Discrete Mathematics

1 Sum Rule

If a first task can be performed in m ways, while a second task can be performed in n ways, and both cannot be done simultaneously, then performing either task can be done in $m + n$ ways.

Example: Discrete Mathematics Book

A library has:

- 40 copies of Rosen

- 30 copies of another author

In how many ways can a book be selected?

$$40 + 30 = 70 \text{ ways.}$$

Example: Rolling Two Dice

In how many ways can we get a sum of 6, 7, or 8 by rolling two dice?

$$6 = 1 + 5, \ 2 + 4, \ 3 + 3, \ 4 + 2, \ 5 + 1 \qquad \text{(5 ways)}$$
$$7 = 1 + 6, \ 2 + 5, \ 3 + 4, \ 4 + 3, \ 5 + 2, \ 6 + 1 \qquad \text{(6 ways)}$$
$$8 = 2 + 6, \ 3 + 5, \ 4 + 4, \ 5 + 3, \ 6 + 2 \qquad \text{(5 ways)}$$

Total: $5 + 6 + 5 = 16$ ways.

Set Representation:

$$S := \text{all outcomes of throwing two dice} = \{(1,1), (1,2), \ldots, (6,6)\}.$$

1.1 Partition of a Set

A **partition** of a set S is a collection of subsets S_i, where $i = 1, 2, \ldots, k$, such that:

$$\bigcup_{i=1}^{k} S_i = S \quad \text{and} \quad S_i \cap S_j = \emptyset \quad \forall i \neq j.$$

This means that every element of S belongs to exactly one subset S_i, ensuring that there is no overlap. Additionally, the cardinality of S satisfies:

$$|S| = \sum_{i=1}^{k} |S_i|$$

2 Product Rule

If a task can be decomposed into two stages, where the first stage can be done in m ways and the second stage can be done in n ways, the full task can be done in $m \cdot n$ ways.

Example: Casting for a Play

A theater company needs one actor and one actress. There are:

- 8 women

- 6 men

Number of ways to choose a pair:
$$8 \cdot 6 = 48 \text{ ways.}$$

Example: Poker Hands

How many poker hands can be dealt from a deck of 52 cards?

- Without replacement:
$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200.$$

- With replacement:
$$52^5 = 380204032.$$

2.1 Cartesian Product

Given two sets A and B, their **Cartesian product**, denoted as $A \times B$, is defined as the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

This definition naturally extends to multiple sets:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for all } i\}.$$

If A and B are finite sets, the cardinality of their Cartesian product is given by:

$$|A \times B| = |A| \cdot |B|.$$

This means that if A has m elements and B has n elements, then $A \times B$ contains exactly $m \cdot n$ ordered pairs.

Example: Cartesian Product of Two Finite Sets

If $A = \{1, 2\}$ and $B = \{a, b, c\}$, then:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Since $|A| = 2$ and $|B| = 3$, we verify:

$$|A \times B| = 2 \times 3 = 6.$$

This property generalizes to multiple sets:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.$$

Example: Numbers Between 5000 and 10000

How many numbers between 5000 and 10000 can be formed without repeating digits?

$$\text{Choices for the first digit: } 5 \quad (\text{Take } 5, 6, 7, 8 \text{ or } 9).$$

$$\text{Choices for the second digit: } 9 \quad (0 \text{ and remaining digits except } 5).$$

$$\text{Choices for the third digit: } 8, \quad \text{fourth digit: } 7.$$

$$\text{Total: } 5 \cdot 9 \cdot 8 \cdot 7 = 2520.$$

3 Double Counting

Counting the same set in two ways produces the same result.

Example: Subjects Taken by Students

	Algebra	Calculus	Discrete Math	Programming	Total
S_1	X	X	X		3
S_2		X		X	2
S_3	X		X	X	3
S_4	X				1
S_5	X	X	X	X	4
Total	4	3	3	3	13

Table 1: By counting row-wise or column-wise, the result is the same.

Each student chooses 4 subjects. There are six subjects with enrollments: $42, 38, 35, 35, 22, 20$. How many students are there?

$$42 + 38 + 35 + 35 + 22 + 20 = 4n$$

$$192 = 4n \implies n = 48.$$

3.1 Handshaking Lemma

In any meeting, the number of people who shake hands with an odd number of people is even.

Proof:
$$\text{Persons: } p_1, p_2, \ldots, p_n.$$

$$\text{Handshaking pairs: } (p_i, p_j) = (p_j, p_i), \quad \text{and } (p_i, p_i) \text{ does not exist.}$$

Each person in a group shakes hands with k_i other people, where:

$$1 \le k_i \le n - 1.$$

The total number of handshakes is given by:

$$\sum_{i=1}^{n} k_i.$$

Since each handshake contributes to the degree count of two people, the sum must be even. This implies:

- Either all k_i values are even, or

- There is an even number of odd k_i values.

$$\implies \text{Total number of handshakes is even.}$$

4 Pigeonhole Principle

If there are more pigeons than pigeonholes, at least one pigeonhole must have at least two pigeons.

Formal Statement

Given $m, n, p \in \mathbb{N}$, if $np + m$ elements are distributed in n sets, at least one of them contains no fewer than $p + 1$ elements.

4.1 Proof by contradiction

Suppose that in each set there are p_i elements, $i = 1, 2, \ldots, n$.
Furthermore, suppose $p_i \leq p \quad \forall i, \quad$ then

$$\sum_{i=1}^{n} p_i \leq np \leq np + m \quad \#$$

Examples:

- If we choose 5 different numbers from 1 to 8 (inclusive), then there are 2 numbers that add up to 9.

  - $\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\} \implies$ numbers that add to 9.
  - These are partitions of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
  - You can't choose 5 of them such that no two of them add to 9 - we'd end up taking 2 from the same partition which would add to 9.

- Suppose we choose 11 numbers from the set $\{1, 2, \ldots, 20\}$. We express a number n as:

$$n = 2^k m, \quad \text{where } k \geq 0 \text{ and } m \text{ is odd.}$$

  Since $1 \leq m \leq 19$, there are 10 possible odd values for m.

  By the Pigeonhole Principle, if we pick 11 numbers, at least two must share the same m, meaning they are of the form $2^{k_1} m$ and $2^{k_2} m$.

  Thus, we have:
$$\frac{2^{k_1} m}{2^{k_2} m} = 2^{k_1 - k_2} \in \mathbb{Z}$$

  which implies that one number is a multiple of the other.

- Any lossless compression algorithm fails for at least one data file.

  Strings of N bits $\to$ 1 bit to $N - 1$ bits

$$2 + 2^2 + 2^3 + \ldots + 2^{N-1} = \sum_{k=1}^{N-1} 2^k = \frac{1 - 2 \cdot 2^{N-1}}{1 - 2} = 2^N - 1$$

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Example: Passwords with letters and digits

How many passwords can be formed with lengths 6 through 8, with letters and digits, when passwords must include at least one digit?

$$P = P_6 + P_7 + P_8, \quad \text{with restrictions}$$

$$26 \text{ letters} + 10 \text{ digits} - \text{Only letters: } P_6 = 36^6 - 26^6, \quad P_7 = 36^7 - 26^7, \quad P_8 = 36^8 - 26^8$$

$$P = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8) = 2684483063360$$

5 Substraction rule (Inclusion-exclusion principle)

If a task can be carried out in n ways or m ways, then the total number of ways to do it is $n + m -$ number of ways common to both.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Bit Strings

How many bit strings of length 8 are there that begin with 1 or end with 00?

$$2^7 + 2^6 - 2^5 = 160$$

6 Combinations, permutations and variations

- There is 1 instructor and 4 students are to be selected from a group of 10 students.
  $\rightarrow$ Order matters, selection matters. $\rightarrow$ Permutations.

- There is 1 instructor and 4 students are to be selected from a group of 10 students.
  $\rightarrow$ Order does not matter, selection matters. $\rightarrow$ Variations.

- There are 3 instructors and 4 students are to be selected from a group of 10 students.
  $\rightarrow$ Order does not matter, selection does not matter. $\rightarrow$ Combinations.

6.1 Permutations

Let $A \neq \emptyset$ be a set of n elements. A permutation of A is a bijection:

$$\phi : \{1, 2, \ldots, n\} \to A$$

The number of permutations of A is denoted by $P(n)$ and is given by:

$$P_n = n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$$

Example: Permutations with words

How many ways can the letters in the word "BLACKSMITH" be arranged?

$$P_n = 10! = 3628800$$

Now, the same does not apply to the word "MISSISSIPPI" because of the repeated letters. Instead, we need to make each letter unique and divide by the number of ways to arrange the repeated letters:

$$MI_1 S_1 S_2 I_2 S_3 S_4 I_3 P_1 P_2 I_4$$

$$\text{Then,} \quad P_n = \frac{11!}{4! \cdot 4! \cdot 2!} = 34650$$

6.2 Multinomial Coefficients

The number of ways to partition a set of n elements into k disjoint subsets of sizes $n_1, n_2, \ldots, n_k$ is given by the multinomial coefficient:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Example: Multinomial Coefficients

Suppose $n, k \in \mathbb{N}$ and $n = 2k$. Prove that $\frac{n!}{2^k} \in \mathbb{N}$.

$$\text{Let } A = \{x_1, x_1, x_2, x_2, \ldots, x_k, x_k\}$$
$$\text{where } |A| = n = 2k$$
$$\text{Number of ways to arrange the elements of } A = \frac{n!}{2^k}$$

$$\frac{n!}{2^k} = \frac{2k!}{2^k} = (2^k - 1) \cdot (2^k - 2) \cdots 1 \in \mathbb{N}$$

6.3 Variations

Let $A \neq \emptyset$ be a set of n elements. A variation of A is a one-to-one mapping:

$$\phi : \{1, 2, \ldots, r\} \to A \quad \text{where } r < n$$

The number of variations of A is denoted by $V(n, r)$ and is given by:

$$V_n^r = n \cdot (n-1) \cdot (n-2) \cdots (n - r + 1) = \frac{n!}{(n-r)!}$$

Example: Six letter words

How many six-letter words can be formed from the word "BLACKSMITH"? How many of these contain "B"?

$$V_{10}^6 = \frac{10!}{4!} = 90720$$

$$\text{With "B": } 6 \cdot V_9^5 = \frac{9!}{4!} = 60480$$

6.4 Variations with repetitions

Let $A \neq \emptyset$ be a set of n elements. A variation with repetitions of A is a mapping:

$$\phi : \{1, 2, \ldots, r\} \to A$$

The number of variations with repetitions of A is denoted by VR_n^r and is given by:

$$VR_n^r = n^r$$

Example: "Quinielas"

In a "quiniela" game, players must predict the outcome of 15 soccer matches. Each match can end in a win, loss, or draw. How many possible outcomes are there?

$$\text{Let A} = \{\text{win, loss, draw}\} \quad \text{and} \quad |A| = 3$$

$$VR_3^{15} = 3^{15} = 14348907$$

6.5   Combinations

Let $A \neq \emptyset$ be a set of n elements. A combination of elements of A is any subset of A with r elements.

The number of combinations of A is denoted by C_n^r and is given by:

$$C_n^r = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

Example: Lottery

How many combinations are ther in the "lotería primitiva" game?

$$C_{49}^6 = \frac{49!}{6! \cdot 43!} = 13983816$$

Example: Poker Hands

How many poker hands can be dealt from a deck of 52 cards and 2 jokers?
How many of them have exactly 3 aces and no jokers?
How many ace trios can one form (including jokers)?

$$C_{54}^5 = \frac{54!}{5! \cdot 49!} = 25827165$$

$$C_4^3 \cdot C_{48}^2 = 4 \cdot \frac{48!}{2! \cdot 46!} = 6912$$

$$C_4^3 = 4$$

Example: Balls in Boxes

3 balls are to be chosen from a box of seven, where there are 3 red balls, 2 blue balls and 2 white balls. How many times will we get at least 2 red balls?

$$\text{Let } A = \{R_1, R_2, R_3, B_1, B_2, W_1, W_2\}$$

$$C_7^3 = \frac{7!}{3! \cdot 4!} = 35$$
$$C_3^2 \cdot C_4^1 = 3 \cdot 4 = 12$$
$$C_3^3 = 1$$

$$\text{Total: } 12 + 1 = 13$$

Proof: combinatorial

$$S, \quad |S| = n$$

$$A, \quad |A| = r, \quad \overline{A} = S - A = \{x \in S \mid x \notin A\}$$

$$f : \{\text{subsets of } S\} \to \{\text{subsets of } S\}$$

$$f(A) = \overline{A}, \quad \text{where } f \text{ is a bijection}$$

$$\text{Then,} \quad |A| = |\overline{A}| = \binom{n}{r}$$

6.6 Binomial Theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof:

$$(x + y)^n = (x + y)(x + y) \cdots (x + y)$$

$$\text{Expand:} \quad \text{terms of the form } x^{n-k} y^k$$

$$\text{Number of terms:} \quad \binom{n}{k}$$

$$\text{Corollary} : \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\text{Corollary} : \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Example: Binomial Theorem

What is the coefficient of $x^{12} y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$$\binom{25}{k} \cdot (2x)^k \cdot (-3y)^{25-k}$$

$$k = 13 \to \binom{25}{13} \cdot 2^{12} \cdot (-3)^{13}$$

6.7 Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Triangle

$n = 0$									1								
$n = 1$								1		1							
$n = 2$							1		2		1						
$n = 3$						1		3		3		1					
$n = 4$					1		4		6		4		1				
$n = 5$				1		5		10		10		5		1			
$n = 6$			1		6		15		20		15		6		1		
$n = 7$		1		7		21		35		35		21		7		1	
$n = 8$	1		8		28		56		70		56		28		8		1

Which equals the combinations of n elements taken k at a time : $\binom{n}{k}$.

Proof: Identity

$$\text{Set } S = \text{with } n + 1 \text{ elements}$$

$$\text{Let } A = S - \{a\}, \quad |A| = n$$

$$\text{Number of subsets of } S \text{ with } k \text{ elements} =$$

Number of subsets of A with k elements + Number of subsets of A with $k - 1$ elements

6.8   Combinations with repetitions

Example: Fruits

4 fruits are to be chosen from a group of apples, oranges, and bananas. How many ways are there to choose the fruits?

$$CR_3^4 = \binom{3 + 4 - 1}{4} = \binom{6}{4} = 15$$

Example: Coins and Bills

How many ways are there to select five bills from a cash containing $1, $5, $10, $20, $50, and $100 bills?

$$CR_n^r = \binom{n + r - 1}{r}$$

$$CR_6^5 = \binom{6 + 5 - 1}{5} = \binom{10}{5} = 252$$

Example: Cookies

How many ways are there to distribute 6 cookies out of four types?

$$CR_4^6 = \binom{4 + 6 - 1}{6} = \binom{9}{6} = 84$$

Example: Equation

How many solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

$$CR_3^{11} = \binom{3 + 11 - 1}{11} = \binom{13}{11} = 78$$

6.9   Generating permutations and combinations

Lexicographic ordering

$$\{1, 2, 3, \ldots, n\}$$

$$a_1, a_2, a_3, \ldots, a_n \text{ precedes } b_1, b_2, b_3, \ldots, b_n$$

$$\text{for some } k, 1 \leq k \leq n, a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}, a_k < b_k$$

Example: Permutations

$$\{1, 2, 3, 4, 5\}$$

$$23145 \text{ precedes } 23514, \quad \text{since } 2 = 2, 3 = 3, 1 < 5$$

6.10   Permutations

$$a_1, a_2, a_3, \ldots, a_n$$

$$\text{If } a_{n-1} < a_n, \text{ then exchange } a_n \text{ and } a_{n-1}$$

$$\text{If } a_{n-1} > a_n, \text{ then } a_{n-2} < a_{n-1} < a_n, \text{ exchange } a_n \text{ and } a_{n-2}$$

Ways to Distribute Balls into Labeled Boxes

	At most 1 per box	Any number per box	Exactly one per box
k labeled balls	$k \leq n, V_n^k$ or $P(n, k)$	$\dfrac{k!}{k_1! k_2! \ldots k_n!}$	N/A
k unlabeled	$\dbinom{n}{k} = \dfrac{n!}{(n-k)! k!}$	CR_n^k	N/A
Unlimited balls	N/A	N/A	k^n

Example Problem

You have 6 different dice. In how many ways can you give them to 12 students, giving at most 1 each?

$$V_{12}^6 = \frac{12!}{6!}$$

If the dice are identical:

$$\binom{12}{6} = \frac{12!}{6! \cdot 6!}$$

7   Recurrences

Theorem: Let $S_1, S_2, \ldots, S_n$ be sets from the same universe U, defined as:

$$S_i = \{x \in U \mid P_i(x) \text{ is true}\}$$

Then:

$$\left| \bigcap_{i=1}^{n} S_i^c \right| = |U| - \sum_{i=1}^{n} |S_i| + \sum_{i<j} |S_i \cap S_j| - \sum_{i<j<k} |S_i \cap S_j \cap S_k| + \cdots + (-1)^n |S_1 \cap S_2 \cap \cdots \cap S_n|$$

$$\text{where } S_i^C = U - S_i$$

Proof:

Let $x \in U$ be an element of the universe, such that $P_i(x)$ is false $\forall i$.

Then $x \in S_i^c \forall i \implies x \in \bigcap_{i=1}^{n} S_i^C. \implies$ Counts 1 in the left-hand side.

In the right-hand side, $|U|$ is counted once, and 0 in all other terms.

x is part of rS_i sets, $0 \le r \le n$. $\implies$ Counts 0 in the left-hand side.

In the right-hand side, $|U|$ is counted once, and 1 for every intersection $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ as long as all the indices $i_1, i_2, \ldots, i_k$ match some of the r sets to which x belongs.

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \cdots + (-1)^r \binom{r}{r} = 0. \quad \text{Binomial theorem: } (1 + (-1))^r = \sum_{k=0}^{r} \binom{r}{k} \cdot 1^{r-k} \cdot (-1)^k$$

Corollary:

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{i=1}^{n} |S_i| - \sum_{i<j} |S_i \cap S_j| + \sum_{i<j<k} |S_i \cap S_j \cap S_k| - \cdots + (-1)^{n-1} |S_1 \cap S_2 \cap \cdots \cap S_n|$$

Example:

A restaurant offers three different menus. Three customers go in the restaurant and each one chooses one of the menus. How many ways are there for the customers to not get their correct order?

$$\text{Let } A, B, C \text{ be the menus.}$$

$$\text{Ordered: } A, B, C \quad A, C, B \quad B, A, C \quad B, C, A \quad C, A, B \quad C, B, A \implies 3! = 6 \text{ ways.}$$

Derangements:

$$U = \{ \text{ permutations of \{A, B, C\} } \}$$

$$S_i = \{ \text{ permutations where the } i\text{th element is in its correct position } \} = \{x \in U \mid x_i = i\}$$

$$|S_1^c \cap S_2^c \cap S_3^c| = |U| - |S_1| - |S_2| - |S_3| + |S_1 \cap S_2| + |S_1 \cap S_3| + |S_2 \cap S_3| - |S_1 \cap S_2 \cap S_3| =$$

$$= 6 - 3! + 2! - 1! = 2$$

The probability that no customer gets their correct order is $\dfrac{2}{3!} = \dfrac{1}{3}$

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7.1 Derangements of n elements

n people order each one a different menu. In how many ways can they all get the wrong order?

$$U = \{ \text{ permutations of } \{1, 2, \ldots, n\} \}$$

$$S_i = \{ \text{ permutations where the } i\text{th element is in its correct position } \} = \{x \in U \mid x_i = i\}$$

$$\left| \bigcap_{i=1}^{n} S_i^c \right| = |U| - \sum_{i=1}^{n} |S_i| + \sum_{i<j} |S_i \cap S_j| - \sum_{i<j<k} |S_i \cap S_j \cap S_k| + \cdots + (-1)^n |S_1 \cap S_2 \cap \cdots \cap S_n| =$$

$$= |U| - |S_1| - |S_2| - \cdots - |S_n| + |S_1 \cap S_2| + \cdots + (-1)^n |S_1 \cap S_2 \cap \cdots \cap S_n| =$$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right) \cong d(n)$$

$$\text{Prob} = \frac{d(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Computing Derangements

$$d(n) = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\text{Remember: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

Derangements are a truncated version of the exponential function at order n.

$$\left| e^{-1} - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right| \leq \frac{1}{(n+1)!} < \frac{1}{2}$$

$$d(n) = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \quad \text{Prob} = \frac{1}{e} = 0.368$$

7.2 Recurrence relations

We have a sequence $\{x_n\}_{n=0}^{\infty}$
It is generated by a recurrence relation if for all $n \geq r$ there is a function f such that:

$$f(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-r}) = 0$$

A linear recurrence relation of order r is a relation of the form:

$$x_n = a_1(n)x_{n-1} + a_2(n)x_{n-2} + \cdots + a_r(n)x_{n-r} + g(n)$$

Example: Fibonacci sequence

A young pair of rabbits. They don't die and don't reproduce in the first month, but in the second month they have a new pair. From the third month on, they reproduce every month. Find a recurrence relation for the number of rabbits at every month.

$$f(n) = f(n-1) + f(n-2)$$

Month	Young	Can Reproduce	Total Pairs $f(n)$	Explanation
1	1	0	1	Start with one pair of rabbits.
2	1	0	1	They mature and can reproduce next month.
3	1	1	2	The original pair reproduces, adding one new pair.
4	2	1	3	The original pair reproduces again, and the new pair from month 3 matures.
5	3	2	5	Both pairs from month 3 and 4 reproduce, adding two new pairs.
6	5	3	8	The pairs from month 4 and 5 reproduce, adding three new pairs.

Table 2: Chronological explanation of the Fibonacci sequence in the context of rabbit pairs.

Example: Hanoi Towers

Figure 1: Illustration of the Hanoi Towers problem with disks.

The Hanoi Towers problem consists of three rods and n disks of different sizes. The disks are stacked in decreasing order of size on the first rod. The goal is to move all the disks to the third rod, following the rules:

- Only one disk can be moved at a time.

- A disk can only be placed on top of a larger disk.

Let $h(n)$ be the number of moves required to move n disks from the first to the third rod. The recurrence relation is:

$$h(n) = 2h(n-1) + 1 = 2(2h(n-2) + 1) + 1 = 2^2 h(n-2) + 2 + 1 =$$

$$\cdots = 2^n h(0) + 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 1$$

$$h(n) = 2^n - 1$$

7.3 Derangements as recurrence

Figure 2: Graph representation of a derangement.

- Suppose $4 \to 1$ $d(n-2)$ ways to derange the remaining elements.

- Suppose $4 \nrightarrow 1$ $d(n-1)$ ways to derange the remaining elements.

$$d(n) = (n-1)(d(n-1) + d(n-2))$$

Example:

Find a recurrence relation for the number of bit strings of length n that do not contain two consecutive 0s.

Consider $n \geq 3$. Build the string from strings of length $n-1$.

- If the last digit is 1, there are $d(n-1)$ ways to build the string.

- If the last digit is 0, the second-to-last digit must be 1. There are $d(n-2)$ ways to build the string.

$$d(n) = d(n-1) + d(n-2) \quad \text{which is the Fibonacci sequence again!}$$

8 Linear Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = \{a_n\}_{n=1}^{\infty}$$

With $c_i \in \mathbb{R}$, $c_k \neq 0$ and $k \geq 1 = $ order of the recurrence relation

$$1. \quad a_n = r^n, \quad r \in \mathbb{R}, \quad r \neq 0$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0 \quad \to \text{Characteristic equation}$$

2. Superposition principle: the sum of solutions is also a solution

$$\{s_n\}, \{t_n\} \text{ are solutions} \to \{s_n + t_n\} \text{ is also a solution}$$

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k}$$

$$t_n = c_1 t_{n-1} + c_2 t_{n-2} + \cdots + c_k t_{n-k}$$

$$s_n + t_n = c_1(s_{n-1} + t_{n-1}) + c_2(s_{n-2} + t_{n-2}) + \cdots + c_k(s_{n-k} + t_{n-k})$$

8.1 The $k = 2$ case

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad c_2 \neq 0$$

$$a_0 = \alpha_0, \quad a_1 = \alpha_1, \quad \text{initial conditions}$$

$$\text{Then,} \quad r^2 - c_1 r - c_2 = 0 \implies \text{solutions } r_1 \neq r_2$$

General solution:

$$a_n = \beta_1 r_1^n + \beta_2 r_2^n$$

Use the initial conditions to find β_1 and β_2.

$$a_0 = \beta_1 + \beta_2 = \alpha_0 \quad \rightarrow \beta_2 = \alpha_0 - \beta_1$$

$$a_1 = \beta_1 r_1 + \beta_2 r_2 = \alpha_1 \quad \rightarrow \alpha_1 = \beta_1 r_1 + (\alpha_0 - \beta_1) r_2$$

$$\alpha_1 - \alpha_0 r_2 = \beta_1 (r_1 - r_2) \quad \rightarrow \beta_1 = \frac{\alpha_1 - \alpha_0 r_2}{r_1 - r_2}$$

$$\text{Then,} \quad \beta_2 = \alpha_0 - \frac{\alpha_1 - \alpha_0 r_2}{r_1 - r_2} = \frac{\alpha_0 r_1 - \alpha_1}{r_1 - r_2}$$

Example:

$$a_n = a_{n-1} + 2a_{n-2}, \quad a_0 = 2, \quad a_1 = 7$$

$$r^2 - r - 2 = 0 \quad \rightarrow r_1 = 2, \quad r_2 = -1$$

$$a_n = \beta_1 2^n + \beta_2 (-1)^n$$

$$2 = \beta_1 + \beta_2 \quad \rightarrow \beta_2 = 2 - \beta_1$$

$$7 = 2\beta_1 - \beta_2 \quad \rightarrow 7 = 2\beta_1 - (2 - \beta_1) \quad \rightarrow \beta_1 = 3$$

$$\text{Then,} \quad \beta_2 = 2 - 3 = -1$$

$$a_n = 3 \cdot 2^n - (-1)^n$$

Example: Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

$$r^2 - r - 1 = 0 \quad \rightarrow r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}$$

$$0 = \beta_1 + \beta_2 \quad \rightarrow \beta_2 = -\beta_1$$

$$1 = \beta_1 \frac{1 + \sqrt{5}}{2} - \beta_1 \frac{1 - \sqrt{5}}{2} \quad \rightarrow \beta_1 = \frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$$

Special case: $r_1 = r_2$

$$r^2 - c_1 r - c_2 = 0 \quad \to \text{With } r_1 = r_2 = r$$

Then, the general solution is $a_n = \beta_1 r^n + \beta_2 n r^n$

$$a_0 = \beta_1 = \alpha_0 \quad \to \beta_1 = \alpha_0$$

$$a_1 = \beta_1 r + \beta_2 r = \alpha_1 \quad \to \beta_2 = \frac{\alpha_1 - \alpha_0 r}{r}$$

Example:

$$a_n = 6a_{n-1} - 9a_{n-2}, \quad a_0 = 1, \quad a_1 = 6$$

$$r^2 - 6r + 9 = 0 \quad \to r = 3$$

$$a_n = \beta_1 3^n + \beta_2 n 3^n$$

$$1 = \beta_1 \quad \to \beta_1 = 1$$

$$6 = 3\beta_1 + 3\beta_2 \quad \to 6 = 3 + 3\beta_2 \quad \to \beta_2 = 1$$

$$a_n = 3^n + n3^n$$

8.2 Theorem: Solution of degree k recurrence relations with constant coefficients

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad c_i \in \mathbb{R}, \quad c_k \neq 0$ be a linear recurrence relation with constant coefficients. Consider the characteristic equation:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

Assume it has different roots $r_1, r_2, \ldots, r_t$, with multiplicities $m_1, m_2, \ldots, m_t$, with $m_1 + m_2 + \cdots + m_t = k$. Then, the general solution is:

$$a_n = (\beta_{10} + \beta_{11} n + \beta_{12} n^2 + \cdots + \beta_{1m_1} n^{m_1 - 1}) r_1^n + \cdots + (\beta_{t0} + \beta_{t1} n + \beta_{t2} n^2 + \cdots + \beta_{tm_t} n^{m_t - 1}) r_t^n$$

$$a_n = \sum_{i=1}^{t} \sum_{j=1}^{m_i} \beta_{ij} n^{j-1} r_i^n$$

Given k initial conditions $a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_{k-1} = \alpha_{k-1}$, the coefficients β_{ij} can be found by solving the system of linear equations.

Example:

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}, \quad a_0 = 1, \quad a_1 = -2, \quad a_2 = -1$$

$$r^3 + 3r^2 + 3r + 1 = 0 \quad \to r = -1, \text{ with multiplicity 3}$$

$$a_n = (\beta_1 + \beta_2 n + \beta_3 n^2)(-1)^n$$

$$1 = \beta_1 \quad \to \beta_1 = 1$$

$$-2 = (1 + 2\beta_2 + \beta_3)(-1) \quad \to 1 = \beta_2 + \beta_3 \quad \to \beta_2 = 3$$

$$-1 = (1 + 2\beta_2 + 4\beta_3) \quad \to -1 = 1 + 2\beta_2 + 4\beta_3 \quad \to \beta_3 = -2$$

$$a_n = 1 + 3n - 2n^2(-1)^n$$

8.3   Nonhomogeneous Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

With $F(n)$ a function of n

Theorem: The general solution of the nonhomogeneous case is:

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Where $a_n^{(h)}$ is the general solution of the homogeneous case, with $F(n) = 0$, and $a_n^{(p)}$ is a particular solution of the nonhomogeneous case.

Example:

$$a_n = 3a_{n-1} + 2n, \quad a_1 = 3$$
$$\text{Homogeneous:} \quad r - 3 = 0 \quad \to r = 3$$
$$\text{General solution:} \quad a_n^{(h)} = \beta 3^n$$

$$\text{Particular solution: assume } p_n = cn + d$$
$$cn + d = 3(c(n-1) + d) + 2n$$
$$cn + d = 3cn - 3c + 3d + 2n$$
$$c = 3c + 2 \implies c = -1$$
$$d = 3c + 3d \implies d = \frac{-3}{2}$$

$$\text{Particular solution:} \quad p_n = -n - \frac{3}{2}$$
$$\text{General solution:} \quad a_n = \beta 3^n - n - \frac{3}{2}$$

$$a_1 = 3 = \beta 3 - 1 - \frac{3}{2}$$
$$3\beta = \frac{11}{2} \implies \beta = \frac{11}{6}$$

$$a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$$

Example:

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$
$$r^2 - 5r + 6 = 0 \quad \to r = 2, 3$$

$$a_n^{(h)} = \beta_1 3^n + \beta_2 2^n$$

$$p_n = c7^n = 5(c7^{n-1}) - 6(c7^{n-2}) + 7^n$$
$$49c = 35c - 6c + 49 \quad \to 20c = 49 \quad \to c = \frac{49}{20}$$

$$a_n = \beta_1 3^n + \beta_2 2^n + \frac{49}{20} 7^n$$

$$a_0 = 1, \quad a_1 = 2$$

$$1 = \beta_1 + \beta_2 + \frac{49}{20} \quad \to \beta_1 + \beta_2 = -\frac{29}{20}$$

$$2 = 3\beta_1 + 2\beta_2 + \frac{49}{20} \quad \to 3\beta_1 + 2\beta_2 = -\frac{11}{20}$$

$$\beta_1 = -\frac{9}{20}, \quad \beta_2 = -\frac{1}{4}$$

$$a_n = -\frac{9}{20} 3^n - \frac{1}{4} 2^n + \frac{49}{20} 7^n$$

8.4 Theorem: Particular solutions

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$ be a linear recurrence relation with constant coefficients and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$. Then, a particular solution if s is not a root is:

$$p_n = (d_t n^t + d_{t-1} n^{t-1} + \cdots + d_1 n + d_0) s^n$$

If s is a root of the characteristic equation with multiplicity m, then the particular solution is:

$$p_n = (d_t n^t + d_{t-1} n^{t-1} + \cdots + d_1 n + d_0) s^n n^m$$

Example:

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

$$F(n) = 3^n, \quad F(n) = n3^n, \quad F(n) = n^2 2^n, \quad F(n) = (1+n)2^n$$

$$r^2 - 6r + 9 = 0 \quad \to r = 3 \text{ with multiplicity } 2$$

For $F(n) = 3^n$:

$$p_n = d_0 n^2 3^n$$

For $F(n) = n3^n$:

$$p_n = n^2 (d_1 n + d_0) \cdot 3^n$$

For $F(n) = n^2 2^n$:

$$p_n = (d_2 n^2 + d_1 n + d_0) 2^n$$

For $F(n) = (1+n)2^n$:

$$p_n = (d_1 n + d_0) 2^n$$

8.5 The $r = 1$ case

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) \cdot 1^n, \quad \text{with } r = 1 \text{ a root}$$

Example:

$$a_n = \sum_{k=1}^{n} k$$

$$a_n = a_{n-1} + n, \quad a_1 = 1$$

$$r - 1 = 0 \quad \rightarrow r = 1, \quad x_h = c1^n = c$$

$$x_p = (p_1 n + p_0) \cdot n = p_1 n^2 + p_0 n$$

$$p_1 n^2 + p_0 n = p_1 (n-1)^2 + p_0 (n-1) + n$$

$$p_1 n^2 + p_0 n = p_1 n^2 - 2p_1 n + p_1 + p_0 n - p_0 + n$$

$$0n + 0 = (1 - 2p_1)n + (p_1 - p_0)$$

$$p_1 = \frac{1}{2}, \quad p_0 = \frac{1}{2}$$

$$\implies a_n = c + \frac{n^2}{2} + \frac{n}{2}, \quad a_1 = 1 \implies c = 0$$

$$a_n = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

9 Generating Functions

The generating function for a sequence $\{a_0, a_1, a_2, \dots\}$ of real numbers is the infinite formal series:

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots a_k x^k = \sum_{k=0}^{\infty} a_k x^k$$

Example:

$$a_k = k + 1 \quad \rightarrow \quad G(x) = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{k=0}^{\infty} (k+1)x^k$$

$$a_k = 2^k \quad \rightarrow \quad G(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} 2^k x^k$$

9.1 Finite sequence

$$\{a_0, a_1, a_2, \dots, a_k\} \quad \rightarrow \quad G(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$$

Which is like $a_i = 0$ for $i \geq k + 1$.

Example:

$$\{1, 1, 1, 1, 1, 1\} \quad \rightarrow \quad G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{x^6 - 1}{x - 1}$$

Example:

$$a_k = \binom{n}{k} \quad \rightarrow \quad G(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

9.2 Frequent and important generating functions

1.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k, \text{converges for } |x| < 1$$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots = \sum_{k=0}^{\infty} a^k x^k, \text{converges for } |x| < \frac{1}{|a|}$$

2.

$$\begin{cases} G_1(x) = \sum_{k=0}^{\infty} a_k x^k \\ G_2(x) = \sum_{k=0}^{\infty} b_k x^k \end{cases} \implies \begin{cases} G_1(x) + G_2(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \\ G_1(x) \cdot G_2(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k \end{cases}$$

3.

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{1-x} \cdot \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \cdot \sum_{j=0}^{\infty} x^j = \sum_{k=0}^{\infty} (k+1) x^k$$

4.

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} x^k \right) = \sum_{k=0}^{\infty} k x^{k-1} = \sum_{k=0}^{\infty} (k+1) x^k$$

5.

Definition: $\binom{u}{k} = \begin{cases} \dfrac{u(u-1)\ldots(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$ for $u \in \mathbb{R}$ and $k \in \mathbb{N}$

If $u = -3$ and $k = 2$ then $\binom{-3}{2} = \dfrac{-3 \cdot -4}{2} = 6$

If $u = \dfrac{1}{2}$ and $k = 3$ then $\binom{\frac{1}{2}}{3} = \dfrac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}}{6} = \dfrac{1}{16}$

Special case: u is a negative integer $u = -n, \quad n \in \mathbb{N}$

$$\binom{-n}{r} = \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!} = \frac{(-1)^r n(n+1)\ldots(n+r-1)}{r!} =$$

$$= (-1)^r \binom{n+k-1}{k} = (-1)^r CR_n^r$$

Extended Binomial Theorem: $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$

Special case: $u = -n$ and $n \in \mathbb{N}$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

9.3 Applications of generating functions

Example:

Find the number of integer solutions of $x_1 + x_2 + x_3 = 17$ when $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, and $3 \leq x_3 \leq 7$.

$$G(x) = (x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)$$

We look for the coefficient of x^{17} in $G(x)$.

$$x^5 \cdot x^5 \cdot x^7 + x^5 \cdot x^6 \cdot x^6 + x^4 \cdot x^6 \cdot x^7 \rightarrow 3 \text{ solutions}$$

Example:

In how many ways can we distribute 8 identical cookies among 3 children so that each child recieves at least 2 and and no more than 4 cookies?

$$G(x) = (x^2 + x^3 + x^4)^3 = x^6(1 + x + x^2)^3 = x^6(1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6)$$

$$\text{Coefficient of } x^8 = 6 \text{ ways}$$

Example:

In how many ways can we gather r euros using coins of 1 and 2 and notes of 5?

$$G(x) = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^5 + x^{10} + \ldots)$$

For $r = 7$:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)(1 + x^2 + x^4 + x^6)(1 + x^5) =$$

$$x^7 \cdot 1 \cdot 1 + x \cdot x^6 \cdot 1 + x^5 \cdot x^2 \cdot 1 + x^3 \cdot x^4 \cdot 1 + x^2 \cdot x^5 \cdot 1 + x^4 \cdot x^3 \cdot 1 + x^6 \cdot x^2 \cdot 1 = 6 \text{ ways}$$

Same, but now order matters:
To insert n coins or notes adding up to r euros:

$$(x + x^2 + x^5)^n, \quad \text{coefficient of } x^r$$

$$G(x) = 1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots = \frac{1}{1 - (x + x^2 + x^5)}$$

Example:

Suppose we have 3 green balls, 3 blue balls, 2 red balls and 1 white ball. How many ways are there to choose four of them?

$$x_g + x_b + x_r + x_w = 4$$

$$G(x) = (1 + x + x^2 + x^3)(1 + x + x^2 + x^3)(1 + x + x^2)(1 + x) = (1 + x + x^2 + x^3)^2(1 + x + x^2)(1 + x)$$

$$G(x) = \frac{1 - x^4}{1 - x} \cdot \frac{1 - x^4}{1 - x} \cdot \frac{1 - x^3}{1 - x} \cdot \frac{1 - x^2}{1 - x} = \frac{(1 - x^4)^2(1 - x^3)(1 - x^2)}{(1 - x)^4} =$$

$$= (1 - 2x^4)(1 - x^3)(1 - x^2) = (1 - x^2 - x^3 - 2x^4 + \ldots)(1 + 4x + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \binom{7}{4}x^4 + \ldots) =$$

$$\binom{7}{4} - \binom{5}{2} - 4 - 2 = 19$$

$$\frac{1}{(1 - x)^4} = \sum_{n=0}^{\infty} \binom{n + 3}{3} x^n = \binom{7}{4}$$

Example:

18 candies in 5 boxes with none empty, an even number of candies in each box, with no more than 10 candies in thee first two and 2 to 6 candies in the last three.

$$G(x) = (x^2 + x^4 + x^6 + x^8 + x^{10})^2 (x^2 + x^4 + x^6)^3 =$$

$$= x^4(1+x^2+x^4+x^6+x^8)^2 x^6(1+x^2+x^4)^3 = x^{10}\frac{(1-x^{10})^2}{(1-x^2)^2}\frac{(1-x^6)^3}{(1-x^2)^3} = x^{10}\frac{(1-x^{10})^2(1-x^6)^3}{(1-x^2)^5}$$

$$\text{Coefficient of } x^8: \quad (1-x^{10})^2(1-x^6)^3 \to (1-3x^6+3x^{12}-x^{18}) \to (1-3x^6)$$

$$\frac{1}{(1-x^2)^5} = \sum_{n=0}^{\infty}\binom{n+4}{n}x^{2n}$$

$$(1-3x^6)\left(1+5x^2+\binom{6}{2}x^4+\binom{7}{3}x^6+\binom{8}{4}x^8+\dots\right) \to -15+\binom{8}{4} = 55$$

9.4 Recurrence relations and generating functions

Example:

$$a_k = 3a_{k-1}, \quad a_0 = 2$$

$$r = 3 \to a_k = c3^k \to a_k = 2 \cdot 3^k$$

Now with:

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k =$$

$$= 2 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k = 2, \quad \text{since } a_k = 3a_{k-1}$$

$$G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty} 3^k x^k \to a_k = 2 \cdot 3^k$$

Example:

$$a_n = 8a_{n-1} + 10^{n-1}, \quad a_1 = 9, \quad n \geq 0$$

$$G(x) = \sum_{n=1}^{\infty} a_n x^n$$

Introduction of a_0: $a_1 = 8a_0 + 10^0 = 9 \to a_0 = 1$

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$G(x) - a_0 = 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} =$$

$$= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n = 8xG(x) + \frac{x}{1-10x}$$

$$G(x) - a_0 = 8xG(x) + \frac{x}{1 - 10x} = \frac{1 - 9x}{1 - 10x}$$

$$G(x) = \frac{1 - 9x}{(1 - 10x)(1 - 8x)}$$

$$G(x) = \frac{A}{1 - 8x} + \frac{B}{1 - 10x} = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right) =$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \rightarrow a_n = \frac{1}{2}(8^n + 10^n)$$

Where $A = \frac{1}{2}$ and $B = \frac{1}{2}$, since

$$1 - 9x = A(1 - 10x) + B(1 - 8x) \rightarrow 1 = A + B, \quad -9 = -10A - 8B$$

$$9 = 10A + 8 - 8A \rightarrow 1 = 2A \rightarrow A = \frac{1}{2}, \quad B = \frac{1}{2}$$

$$\textbf{Homogeneous: } r = 8 \rightarrow a_n^h = c8^n$$

$$a_p = D \cdot 10^n \rightarrow 10D = 8D + 1 \rightarrow D = \frac{1}{2}$$

$$a_n = c8^n + \frac{1}{2} \cdot 10^n$$

$$a_1 = 9 = c8 + 5 \rightarrow c = \frac{1}{2}$$

Example:

$$a_n = 3a_{n-1} + n, \quad a_0 = 1, \quad n \geq 1$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n x^n = 3a_{n-1}x^n + nx^n$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1}x^n + \sum_{n=1}^{\infty} nx^n$$

$$G(x) - a_0 = 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} nx^{n-1} = 3x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} nx^n$$

$$G(x) - 1 = 3xG(x) + x\sum_{n=0}^{\infty} nx^n = 3xG(x) + x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = 3xG(x) + x\frac{d}{dx}\left(\frac{1}{1 - x}\right)$$

$$G(x) - 1 = 3xG(x) + x\frac{1}{(1 - x)^2} = 3xG(x) + \frac{x}{(1 - x)^2}$$

$$G(x) = \frac{1}{1 - 3x} + \frac{x}{(1 - x)^2(1 - 3x)} = \frac{A}{1 - x} + \frac{B}{(1 - x)^2} + \frac{C}{1 - 3x}$$

$$x = A(1 - x)(1 - 3x) + B(1 - 3x) + C(1 - x)^2$$

$$\begin{cases} 1 = B(1 - 3) \rightarrow B = -\frac{1}{2} \\ \frac{1}{3} = C\frac{4}{9} \rightarrow C = \frac{3}{4} \\ 0 = A + B + C \rightarrow A = -\frac{1}{4} \end{cases}$$

$$G(x) = \frac{7}{4}\frac{1}{1-3x} - \frac{1}{2}\frac{1}{(1-x)^2} - \frac{1}{4}\frac{1}{1-x}$$

$$G(x) = \sum_{n=0}^{\infty} \left(\frac{7}{4}3^n - \frac{1}{2}(n+1) - \frac{1}{4} \right)$$

Example:

$$a_{n+2} - 5a_{n+1} + 6a_n = 2, \quad x_0 = 3, \quad x_1 = 7, \quad n \geq 0$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = 3 + 7x + \sum_{n=2}^{\infty} a_n x^n$$

Multiply by x^{n+2}:

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$$

Sum over all allowed n:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}$$

Match the power of x with the index of a_n extracting as many factors x as needed:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6x^2\sum_{n=0}^{\infty} a_n x^n = 2x^2\sum_{n=0}^{\infty} x^n$$

Shift the index of the sums:

$$\sum_{n=2}^{\infty} a_n x^n - 5x\sum_{n=1}^{\infty} a_n x^n + 6x^2\sum_{n=0}^{\infty} a_n x^n = 2x^2\sum_{n=0}^{\infty} x^n$$

Identify functions (use initial conditions as needed):

$$G(x) - 3 - 7x = -5x(G(x) - 3) + 6x^2 G(x) = \frac{2x^2}{1-x}$$

Solve for $G(x)$:

$$G(x)(1 - 5x + 6x^2) = 3 + 7x - 15x + \frac{2x^2}{1-x}$$

$$G(x) = \frac{3 - 8x}{1 - 5x + 6x^2} + \frac{2x^2}{(1-x)(1 - 5x + 6x^2)}$$

$$G(x) = \frac{3 - 8x}{(1-2x)(1-3x)} + \frac{2x^2}{(1-x)(1-2x)(1-3x)}$$

$$G(x) = \frac{2}{1-3x} + \frac{1}{1-x} = 2\sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} x^n$$

$$a_n = 2 \cdot 3^n + 1$$

Example:

A person has 6 friends: $\{A, B, C, D, E, F\}$. He wants to invite 4 of them home, but A and B cannot come together, and neihter can E and F, because of personal conflicts. If the invitation is made at random, what is the possibiliy that the dinner goes well?

$$P = \frac{\text{number of good outcomes}}{\text{total number of groups of 4 friends}}$$

$$\text{Total number of groups of 4 friends: } \binom{6}{4} = 15$$

$$S_{AB} = \text{number of groups with A and B together} = \binom{4}{2} = 6$$

$$S_{EF} = \text{number of groups with E and F together} = \binom{4}{2} = 6$$

$$S_{ABEF} = \text{number of groups with A and B and E and F together} = \binom{2}{2} = 1$$

$$\text{Number of good outcomes: } 15 - 6 - 6 + 1 = 4$$

$$P = \frac{4}{15}$$

Example:

In how many ways can we order the letters A, E, M, O, Y, U in 6-letter strings without letter repetition that do not contain the word "ME" or "YOU"?

$$\text{Total number of strings: } 6! = 720$$

$$S_{ME} = \text{number of strings with "ME"} = 5! = 120$$

$$S_{YOU} = \text{number of strings with "YOU"} = 4! = 24$$

$$S_{MEYOU} = \text{number of strings with "ME" and "YOU"} = 3! = 6$$

$$\text{Number of good strings: } 720 - 120 - 24 + 6 = 582$$

Example:

A message is made of 12 symbols in a specified order. To transmit it, you have to send 45 blanks along with the symbols, and every two symbols must be separated by at least 3 blanks. How many different messages can be sent?

$$CR_{11}^{12} = \binom{11 + 12 - 1}{12} = \binom{22}{12} = 646646$$

Example:

How many subsets of 4 elements can be formed from elements of $\{1, 2, 3, \ldots, 13, 14, 15\}$ such that no subset contains two consecutive numbers?

$$\text{Subsets look like: } \{n_1, n_2, n_3, n_4\} \quad 1 \leq n_i \leq 15$$

$$\text{For non-consecutive numbers: } \{n_1, n_2, n_3, n_4\} \quad n_2 - n_1 \geq 2, \quad n_3 - n_2 \geq 2, \quad n_4 - n_3 \geq 2$$

$$\begin{cases} x_1 = n_1 \geq 1 \\ x_2 = n_2 - n_1 \geq 2 \\ x_3 = n_3 - n_2 \geq 2 \\ x_4 = n_4 - n_3 \geq 2 \end{cases} \implies n_4 = x_4 + x_3 + x_2 + x_1 \leq 15$$

$$\begin{cases} y_1 = x_1 - 1 \geq 0 \\ y_2 = x_2 - 2 \geq 0 \\ y_3 = x_3 - 2 \geq 0 \\ y_4 = x_4 - 2 \geq 0 \end{cases} \implies y_4 + y_3 + y_2 + y_1 \leq 8$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 8$$

$$CR_5^8 = \binom{8 + 5 - 1}{8} = \binom{12}{8} = 495$$

Example:

How many ways are there to choose 25 objects from 7 categories so that there are no more than 6 and no fewer than 2 objects in each category?

$$G(x) = (x^2 + x^3 + x^4 + x^5 + x^6)^7 = x^{14}(1 + x + x^2 + x^3 + x^4)^7 = x^{14}\left(\frac{1 - x^5}{1 - x}\right)^7$$

We need the coefficient of x^{11}:

$$(1 - x^5)^7 = 1 - 7x^5 + 21x^{10} - \ldots$$

$$\left(\frac{1}{1-x}\right)^7 = \sum_{n=0}^{\infty}\binom{n+6}{n}x^n = \binom{6}{0} + \binom{7}{1}x + \cdots + \binom{12}{6}x^6 + \cdots + \binom{17}{11}x^{11} + \ldots$$

So the coefficient of x^{11} is:

$$\binom{17}{11} - 7\binom{12}{6} + 21\binom{7}{1} = 6055$$

Example:

$$a_{n+2} - 2a_{n+1} + a_n = 2^n, \quad a_0 = 1, \quad a_1 = 2$$

$$\text{Homogeneous: } r^2 - 2r + 1 = 0 \to r = 1 \to a_n^h = c_1 + c_2 n$$

$$a_n = c_1 + c_2 n + a_n^p$$

$$\text{Particular: } a_{n+2} - 2a_{n+1} + a_n = 2^n \to a_n^p = 2^n$$

$$a_n = c_1 + c_2 n + 2^n$$

$$a_0 = 1 = c_1 + 2^0 \to c_1 = 0$$

$$a_1 = 2 = c_2 + 2^1 \to c_2 = 0$$

$$a_n = 2^n$$

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Example:

Given the recurrence relation:

$$a_{n+2} - 2a_{n+1} + a_n = 2^n, \quad a_0 = 1, \quad a_1 = 2$$

Multiplying both sides by x^n and summing over all n:

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

Defining the generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

We rewrite the summations:

$$\sum_{n=0}^{\infty} a_{n+2}x^n = G(x) - a_0 - a_1 x$$

$$\sum_{n=0}^{\infty} a_{n+1}x^n = x\sum_{n=1}^{\infty} a_n x^{n-1} = x(G(x) - a_0)$$

Using these, we derive:

$$G(x) - 1 - 2x - 2x(G(x) - 1) + x^2 G(x) = x^2 \sum_{n=0}^{\infty} 2^n x^n = \frac{x^2}{1 - 2x}$$

$$G(x)\left[1 - 2x + x^2\right] - 1 = \frac{x^2}{1 - 2x}$$

$$G(x) = \frac{1}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2} = \frac{1}{1-2x}$$

Since:

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$

we conclude:

$$a_n = 2^n$$

Example:

1. How many words can be formed with at most 5 letters of WATER?

2. How many of them do not contain W?

3. How many of them contain neither W nor E?

4. How many of them contain both W and E?

For (i)

$$P_5 + V(5,4) + V(5,3) + V(5,2) + V(5,1)$$

$$\frac{5!}{0!} + \frac{5!}{4!} + \frac{5!}{2!} + \frac{5!}{3!} + \frac{5!}{4!}$$

For (ii):

$$V(4,1) + V(4,2) + V(4,3) + P_4$$

For (iii):
$$V_3^1 + V_3^2 + P_3$$

For (iv):
$$P_2 + 3P_3 + \binom{3}{2}P_4 + P_5$$

Example:

A soccer team has 3 Goalkeepers, 8 Defenders, 6 Midfielders, and 4 Forwards. The coach wants to pick up a team with 1 Goalkeeper, 5 Defenders, 3 Midfielders, and 2 Forwards. However, he does not want players MA and FB to play together.

Total Teams:
$$\binom{3}{1}\binom{8}{5}\binom{6}{3}\binom{4}{2}$$

Forbidden Teams:
$$\binom{3}{1}\binom{8}{5}\binom{5}{2}\binom{3}{1}$$

Also Consider:
$$\binom{5}{2}\binom{3}{2} \quad \text{(with A, not B)}$$

$$\binom{5}{3}\binom{3}{1} \quad \text{(with B, not A)}$$

$$\binom{5}{3}\binom{3}{2} \quad \text{(neither)}$$

Example:

Show that in any set of 9 natural numbers, none of which has a prime factor larger than 5, there must be two whose product is a perfect square.
We express a number n in terms of its prime factorization:

$$n = 2^x 3^y 5^z$$

Thus, allowed numbers can be characterized by the tuple $(x, y, z) \mod 2$.
If we have two numbers:
$$n = 2^{x_1} 3^{y_1} 5^{z_1}, \quad m = 2^{x_2} 3^{y_2} 5^{z_2}$$

Then their product:
$$n \cdot m = 2^{x_1+x_2} 3^{y_1+y_2} 5^{z_1+z_2}$$

must be a perfect square, meaning $x_1 + x_2$, $y_1 + y_2$, and $z_1 + z_2$ must be even.

$$\begin{cases} x_1 + x_2 \equiv 0 \mod 2 \\ y_1 + y_2 \equiv 0 \mod 2 \\ z_1 + z_2 \equiv 0 \mod 2 \end{cases}$$

Since x, y, z can each take values in $\{0, 1\}$, there are $2^3 = 8$ possible different tuples (x, y, z) mod 2. By the pigeonhole principle, in any set of 9 numbers, at least two must share the same tuple, ensuring their product is a perfect square.

Example:

We want to make perfumes and we have 5 ingredients:
Vanilla, Lavender, Orange, Wood, Grass.

1. How many of them do not contain Vanilla?

2. How many do not contain Vanilla or Wood?

3. How many of them contain both Vanilla and Wood?

4. How many perfumes can we make with up to 5 essences?

The number of perfumes with up to 5 ingredients is given by 1 less than the solutions of

$$n_v + n_l + n_o + n_w + n_g + v \le 5$$

$$n_v + n_l + n_o + n_w + n_g + v = 5$$

$$CR(6,5) - 1 = 251$$

(i)
$$1 \le n_l + n_o + n_w + n_g \le 5$$

$$CR(5,5) - 1 = 125$$

(ii)
$$1 \le n_l + n_o + n_g \le 5$$

$$CR(4,5) - 1 = 55$$

(iii)
$$251 - 125 - 125 + 55 = 56$$

Example:

How many ways are there to order w white balls and b black balls in a row such that there are exactly $k + 1$ groups of black balls?
Using combinatorial grouping:

$$CR_{k+1}^{b-(k+1)} = \binom{b-1}{b-k-1}$$

White balls separate groups, so:

$$\binom{w+1}{w-k}$$

10 Correspondences

Correspondences are relationships between elements of two sets A and B.
For example, let:

$$A := \{\text{cities of the world}\}, \quad B := \{\text{countries of the world}\}$$

$$x \in A, \quad y \in B, \quad xRy \iff x \text{ is the capital of } y$$

$H(x, y)$ means that x is something with respect to y.

$$\textbf{Defining set:} \quad \{x \in U \mid H(x, y)\}$$

10.1   Definition

Given two sets A and B, a correspondence of relation between A and B is a subset R of $A \times B$, defined as:
$$R := \{(a, b) \in A \times B \mid H(a, b)\}$$

Example:
$$A := \{1, 2, 3, 4, 5, 6\}, \quad R := \{(m, n) \in A \times A \mid m \text{ divides } n\}$$
$$(1, 3) \in R, \quad (2, 4) \in R, \quad (3, 5) \notin R$$

Example:
$$f(x) : \mathbb{R} \to \mathbb{R}^+, \quad x \to x^2$$
$$f := \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid y = x^2\}$$

10.2   Representation of correspondences

1. **Arrow diagram:** R is represented by arrows from elements of A to elements of B.
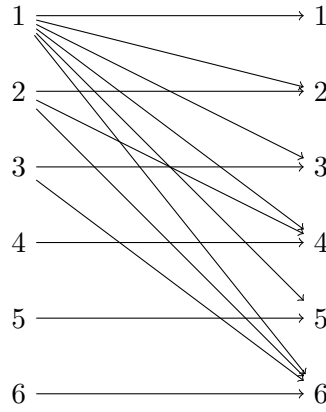
Figure 3: Arrow diagram for the relation R where m divides n.

2. **Binary matrix:** R is represented by a matrix M of dimensions $n \times n$ where $n = |A| = |B|$.

3. **Directed multigraph:** R is represented by a directed multigraph where the vertices are elements of A and B.

10.3   Domain and range of correspondences

Let R be a correspondence between A and B, then:

1. The **domain** of R is the set of all elements of A that are related to some element of B.
$$dom(R) = \{a \in A \mid (a, b) \in R \text{ for some } b \in B\} \subseteq A$$

2. The **range** of R is the set of all elements of B that are related to some element of A.
$$ran(R) = \{b \in B \mid (a, b) \in R \text{ for some } a \in A\} \subseteq B$$

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	0	1	0	1
3	0	0	1	0	0	1
4	0	0	0	1	0	1
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Figure 4: Binary matrix representation of the relation R where m divides n.
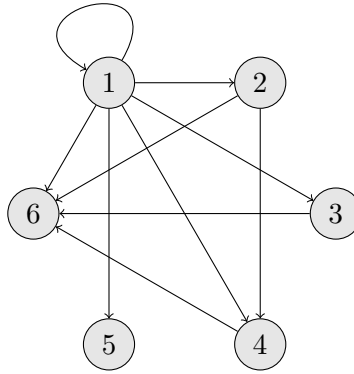
Figure 5: Directed multigraph for the relation R where m divides n.

Example:

$$A := \{1, 2, 3, 4, 5, 6\}, \quad mRn \iff 2 \text{ divides } (m - n)$$
$$m - n = 2k, \quad k \in \mathbb{Z} \to m = n(mod2)$$

$$(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5),$$
$$(4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)$$

10.4   Special relations

1. **Identity relation:** $I \subseteq A \times A$

$$x, y \in A, \quad xIy \iff x = y$$

2. **Universal correspondence:** $U = A \times B$

3. **Inverse correspondence:** If there exists $R \subseteq A \times B$, then: $R^{-1} \subseteq B \times A$

$$R := \{(b, a) \in B \times A \mid (a, b) \in R\}$$

10.5 Properties of correspondences

1. **Reflexivity:** $R \subseteq A \times A$ is reflexive if and only if $(a, a) \in R$ for all $a \in A$.

2. **Symmetry:** $R \subseteq A \times A$ is symmetric if and only if $(a, b) \in R \Leftrightarrow (b, a) \in R$.

3. **Antisymmetry:** $R \subseteq A \times A$ is antisymmetric if and only if whenever
$(a, b) \in R$ and $(b, a) \in R \implies a = b$.

4. **Transitivity:** $R \subseteq A \times A$ is transitive if and only if whenever
$(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$.

Example:

$$A := \{a, b, c, d\}, \quad R := \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (b, d), (d, d)\}$$

1. Reflexive: $(c, c) \notin R \implies R$ is not reflexive.

2. Symmetric: $(a, c) \in R$, but $(c, a) \notin R \implies R$ is not symmetric.

3. Antisymmetric: $(a, b) \in R, (b, a) \in R$, but $a \neq b \implies R$ is not antisymmetric.

4. Transitive: $(a, b) \in R, (b, d) \in R$, but $(a, d) \notin R \implies R$ is not transitive.

Example:

$$A := \{1, 2, 3, 4, 5, 6\}, \quad R := \{(m, n) \mid m \text{ divides } n\}$$

1. Reflexive: $\forall m \in A \implies m$ divides $m \implies R$ is reflexive.

2. Symmetric: m divides $n \implies n$ does not divide $m \implies R$ is not symmetric.

3. Antisymmetric: m divides n, n divides $m \implies m = n \implies R$ is antisymmetric.

4. Transitive: m divides n, n divides $p \implies m$ divides $p \implies R$ is transitive.

Example:

$$A = \mathbb{N} \times \mathbb{N} = \{(a, b) \mid a, b \in \mathbb{N}\}, \quad R = \{((a, b), (c, d)) \mid a + d = b + c\}$$

1. Reflexive: $(a, b) + (b, a) = 2b = 2a \implies R$ is reflexive.

2. Symmetric: $(a, b) + (c, d) = (c, d) + (a, b) \implies R$ is symmetric.

3. Antisymmetric: $(a, b) + (c, d) = (c, d) + (a, b) \nRightarrow (a, b) = (c, d) \implies R$ is not antisymmetric.

4. Transitive: $(a, b) + (c, d) = (c, d) + (e, f) \implies R$ is transitive.

10.6 Operations with correspondences

1. **Union:** $R \cup S = \{(a, b) \in A \times A \mid (a, b) \in R \text{ or } (a, b) \in S\}$

2. **Intersection:** $R \cap S = \{(a, b) \in A \times A \mid (a, b) \in R \text{ and } (a, b) \in S\}$

Both Union and Intersection preserve reflexivity and symmetry:

1. If R and S are reflexive, then

$$\forall a \in A \quad (a, a) \in R \text{ and } (a, a) \in S \implies (a, a) \in R \cup S \text{ and } (a, a) \in R \cap S$$

2. R and S are symmetric

$$\text{If } (a, b) \in R \implies (b, a) \in R \text{ and } (c, d) \in S \implies (d, c) \in S$$

However, only Intersection preserves antisymmetry and transitivity.

10.6.1 Counterexamples for the union

1. **Antisymmetry:** $R = \{(a,b)\}$ and $S = \{(b,a)\} \implies$ antisymmetric, but $R \cup S = \{(a,b), (b,a)\}$ is not antisymmetric

2. **Transitivity:** $R = \{(a,b)\}$ and $S = \{(b,c)\} \implies$ transitive, but $R \cup S = \{(a,b), (b,c)\}$ is not transitive

10.7 Types of relations

1. **Equivalence relation:** A set $R \subseteq A \times A$ is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

2. **Order relation:** A set $R \subseteq A \times A$ is an order relation if and only if it is reflexive, antisymmetric, and transitive.

10.7.1 Equivalence relations

Example:

$$A := \{\text{people living in Spain}\}, \quad R := \{(x,y) \mid x \text{ and } y \text{ live in the same municipality}\}$$

1. Reflexive: x lives in the same municipality as x.

2. Symmetric: If x lives in the same municipality as y, then y lives in the same municipality

3. Transitive: If x lives in the same municipality as y and y lives in the same municipality as z, then x lives in the same municipality as z.

Example:

$$A := \mathbb{R}^2 - \{(0,0)\}, \quad R := \{(x,y) \mid x \text{ and } y \text{ are on the same line through the origin}\}$$

1. Reflexive: $(0,0)$ is on the same line through the origin as $(0,0)$.

2. Symmetric: If x is on the same line through the origin as y, then y is on the same line through the origin as x.

3. Transitive: If x is on the same line through the origin as y and y is on the same line through the origin as z, then x is on the same line through the origin as z.

10.7.2 Equivalence class

Let $A, \quad R \subset A \times A$ be an equivalence relation, then the class of equivalence of $x \in A$ is defined as

$$[x] = \{y \in A \mid (x,y) \in R\}$$

The set of classes of equivalence of R is a covering of A

$$A = \bigcup_{x \in A} [x], \quad \text{remember } x \in [x] \text{ because of reflexivity and also } xRx \quad \forall x \in A$$

Furthermore, $\bigcup_{x \in A} [x]$ is a partition of A

Theorem: Let A be a set, R an equivalence,

$$xRy \iff [x] = [y]$$

Proof: $[x] = [y]$
We need to prove:
$$\begin{cases} [x] \subseteq [y] \\ [y] \subseteq [x] \end{cases}$$

$$\text{where } [x] = \{z \in A \mid xRz\}, \quad [y] = \{z \in A \mid yRz\}$$

$[x] \subseteq [y]$:
Let $z \in [x]$, then $xRz \implies xRy \implies yRz \implies z \in [y]$.
Let $z \in [y]$, then $yRz \implies yRx \implies xRz \implies z \in [x]$.
The classes of equivalence form a partition of A.

$$\forall x \in A, \quad x \in [x]$$

If $[x]$ and $[y]$ have a common element, $z \in [x] \cap [y]$, then $[x] = [y]$.

10.8   Quotient set

Let A be a set, R an equivalence relation on A, then the quotient set of A by R is defined as:

$$A/R = \{[x] \mid x \in A\}$$

which means that A/R is the set of all classes of equivalence of A with respect to R.

Example:

$$A = \{a, b, c, d, e\}$$
$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}$$

$$A/R = \{\{a,b,c\},\{d\},\{e\}\}, \quad \text{where } [a] = [b] = [c]$$
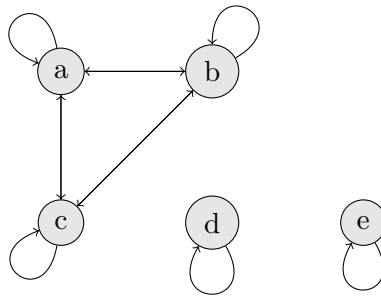
Figure 6: Directed multigraph for the relation R where a, b, and c are related to each other, and d, and e are related to themselves.

10.8.1   Example: Modular arithmetic

$$a, b \in \mathbb{Z}, \quad a \equiv b \mod n \iff a - b = kn, \quad k \in \mathbb{Z}, \quad n \in \mathbb{N}$$

$$aRb \iff a \equiv b \mod n \text{ is an equivalence relation}$$

$$\textbf{Reflexive: } a = a + 0n \implies a \equiv a \mod n \implies aRa$$

$$\textbf{Symmetric: } a \equiv b \mod n \implies a - b = kn$$

$$\implies b - a = (-k)n \implies b \equiv a \mod n \implies aRb \implies bRa$$

$$\textbf{Transitive: } \begin{cases} a - b = kn \\ b - c = ln \end{cases} \implies a - c = (k + l)n \implies a \equiv c \mod n \implies aRc$$

10.8.2 Equivalence classes: quotient set of $\mathbb{Z}$ by $n\mathbb{Z}$

$$[0] = \{x \in \mathbb{Z} \mid 0 - x = kn, \quad k \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid x = kn, \quad k \in \mathbb{Z}\}$$

For $n \geq 2$:

$$[1] = \{x \in \mathbb{Z} \mid 1 - x = kn, \quad k \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid 1 - x = kn, \quad k \in \mathbb{Z}\}$$

$$\vdots$$

$$[n] = \{x \in \mathbb{Z} \mid n - x = kn, \quad k \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid x = kn, \quad k \in \mathbb{Z}\}$$

$$\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$$
$$xy = (a + kn)(b + ln) = ab + (al + bk + kln)n$$

In modular arithmetic:

$$[a] + [b] = [a + b], \quad [a] \cdot [b] = [a \cdot b]$$

$$x \in [a], \quad y \in [b] \implies x = a + kn, \quad y = b + ln$$

$$x + y = a + b + (k + l)n \implies x + y \in [a + b]$$

10.9 Order relations

Let A be a set, $R \subseteq A \times A$ an order relation. If R is reflexive, antisymmetric, and transitive, then R is an order relation.

Example:

$$R, \quad xRy \iff x \leq y$$

1. Reflexive: $x \leq x$

2. Antisymmetric: $x \leq y, y \leq x \implies x = y$

3. Transitive: $x \leq y, y \leq z \implies x \leq z$

Example:

Let A be a set, $P(A)$ the set of all subsets of A.

$$B, C \in P(A), \quad B \subseteq C \iff B \leq C$$

1. Reflexive: $B \subseteq B$

2. Antisymmetric: $B \subseteq C, C \subseteq B \implies B = C$

3. Transitive: $B \subseteq C, C \subseteq D \implies B \subseteq D$

Example:

$$A = \{a, b, c\}, \quad P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$B, C \in P(A), \quad B \subseteq C \iff B \leq C$$

$$\{a\} \leq \{a, b\}, \quad \text{but } \{a\} \nleq \{b, c\}$$

10.10   Theorem:

Let A be a set, with $\leq$ an order relation on A. Then:

$$B \subseteq A \implies B \text{ is also an ordered set}$$

1. **Reflexive:** $x \in B \implies x \leq x$

2. **Antisymmetric:** $x \leq y, y \leq x \implies x = y$

10.11   Notable elements in an order

Let A be a set, $\leq$ an order relation on A and $B \subseteq A$. Then:

1. $x \in A$ is an **upper bound** of B if:

$$\forall y \in B \implies y \leq x, \quad \text{(B is bounded from above)}$$

2. $x \in A$ is a **lower bound** of B if:

$$\forall y \in B \implies x \leq y, \quad \text{(B is bounded from below)}$$

Note: if B is bounded from above and below, then B is bounded.

3. $x \in A$ is the **supremum** of B if:

$$x \leq y, \quad \forall y \in B, \quad \text{where } y \text{ is any upper bound of } B$$

In case $x \in B$, then x is the maximum of B.

4. $x \in A$ is the **infimum** of B if:

$$y \leq x, \quad \forall y \in B, \quad \text{where } y \text{ is any lower bound of } B$$

In case $x \in B$, then x is the minimum of B.

5. $y \in B$ is a **maximal element** of B if:

$$\nexists x \in B \text{ such that } y \leq x$$

6. $y \in B$ is a **minimal element** of B if:

$$\nexists x \in B \text{ such that } y \geq x$$

Theorem: The supremum and infimum of a set are unique. Let γ, γ' be suprema of B, then:

$$\gamma \leq \gamma' \text{ and } \gamma' \leq \gamma \implies \gamma = \gamma'$$

Example:

$$A = \{a, b, c\}, \quad W = P(A) - A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

$$W \subseteq P(A), \quad \emptyset \subset w \in W \implies \emptyset \leq w \quad \forall w \in W$$

$$\implies \emptyset \text{ is a lower bound of } W, \quad \emptyset \text{ is the infimum of } W, \quad \emptyset \text{ is the minimum of } W$$

Also, A is an upper bound of W, A is the supremum of W, but A is not the maximum of W. The empty set is also a minimal element of W. However, there are three maximal elements in W: $\{a, b\}, \{a, c\}, \{b, c\}$. (Note that A contains the but $A \notin W$).

Example:

$$A = \mathbb{N}, m \leq n \text{ if } m \text{ divides } n, \quad B = \{2, 3, 4, 5, 6, 7, 8\}$$

$$\text{The minimals of } B \text{ are } 2, 3, 5, 7, \text{ and the maximals are } 5, 6, 7, 8$$

Observe that 5 and 7 are both maximal and minimal elements of B.

$$\text{The supremum of } B \text{ is the m.c.m. of all elements of } B = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840,$$

$$\text{and the infimum is the l.c.d. of all elements of } B = 1$$

10.12   Total order

Let A be a set, $\leq$ an order relation on A. $\leq$ is a total order if and only if:

$$\forall x, y \in A \quad x \leq y \text{ or } y \leq x$$

Examples:

The usual order of $\mathbb{N}$ is a total order.
The alphabetical order of words is a total order.

10.13   Types of order relations

1. **Well order:** A set is well oredered if $\forall a, b \in A$, $a \leq b$ or $b \leq a$.

2. **Good order:** A set A is good ordered if every non-empty subset of A has a minimum.

$\mathbb{N}$ is well ordered with respect to the usual order $\leq$. $\mathbb{R}$ and $\mathbb{Q}$ are not well ordered.

$$\text{Good order} \implies \text{Total order} \implies \text{Order}$$

Let A be a well ordered set, then every subset of A has a minimum. In particular $\forall a, b \in A$, $\{a, b\} \subseteq A$ has a minimum which is the minimum of a and b.

10.13.1   Theorem:

Let A be an ordered set under $\mathcal{R}$, and B an ordered set under $\mathcal{S}$. Then:

1. A is also an ordered set under $\mathcal{R}^{-1}$.

2. $(A \times B, \mathcal{R} \times \mathcal{S})$ is also an ordered set, where:

$$(a, b)\mathcal{R} \times \mathcal{S}(c, d) \iff \begin{cases} a\mathcal{R}c \\ b\mathcal{S}d \end{cases}$$

3. $(A \times B, L)$, where L is the lexicographic order, is also an ordered set.

$$(a, b)L(c, d) \iff \begin{cases} a\mathcal{R}c & \text{if } a = c \\ b\mathcal{S}d & \text{otherwise} \end{cases}$$

Examples:

$$\mathbb{N} \times \mathbb{N}, \quad \leq \text{ is the usual order}$$

$$(1,1) \leq (1,2), \quad (2,1)/(1,2)$$

$$\mathbb{N} \times \text{ dictionary}, \quad \leq \text{ is the alphabetical order}$$

$$(1,\text{fish}) \leq (3,\text{whale}), \quad (2,\text{shark})/(7,\text{bird})$$

3. **Topological order:** **Theorem:** Let A be a set, $\leq$ an order relation on A. Then there exists an order $\preceq$ that is total and is compatible with the order $\leq$.

$$\forall a,b \in A, \quad a \leq b \implies a \preceq b$$

Proof: Let $a_1 \in A$, a_1 is a minimal element of A with respect to $\leq$. Then, this will be the smallest element of A with respect to $\preceq$ (it is the minimum of A). Now we look for a minimal element of $A - \{a_1\}$ with respect to $\leq$, a_2 such that $a_2 \preceq a_1$. We continue this process until we have a total order of A.

$$a_1 \preceq a_2 \preceq a_3 \preceq \ldots \preceq a_n, \quad \text{where } n = |A|$$

10.14 Lattices

Let A be a set, $\leq$ an order relation on A. Then $(A, \leq)$ is a lattice if and only if both the supremum and infimum of any two elements of A exist and belong to A.

Examples:

Let $A = \{a, b, c\}$, under the usual order $\leq$ and consider $P(A)$ with an order given by inclusion $\subseteq$ (e.g. $\{a\} \subseteq \{a, b\}$). Then $(P(A), \subseteq)$ is a lattice.

$$s_1, s_2 \in P(A), \quad inf\{s_1, s_2\} = s_1 \cap s_2 \in P(A), \quad sup\{s_1, s_2\} = s_1 \cup s_2 \in P(A)$$

$\mathbb{N}$ with $n \leq m$ if n divides m is a lattice.

$$inf\{n, m\} = gcd(n, m) \in \mathbb{N}, \quad sup\{n, m\} = lcm(n, m) \in \mathbb{N}$$

10.15 Hasse diagram

Let A be a set, $\leq$ an order relation on A. For $a, b \in A$, if $a < b$, and $\nexists\, c \in A$ such that $a < c < b$, then we say that b covers a.

$$\text{Meaning of } a < b \quad \begin{cases} a \leq b \\ a \neq b \end{cases}$$

In a Hasse diagram:

1. The elements of A are represented by points.

2. If $a < b$, then a lies in a line below the line of b.

3. If b covers a, then there is a line connecting a and b.

Example:

$$A = \{1, 2, 3, 4, 5, 6\}, \quad \text{division order}$$
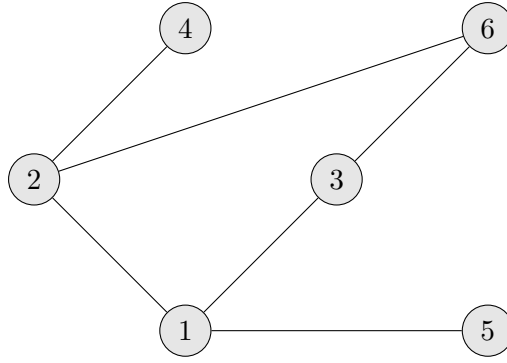
Figure 7: Hasse diagram for the set $A = \{1, 2, 3, 4, 5, 6\}$ with the division order.
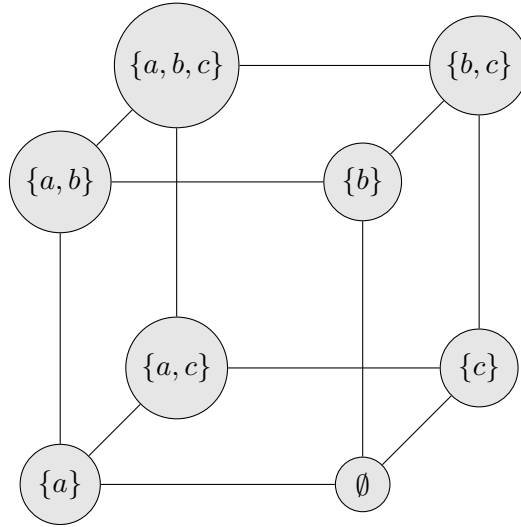
Figure 8: Hasse diagram for the set $P(A)$ with the inclusion order, where $A = \{a, b, c\}$.

Example:

$$P(A), \quad \subseteq \text{ order}, \quad A = \{a, b, c\}$$

10.16 Isomorphism of relations

Let A, B be sets, R_A, R_B relations on A and B, respectively. Then A and B are isomorphic if and only if there exists a bijective function $f : A \to B$ such that:

$$\text{If } a, a' \in A, \quad a R_A a' \iff f(a) R_B f(a')$$

Example:

$$B = \{x_1 x_2 x_3 \mid x_i \in \{0, 1\}\}$$

$$x_1 x_2 x_3 \leq y_1 y_2 y_3 \iff \begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 \\ x_3 \leq y_3 \end{cases}$$

$$100 \leq 101, \quad \leq 110/101$$

$$x_1 = 0 \text{ if } a \notin W \in P(A), \quad x_1 = 1 \text{ if } a \in W \in P(A)$$
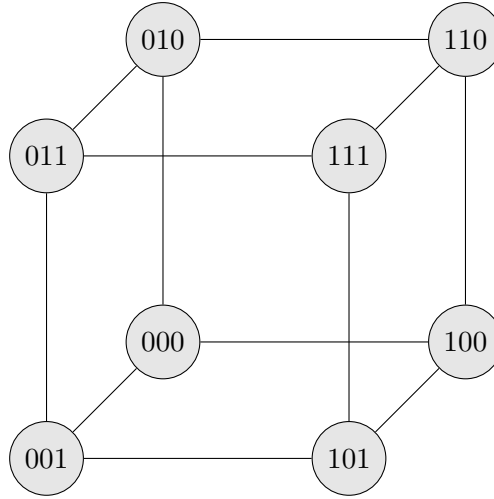
Figure 9: Hasse diagram for the set $B = \{x_1 x_2 x_3 \mid x_i \in \{0,1\}\}$ with the binary order.

10.17 Domain and range of a relation

Obtain the domain and range of the correspondence in $\mathbb{N}$ defined by $xRy \iff 2x + y = 16$.
A correspondence or relation is a subset of $A \times B$. The domain of a relation $R \subseteq A \times B$ is the set of all elements of A that are related to some element of B. The range of R is the set of all elements of B that are related to some element of A.

$$2x + y = 16 \implies y = 16 - 2x$$
$$x = 1 \implies y = 14 \implies (1, 14) \in R, \quad 1R14$$
$$x = 2 \implies y = 12 \implies (2, 12) \in R, \quad 2R12$$
$$x = 3 \implies y = 10 \implies (3, 10) \in R, \quad 3R10$$
$$x = 4 \implies y = 8 \implies (4, 8) \in R, \quad 4R8$$
$$x = 5 \implies y = 6 \implies (5, 6) \in R, \quad 5R6$$
$$x = 6 \implies y = 4 \implies (6, 4) \in R, \quad 6R4$$
$$x = 7 \implies y = 2 \implies (7, 2) \in R, \quad 7R2$$

The domain of R is $\{1, 2, 3, 4, 5, 6, 7\}$ and the range is $\{2, 4, 6, 8, 10, 12, 14\}$.

Example:

Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (a, b), (a, c), (b, b), (b, c)\}$. Add elements to R to make it:

1. **Reflexive:** Add $(c, c), (d, d)$

2. **Symmetric:** Add $(b, a), (c, a), (c, b)$

3. **Antisymmetric:** No need to add anything, as it is already antisymmetric (no $x \neq y$ such that xRy and yRx)

4. **Transitive:** $(a, b), (b, c) \implies (a, c)$, which is already in R

Example:

Consider the correspondence $R \subseteq \mathbb{N} \times \mathbb{N}$:

$$(m, n)R(p, q) \iff m + q = n + p$$

1. Proof that R is an equivalence relation.

2. Find the equivalence classes of $(1, 1)$, $(2, 1)$, $(3, 1)$, $(1, 2)$ and $(1, 3)$.

3. Find the quotient set $\mathbb{N} \times \mathbb{N}/R$ and try to identify it with a known set.

1. **Reflexive:** $m + n = n + m$

2. **Symmetric:** $m + q = n + p \implies n + p = m + q$

3. **Transitive:** $m + q = n + p, \quad n + p = r + s \implies m + q = r + s$

$$
\begin{aligned}
(1,1)R(n,n) &\implies [1,1] = \{(1,1),(2,2),(3,3),\ldots\} \\
(2,1)R(n+1,n) &\implies [2,1] = \{(2,1),(3,2),(4,3),\ldots\} \\
(3,1)R(n+2,n) &\implies [3,1] = \{(3,1),(4,2),(5,3),\ldots\} \\
(1,2)R(n-1,n) &\implies [1,2] = \{(1,2),(2,3),(3,4),\ldots\} \\
(1,3)R(n-2,) &\implies [1,3] = \{(1,3),(2,4),(3,5),\ldots\}
\end{aligned}
$$

$$\mathbb{N} \times \mathbb{N}/R = \{[n, n+k], \quad \forall k \in \mathbb{Z}, \quad k > (-n)\} = [k] = \mathbb{Z}$$

Example:

Let A be a set and let R be a relation given by:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

1. Prove that R is an order relation in A.

2. Find the domain and range of R.

3. Draw the Hasse diagram of R.

4. Find a topological order that contains R.

1. **Reflexive:** We have 1s in the diagonal, so it is reflexive.

2. **Antisymmetric:** For every 1 in the matrix, there is a 0 in the symmetric position and vice versa, so it is antisymmetric.

3. **Transitive:** By enumerating the elements, we can see that it is transitive.

$$\text{Domain: } \{a, b, c, d, e\}, \quad \text{Range: } \{a, b, c, d, e\}$$

$$\text{Minimals: } a \text{ and } e, \quad \text{Maximal: } d$$

$$\text{Topological order: } e \preceq a \preceq b \preceq c \preceq d$$
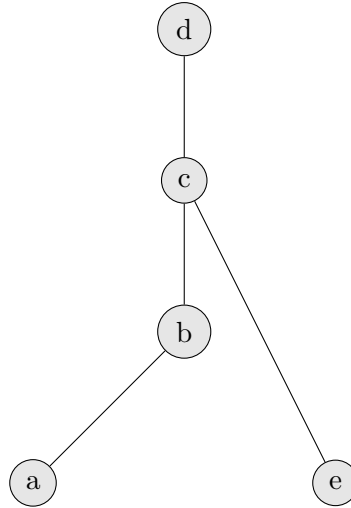
Figure 10: Hasse diagram for the set $A = \{a, b, c, d, e\}$ with the relation R.

Example:

Let $A = \{2, 3, 5, 6, 10, 30\}$ with the order relation $n \leq m$ if n divides m.

1. Is A totally ordered?

2. Hasse diagram of A.

3. Find the maximum, minimum, supremum and infimum of $A \subset \mathbb{N}$.

An order is total if for every pair of elements $n, m \in A$, $n \leq m$ or $m \leq n$.

$$2 \nmid 3, \quad 3 \nmid 2 \implies \text{not totally ordered}$$

$$\text{Maximals: } 30, \quad \text{Minimals: } 2, 3, 5$$
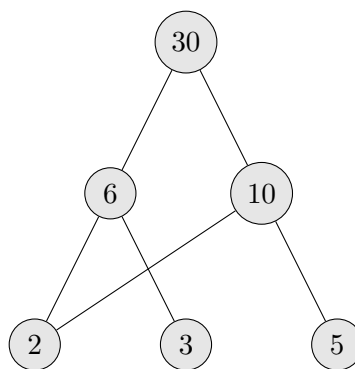
Figure 11: Hasse diagram for the set $A = \{2, 3, 5, 6, 10, 30\}$ with the division order.

$$\text{Supremum and maximum: } 30, \quad \text{Infimum but not minimum: } 1$$

Example:

Find the maximals, minimals, maximum and minimum of:
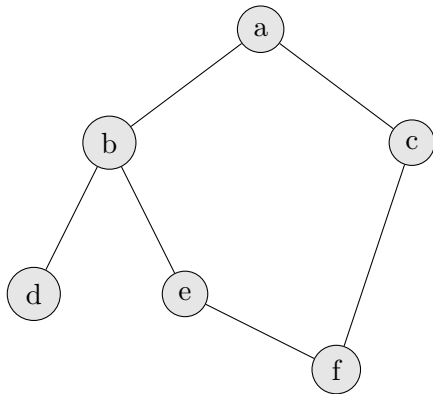
Figure 12: Hasse diagram for the set $A = \{a, b, c, d, e, f\}$ with the relation R.
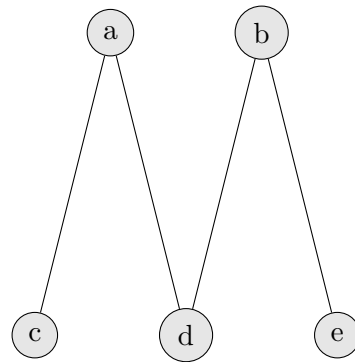
Figure 13: Another Hasse diagram for a different set $B = \{a, b, c, d, e\}$.

For the first diagram:

$$\text{Maximals and maximum: } a, \quad \text{Minimals: } d, f$$

For the second diagram:

$$\text{Maximals: } a, b, \quad \text{Minimals: } c, d, e$$

Example:

Find the upper bounds, lower bounds, supremum and infimum of the set $B = \{4, 5, 6\}$.
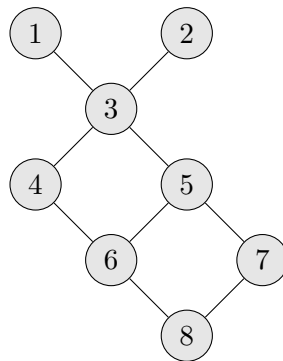
Figure 14: Hasse diagram for the set $B = \{4, 5, 6\}$ with the relation R.

$$\text{Upper bounds: } 1, 2, 3 \quad \text{Lower bounds: } 8, 6$$

$$\text{Supremum but not maximum (does not belong to } B\text{): } 3, \quad \text{Infimum and minimum: } 6$$

11   Graph theory

A graph is a set of vertices V and a set of edges $E \subseteq V \times V$. An edge is a pair of vertices.
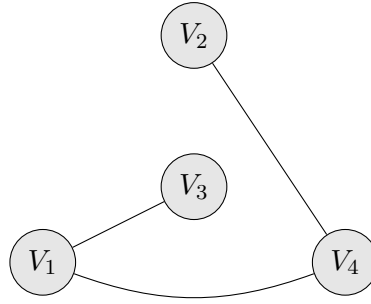
Figure 15: Graph with vertices $V = \{V_1, V_2, V_3, V_4\}$ and edges $E = \{(V_1, V_3), (V_1, V_4), (V_2, V_4)\}$.

11.1   Directed graphs

In a directed graph, the edges have a direction. We use the notation of the cartesian product (v_i, v_j) to indicate that there is an edge from v_i to v_j.

11.1.1   Social networks
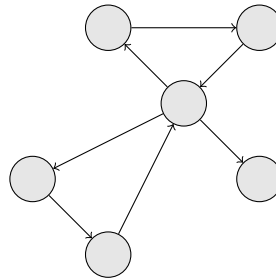
Figure 16: Directed graph representing a social network.

11.2   Multigraphs

A multigraph allows for multiple edges between the same pair of vertices and loops (edges that start and end at the same vertex).

11.3   Weighted graphs

In a weighted graph, each edge has a weight associated with it.
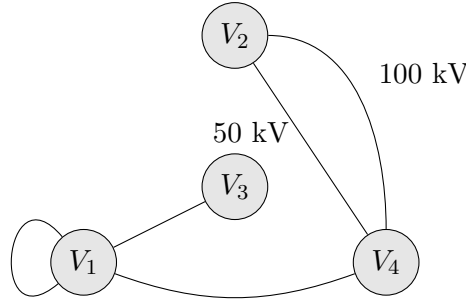
Figure 17: Graph with vertices $V = \{V_1, V_2, V_3, V_4\}$ and edges $E = \{(V_1, V_3), (V_1, V_4), (V_2, V_4)\}$.

11.3.1 Signed graphs

In a signed graph, each edge has a weight that equals 1 or -1.

11.4 Basic terminology and results

$$G(V, E), \quad \text{where } V = \{v_1, v_2, \ldots, v_n\}, \quad E = \{(v_i, v_j) \mid v_i, v_j \in V\}$$

Two vertices $v_i, v_j \in G$ are adjacent (neighbors) if there is an edge between them (they form a pair). In a directed graph, we say that v_i is adjacent to v_j if there is an edge from v_i to v_j. Given a vertex $v_i \in V$, $G = (V, E)$ an undirected graph, the set of neighbors of v is called the neighborhood of v and is denoted by $N(v)$.

$$A \subseteq V, \quad N(A) = \bigcup_{i=1}^{k} N(v_i)$$

$$A = \{v_1, v_2, v_3\}$$

The degree of a vertex v_i is the number of edges incident to v_i. In a directed graph, we have the in-degree and out-degree of a vertex.

$$\deg(v) = |N(v)|$$
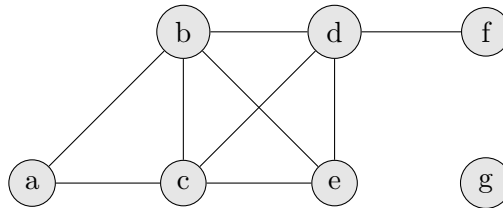
Figure 18: Graph with vertices $V = \{a, b, c, d, e, f, g\}$ and edges
$E = \{(a, c), (a, b), (b, d), (c, e), (d, f), (b, e), (c, d), (b, c), (d, e)\}$.

a is adjacent to b and c, so $N(a) = \{b, c\}$.
Taking $A = \{b, c, e\}$, $N(A) = \{a, b, c, d, e\}$.
Taking $B = \{a, f\}$, $N(B) = \{b, c, d, e\}$.

$$\deg(a) = 2, \quad \deg(b) = 4, \quad \deg(c) = 3, \quad \deg(d) = 3, \quad \deg(e) = 3, \quad \deg(f) = 1, \quad \deg(g) = 0$$

f is called a leaf vertex, or a pendant node (a vertex with degree 1).

11.5 Handshaking theorem

Let $G = (V, E)$ be a graph. The sum of the degrees of all vertices is equal to twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

11.5.1 Theorem

Every undirected graph has an even number of vertices with odd degree.

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = 2|E| \implies \text{odd} + \text{even} = \text{even}$$

For a directed graph, the sum of the in-degrees is equal to the sum of the out-degrees, where the in-degree $(\deg^-)$ of v_i is the number of nodes that v_i is adjacent from and the out-degree $(\deg^+)$ is the number of nodes that v_i is adjacent to.

11.5.2 Handshaking theorem for directed graphs

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

11.6 Special graphs (undirected)

1. **Complete graph (K_n):** A graph where every pair of vertices is adjacent. They are also called cliques.

$$\text{Set of vertices } V, \quad |V| = n, \quad E = V \times V$$
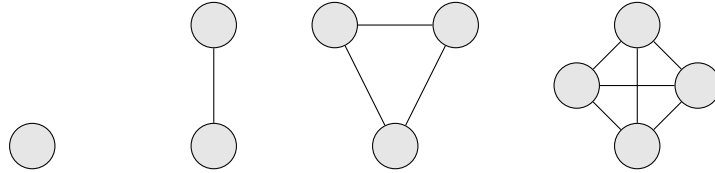
Figure 19: Complete graphs K_1, K_2, K_3, and K_4 aligned horizontally.

2. **Cycle graph $(C_n, n \geq 3)$:** A graph where the vertices form a cycle.

$$V = \{v_1, v_2, \ldots, v_n\}, \quad E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$$
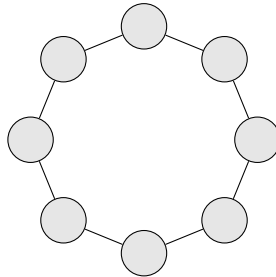
Figure 20: Cycle graph C_8 representing an octagon.

3. **Wheel graph (W_n):** A cycle graph with an additional vertex that is adjacent to all other vertices.
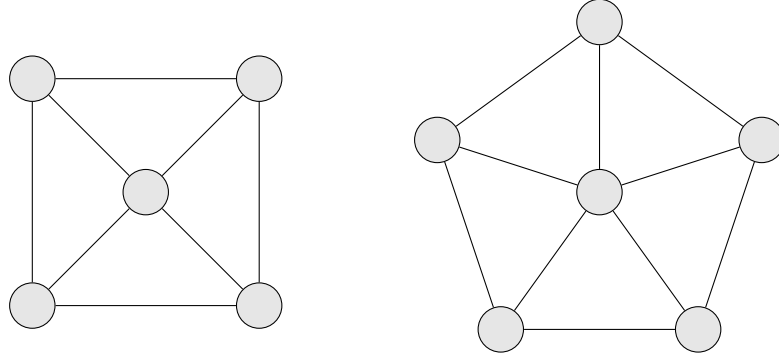
Figure 21: Wheel graphs W_4 and W_5.

4. **Hypercube graph (Q_n):** A graph where each vertex represents a binary string of length n and two vertices are adjacent if their binary strings differ in exactly one bit.

$$E = \{(s_i, s_j) \mid s_i, s_j \in V, \text{ and } s_i, s_j \text{ differ in exactly one bit (position)}\}$$
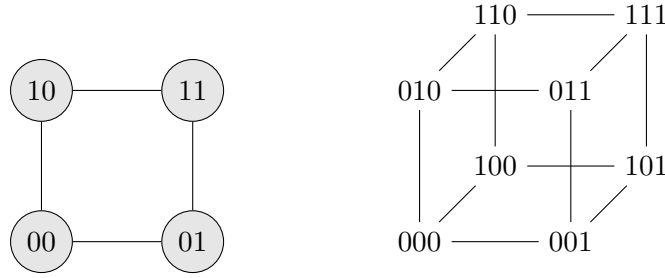
Figure 22: Hypercube graphs Q_2 and Q_3.

11.7 Bipartite graphs (undirected)

A graph $G = (V, E)$ is bipartite if the set of vertices can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2.

$$V = V_1 \cup V_2, \quad V_1 \cap V_2 = \emptyset, \quad \forall (v_i, v_j) \in E, \quad v_i \in V_1, \quad v_j \in V_2$$
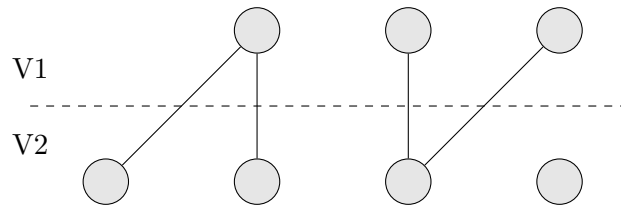
Figure 23: Bipartite graph with vertices $V = \{1, 2, 3, 4, 5, 6, 7\}$ and edges $E = \{(1, 2), (2, 3), (4, 5), (5, 6)\}$.

11.7.1 Theorem

A graph is bipartite if and only if every vertex can be assigned one of two colors such that no two adjacent vertices have the same color.
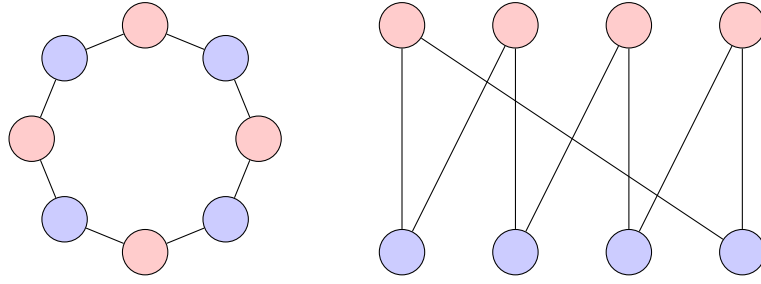
Figure 24: Bipartite graph C_8 (octagon) and alternative representation with colored vertices.

$K_{n,m}$ is the complete bipartite graph with n vertices in V_1, m vertices in V_2 and all possible edges between V_1 and V_2.
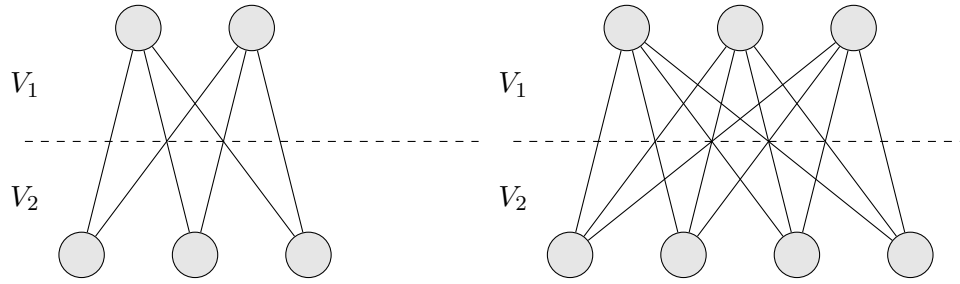
Figure 25: Complete bipartite graphs $K_{2,3}$ and $K_{3,4}$.

11.7.2 Matching

A matching in a bipartite graph is a subset of edges such that no two edges share a common vertex.
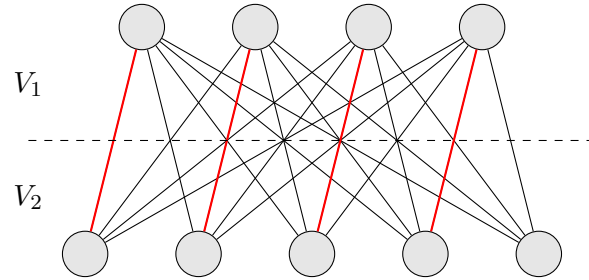
Figure 26: Complete bipartite graph with a matching highlighted in red.

The maximum matching is the matching with the maximum number of edges. The complete matching is the matching where all vertices in V_1 have an edge.

12 Graph Operations

12.1 Subgraph

A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. We call the subgraph proper if $V' \neq V$ and $E' \neq E$.

$$G = (V, E), \quad \text{Subgraph induced by } V' \subseteq V \text{ is } G' = (V', E'), \quad E' = \{(u, v) \in E \mid u, v \in V'\}$$
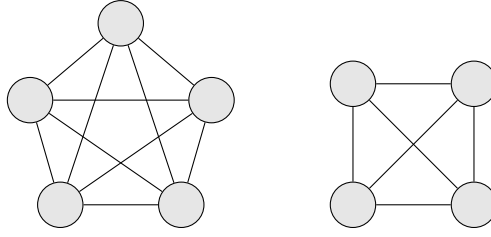
Figure 27: Complete graph K_5 and its induced graph.

12.2   Other operations

1. **Remaining a single edge:** $G = (V, E) \implies G' = (V, E')$, where $E' = E - \{e\}$, $e \in E$.

2. **Adding an edge:** $G = (V, E) \implies G' = (V, E')$, where $E' = E \cup \{e\}$, $e \notin E$. This is not a subgraph.

   Some operations can be done for more than one edge:

   $$(V, E) \implies (V, E') \text{ where } E' = E - \{e_1, e_2, \ldots, e_k\} \text{ or } E' = E \cup \{e_1, e_2, \ldots, e_k\}$$

3. **Edge contractions:**   removing an edge $e = (u, v)$ and merging the vertices u and v into a single vertex w. The new graph is denoted as $G' = (V', E')$, where $V' = V - \{u, v\} \cup \{w\}$ and E' contains all edges from E except for those incident to u and v.
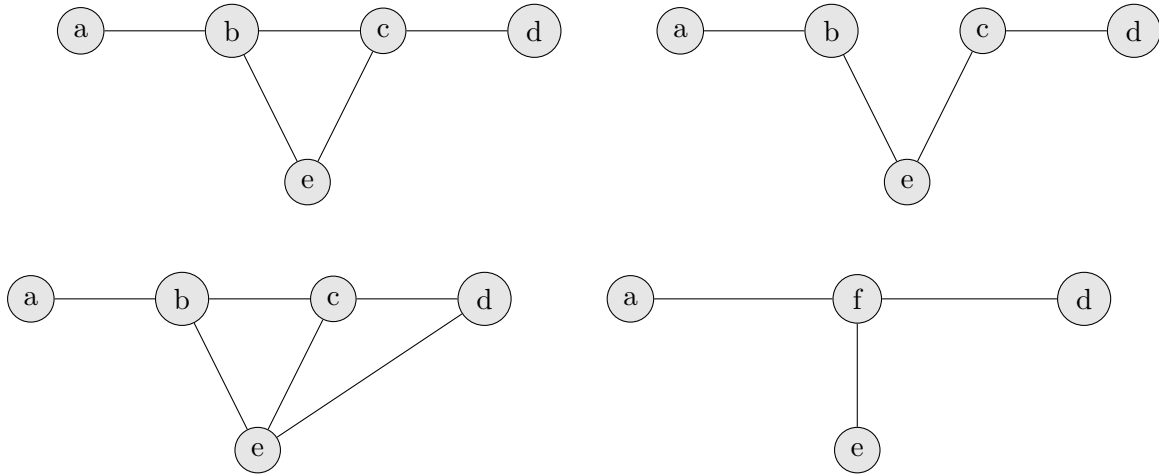
Figure 28: Operations on a graph: edge contraction and adding and removing edges.

12.3   Graph representation

1. **Incidence list:** List vertices and for each vertex, list the edges incident to it.

Vertex	Neighbors
a	(c,d)
b	(d,e)
c	(a,e)
d	(a,b)
e	(b,c)
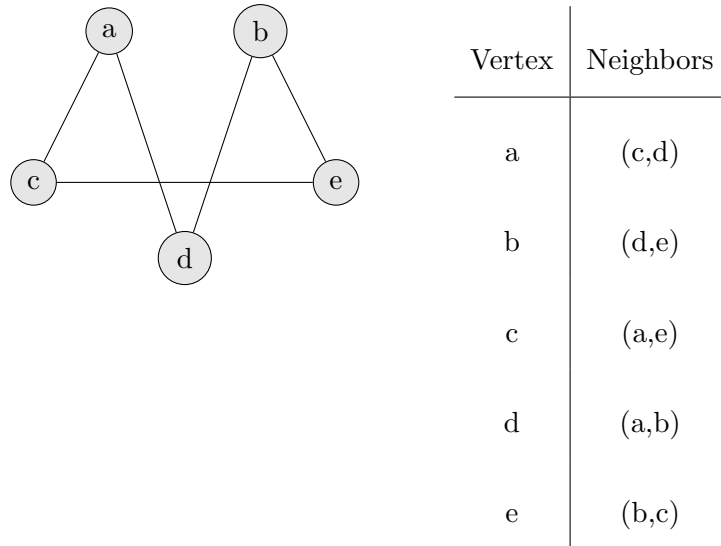
Figure 29: Graph representation using incidence list. Useful for sparse graphs.

This representation is also useful for directed graphs, where the edges are directed from one vertex to another:

Initial Vertex	Terminal Vertex
a	b
b	-
c	a
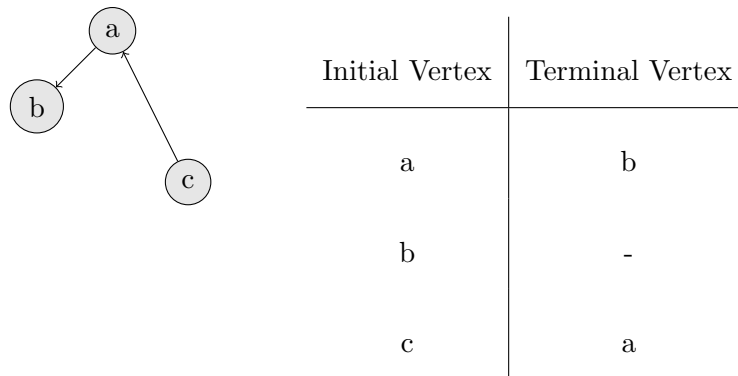
Figure 30: Directed graph representation using incidence list.

2. **Adjacency matrix:** A square matrix where the element at row i and column j indicates whether there is an edge between vertex i and vertex j.

$$(A)_{ij} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$
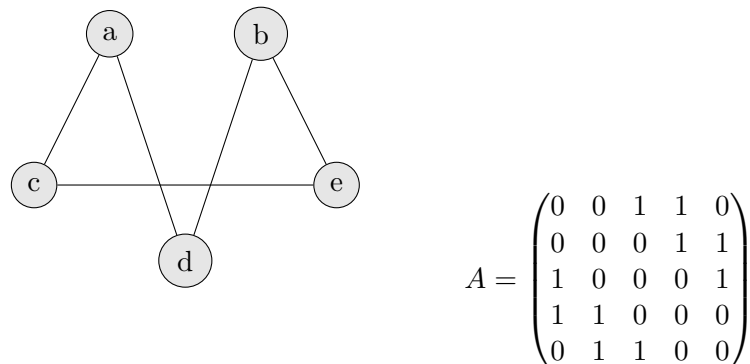
Figure 31: Graph representation using adjacency matrix. Useful for dense graphs.

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For directed graphs, the adjacency matrix is not symmetric. The element at row i and column j indicates whether there is a directed edge **from** vertex i to vertex j.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
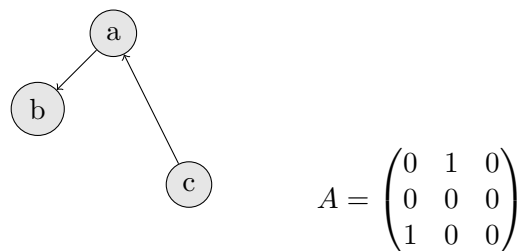
Figure 32: Directed graph representation using adjacency matrix.

For multigraphs, the adjacency matrix can be modified to account for multiple edges between vertices. The element at row i and column j indicates the number of edges between vertex i and vertex j.

3. **Incidence matrix:** A matrix where the columns represent edges and the rows represent vertices. The element at row i and column j indicates whether edge j is incident to vertex i. For undirected graphs, the element is 1 if the edge is incident to the vertex, and 0 otherwise. For directed graphs, the element is 1 if the edge is directed towards the vertex, -1 if it is directed away from the vertex, and 0 otherwise.

$$I = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
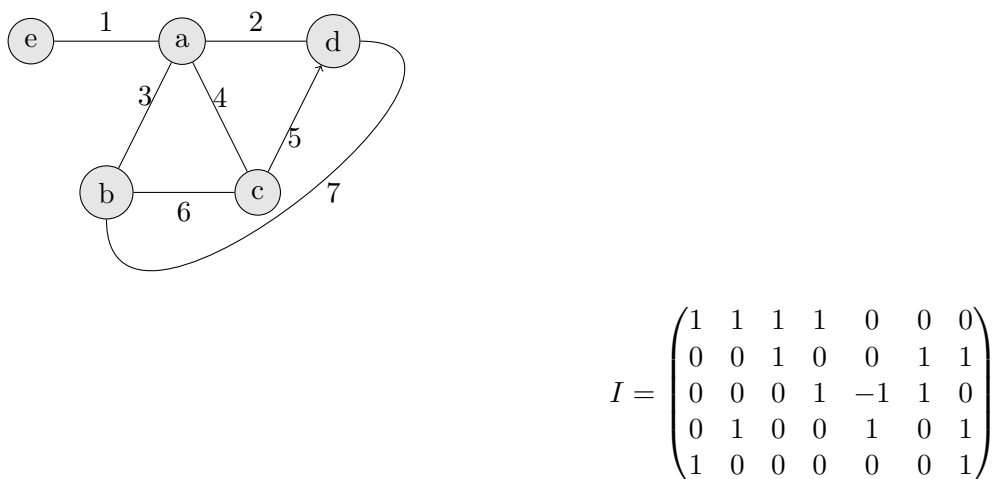
Figure 33: Graph representation using incidence matrix.

12.4   Graph isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$. This means that the two graphs have the same structure, even if their representations are different. Then, f is called an isomorphism.
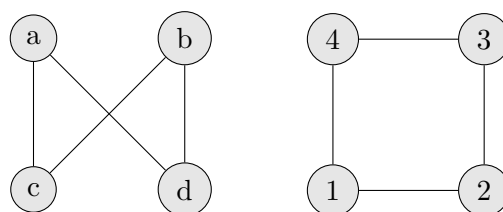
Figure 34: Two isomorphic graphs.

The isomorphism can be represented as a mapping between the vertices of the two graphs. For example, in the above graphs, we can have the following mapping:

$$a \mapsto 1$$
$$b \mapsto 3$$
$$c \mapsto 4$$
$$d \mapsto 2$$

12.5   Graph invariants

A graph invariant is a property of a graph that remains unchanged under isomorphism. Some common graph invariants include:

- Number of vertices and edges

- Degree sequence (the list of degrees of the vertices)
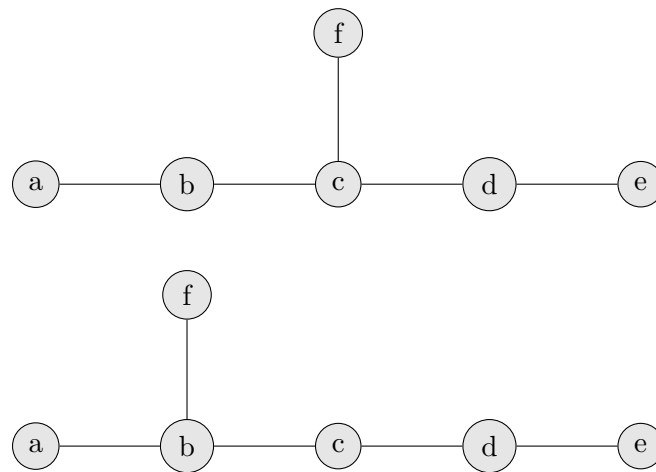
Figure 35: Two non-isomorphic graphs with the same number of vertices and edges, but different degree sequences.

If we try to find an isomorphism between the two graphs in Figure 35, we will find that the degree sequence of the vertices is different. For example, vertex c has two neighbors of degree 2 and one neighbor of degree 1 on the left graph, while on the right graph, vertex c has two neighbors of degree 1 and one neighbor of degree 2. This means that the two graphs are not isomorphic.
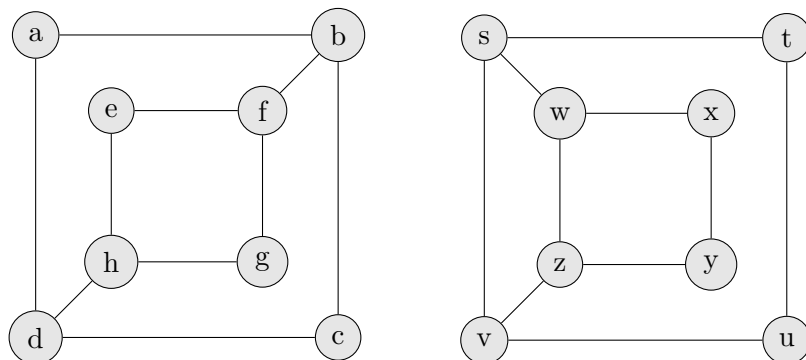
Figure 36: Two non-isomorphic graphs.

In the above example, by listing the vertices and their neighbors, we can see that the two graphs don't have the same degree sequences:

$$a : 2 \qquad\qquad s : 3$$
$$b : 3 \qquad\qquad t : 2$$
$$c : 2 \qquad\qquad u : 2$$
$$d : 3 \qquad\qquad v : 3$$
$$e : 2 \qquad\qquad w : 3$$
$$f : 3 \qquad\qquad x : 2$$
$$g : 2 \qquad\qquad y : 2$$
$$h : 3 \qquad\qquad z : 3$$

The degree sequences are different, so the two graphs are not isomorphic. We can also see that the two graphs have cycles of different lengths.

12.6 Adjacency matrix and isomorphism

Two graphs are isomorphic if their adjacency matrices can be transformed into each other by permuting the rows and columns. This means that the two graphs have the same structure, even if their representations are different.
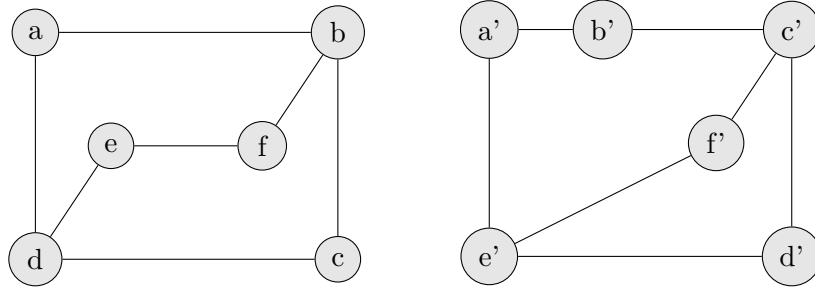
Figure 37: Two isomorphic graphs.

The adjacency matrices of the two graphs are:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The adjacency matrices of the two graphs are not the same, but they can be transformed into each other by permuting the rows and columns. This means that the two graphs are isomorphic.

$$A_2 = P A_1 P^T$$

where P is a permutation matrix that permutes the rows and columns of A_1 to obtain A_2.

$$P = P_{16} P_{23} P_{34} P_{45} P_{51} P_{62}$$

where P_{ij} is the permutation matrix that swaps the i-th and j-th rows and columns of a matrix. The permutation matrix P can be obtained by applying the following permutations to the rows and columns of A_1: This means that the two graphs have the same structure, even if their representations are different.

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12.7 Paths

A path in a graph is a sequence of vertices such that each vertex is connected to the next vertex by an edge. The length of a path is the number of edges in the path. A path is simple if it does not traverse any edge more than once.
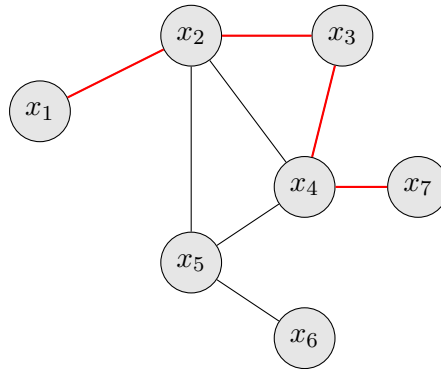
Figure 38: Graph with a marked path. The path is highlighted in red.

In directed graphs, edges have to be traversed from beggining to end.

12.8 Connected graphs

A graph is connected if there is a path between every pair of vertices.

Theorem: This path can be chosen simple.
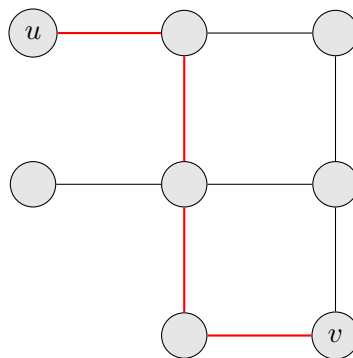
Figure 39: Graph with a marked path. The path is highlighted in red.

12.8.1 Connected components

A connected component of a graph is any connected subgraph that is not a proper subgraph of any other connected subgraph. In other words, a connected component is a "piece" in which you can break a graph. A graph can have multiple connected components.

12.8.2 Vertex and Edge cuts

In a connected graph, a vertex or an edge is a cut if removing it disconnects the graph.
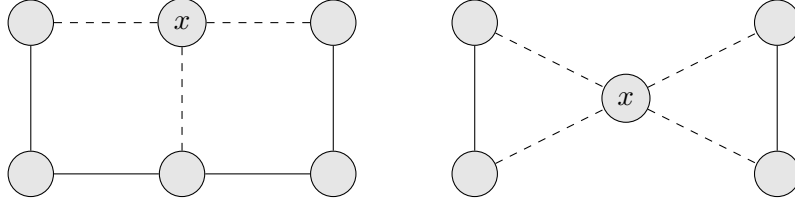
Figure 40: Graphs showing a vertex cut. The vertex x is not a cut vertex in the first graph, as removing it does not disconnect the graph. In the second graph, removing x disconnects the graph.

12.8.3   Vertex and Edge connectivity

The vertex $\kappa(G)$ or edge $\lambda(G)$ connectivity of a graph is the minimum number of vertices or edges that need to be removed to disconnect the graph.

$$\kappa(G) \leq \lambda(G)$$

For the connected graph in Figure 40, the vertex connectivity is 2, as removing the vertex x is not enough to disconnect the graph. The edge connectivity is 2, as removing any single edge does not disconnect the graph. For the second graph, the vertex connectivity is 1, as removing the vertex x disconnects the graph. The edge connectivity is 2 as well, as removing any one edge does not disconnect the graph.

$$\kappa(G_1) = 2 \quad \lambda(G_1) = 2 \quad \kappa(G_2) = 1 \quad \lambda(G_2) = 2$$

For the complete graph K_n, the vertex and edge connectivity are both $n - 1$.

$$\kappa(K_n) = n - 1 \quad \lambda(K_n) = n - 1$$

Theorem: Let $G = (V, E)$ be a connected graph. Then, $|V| = n \implies |E| \leq n - 1$. If $|E| < n - 1$, then G can not be connected.

Example

Provide an example of:

- A regular bipartite graph.

- A regular graph of degree 3 with 9 vertices.

- A graph with n vertices and $\dfrac{(n-1)(n-2)}{2}$ edges.

- A graph which is isomorphic to its complement.

Note: A regular graph is a graph where every vertex has the same degree.
Note: Given a graph G, the complement of G is a graph $\bar{G}$ with the same vertices as G, but with edges that are not in G. In other words, if $(u, v) \in E(G)$, then $(u, v) \notin E(\bar{G})$ and vice versa.
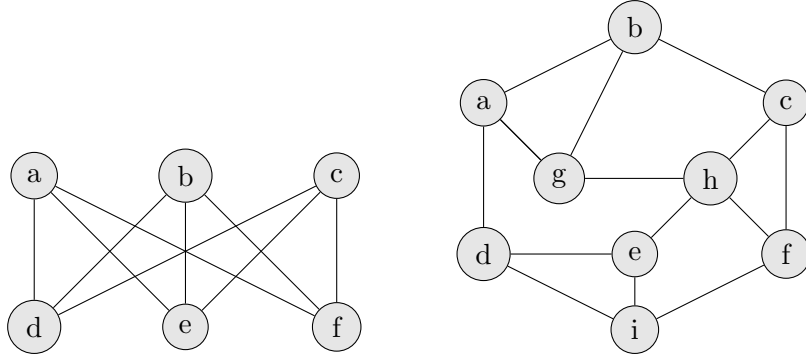
Figure 41: A regular bipartite graph and an attempt of a regular graph of degree 3 with 9 vertices. The second graph is not regular, as the vertex h has degree 4. It is not possible to create a regular graph of degree 3 with 9 vertices.
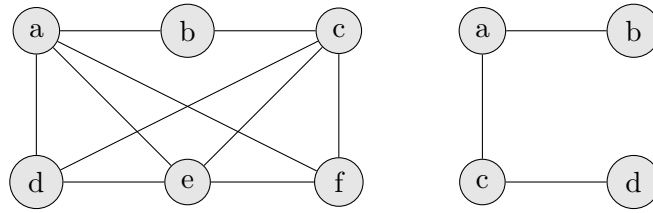
Figure 42: A graph with $n = 6$ vertices and $\dfrac{(n-1)(n-2)}{2} = 10$ edges and a graph which is isomorphic to its complement.

Example

Calculate the number of vertices of:

- A regular graph of degree 2 with 9 edges.

- A regular graph with 6 edges.

- A graph with 10 edges, 2 vertices of degree 4 and the rest of degree 3.

By the handshaking lemma, we know that the sum of the degrees of all vertices is equal to twice the number of edges. This means that if we have a graph with n vertices then:

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

where $d(v_i)$ is the degree of vertex i.
So we can establish the following equation for a regular graph of degree d:

$$|V| = \frac{2|E|}{d}$$

For the first case, we have:

$$|V| = \frac{9 \cdot 2}{2} = 9$$

For the second case, we have:

$$|V| = \frac{6 \cdot 2}{d} \implies d = 3 \text{ or } 2 \implies |V| = 4 \text{ or } 3$$

For the third case, we have:

$$20 = 2 \cdot 4 + (n-2) \cdot 3 \implies 20 = 8 + 3n - 6 \implies n = 6$$

Example

Prove that the number of nodes of a self-complementary graph is either $4k$ or $4k + 1$ for some integer k.

Proof: Let G be a self-complementary graph with n vertices. Then, the number of edges in G is equal to the number of edges in its complement $\bar{G}$. This means that:

$$|E| + |\bar{E}| = \binom{n}{2} \implies 2|E| = \binom{n}{2} \implies |E| = \frac{n(n-1)}{4}$$

If n is even, then $n(n-1)$ is divisible by 4. If n is odd, then $n(n-1)$ is divisible by 2, but not by 4. This means that n must be of the form $4k$ or $4k + 1$ for some integer k.

Example

Conclude wether the following graphs are isomorphic or not:
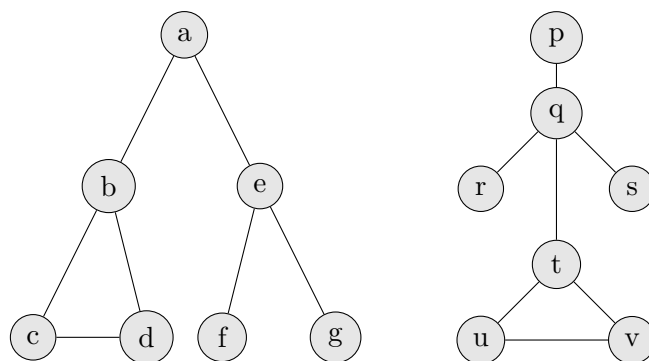
Figure 43: Two graphs.

The two graphs are not isomorphic. We can see this by looking at the degree sequences of the vertices in each graph and realising that they are different.

If we consider the following graph:
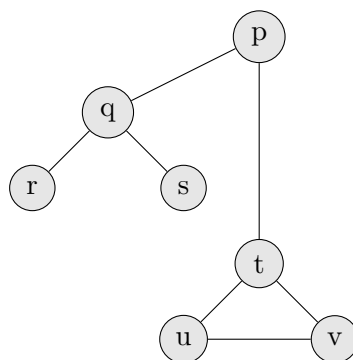
We can see that the first graph is isomorphic to this one. We can see this by looking at the degree sequences of the vertices in each graph and mapping their vertices in the following way:

$$a \mapsto p \qquad\qquad e \mapsto q$$
$$b \mapsto t \qquad\qquad f \mapsto r$$
$$c \mapsto u \qquad\qquad g \mapsto s$$
$$d \mapsto v$$

12.9 Paths in directed graphs

A directed graph is strongly connected if for every pair of vertices a and b, there is a directed path from a to b and a directed path from b to a.
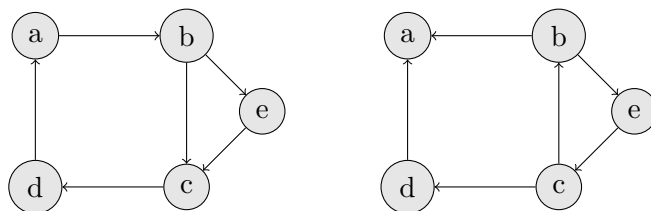
Figure 44: Two directed graphs, the first one is strongly connected and the second one is weakly connected.

The first graph is strongly connected, as there is a directed path from every vertex to every other vertex. The second graph is weakly connected, as there is a directed path from b to a, but not from a to b.

12.10 Strong Connected Components

A strongly connected component of a directed graph is a maximal strongly connected subgraph. In other words, a strongly connected component is a "piece" in which you can break a directed graph. A directed graph can have multiple strongly connected components.

For example, in the second directed graph in Figure 44, the strongly connected components are:

$$\{a\}, \quad \{d\}, \quad \{b, c, e\}$$

12.11 Paths and Isomorphisms

Paths and cycles are another invariant under isomorphisms:
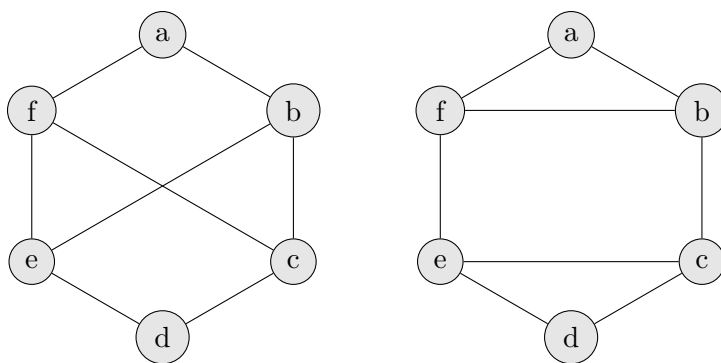
Figure 45: Two non-isomorphic graphs. The first graph does not have any cycle of length 3, while the second graph has a cycle of length 3.

Cycles and paths should be preserved under isomorphisms. This means that if two graphs are isomorphic, then they should have the same number of cycles and paths of the same length.

12.12 Theorem: Counting vertices in paths

Let $G = (V, E)$ be a graph and A be its adjacency matrix with respect to the ordering $\{v_1, v_2, \ldots, v_n\}$. Then, the number of paths of length k from vertex v_i to vertex v_j is given by the (i, j)-th entry of the matrix A^k.

Proof by induction: The theorem is obviously true for $k = 1$. Now, let us assume that the theorem is true for $k = n$. Then, we have:

$$A^n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

now let us consider A^{n+1}:

$$A^{n+1} = A^n \cdot A, \quad \text{notation: } A^n_{ij} = b_{ij} \text{ and } A_{ij} = a_{ij}$$

$$A^{n+1}_{ij} = \sum_{m=1}^{n} b_{im} a_{mj}, \quad \text{where } a_{mj} \text{ equals 1 if there is an edge from } m \text{ to } j \text{ and 0 otherwise.}$$

This means that A^{n+1}_{ij} counts the number of paths of length $n + 1$ from vertex i to vertex j.

Example

Consider C_4:
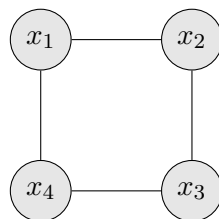
Figure 46: Graph C_4.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

One can observe that entries in the matrix A^2 are equal to the number of paths of length 2 from vertex i to vertex j. For example, the entry $A^2_{12} = 0$, which means that there is no path of length 2 from vertex x_1 to vertex x_2. The entry $A^3_{12} = 4$, which means that there are two paths of length 3 from vertex x_1 to vertex x_2.
A^3 has all 0s in the diagonal, which means that there are no paths of length 3 from vertex x_i to itself. This is because C_4 is a cycle and the only way to return to the same vertex is to go through all other vertices.

13   Euler and Hamilton paths

Euler paths are paths that visit every edge of a graph. A Hamiltonian path is a path that visits every vertex of a graph. Euler graphs mark the beggining of graph theory, as they were the first graphs to be studied. In 1735, Euler studied the Seven Bridges of Königsberg problem, which asked if it was possible to walk through the city of Königsberg and cross each of its seven bridges exactly once. Euler proved that it was not possible to do so, as the graph representing the city was not connected. This is a graph representation of the problem:
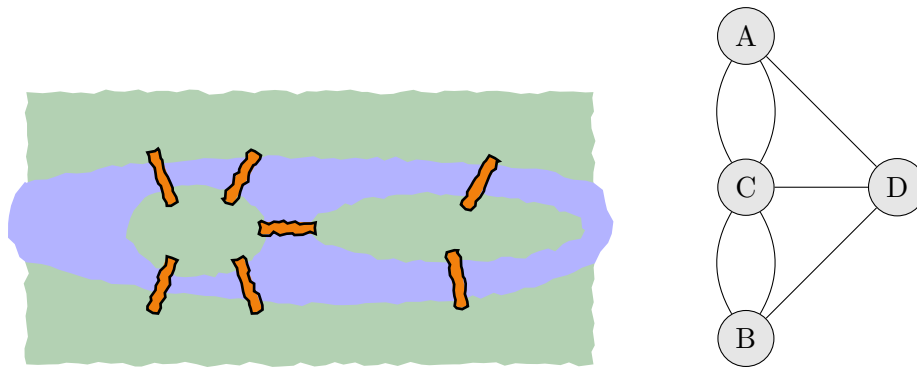
Figure 47: Königsberg bridges and its graph representation.

13.1   Eulerian cycles

An Eulerian cycle is a cycle that visits every edge of a graph exactly once.

Theorem: An eulerian cycle exists in a multigraph if and only if all vertices have even degree and the graph is connected. An Eulerian path exists in a multigraph if and only if exactly 2 vertices have odd degree and the graph is connected.

Example

Consider the following graph:
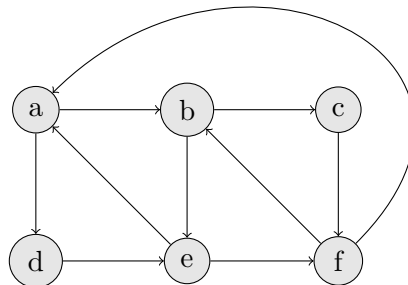
Figure 48: A graph with an Eulerian cycle.

13.2   Hamiltonian cycles

A graph has a Hamiltonian cycle if there is a cycle that visits every vertex of the graph exactly once. A Hamiltonian path is a path that visits every vertex of the graph exactly once.

A graph with leaves (vertices with degree 1) can not have a Hamiltonian cycle. This is because a Hamiltonian cycle must visit every vertex of the graph, and if there are leaves, then it is not possible to return to the starting vertex.

For example, K_n has a Hamiltonian cycle, as it is a complete graph and every vertex is connected to every other vertex. Consider K_5:
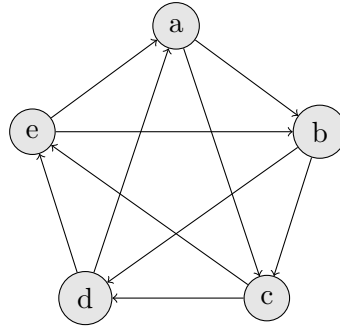
Figure 49: The complete graph K_5 with an Eulerian cycle and actually 2 Hamiltonian cycles.

13.3 Dirac's theorem

Let $G = (V, E)$ be a simple graph with $n = |V| \geq 3$. Then, if $\deg(v_i) \geq \frac{n}{2}$ for every vertex $v_i \in V$, then G has a Hamiltonian cycle.

13.4 Ore's theorem

Let $G = (V, E)$ be a simple graph with $n = |V| \geq 3$. Then, if $\deg(v_i) + \deg(v_j) \geq n$ for every non-adjacent pair of vertices $v_i, v_j \in V$, then G has a Hamiltonian cycle.

Vertices of degree 2 in a Hamiltonian graph must have their two edges included in the Hamiltonian cycle. This is because if we remove the edge from the Hamiltonian cycle, then we can not return to the starting vertex.

14 Shortest paths

14.1 Dijkstra's algorithm
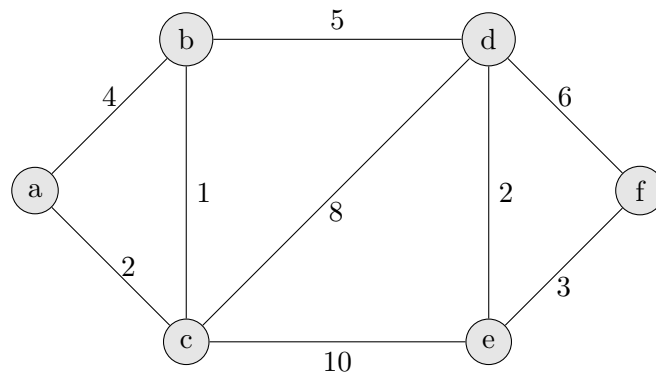
Figure 50: A graph with weights.

Dijkstra's algorithm is a greedy algorithm that finds the shortest path from a source vertex to all other vertices in a weighted graph. The algorithm works by maintaining a set of vertices whose shortest distance from the source is known, and repeatedly adding the vertex with the smallest distance to this set until all vertices have been added. In this case, for finding the shortest distance between a and f, the algorithm works as follows:

1. $a, 0$, other vertices are at an infinite distance.

2. We have the following vertices: $(b, 4, a)$, $(c, 2, a)$. We choose the one with the smallest distance, which is c.

3. Look for the vertex with the smallest distance (greedy algorithm), we list available vertices: $(b, 3, c)$, $(d, 10, c)$ and $(e, 12, c)$ and choose the one with the smallest distance, which is b.

4. Now we add $(d, 8, b)$.

5. We have the following vertices: $(e, 10, d)$ and $(f, 14, d)$. We choose e.

6. Now we add $(f, 13, e)$.

In the end the shortest path from a to f is $a \rightarrow c \rightarrow b \rightarrow d \rightarrow e \rightarrow f$ with a total distance of 13. Now consider the following variation of the graph:
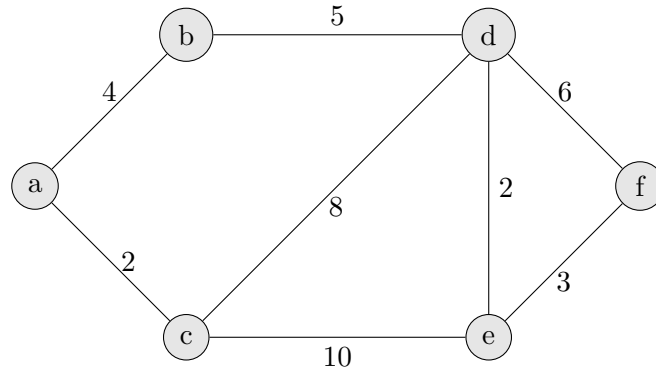
Figure 51: A graph with weights.

In this case, the algorithm works as follows:

1. $a, 0$, other vertices are at an infinite distance.

2. We have the following vertices: $(b, 4, a)$, $(c, 2, a)$. We choose the one with the smallest distance, which is c.

3. Look for the vertex with the smallest distance (greedy algorithm), we list available vertices: $(d, 10, c)$ and $(e, 12, c)$ and choose the one with the smallest distance, which is b.

4. Now we add $(d, 9, b)$.

14.2 Planar graphs

A graph is planar if it can be drawn on a plane without any edges crossing.
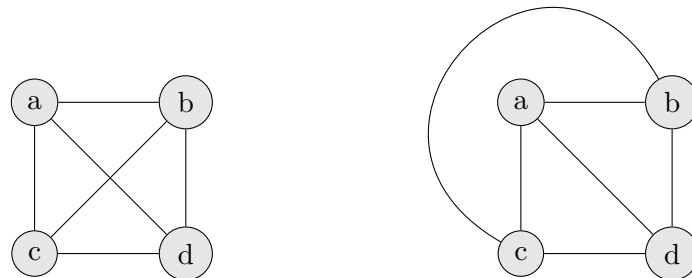
Figure 52: Complete graph K_4 and its planar representation.

Example
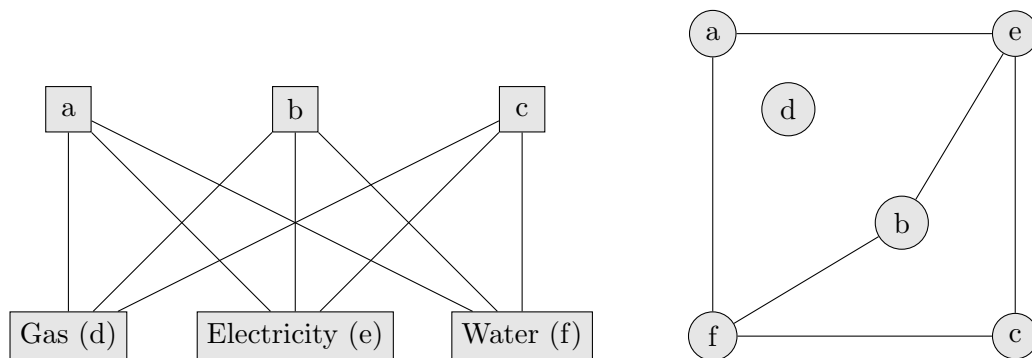
Figure 53: There is no planar representation of $K_{3,3}$.

14.3 Euler's formula

Let $G = (V, E)$ be a connected planar graph with $v = |V|$ vertices and $e = |E|$ edges, that divides the plane in r regions. Then, the following equation holds:

$$v - e + r = 2$$

This formula is used to calculate the number of regions in a planar graph. For example, consider the following graph:
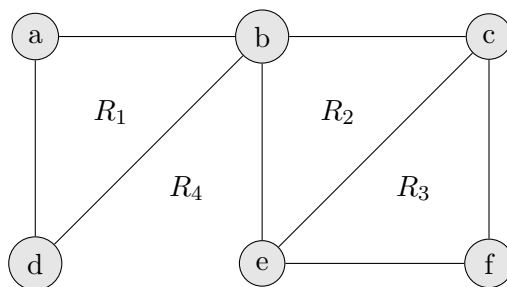
Figure 54: A planar graph with 3 regions.

In this case, we have $v = 6$, $e = 8$ and $r = 4$. We can see that the graph is connected and planar, so we can apply Euler's formula:

$$6 - 8 + 4 = 2$$

This means that the graph has 4 regions. We can see that the graph is divided into 4 regions: R_1, R_2, R_3 and R_4.

14.3.1 Proof of Euler's formula

Proof by induction on the number of edges. Let G_k be a subgraph of $G = (V, E)$ obtained by adding one edge to G_{k-1}. Then, we have:

$$G_0 : \bigcirc, \quad G_1 : \bigcirc\!-\!\bigcirc, \quad \text{where } v = 2, e = 1, r = 1 \implies 2 - 1 + 1 = 2$$

Assume that this is true for G_k. Now, prove for G_{k+1}. Let G_{k+1} be obtained from G_k by adding an edge e. Then, we have:
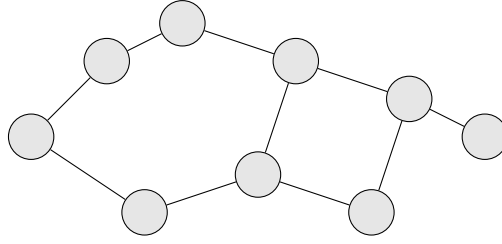
Figure 55: A planar graph with 3 regions.

From this, we have two corollaries:

1. If G is a connected planar graph with $v \geq 3$, then $e \leq 3v - 6$.

2. G has a vertex with degree not larger than 5. This is because of (1).

    $v \geq 3 \implies e \leq 3v - 6$. Let us assume that $\deg(v_i) \geq 6 \quad \forall v_i \in V$. Then, we have:

    $2e \geq 6v \implies e \geq 3v \implies 3v - 6 \geq 3v \implies -6 \geq 0$, which is a contradiction.

3. If $v \geq 3$ and there are no cycles of length 3, then $e \leq 2v - 4$.

14.4 Kuratowski's theorem

Let $G = (V, E)$ be a graph. Then, G is non-planar if and only if it contains a subgraph that is homeomorphic to $K_{3,3}$ or K_5.

14.4.1 Elementary subdivision of a graph

An elementary subdivision of a graph is obtained by removing an edge and adding a vertex in its place and adding edges to the new vertex. For example, consider the following graph:
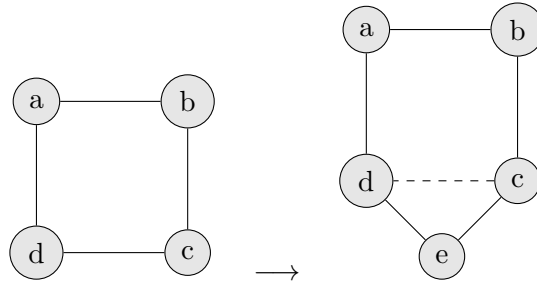
Figure 56: A graph with an elementary subdivision.

Two graphs are homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

14.5   Peterson graphs
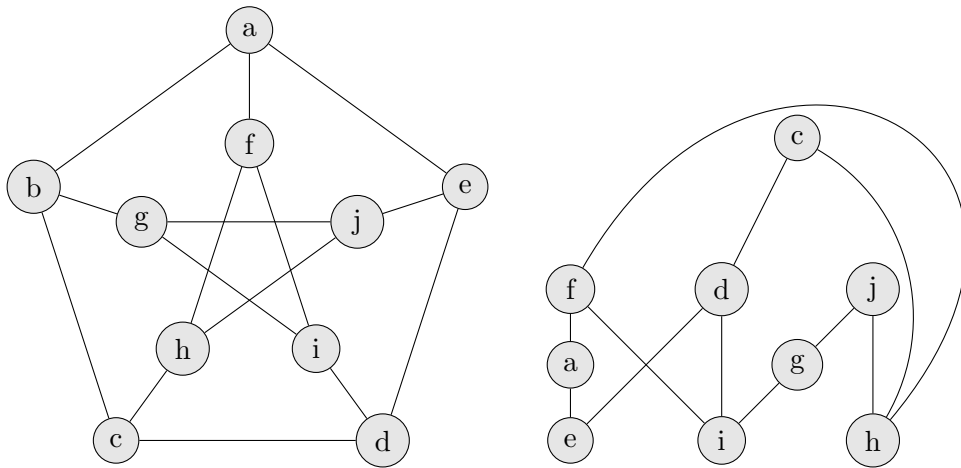
Figure 57: Peterson graph and a subgraph of it

14.6   Graph coloring

A coloring of a graph $G = (V, E)$ is the assignment of a color to each vertex in V such that no two adjacent vertices have the same color.

The chromatic number of a graph is the minimum number of colors needed to color the graph.

14.6.1   Theorem:

The chromatic number $\chi(G)$ of a planar graph is at most 4. This means that any planar graph can be colored with at most 4 colors such that no two adjacent vertices have the same color.

14.7   Scheduling
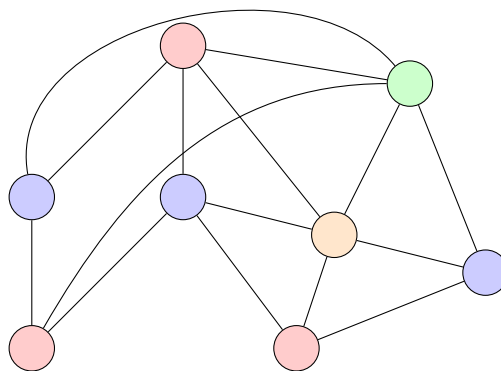
Figure 58: A graph with 4 colors.

Example

Consider the following graphs:
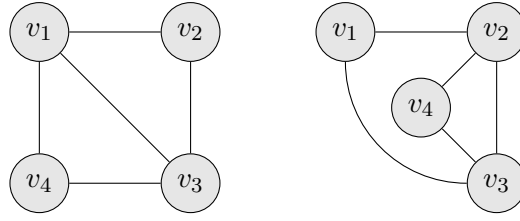
Figure 59: Graphs G_1 and G_2.

$$A_2 = PA_1P^T, \text{ where}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We can establish the following mapping:

$$v_1 \rightarrow v_2$$
$$v_2 \rightarrow v_1$$
$$v_3 \rightarrow v_4$$
$$v_4 \rightarrow v_3$$

Now we can obtain P:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

Consider the following graphs and determine if they are Eulerian or Hamiltonian.:
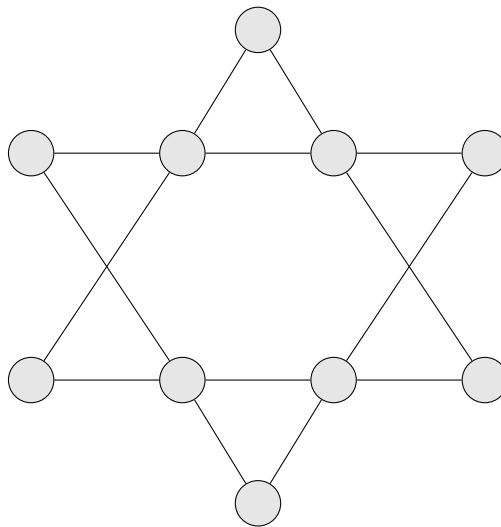
Figure 60: A graph with 10 vertices.

The graph is Eulerian because it has 10 vertices and 12 edges. The degree of each vertex is even, so we can find an Eulerian cycle. However, it is not Hamiltonian.

66
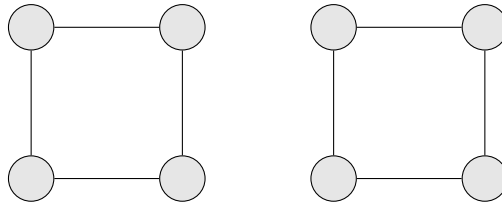
Figure 61: A graph with 8 vertices.

The graph is disconnected so we inmediately know that it is not Eulerian nor Hamiltonian.
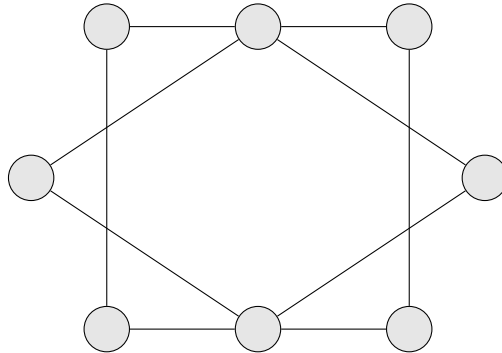
Figure 62: A graph with 8 vertices.

The graph is Eulerian because it has 8 vertices and 10 edges. The degree of each vertex is even, so we can find an Eulerian cycle. However, it is not Hamiltonian.
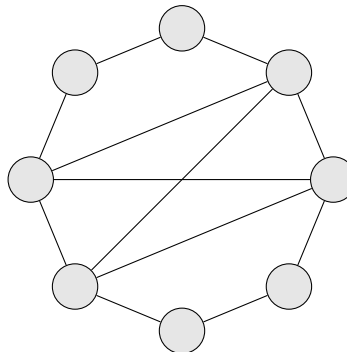
Figure 63: A graph with 8 vertices.

The graph is Eulerian because it has 8 vertices and 12 edges. The degree of each vertex is even, so we can find an Eulerian cycle. It is also Hamiltonian because it has a cycle that visits all vertices once.
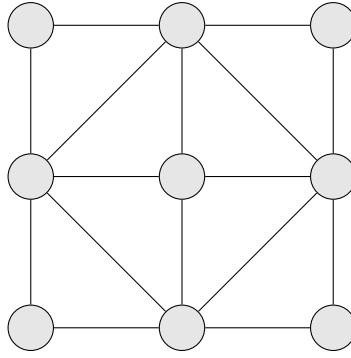
Figure 64: A graph with 9 vertices.

The graph is not Eulerian because the degree of more than 2 vertices is odd, so we can't find an Eulerian cycle. It is not Hamiltonian.

Example

a) Prove that cycles in bipartite graphs are even in length.

b) Prove that no bipartite graph with an odd number of vertices can be Hamiltonian.

c) Use the above to prove that the following graph is not Hamiltonian:
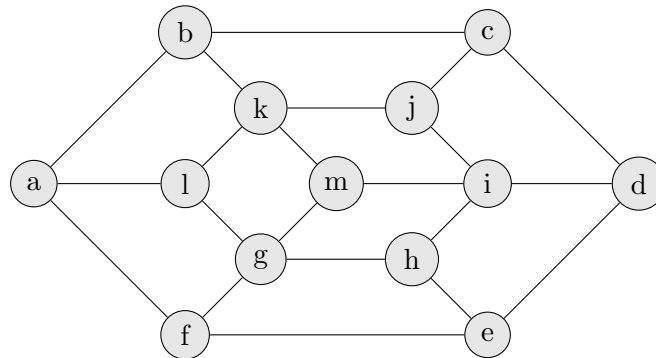
Figure 65: A graph with an odd number of vertices.

To prove (a), coming back to the starting vertex requires to the other subset of vertices. This means that the cycle must have an even number of vertices. To prove (b), we can use the fact that a bipartite graph with an odd number of vertices cannot have a cycle of odd length. Since a Hamiltonian cycle must visit all vertices, it cannot exist in a bipartite graph with an odd number of vertices.

Now for (c), we can write the graph as a bipartite graph with 13 vertices. Since it has an odd number of vertices, it cannot be Hamiltonian:
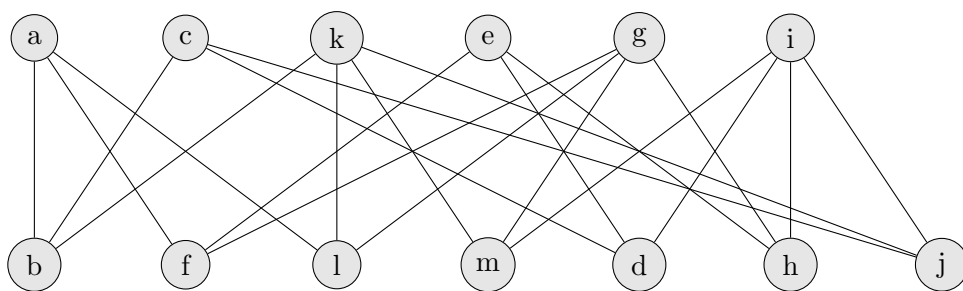
Figure 66: A bipartite graph with 13 vertices.

Example

Find the shortest path from vertex a to vertex d in the following graph:
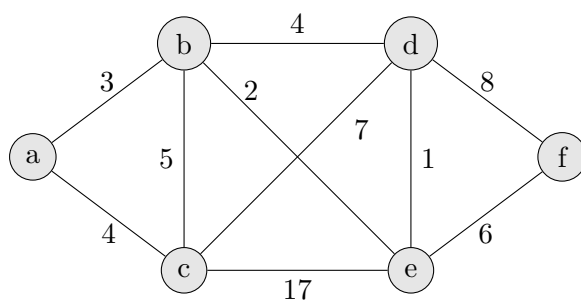
Figure 67: A graph with 6 vertices.

$$
\begin{aligned}
\mathbf{a(-,0)} &\to & a(-,0) \\
\mathbf{b(a,3)} &\to & b(a,3) \\
c(a,4) &\to & \mathbf{c(a,4)} \\
d(-,\infty) &\to & d(b,7) \to & \mathbf{d(e,6)} \\
e(-,\infty) &\to & \mathbf{e(b,5)} \to & e(b,5) \\
f(-,\infty) &\to & f(-,\infty) \to & f(e,11) \to & \mathbf{f(e,11)}
\end{aligned}
$$

So the shortest path from vertex a to vertex d is:

$$a \to b \to e \to f, \quad \text{with a total length of 11.}$$

Example

A connected planar graph has 30 edges and a planar representation of the graph divides the plane in 20 regions. What is the number of vertices in the graph?

$$v - 30 + 20 = 2 \implies v = 12.$$

Example

A connected graph has 8 vertices, each of degree 3. How many regions does a planar representation of the graph have?

$$\sum_{i=1}^{v} d(v_i) = 2e \implies 3 \cdot 8 = 2e \implies e = 12.$$

$$v - e + r = 2 \implies 8 - 12 + r = 2 \implies r = 6.$$

Example

How many colors are needed to color the vertices of the following graph?
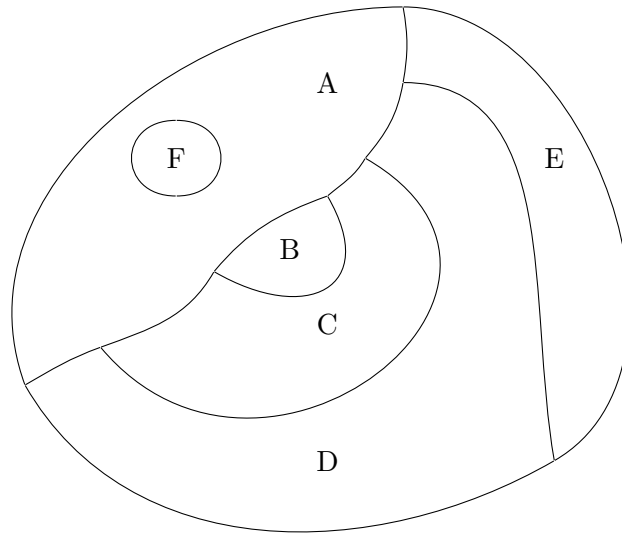
Figure 68: A graph with 6 regions.
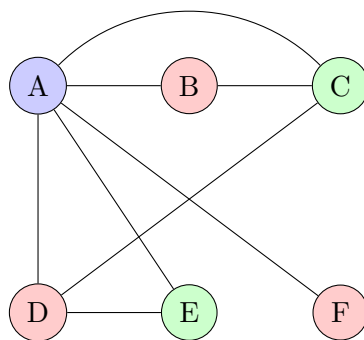
Figure 69: A representation of the graph.

In the end we need 3 colors to color the graph.

15 Trees

A tree is an undirected connected graph with no circuits.

Theorem A connected, undirected graph is a tree if there is a unique path between any pair of nodes.
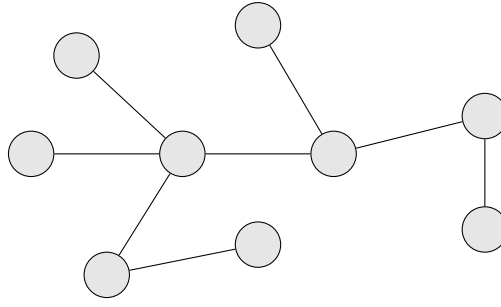
Figure 70: A tree.

15.1 Rooted trees

A rooted tree is a tree with a distinguished vertex called the root. The vertices of the tree are divided into levels, where the root is at level 0, its children are at level 1, and so on.
The previous tree can be represented as a rooted tree as follows:
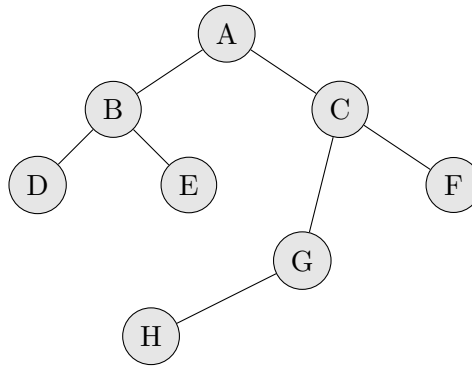
Figure 71: A rooted tree.

In this case, the root is vertex A, C is a child of A and a parent of F and G. D and E are siblings. H is a leaf, which means it has no children. The rest of the vertices are internal nodes. A rooted tree is m-ary if each internal node has at most m children. A rooted tree is full m-ary if each internal node has exactly m children. A binary tree is a full 2-ary tree.
In ordered rooted trees, the order is given from up to down and left to right. In unordered rooted trees, the order is not important.

15.2 Properties of trees

Theorem   A tree with n vertices has $n-1$ edges.

Proof   We can prove this by induction. For the base case, a tree with 1 vertex has 0 edges. Now, assume that a tree with k vertices has $k-1$ edges. A tree with $k+1$ vertices has at least 1 leaf, connected by one edge to the rest of the tree. By removing the leaf, we have a tree with k vertices and $k-1$ edges. Adding the leaf back gives us k edges. Therefore, a tree with $k+1$ vertices has k edges.

Theorem   A full m-ary tree contains $n = mi + 1$ vertices, where i is the number of internal nodes.

Theorem A full m-ary tree with n vertices has $i = \dfrac{n-1}{m}$ internal nodes and $l = \dfrac{(m-1)n+1}{m}$ leaves. If a tree has i internal nodes, it has $l = (m-1)i + 1$ leaves and if it has l leaves, it has $i = \dfrac{l-1}{m-1}$ internal nodes and $n = \dfrac{ml+1}{m-1}$ vertices.

15.2.1 Height

The height of a full m-ary tree is the length of the longest path from the root to a leaf. The maximum number of leaves in a tree with height h is m^h.
A tree:

1. is connected

2. has no cycles

3. has a unique path between any two edges

4. has $n-1$ edges

To show that a graph is a tree, we only need to prove two of properties 1-3.

15.3 Spanning trees

Given a graph $G = (V, E)$, a spanning tree of G is a subgraph of G that is a tree and contains all vertices.
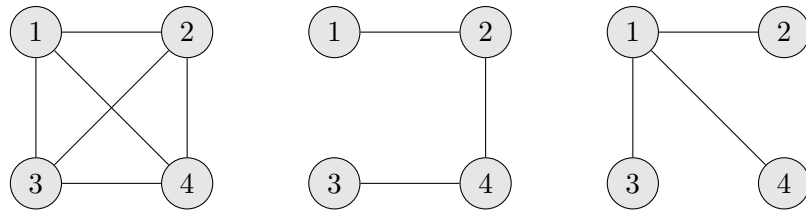
Figure 72: Complete graph K_4 and its spanning trees.

Theorem A simple graph is connected if and only if it has a spanning tree. A spanning tree connects all vertices, so there exists a path between any two vertices. Therefore, it is connected. In order to find a spanning tree, we first check if it is a tree. If it is, then it is a spanning tree. If it is not, remove an edge and check again.

15.4 Depth-first search

Consider the following graph:
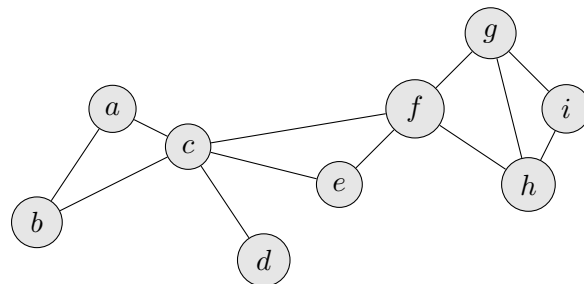
Figure 73: A graph.

We can use depth-first search to find a spanning tree. Starting from vertex a, we visit its neighbors in the following order: b, c, d, e, f, g, h, and i. The resulting spanning tree is shown below:
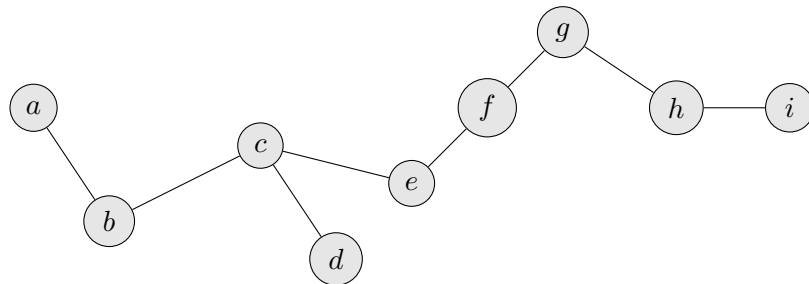

Figure 74: A spanning tree of the graph in Figure 73.

15.5 Breadth-first search

Breadth-first search is another algorithm for finding a spanning tree. It starts from a vertex and explores all its neighbors before moving on to the next level of neighbors. The resulting spanning tree is shown below:
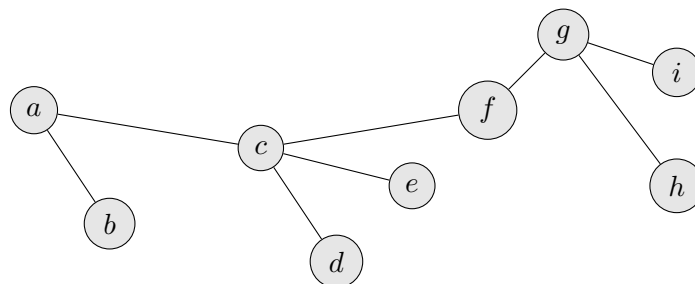
Figure 75: A spanning tree found using breadth-first search.

15.6 Minimum spanning trees

For weighted graphs, the minimum spanning tree is the one with the lowest sum of weights.

15.6.1 Prim algorithm
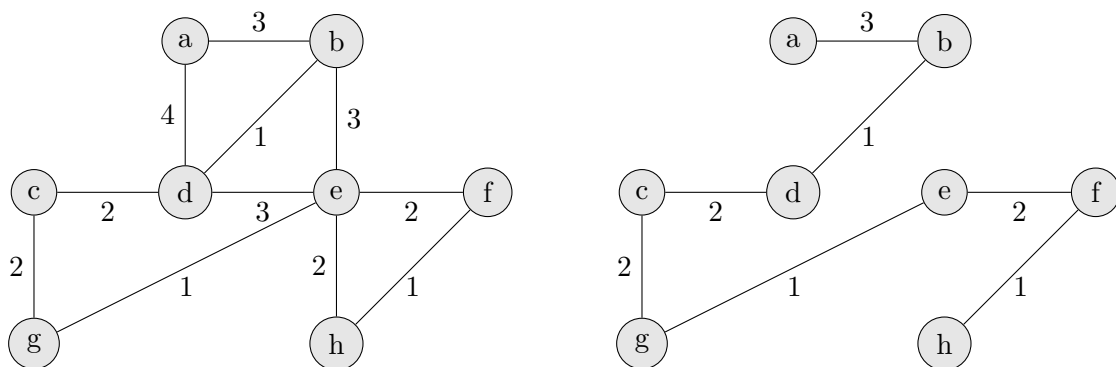
Figure 76: A graph with weights on the edges and its minimum spanning graph.

1. Start with a vertex and add it to the tree.

2. Find the edge with the lowest weight that connects a vertex in the tree to a vertex outside the tree.

3. Add the edge and the vertex to the tree.

4. Repeat until all vertices are included in the tree.
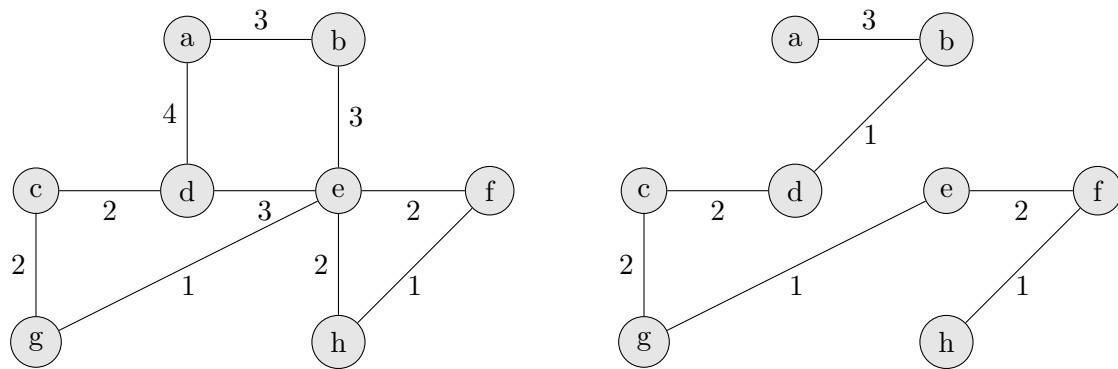
15.7 Kruskal algorithm

Figure 77: A graph with weights on the edges and its minimum spanning graph.

1. Sort the edges in ascending order of weight.

2. Start with an empty tree and add the edges in order, as long as they do not create a cycle.

3. Repeat until all vertices are included in the tree.

Exercise

How many vertices does a full 5-ary tree with 100 internal vertices have?
How many leaves does a full 3-ary tree with 100 vertices have?

$$n = mi + 1, \quad n_1 = 5 \cdot 100 + 1 = 501$$

$$l = n - i, \quad i = \frac{n-1}{m} \implies l = n - \frac{n-1}{m} = \frac{n(m-1)+1}{m}$$
$$l_2 = \frac{100 \cdot 2 + 1}{3} = \frac{201}{3} = 67$$

Exercise

Each person sends a letter to five people or does not send any letter. Suppose that 10000 people send letters. How many people recieve it and how many do not send it?
We have a fll 5-ary tree:

$$n = 5 \cdot 10000 + 1 \begin{cases} 50000 & \text{root started the chain} \\ 50001 & \text{root recieved the letter} \end{cases}$$

$$l = n - i \implies l = 50000 - 10000 = 40000$$

Exercise

Show that any tree has chromatic number 2.

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Proof Any tree can be rooted. Then, nodes at a level are not connected to each other. Therefore, we can color the nodes at even levels with one color and the nodes at odd levels with another color. This way, we can color the tree with 2 colors.

Exercise

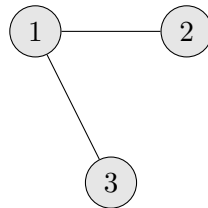How many non-isomorphic trees do K_3, K_4, and K_5 have? For K_3, we have 1 tree:

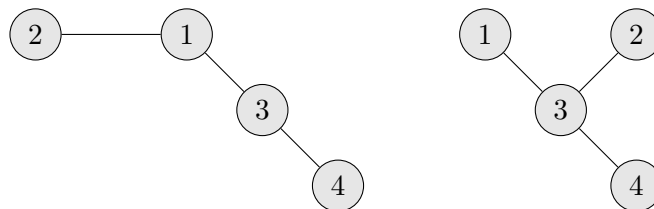Figure 78: A non-isomorphic tree of K_3.

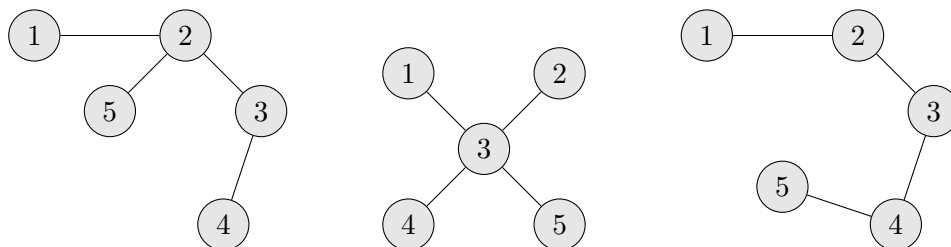For K_4, we have 2 trees:

Figure 79: Non-isomorphic trees of K_4.

For K_5, we have 3 trees:

Figure 80: Non-isomorphic trees of K_5.

Exercise

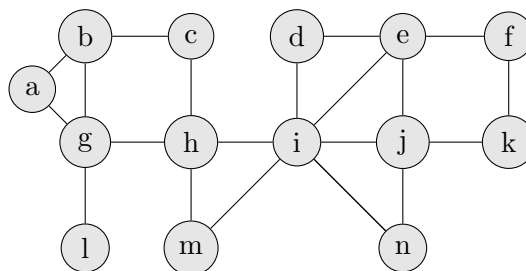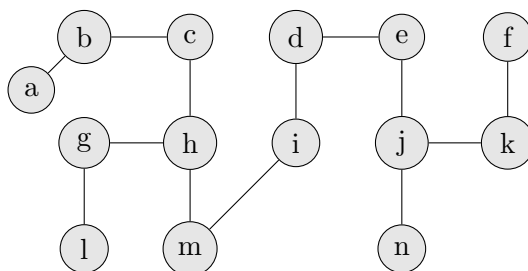Find a spanning tree of the following graph using both depth-first search and breadth-first search:

Figure 81: A graph.

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Starting from vertex a, we can find a spanning tree using depth-first search:

$$a \to b \to c \to h \to m \to i \to d \to e \to j \to k \to f$$

but we are missing g, l and n. So we can add them to the tree:

$$a \to b \qquad \to c \to h \qquad \to m \to i \qquad \to d \to e \qquad \to j \to k \qquad \to f$$
$$\searrow g \qquad\qquad\qquad\qquad\qquad \searrow n$$
$$\searrow l$$

Figure 82: A spanning tree of the graph in Figure 81.

Now using breadth-first search, we can find a spanning tree starting from vertex a:
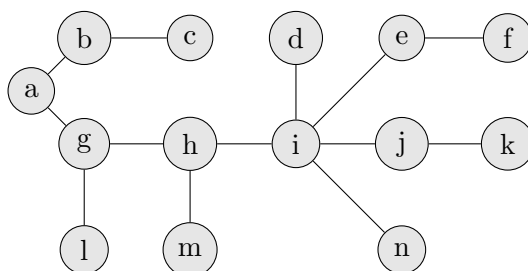

Figure 83: A spanning tree of the graph in Figure 81.
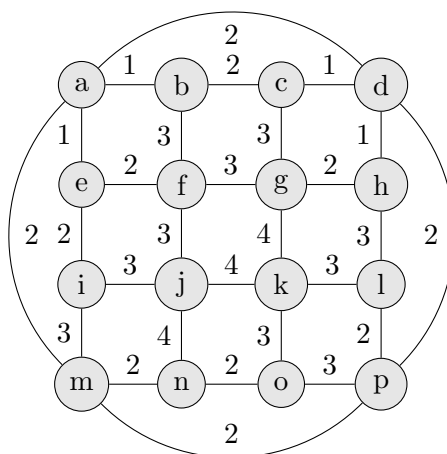
Exercise

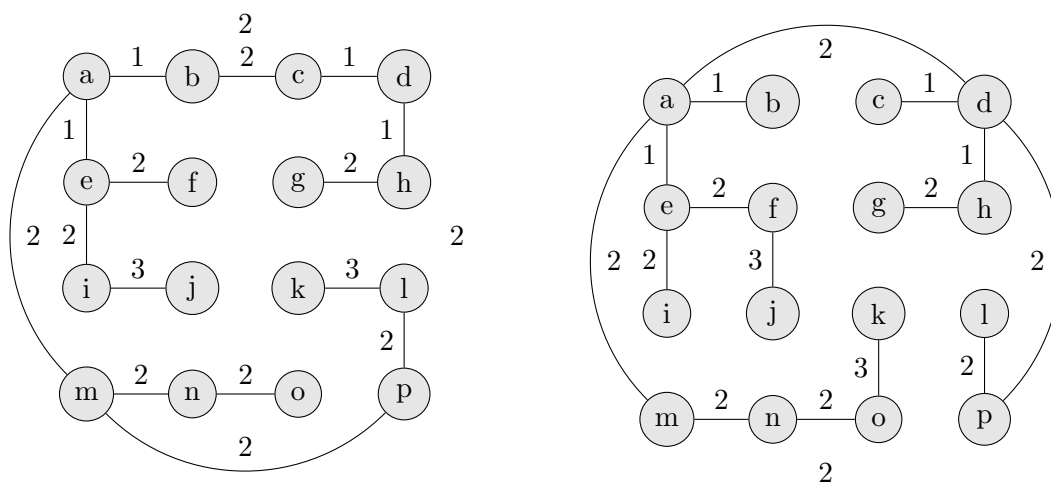Consider the following graph:

Figure 84: A graph.

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Figure 85: Minimum spanning trees of the graph with total weight 28, using Prim's and Kruskal's algorithms.