Principles of Math. Analysis: Self-Evaluation 1

Nuño Cambero García

20-10-2025

Problem 1

a) **Given the space $X = \{1, 2, 3, 4, 5\}$, build the σ-algebra generated by the collection of sets:**

$$\mathcal{E} = \{\{1\}, \{1, 2\}, \{1, 5\}\}.$$

To build the σ-algebra generated by $\mathcal{E}$, we need to include all possible unions, intersections, and complements of the sets in $\mathcal{E}$. The generated σ-algebra will include:

- The empty set: $\emptyset$, and the whole set: $X = \{1, 2, 3, 4, 5\}$
- The sets in $\mathcal{E}$: $\{1\}, \{1, 2\}, \{1, 5\}$, and its complements: $\{2, 3, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4\}$
- Unions of the sets in $\mathcal{E}$: $\{1, 2, 5\}$, and the complement $\{3, 4\}$

After considering all combinations, the σ-algebra generated by $\mathcal{E}$ is:

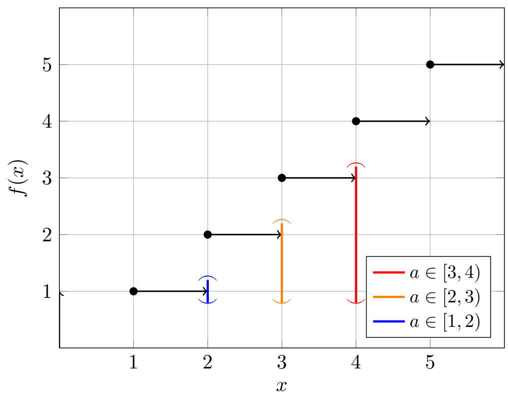
$$\sigma(\mathcal{E}) = \{\emptyset, \{1\}, \{1, 2\}, \{1, 5\}, \{1, 2, 5\}, \{3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4\}, X\}.$$

b) **Study if the function $f : X \to \mathbb{R}$ defined by $f(x) = x$ is measurable.**

To determine if the function f is measurable, we need to check if the preimage of every Borel set in $\mathbb{R}$ is in the σ-algebra generated by $\mathcal{E}$. Let us try with $\mathcal{B} = (-\infty, a]$ for various values of a:

- For $a < 1$: $f^{-1}((-\infty, a]) = \emptyset \in \sigma(\mathcal{E})$
- For $1 \leq a < 2$: $f^{-1}((-\infty, a]) = \{1\} \in \sigma(\mathcal{E})$
- For $2 \leq a < 3$: $f^{-1}((-\infty, a]) = \{1, 2\} \in \sigma(\mathcal{E})$
- For $3 \leq a < 4$: $f^{-1}((-\infty, a]) = \{1, 2, 3\} \notin \sigma(\mathcal{E})$

Since there exists a Borel set (for example, $(-\infty, 4]$) whose preimage is not in $\sigma(\mathcal{E})$, the function f is not measurable.



Problem 2

Consider a mapping $f : X \to Y$ where $(Y, \mathcal{A})$ is a measurable space and prove that

$$\mathcal{A}' = \{f^{-1}(E) : E \in \mathcal{A}\}$$

is a σ-algebra in X.

To prove that $\mathcal{A}'$ is a σ-algebra in X, we need to verify the three properties of a σ-algebra:

- **Contains the empty set:** Since $\emptyset \in \mathcal{A}$, we have $f^{-1}(\emptyset) = \emptyset \in \mathcal{A}'$.
- **Closed under complements:** If $A \in \mathcal{A}'$, then there exists $E \in \mathcal{A}$ such that $A = f^{-1}(E)$. The complement of A in X is:

  $$A^c = X \setminus A = X \setminus f^{-1}(E) = f^{-1}(Y \setminus E) = f^{-1}(E^c).$$

  Since $E^c \in \mathcal{A}$, it follows that $A^c \in \mathcal{A}'$.
- **Closed under countable unions:** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{A}'$. Then, for each n, there exists $E_n \in \mathcal{A}$ such that $A_n = f^{-1}(E_n)$. The countable union of these sets is:

  $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(E_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right).$$

  Since $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}'$.

Since $\mathcal{A}'$ satisfies all three properties of a σ-algebra, we conclude that $\mathcal{A}'$ is indeed a σ-algebra in X.

Problem 3

Use the definition to prove that if $(X, \mathcal{A}, \mu)$ is a measure space and the sets $\{A_n\}_{n \in N}$ are in $\mathcal{A}$ and satisfy:

$$A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A_n \supseteq \dots, \quad \textbf{with } \mu(A_1) < \infty,$$

then,

$$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n).$$

We can define a new sequence of sets $B_n = A_1 \setminus A_n$. With A_n a decreasing sequence, B_n is an increasing sequence of sets and we have:

$$\bigcap_{n=1}^{\infty} A_n = A_1 \setminus \bigcup_{n=1}^{\infty} B_n.$$

By the properties of measures, we know that:

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} \mu(B_n).$$

Since $B_n = A_1 \setminus A_n$, we have:

$$\mu(B_n) = \mu(A_1) - \mu(A_n).$$

Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} [\mu(A_1) - \mu(A_n)] = \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

Now, substituting back into the expression for $\mu\left(\bigcap_{n=1}^{\infty} A_n\right)$, we get:

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(A_1) - [\mu(A_1) - \lim_{n\to\infty} \mu(A_n)] = \lim_{n\to\infty} \mu(A_n).$$

Problem 4

On the measurable space $(X, \mathcal{A})$, we consider two points $x_0, x_1 \in X$ and define the function

$$\mu(A) = \begin{cases} 1, & \textbf{if } x_0 \in A \textbf{ or } x_1 \in A \textbf{ (only one of them)}, \\ 2, & \textbf{if } x_0, x_1 \in A \textbf{ (both)}, \\ 0, & \textbf{if } x_0 \notin A \textbf{ and } x_1 \notin A. \end{cases}$$

Prove that μ is a measure on $\mathcal{A}$.

To prove that μ is a measure on $\mathcal{A}$, we need to verify the following properties:

- **Non-negativity:** By definition, $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.
- **Null empty set:** We have $\mu(\emptyset) = 0$ since neither x_0 nor x_1 is in the empty set.
- **Countable additivity:** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{A}$. We need to show that:

  $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

  We consider the following cases:

  - If neither x_0 nor x_1 is in any of the A_n, then $\mu(A_n) = 0$ for all n, and thus both sides equal 0.
  - If exactly one of x_0 or x_1 is in one of the A_n, say $x_0 \in A_k$ for some k, then $\mu(A_k) = 1$ and $\mu(A_n) = 0$ for all $n \neq k$. Thus, the left side equals 1, and the right side also equals 1.
  - If both x_0 and x_1 are in different A_n, say $x_0 \in A_k$ and $x_1 \in A_m$ for $k \neq m$, then $\mu(A_k) = 1$ and $\mu(A_m) = 1$, while $\mu(A_n) = 0$ for all $n \neq k, m$. Thus, the left side equals 2, and the right side also equals 2.
  - If both x_0 and x_1 are in the same set A_k, then $\mu(A_k) = 2$ and $\mu(A_n) = 0$ for all $n \neq k$. Thus, the left side equals 2, and the right side also equals 2.

Since all three properties are satisfied, we conclude that μ is indeed a measure

Problem 5

We consider the function:

$$g(x) = \arctan[x], \quad x \in \mathbb{R},$$

where $[x]$ is the integer part of x.

a) **Prove that g is a distribution function for some Lebesgue-Stieltjes measure μ_g.**

b) **Obtain the measure of the following sets, approximating the infinite sets by an increasing sequence of sets:**

$$(0,1), \quad [-1,1], \quad A = \{x : |x^2 - 1| < 1\}, \quad (1,\infty), \quad \mathbb{R}.$$

For g to be a distribution function, it must be non-decreasing, right-continuous. The function $\arctan[x]$ is non-decreasing since $[x]$ is non-decreasing and arctan is an increasing function. It is right-continuous because $[x]$ is right-continuous, and arctan is continuous. Thus, g is a distribution function for some Lebesgue-Stieltjes measure μ_g.
To find the measure of the given sets, we use the properties of the Lebesgue-Stieltjes measure associated with g:

- For the interval $(0,1)$:

$$\mu_g((0,1)) = g(1) - g(0) = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

- For the interval $[-1,1]$:

$$\mu_g([-1,1]) = g(1) - g(-1) = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

- For the set $A = \{x : |x^2 - 1| < 1\}$: This set can be rewritten as $(-\sqrt{2}, \sqrt{2}) \sim (-1,1)$:

$$\mu_g(A) = g(1) - g(-1) = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

- For the interval $(1,\infty)$:

$$\mu_g((1,\infty)) = \lim_{x\to\infty} g(x) - g(1) = \frac{\pi}{2} - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

- For the entire real line $\mathbb{R}$:

$$\mu_g(\mathbb{R}) = \lim_{x\to\infty} g(x) - \lim_{x\to-\infty} g(x) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$