Exercises in Principles of Mathematical Analysis

Exercise 1.1.1

Prove that $\mathbb{Q}$ has zero measure.
A set $Q \subset \mathbb{R}$ has measure zero if and only if:

$$Q \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{where } A_j \text{ are intervals and } \sum_{j=1}^{\infty}|A_j| < \varepsilon, \forall \epsilon > 0.$$

Since $\mathbb{Q}$ is countable, we can enumerate its elements as $\{q_1, q_2, q_3, \ldots\}$.
We start with:

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1/2}{1 - 1/2} = 1.$$

Then, we can multiply this series by any $\epsilon > 0$ to get:

$$\sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \varepsilon.$$

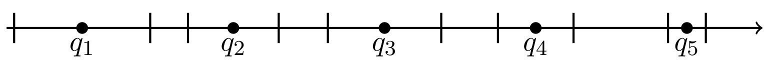
For each rational number q_j, we can construct an interval A_j centered at q_j with length $\dfrac{\varepsilon}{2^j}$:

$$A_j = \left(q_j - \frac{\varepsilon}{2^{j+1}}, q_j + \frac{\varepsilon}{2^{j+1}}\right).$$

Thus, we have:

$$\mathbb{Q} \subset \bigcup_{j=1}^{\infty} A_j, \quad \text{and } \sum_{j=1}^{\infty}|A_j| = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Since ε can be made arbitrarily small, we conclude that $\mathbb{Q}$ has measure zero.

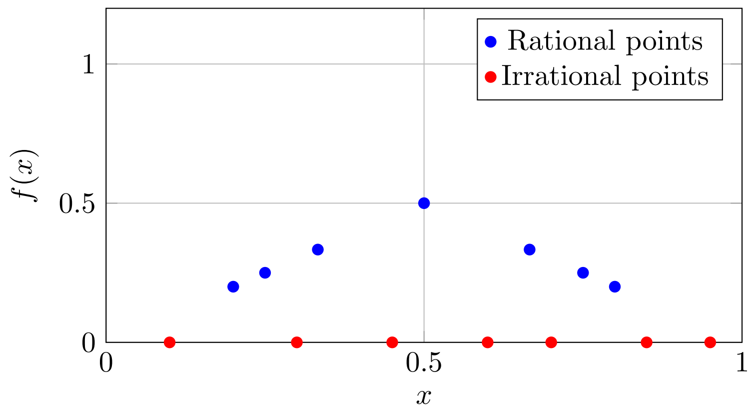


Exercise 1.1.2

For the following function defined on $[0, 1]$:

$$f(x) = \begin{cases} \frac{1}{q} & \textbf{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z}, q \neq 0 \\ 0 & \textbf{if } x \notin \mathbb{Q} \end{cases}$$

1. **Show that f is discontinuous only at the rational points.**
2. **Prove that f is Riemann integrable.**



If x is irrational, then

$$\lim_{x \to x_0} f(x) \stackrel{?}{=} 0 = f(x_0),$$

$\forall \varepsilon > 0$, we want $|f(x) - 0| = |f(x)| < \varepsilon$ if x is close enough to x_0 i.e. $x \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.
For example, take $\varepsilon = \frac{1}{3}$, then $|f(x)| < \frac{1}{3}$ if $f(x) = \frac{1}{q} < \frac{1}{3} \Rightarrow q > 3$. So, except for the rationals $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, we have $|f(x)| < \varepsilon = \frac{1}{3}$. We can do this for any $\varepsilon > 0$ by choosing $q > \frac{1}{\varepsilon}$, so

$$\lim_{x \to x_0} f(x) = 0.$$

So it is continuous at every irrational point. On $\mathbb{Q}$, the value of $f(x)$ is non-zero, so it is discontinuous at every rational point.
Finally, since the set of discontinuities has measure zero ($|\mathbb{Q}| = 0$), f is Riemann integrable.

Exercise 1.1.4

Consider the sequence of functions given by:

$$f_n(x) = \begin{cases} 1 & \textbf{if } |x| \leq n \\ 0 & \textbf{if } |x| > n \end{cases}$$

Obtain the limit $\lim_{n\to\infty} f_n(x)$ and study whether the convergence is uniform or not.

$$\sup_{n\in\mathbb{N}} |f_n(x) - f(x)| = 1 \neq 0, \quad \forall x \in \mathbb{R}.$$

So the convergence is not uniform.

Exercise 1.1.5

Prove that the following series converges in $[0,1]$. Is the convergence uniform?

$$\sum_{n=0}^{\infty} x(1-x)^n = x\sum_{n=0}^{\infty}(1-x)^n = \begin{cases} \dfrac{x}{1-(1-x)} = \dfrac{x}{x} = 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

If $|1-x| < 1$, then the series converges. This is true for all $x \in [0,1]$. The convergence is not uniform since f is not continuous:

$$f_N(x) = \sum_{n=0}^{N} x(1-x)^n \overset{N\to\infty}{\longrightarrow} f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Exercise 1.1.3

Prove that if a function $f : [a,b] \to \mathbb{R}$ is monotonus, then it:

1. **is bounded**
2. **is Riemann integrable.**

Suppose f is monotonically increasing. Then,

$$f(x) \in [f(a), f(b)], \quad \forall x \in [a,b].$$

So f is bounded.

Now, we build $U_f(P)$ and $L_f(P)$ for a partition $P = \{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \ldots, b\}$.

$$U_f(P) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n},$$

$$L_f(P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n}.$$

Then,

$$U_f(P) - L_f(P) = \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \frac{b-a}{n}(f(b) - f(a)) \overset{n\to\infty}{\longrightarrow} 0.$$

So f is Riemann integrable.