# Vector Calculus

## 1 Euclidean space

We define the euclidean space in $\mathbb{R}^N, N \geq 1$ using cartesian coordinates.
Any element $x \in \mathbb{R}^N, \quad x = (x_1, x_2, \dots, x_N), \quad x \in \mathbb{R}$

### Standard basis

$$e_1 = (1, 0, \dots, 0), \dots, e_N = (0, 0, \dots, 1)$$

$$\text{Then, } x = \sum_{j=1}^{N} x_j \cdot e_j$$

In particular, $B_{\mathbb{R}^3} = \{i, j, k\}$, the canonical basis.

### Properties

- Addition: $\quad (x_1, \dots, x_N) + (y_1, \dots, y_N) = (x_1 + y_1, \dots, x_N + y_N)$
- Multiplication by a scalar $\lambda \in \mathbb{R}$: $\quad \lambda x = \lambda(x_1, \dots, x_N) = (\lambda x_1, \dots, \lambda x_N)$
- Associative: $\quad \lambda, \mu \in \mathbb{R}, \quad (\lambda\mu)x = \lambda(\mu x)$
- Additive Identity: $\quad$ There exists a vector $\overline{0} = (0, \dots, 0)$ such that $x + \overline{0} = x$
- Additive Inverse: $\quad \forall x = (x_1, \dots, x_N), \exists \ \overline{x} = (-x_1, \dots, -x_N)$ such that $x + \overline{x} = \overline{0}$
- Distributive Property (over vector addition):

$$\lambda\left((x_1, \dots, x_N) + (y_1, \dots, y_N)\right) = \lambda(x_1, \dots, x_N) + \lambda(y_1, \dots, y_N)$$

- Distributive Property (over scalar addition):

$$(\lambda + \mu)(x_1, \dots, x_N) = \lambda(x_1, \dots, x_N) + \mu(x_1, \dots, x_N)$$

- Scalar Multiplication Identity: $\quad 1 \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)$
- Zero Scalar Multiplication: $\quad 0 \cdot (x_1, \dots, x_N) = (0, \dots, 0)$

### Norm

The euclidean space in $\mathbb{R}^N$ is a normal space with an associated norm function.

$$\| \cdot \| : \mathbb{R}^N \to \mathbb{R}_+ \cup \{0\}$$

$$x = (x_1, \dots, x_N) \to \|x\| = \sqrt{x_1^2 + \cdots + x_N^2}$$

### Properties

The norm satisfies the following properties:

(a) $\forall x \in \mathbb{R}^N$

- $\|x\| > 0 \iff x \neq 0$
- $\|x\| = 0 \iff x = 0$

(b) $\|\lambda x\| = |\lambda| \, \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$

- Triangular inequality.

### Remark: Distance

We can define the distance between two elements in $\mathbb{R}^N$ as

$$dist(x, y) = \|x - y\| = \|y - x\|$$
$$dist(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

- $dist(x, y) = \|x - y\| > 0$ if $x \neq y$, and $dist(x, y) = 0$ if $x = y$
- $dist(x, y) = \|x - y\| = \| - (y - x)\| = \| - 1\| \cdot \|y - x\| = dist(y, x)$
- $dist(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = dist(x, z) + dist(z, y)$

### Remark

For $\mathbb{R}$ such a distance is the absolute value, $| \cdot | : \mathbb{R} \to \mathbb{R}$

# 2 Inner or scalar product

Let $x, y$ be two vectors in $\mathbb{R}^N$, then

$$x \cdot y = x_1 y_1 + \cdots + x_N y_N$$
$$x \cdot y = \langle x, y \rangle = (x, y)$$

## 2.1 Properties

The inner product satisfies the following properties:

- $\forall x \in \mathbb{R}^N \ \langle x, x \rangle \geq 0$

  $\langle x, x \rangle = 0$ if $x = 0$
- Symmetric: $\langle x, y \rangle = \langle y, x \rangle$
- Bilinear: $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

## 2.2 Cauchy-Schwartz inequality

$$|x \cdot y| \leq \|x\| \|y\|$$

**Proof** If $y = \lambda x$, $|\langle x, \lambda y\rangle| = |\lambda|\|x\|^2 = \|x\||\lambda|\|x\| = \|x\|\|y\|$
If $y \neq \lambda x$ (x and y are linearly independent).
Assume $z = \lambda x + y$

$$0 \leq \langle z, z\rangle = \langle \lambda x + y, \lambda x + y\rangle = \langle \lambda x, \lambda x\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + \langle y, y\rangle$$
$$\text{Since } \|x\|^2 > 0, \quad = \lambda^2\|x\|^2 + 2\lambda\langle x, y\rangle + \|y\|^2$$

If we represent it as a parabola in function of $\lambda$, it has no roots.

$$\lambda = \frac{-2\langle x, y\rangle \pm \sqrt{4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2}}{2\|x\|^2}$$

So the discriminant $\leq 0$

$$4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\implies |\langle x, y\rangle| \leq \|x\|\|y\|$$

## 2.3 Theorem

$$\langle x, y\rangle = \|x\|\|y\|\cos\varphi$$

Writing $x$ and $y$ in polar coordinates:

$$x_1 = \|x\|\cos\alpha, \quad x_2 = \|x\|\sin\alpha$$
$$y_1 = \|y\|\cos\beta, \quad y_2 = \|y\|\sin\beta$$

$$\begin{aligned}\langle x, y\rangle = x_1y_1 + x_2y_2 &= \|x\|\cos\alpha\|y\|\cos\beta + \|x\|\sin\alpha\|y\|\sin\beta \\ &= \|x\|\|y\|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= \|x\|\|y\|\cos(\alpha - \beta) = \|x\|\|y\|\cos\varphi\end{aligned}$$

## Remark

$$\cos\varphi = \frac{\langle x, y\rangle}{\|x\|\|y\|}$$
$$\text{Then } x \perp y \iff \langle x, y\rangle = 0$$

## Examples:

- $C((a, b)) \cong$ continuous functions in $(a, b)$

  $$f, g \in C((a, b)), \quad then\,\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

  We can add some weights:

  $$\langle f, g\rangle = \int_a^b w(t)f(t)g(t)\,dt$$

  An example:

  $$\langle f, g\rangle = \int_a^b e^{-t}f(t)g(t)\,dt$$

- We might have an orthogonal family (with infinite elements) of functions.

$$\{\cos nx, \sin nx\}_{n\in\mathbb{Z}} \quad \text{in } [0, 2\pi]$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx\, dx$$

$$\int_0^{2\pi} \cos nx \sin nx\, dx = \int_0^{2\pi} \cos nx \cos mx\, dx = 0 \quad \text{for } n \neq m$$

# 3 Vector Product (Only in $\mathbb{R}^3$)

Take $x, y \in \mathbb{R}^3$

$$x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

Recall:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

## 3.1 Triple product and properties

We take the triple product

$$a \cdot (b \times c) = (a_1, a_2, a_3) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $u \in \text{span}\{b, c\}$ then $a \cdot (b \times c) = 0$ and $a, b, c$ are coplanar if $a \cdot (b \times c) = 0$

## 3.2 Geometric Interpretation

- The magnitude of $x \times y$ represents the area of the parallelogram formed by $x$ and $y$.
- The direction of $x \times y$ is perpendicular to the plane spanned by $x$ and $y$, following the right-hand rule.
- The cross product satisfies: $x \times y = -(y \times x)$.

# 4 Topology of $\mathbb{R}^n$

Definition of open spaces: we define an open ball in $\mathbb{R}^n$ centered at $x_0$ and of radius $R$.

$$\text{Denoted by} \quad B_R(x_0) = \{x \in \mathbb{R}^n : \text{dist}(x, x_0) < R\}$$

This set includes all points in $\mathbb{R}^n$ whose distance from $x_0$ is less than $R$. Open sets are the building blocks of topological spaces, and they help define concepts such as convergence, continuity, and compactness.

## 4.1 Open set

A set $A \subset \mathbb{R}^n$ is open if $\forall x \in A, \quad \exists R > 0$ such that $B_R(x) \subset A$.
For example:

$$(x, y) \in \mathbb{R}^2, \quad A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 3\}$$

## 4.2 Closed set

A set $A \subset \mathbb{R}^n$ is closed if its complement is open.

$$A \subset \mathbb{R}^n \text{ is closed if } \mathbb{R}^n \setminus A \text{ is open.}$$

## 4.3 Boundary of a set

The boundary of a set $A \subset \mathbb{R}^n$ denoted by $\partial A$:

$$\partial A = \{x \in \mathbb{R}^n : \forall R > 0, \quad B_R(x) \cap A \neq \emptyset \text{ and } B_R(x) \cap (\mathbb{R}^n \setminus A) \neq \emptyset\}$$

Following the previous example, the boundary of $A$ is the circle of radius 3 centered at the origin:

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 3\}$$

### Remark

A set $A \subset \mathbb{R}^n$ is closed if and only if it contains its boundary.

### Example:

$$D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1 \text{ and } x > 0\}$$

$$D^C = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 1 \text{ or } x \leq 0\}$$

$$\partial D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1 \text{ and } x > 0\}$$

$D$ is open, $D^C$ is closed, and $\partial D$ is the semicircle of radius 1 centered at the origin.

### Example:

$$S = \{x = 1 \text{ and } 1 < y \leq 2\}$$

$$S^C = \{x \neq 1 \text{ or } y \leq 1 \text{ or } y > 2\}$$

$$\partial S = \{x = 1 \text{ and } y = 1 \text{ or } y = 2\}$$

$S$ is neither open nor closed, as $S^C$, and $\partial S$ is the line segment from $(1, 1)$ to $(1, 2)$.

### 4.4 Compact set

A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Example:**

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$

$A = \partial A$ so $A$ is closed. $A$ is also bounded, as all points in $A$ are contained within the circle of radius 2 centered at the origin.
$\implies A$ is compact.

**Example: (Exercise 11a)**

$$A = xy - \text{plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3\}$$

This set is closed, and $B_R(x, R) \cap A = \emptyset$ and $B_R(x, R) \cap (\mathbb{R}^3 \setminus A) = \emptyset$.
Therefore, $A$ is will not be compact.

**Example: (Exercise 11b)**

$$A = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$$

For any point $(x, y) \in A$, we can find an open ball $B_R(x, R) \subset B$ that does not intersect the $x$ or $y$ axes.

$$R = \min\{x, y\}$$

### 4.5 Ball in $\mathbb{R}^n$

For a ball at any part of radius $r$ in $\mathbb{R}^n$:

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2$$

$$\text{dist}(x - a) = \|x - a\| = r$$

**Example: (Exercise 1a)**

Sphere centered at (0,1,-1) with $r = 4$

$$(x - 0)^2 + (y - 1)^2 + (z + 1)^2 = 16$$

Intersection with the $x, y, z$-planes:

$$\text{If } z = -1, \quad x^2 + (y - 1)^2 = 16$$

$$\text{If } y = 1, \quad x^2 + (z + 1)^2 = 16$$
$$\text{If } x = 0, \quad (y - 1)^2 + (z + 1)^2 = 16$$

**Example: (Exercise 1b)**

Sphere going through the origin and centered at $(1, 2, 3)$:

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = r^2$$

$$\text{dist}(0, (1, 2, 3)) = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

So the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

### Example: (Exercise 1c)

Find center and radius of the sphere:

$$x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$$

$$(x+1)^2 + (y+4)^2 + (z-2)^2 = 49$$

So the center is $(-1, -4, 2)$ and the radius is 7.

### Example: (Exercise 3)

Check if the vectors are orthogonal:

$$a = (-5, 3, 7), \quad b = (6, -8, 2)$$

$$a \cdot b = -30 - 24 + 14 = -40! = 0$$

So $a$ and $b$ are not orthogonal.

### Example: (Exercise 4a)

Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is:

$$d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Let } a = R - Q, \quad b = P - Q$$

$$\text{Then } d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Area of parallelogram} = \|a \times b\| = \|a\| \cdot d$$

### Example: (Exercise 4b)

Use the formula in part (a) to find the distance from the point $P(1,1,1)$ to the line through $Q(0,6,8)$ and $R(-1,4,7)$.

$$\mathbf{a} = \overrightarrow{QR} = R - Q = (-1-0, 4-6, 7-8) = (-1, -2, -1)$$

$$\mathbf{b} = \overrightarrow{QP} = P - Q = (1-0, 1-6, 1-8) = (1, -5, -7)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & -1 \\ -5 & -7 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & -1 \\ 1 & -7 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -2 \\ 1 & -5 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = (9, -8, 7)$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{9^2 + (-8)^2 + 7^2} = \sqrt{81 + 64 + 49} = \sqrt{194}$$

$$\|\mathbf{a}\| = \sqrt{(-1)^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|} = \frac{\sqrt{194}}{\sqrt{6}} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$$

**Example: (Exercise 5)**

Calculate the volume of the parallelepiped with edges adjacent to $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$, where

$$P(1,1,1), \quad Q(2,0,3), \quad R(4,1,7), \quad S(3,-1,-2).$$

The volume of the parallelepiped can be calculated using the scalar triple product of the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2-1, 0-1, 3-1) = (1,-1,2) \\
\overrightarrow{PR} &= R - P = (4-1, 1-1, 7-1) = (3,0,6) \\
\overrightarrow{PS} &= S - P = (3-1, -1-1, -2-1) = (2,-2,-3)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PR} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} &= \mathbf{i}(0\cdot(-3) - 6\cdot(-2)) - \mathbf{j}(3\cdot(-3) - 6\cdot 2) + \mathbf{k}(3\cdot(-2) - 0\cdot 2) \\
&= \mathbf{i}(0+12) - \mathbf{j}(-9-12) + \mathbf{k}(-6-0) \\
&= 12\mathbf{i} + 21\mathbf{j} - 6\mathbf{k} \\
&= (12, 21, -6)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) &= (1,-1,2)\cdot(12,21,-6) \\
&= 1\cdot 12 + (-1)\cdot 21 + 2\cdot(-6) \\
&= 12 - 21 - 12 \\
&= -21
\end{aligned}$$

The volume of the parallelepiped is the absolute value of this scalar triple product:

$$\text{Volume} = |-21| = 21$$

**Example: (Exercise 6)**

Use the scalar product to check if the following vectors are coplanar: $a = 2i + 3j + k, b = i - j$ and $c = 7i + 3j + 2k$.
The vectors are coplanar if the scalar triple product of the vectors is zero.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} \quad = (-2,-2,10)$$

$$\begin{aligned}
\mathbf{a} \cdot (-2,-2,10) &= (2\times -2) + (3\times -2) + (1\times 10) \\
&= -4 - 6 + 10 = 0
\end{aligned}$$

Since the scalar triple product is zero, the vectors are coplanar.

# 5 Functions of several variables

A function $f : A \to B$ is a correspondence between two sets $A$ and $B$ such that each element in $A$ is associated with exactly one element in $B$.

**Example:**

$$f(x,y) = x^2 + y^2 \quad \text{, where}$$
$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \to f(x,y) = x^2 + y^2$$

The formula for that function:

$$z = x^2 + y^2$$

The graph results in a paraboloid (surface).

### 5.1 Domain for a function $f$

The domain is the set of points where the function is well defined.

### 5.2 Image of a function $f$

The image is the set of points in $B$ that are associated with points in $A$.

**Example: (Exercise 8)**

- The function
  $$x^2 + 2z^2 = 1$$
  is a cylinder with an elipse section.
- The function
  $$x^2 - y^2 + z^2 = 1$$
  - If $y = k$, then $x^2 + z^2 = 1 + k^2$ is a circle.
  - If $z = 0$, then $x^2 - y^2 = 1$ is a hyperbola.
  - If $x = 0$, then $z^2 - y^2 = 1$ is a hyperbola.

### 5.3 Types of functions

- Scalar functions: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, \ldots, x_n) \in \mathbb{R}$.
- Vector functions: $f : \mathbb{R}^n \to \mathbb{R}^m, \quad f(x_1, \ldots, x_n) \in \mathbb{R}^m$.
  If $f : \mathbb{R}^n \to \mathbb{R}^n$, then $f$ is a vector field.

**Example:**

Paramatric equations for a line in $\mathbb{R}^3$:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

### 5.4 Level curves

The level curves of a function $f : \mathbb{R}^N \to \mathbb{R}$ are the curves in the domain of $f$ where $f(x,y) = k$ for some constant $k$.

$$(x_1, \ldots, x_N) \to f(x_1, \ldots, x_N) \in \mathbb{R}$$
$$f(x,y) = c, \quad c \in \mathbb{R}$$

The graph of a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$.

**Example:**

$$f(x, y) = x^2 + y^2$$

$$x^2 + y^2 = c, \quad c \in \mathbb{R}$$

The level curves are circles centered at the origin. They can be seen as the projection of the surface onto the $xy$-plane (from above).

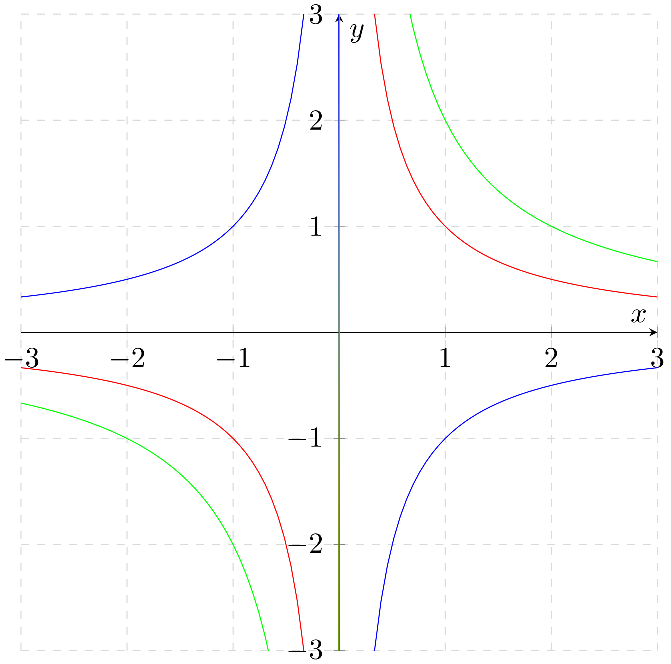
### 5.5 Remark

In $\mathbb{R}^3$, the level curves of a function $f : \mathbb{R}^3 \to \mathbb{R}$ are the curves in the domain of $f$ where $f(x, y, z) = c$ for some constant $c \in \mathbb{R}$.
They allow us to visualize a 3D graph of a function in 2D.
If the level curves are very close to each other, then the function is steep.
Two level curves never touch.

**Example:**

Find the level curves of the function $f(x, y) = xy$.

$$xy = c, \quad c = 1, -1, 2.$$

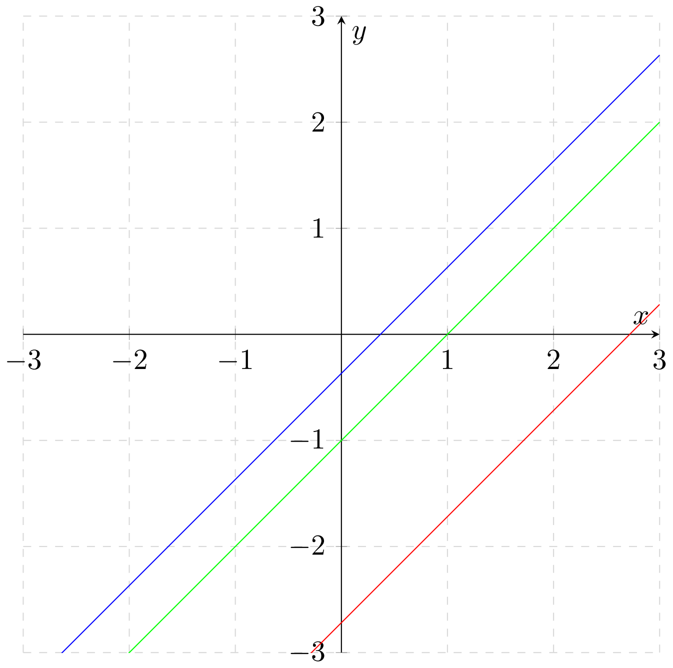



The level curves are a family of hyperbolas.

**Example:**

Find the level curves of the function $f(x, y) = log(x - y)$.

$$log(x - y) = c, \quad c = 1, -1, 0.$$



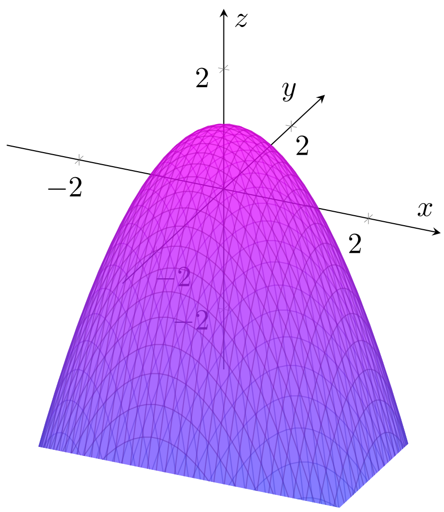

The level curves are a family of straight lines.

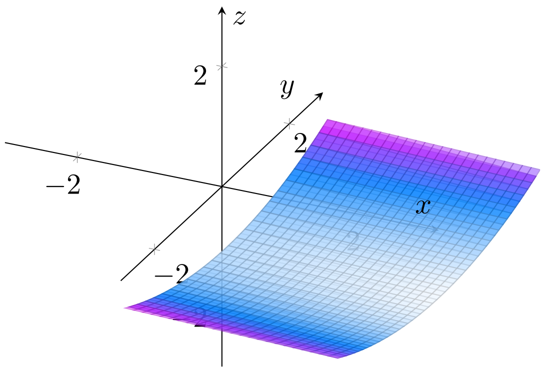
**Example:**

Find the level curves of the functions

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{and} \quad g(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$$

The level curves for $f$ are spheres centered at the origin.
The level curves for $g$ are ellipsoids centered at the origin.






### 5.6 Graph of a function

$$\{(x, f(x)), x \in Dom(f)\}, \quad \text{where } f = 9y^2 + 4z^2 = x^2 + 36$$

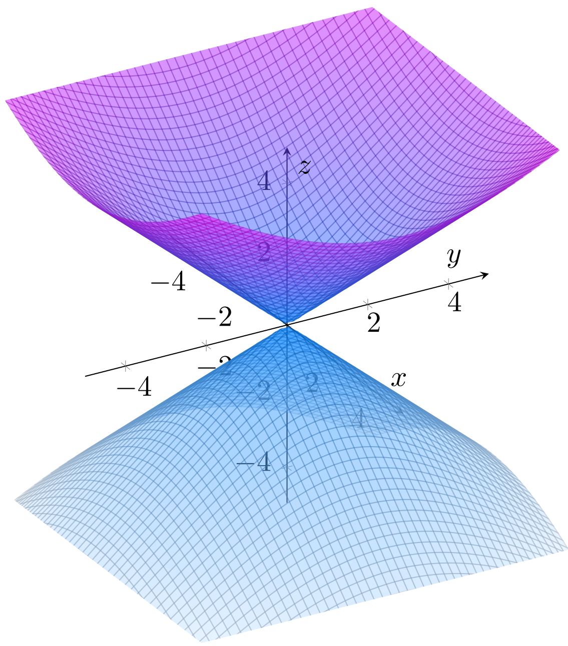
Intersection with the $x, y, z$-planes:

$$\text{If } z = 0, \quad 9y^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad 4z^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } x = 0, \quad 9y^2 + 4z^2 = 36 \to \text{Ellipse}$$

#### Example:

Plot the function $f = x^2 + 4y^2 = z^2$.

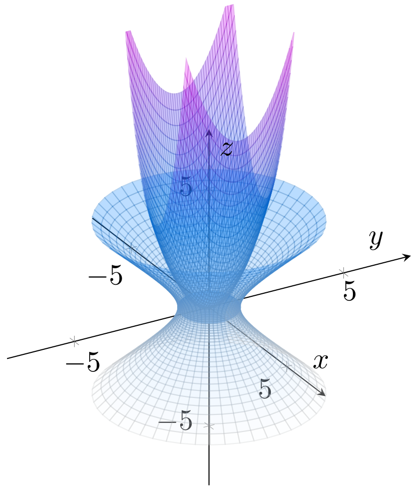
$$\text{If } z = 0, \quad x^2 + 4y^2 = 0 \to x = 0, y = 0$$
$$\text{If } y = 0, \quad x^2 = z^2 \to x = z, x = -z$$
$$\text{If } x = 0, \quad 4y^2 = z^2 \to y = z/2, y = -z/2$$
$$\text{If } z = k, \quad x^2 + 4y^2 = k^2 \to \text{Ellipses}$$




The graph is a cone.

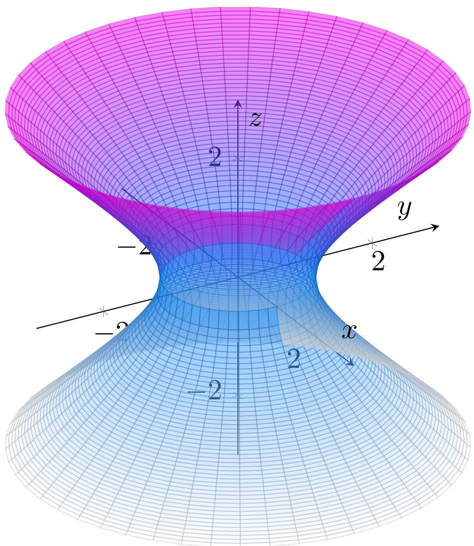
**Problem 8**

$$x^2 + y^2 + 9z^2 = 1 \to \text{Ellipsoid}$$
$$x^2 - y^2 + z^2 = 1 \to \text{Hyperboloid of one sheet}$$
$$y = 2x^2 + z^2 \to \text{Paraboloid}$$




**Example:**

$$-x^2 + y^2 - z^2 = 1 \to \text{Hyperboloid}$$
$$\text{If } z = 0, \quad -x^2 + y^2 = 1 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad -x^2 - z^2 = 1 \to \text{No solution}$$
$$\text{If } x = 0, \quad y^2 - z^2 = 1 \to \text{Hyperbola}$$

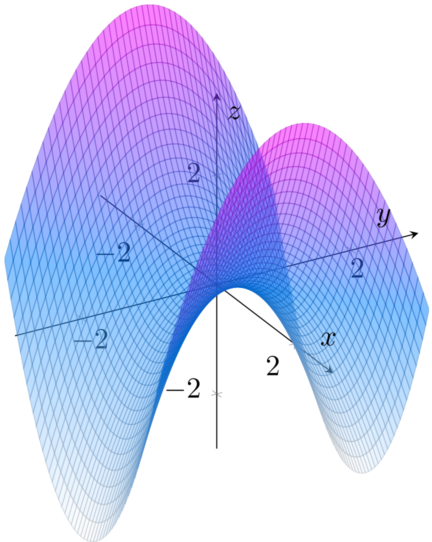



**Example:**

$$z = x^2 - y^2 \to \text{Paraboloid}$$

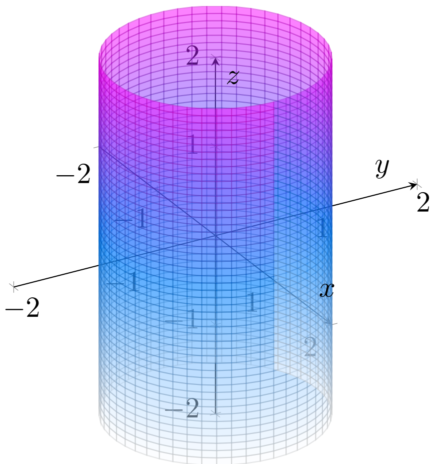
$$\text{If } z = 0, \quad x^2 - y^2 = 0 \to \text{Hyperbola}$$
$$\text{If } z = k, \quad x^2 = y^2 + k \to \text{Hyperbola}$$

$$\text{If } y = 0, \quad z = x^2 \to \text{Parabola}$$
$$\text{If } x = 0, \quad z = -y^2 \to \text{Parabola}$$




**Example:**

Plot the function $x^2 + y^2 = 1$.




# 6 Cartesian coordinates in $\mathbb{R}^N$

In $\mathbb{R}^2$, the Cartesian coordinates are $(x, y)$.
In $\mathbb{R}^3$, the Cartesian coordinates are $(x, y, z)$.

## 6.1 Polar coordinates in $\mathbb{R}^2$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

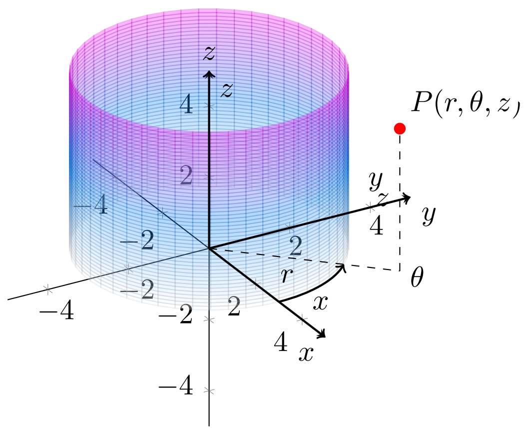
Inverse transformation:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

**Lemma** Let $A = (0, \inf) \times (0, 2\pi)$.
The function $g : A \to \mathbb{R}^2$ defined by $g(r, \theta) = (r\cos(\theta), r\sin(\theta))$ is a bijection, continuous in a ball $B(0, \alpha)$ such that $\{g(r, \theta), 0 < r < \alpha, 0 - leq\theta < 2\pi\}$ is a subset of $B(0, \alpha)$.
To see if the function is one-to-one, assume that $g(r_1, \theta_1) = f(r_2, \theta_2)$ for $r_1, r_2 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
Then $r_1\cos(\theta_1) = r_2\cos(\theta_2)$ and $r_1\sin(\theta_1) = r_2\sin(\theta_2)$. This implies that $r_1 = r_2$, since $r_1 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
As a consequence $\theta_1 = \theta_2$ so that $g$ is one-to-one.
Now taking $(x, y) \in B(0, \alpha)$, and $r = \sqrt{x^2 + y^2} > 0$.
Then, the part $(\frac{x}{2}, \frac{y}{2})$ is in $B(0, 1)$.
Therefore, there exists $\theta \in [0, 2\pi)$ such that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$.
Which implies that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. So $g$ is onto.

## 6.2 Cylindrical coordinates in $\mathbb{R}^3$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$




**Example:**

Transform into cartesian coordinates:

$$(3, \frac{\pi}{2}, 1) = (3\cos(\frac{\pi}{2}), 3\sin(\frac{\pi}{2}), 1) = (0, 3, 1)$$

**Example:**

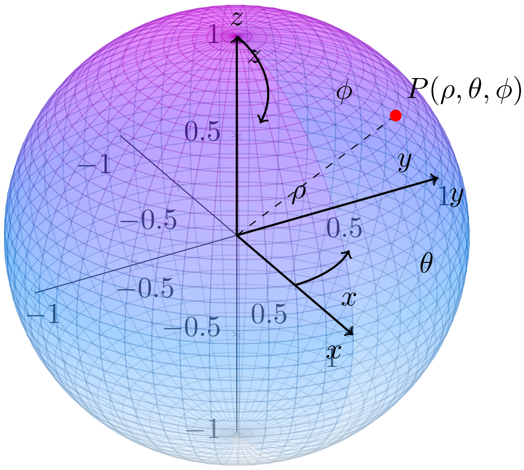
Transform into cartesian coordinates:

$$(4, -\frac{\pi}{3}, 5) = (4\cos(-\frac{\pi}{3}), 4\sin(-\frac{\pi}{3}), 5) = (2, -2\sqrt{3}, 5)$$

## 6.3 Spherical coordinates in $\mathbb{R}^3$

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi), \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$\rho$ is the distance from the origin, $\phi$ is the angle from the $z$-axis, and $\theta$ is the angle from the $x$-axis.




### Example: (Problem 6)

Transform into spherical coordinates:

$$(1, 0, 0) = (1\sin(\phi)\cos(\theta), 1\sin(\phi)\sin(\theta), 1\cos(\phi)) = (0, 0, 1)$$

$$(2, \frac{\pi}{3}, \frac{\pi}{4}) = (2\sin(\frac{\pi}{4})\cos(\frac{\pi}{3}), 2\sin(\frac{\pi}{4})\sin(\frac{\pi}{3}), 2\cos(\frac{\pi}{4})) = (\sqrt{3}, 1, \sqrt{2})$$

## 6.4 Parametric equation of a line in $\mathbb{R}^3$

Having two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the parametric equation of the line passing through $P_1$ and $P_2$ is:

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

### Example: (Problem 14)

Find the parametric equation of the line passing through $P(1, 0, 1)$ and $Q(2, 3, 1)$.

$$\begin{cases} x = 1 + t(2 - 1) = 1 + t \\ y = 0 + t(3 - 0) = 3t \\ z = 1 + t(1 - 1) = 1 \end{cases}$$

## 6.5 Parametric equation of a plane in $\mathbb{R}^3$

For a plane in parametric form:

$$\begin{cases} x = x_0 + a_1 t + a_2 s \\ y = y_0 + b_1 t + b_2 s \\ z = z_0 + c_1 t + c_2 s \end{cases}$$

**Remark** A line in $\mathbb{R}^3$ is a manifold of dimension 1.
A plane in $\mathbb{R}^3$ is a manifold of dimension 2.

### Example: (Problem 9)

Write $x^2 + y^2 + z^2 = 4$ (Sphere) in parametric form.

$$\begin{cases} x = \lambda \\ y = \mu \\ z = \sqrt{4 - \lambda^2 - \mu^2} \end{cases} \qquad \text{where } \lambda^2 + \mu^2 \leq 4$$

Using spherical coordinates:

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi, \quad \rho = 2$$

$$\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi) + \rho^2 \cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi) = 4$$
$$\rho^2 = 4 \to \rho = 2$$

### Example: (Problem 11)

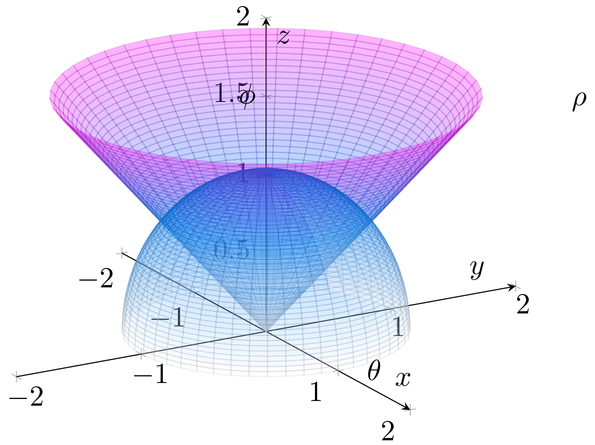
A solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Becase the solid is inside the cone, we have $z \geq \sqrt{x^2 + y^2}$
Because the solid is below the sphere, we have $x^2 + y^2 + z^2 \leq z$

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$$\sqrt{\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi)} \leq \rho \cos(\phi)$$
$$\rho \sin(\phi) \leq \rho \cos(\phi) \to \tan(\phi) \leq 1 \to \phi \leq \frac{\pi}{4}$$

$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) \leq \rho \cos(\phi)$$
$$\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi) \leq \rho \cos(\phi)$$
$$\rho^2 \leq \rho \cos(\phi) \to \rho \leq \cos(\phi)$$



Solid in cartesian coordinates:

$$\begin{cases} z \geq \sqrt{x^2+y^2} \\ x^2+y^2+z^2 \leq z \end{cases}$$

In spherical coordinates:

$$\begin{cases} \rho \leq \cos(\phi) \\ \phi \leq \frac{\pi}{4} \\ \theta \in [0, 2\pi) \end{cases}$$

## 6.6 Intersection of two bodies

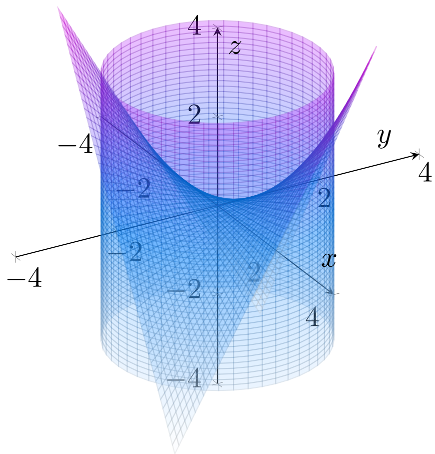
The intersection of two bodies gives (in general) a curve.

### Example: (Problem 12)

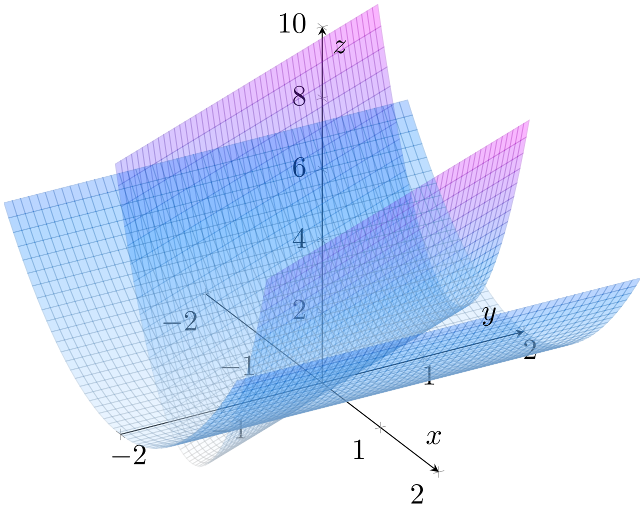
Parametrize the intersection (a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$) of the two bodies:

$$\begin{cases} \text{Cylinder: } x^2+y^2=4 \\ \text{Surface: } z = xy \end{cases}$$

$$\begin{cases} x = 2\cos(t) = \gamma_1(t) \\ y = 2\sin(t) = \gamma_2(t) \\ z = 2\cos(t)\sin(t) = \gamma_3(t) \end{cases} \qquad \text{where } t \in [0, 2\pi)$$




$$\begin{cases} \text{Paraboloid: } z = 4x^2+y \\ \text{Cylinder: } y = x^2 \end{cases} \implies \begin{cases} x = t \\ y = t^2 \\ z = 4t^2+t^4 \end{cases}$$




# 7 Limit of functions

Assume a scalar $f : \mathbb{R}^N \to \mathbb{R}$. We say that the limit of $f(x)$ as $x$ approaches $x_0$ is $L$ and we denote it by:

$$\lim_{x \to x_0} f(x) = L \quad \in \mathbb{R}, \quad x, x_0 \in \mathbb{R}^N$$

## 7.1 Definition of the limit

We say that $\lim_{x \to x_0} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\forall x \in \mathbb{R}^N, \quad 0 < ||x - x_0|| < \delta \implies |f(x) - L| < \epsilon$$

**Theorem** If the limit of $f(x)$ as $x$ approaches $x_0$ exists, then the limit is unique.

**Proof** Argue by contradiction. Assume that there are two limits $L_1$ and $L_2$ such that $L_1 \neq L_2$.

Actually, we can say that:

$$B(L_1, r_1) \cap B(L_2, r_2) = \emptyset$$

Where $B(L_1, r_1)$ is the ball of radius $r_1$ centered at $L_1$ and $B(L_2, r_2)$ is the ball of radius $r_2$ centered at $L_2$.

$$\lim_{x \to x_0} f(x) = L_1, \quad \text{for any } \epsilon_1 > 0, \quad \exists \delta_1 > 0 \text{ such that } |f(x) - L_1| < \epsilon_1 \text{ for } 0 < ||x - x_0|| < \delta_1$$

$$\lim_{x \to x_0} f(x) = L_2, \quad \text{for any } \epsilon_2 > 0, \quad \exists \delta_2 > 0 \text{ such that } |f(x) - L_2| < \epsilon_2 \text{ for } 0 < ||x - x_0|| < \delta_2$$

Indeed,

$$\begin{cases} |f(x) - L_1| < r_1 = \epsilon_1 \\ |f(x) - L_2| < r_2 = \epsilon_2 \end{cases} \implies \text{However, taking } \delta = \min(\delta_1, \delta_2) \text{ so that } ||x - x_0|| < \delta$$

$$\text{such that } \begin{cases} |f(x) - L_1| < \min(r_1, r_2) \\ |f(x) - L_2| < \min(r_1, r_2) \end{cases} \implies L_1 = L_2, \text{ which is a contradiction}$$

## 7.2 Computing limits

Using the definition of the limit, we can compute the limit of a function $f(x)$ as $x$ approaches $x_0$. To do so we must choose the value of $L$ towards the function is going. The process is as follows:

$$\text{For example in } \mathbb{R}^2, \quad \begin{cases} f : \mathbb{R}^2 \to \mathbb{R} \\ (x, y) \to f(x, y) \end{cases}$$

Then:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \epsilon$$

Hence, we must find $\delta \cong \delta(\epsilon)$.

### Example:

Prove that

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) = 0$$

$$\text{Let } \epsilon > 0, \quad \text{we must find } \delta > 0 \text{ such taht } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| < \epsilon$$

$$\text{Since } \left|\sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq 1 \text{ and } \sqrt{x^2 + y^2} \geq 0$$

$$\text{We have } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq \sqrt{x^2 + y^2} < \epsilon$$

$$\text{Therefore, we can choose } \delta = \epsilon$$

### Example:

Prove that

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\text{Let } \epsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } ||f(x) - b|| = |y - b| < \epsilon$$
$$\text{when } ||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

$$\text{We start from } |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta$$
$$\text{Therefore, we can choose } \delta = \epsilon$$

## 7.3 Iterative limits

In 2D,

$$\lim_{x\to a}\left(\lim_{y\to b} f(x, y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x, y)\right)$$

However, this technique only gives us negative answers (non-existence of the limit). Since we are just following one direction.

## 7.4 Approach following families of functions

They might be straight lines, parabolas, etc. around the point of approach.

**Example:**

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

Let y $= \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{x}{|x|\sqrt{1+\lambda^2}} = \frac{1}{\sqrt{1+\lambda^2}}$$

This limit depends on $\lambda$. Therefore, the limit does not exist.

**Example:**

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

Let $y = \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3}{x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x}{1+\lambda^2} = 0$$

Following the family of functions $y = \lambda x$, the limit is 0. However, we cannot confirm that the value of the limit is 0 or that the limit exists using this method.

This method is necessary but not sufficient. It is a good way to check if the limit does not exist.

### Problem 1a

Taking $y = \lambda x$ and knowing that $\sin x \approx x$ for $x \approx 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2+\sin^2(\lambda x)}{2x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x^2+\lambda^2 x^2}{2x^2+\lambda^2 x^2} = \frac{1+\lambda^2}{2+\lambda^2}$$

This limit depends on $\lambda$. Therefore, the limit does not exist.

### Problem 1d

Taking $y = \lambda x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x\lambda x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{\lambda x^2}{\sqrt{x^2(1+\lambda^2)}} = \lim_{x\to 0} \frac{\lambda x}{\sqrt{1+\lambda^2}} = 0$$

This limit does not depend on $\lambda$. Therefore, the limit exists and is 0.
Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{r} = \lim_{r\to 0} r\cos(\theta)\sin(\theta) = 0$$

## 7.5 Polar coordinates

In $\mathbb{R}^2$, the polar coordinates are $(r, \theta)$.

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

**Theorem** Let us consider two functions $f$ and $g$ such that

$$\lim_{x \to x_0} f(x) = 0, \quad \text{and } g \text{ is bounded for } ||x - x_0|| < \delta$$

Then

$$\lim_{x \to x_0} f(x)g(x) = 0$$

### Problem 1e

Taking $y = \lambda x$:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x \to 0} \frac{x^2 \lambda x e^{\lambda x}}{x^4 + 4\lambda^2 x^2} = \lim_{x \to 0} \frac{\lambda x^3 e^{\lambda x}}{x^2(x^2 + 4\lambda^2)} = \lim_{x \to 0} \frac{\lambda x e^{\lambda x}}{x^2 + 4\lambda^2} = 0$$

Now taking $y = x^2$:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x \to 0} \frac{x^2 x^2 e^{x^2}}{x^4 + 4x^4} = \lim_{x \to 0} \frac{x^4 e^{x^2}}{5x^4} = \lim_{x \to 0} \frac{e^{x^2}}{5} = \frac{1}{5}$$

This limit depends on the direction of approach. Therefore, the limit does not exist.

### Problem 1f

Applying generalized spherical coordinates:

$$\begin{cases} x = r \sin(\phi) \cos(\theta) \\ y = \frac{1}{2} r \sin(\phi) \sin(\theta) \\ z = \frac{1}{3} r \cos(\phi) \end{cases}$$

$$\lim_{(x,y,z) \to (0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{r \to 0} \frac{\frac{1}{2} r \sin(\phi) \sin(\theta) \frac{1}{3} r \cos(\phi)}{r^2} = \lim_{r \to 0} \frac{1}{6} \sin(\phi) \cos(\phi) \sin(\theta)$$

This limit depends on $\phi$ and $\theta$. Therefore, the limit does not exist.

## 8 Continuity of functions

A function $f : \mathbb{R}^N \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}^N$ if:

1. $f(x_0)$ is defined.
2. $\lim_{x \to x_0} f(x)$ exists.
3. $\lim_{x \to x_0} f(x) = f(x_0)$

**Theorem** Consider $A$ a subset in $\mathbb{R}^N$ and $f : A \to \mathbb{R}^M$ a vector function.

$$F(x) = (f_1(x), f_2(x), \ldots, f_M(x))$$

If $f_1, f_2, \ldots, f_M$ are continuous at $x_0 \in A$, then $F$ is continuous at $x_0$.

**Theorem** Assume that $f$ and $g$ are continuous at $x_0 \in \mathbb{R}^N$ and $f(x_0)$ respectively. Then, the composition $g \circ f$ is continuous at $x_0$.

$$\text{For example: } f(x) = x^2 y \sin(x + y), \quad f : \mathbb{R}^2 \to \mathbb{R}$$

Since $f$ is a composite function and $x^2, y, \sin(x + y)$ are continuous, then $f$ is continuous.

## Problem 2a

Prove the continuity:

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

If $(x,y) \neq (0,0)$, then $f(x,y)$ is a composition of continuous functions. So that

$$\text{for any } (x_0,y_0) \neq (0,0), \qquad \lim_{(x,y)\to(0,0)} f(x,y) = f(x_0,y_0) = \frac{x_0^2y_0^3}{2x_0^2+y_0^2}$$

At $(0,0)$, the function is well defined: $f(0,0) = 1$, so we must check that the limit exists.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} =$$

Using polar coordinates:

$$\begin{cases} x = \frac{1}{\sqrt{2}} r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$$= \lim_{r\to 0} \frac{\frac{1}{2}r^3\cos^2(\theta)\sin^3(\theta)}{r^2} = \lim_{r\to 0} \frac{1}{2} r\cos^2(\theta)\sin^3(\theta) = 0$$

The limit exists and is equal to 0. However, $f(0,0) = 1 \neq 0$. Therefore, the function is not continuous at $(0,0)$.