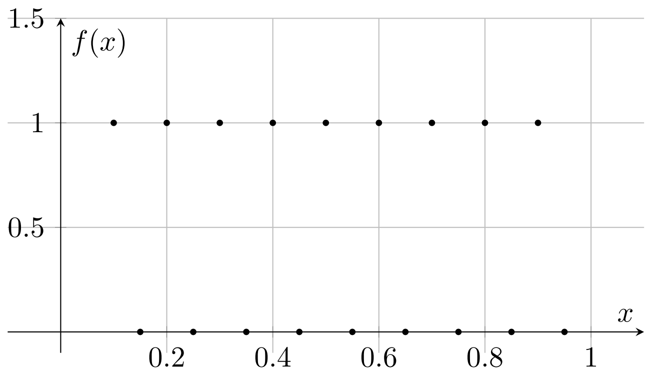
$$B \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

1.2.1 Example: Dirichlet Function

The Dirichlet function:

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

On the interval $[0,1]$:



We see that f is not Riemann integrable since it is discontinuous everywhere.
But consider,

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$$

and define:

$$f_1(x) = \chi_{\{q_1\}}(x) \to \text{integrable on } [0,1]$$
$$f_2(x) = \chi_{\{q_1,q_2\}}(x) \to \text{integrable on } [0,1]$$
$$\vdots$$
$$f_n(x) = \chi_{\{q_1,q_2,\ldots,q_n\}}(x) \to \text{integrable on } [0,1]$$

Then,

$$\lim_{n\to\infty} f_n(x) = \chi_{\mathbb{Q}}(x).$$

1.2.2 Characteristic Function

For any set $A \subset \mathbb{R}$, the characteristic function $\chi_A : \mathbb{R} \to \{0,1\}$ is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$\int_0^1 f_1(x)\,dx = 0 = \int_0^1 f_2(x)\,dx = \ldots = \int_0^1 f_n(x)\,dx = 0.$$

Example

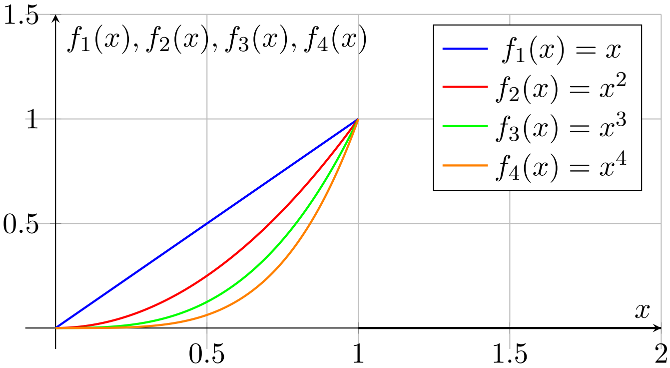
Let

$$f_n(x) = \begin{cases} x^n & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x. \end{cases}$$

Then, with $f_n(x)$ continuous on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \le 1. \end{cases}$$

so we can see that there is a discontinuity at $x = 1$.



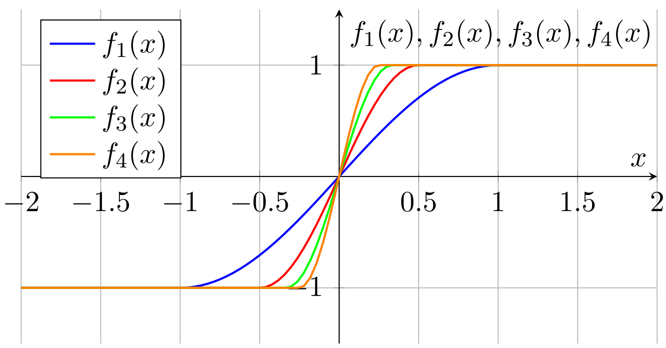
Example

Let

$$f_n(x) = \begin{cases} -1 & \text{if } x \le -\frac{1}{n}, \\ \sin\left(\frac{n\pi x}{2}\right) & \text{if } -\frac{1}{n} \le x \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le x. \end{cases}$$

Then, with $f_n(x)$ continuous and differentiable on $\mathbb{R}$, we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Example

The Dirichlet function is not integrable but it is the limit of a sequence of integrable functions, all with integral equal to zero.
We need to define a new kind of convergence.

1.3 Convergences

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges punctually to a function f on $Dom(f)$ if:

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in Dom(f).$$

$$\forall \epsilon > 0, \quad \forall x \in Dom(f), \quad \exists N(\epsilon, x) \in \mathbb{N} : n > N \implies |f_n(x) - f(x)| < \epsilon.$$

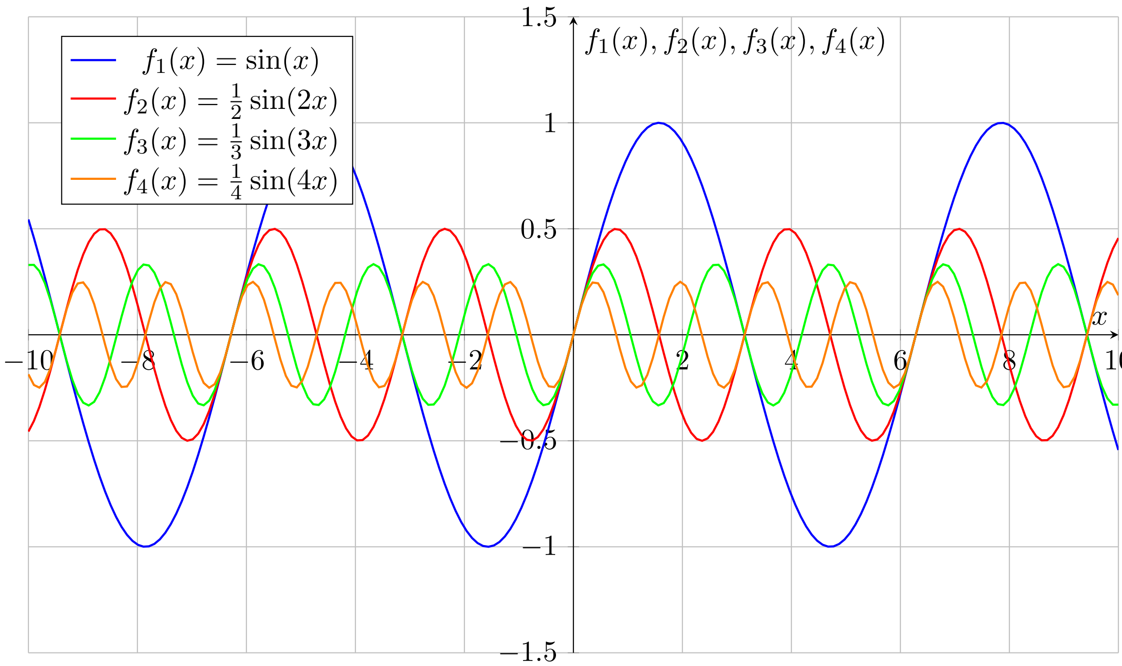
A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to a function f on $Dom(f)$ if:

$$\forall \epsilon, \quad \exists N : n > N \implies |f_n(x) - f(x)| < \epsilon, \quad \forall x \in Dom(f).$$

Example

Let

$$f_n(x) = \frac{1}{n}\sin(nx), \quad x \in \mathbb{R}. \to^{n\to\infty} f(x) = 0.$$



1.3.1 Uniform Convergence

1. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

2. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$ and each f_n is continuous in $[a, b]$, then f is continuous on $[a, b]$.

3. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[a, b]$ that converges punctually to some continuous function f on $[a, b]$ and if the sequence of derivatives $\{f'_n\}_{n\in\mathbb{N}}$ converges uniformly to some continuous function g, then f is differentiable on (a, b) and:

$$f'(x) = g(x) = \lim_{n\to\infty} f'_n(x).$$

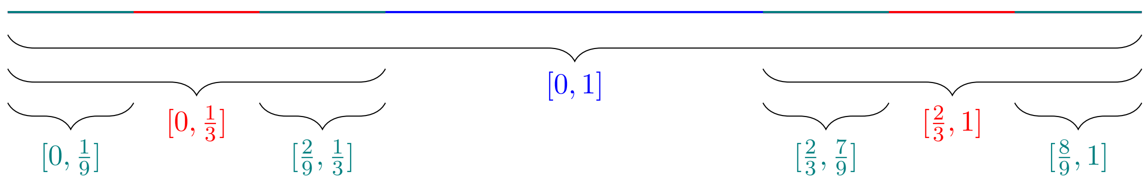
1.3.2 Henri Lebesgue (1875-1941)

How can we count money in bills?

1. Add each amount as the bills come in. (Riemann)
2. Make groups by denomination and count each group. (Lebesgue)

This is the idea behind the Lebesgue integral.

1.4 Cantor Ternary Set



Step 1: $[0, 1]$

Step 2: Remove middle third

Step 3: Remove middle thirds of remaining

The Cantor set C is obtained by removing the open middle third of each remaining interval at each step. Then,

$$C = [0, 1] \setminus J = \bigcup_{n=1}^{\infty} J_n,$$

where J is the union of all removed intervals and J_n is the set remaining after n steps. For the measure of C:

$$|J| = \sum_{n=1}^{\infty} |J_n| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = 1.$$

Thus, the measure of the Cantor set C is:

$$|C| = |[0, 1]| - |J| = 1 - 1 = 0.$$

The Cantor set is not empty; it contains points such as 0, 1, and all endpoints of the removed intervals. It has the following properties:

- It does not contain any intervals.
- It is closed and bounded, hence compact.
- It is a perfect set, which means it is closed and every point is an accumulation point.
- It is uncountable, because there is a bijection between the Cantor set and the interval $[0, 1]$ using ternary representation:

$$\Phi : [0, 1] \to C,$$

where each $x \in C$ is expressed in base 3, and has the form:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0, 2\}.$$

Then, for each $x \in [0, 1]$ we define:

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0, 1\}.$$

We can then define:

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}.$$

Now, $2b_n \in \{0, 2\}$ so $\Phi(x) \in C$. This function is bijective, hence C is uncountable.