

# Vector Calculus

## 1 Euclidean space

We define the euclidean space in  $\mathbb{R}^N, N \geq 1$  using cartesian coordinates.  
Any element  $x \in \mathbb{R}^N$ ,  $x = (x_1, x_2, \dots, x_N)$ ,  $x \in \mathbb{R}$

### Standard basis

$$e_1 = (1, 0, \dots, 0), \dots, e_N = (0, 0, \dots, 1)$$

$$\text{Then, } x = \sum_{j=1}^N x_j \cdot e_j$$

In particular,  $B_{\mathbb{R}^3} = \{i, j, k\}$ , the canonical basis.

### Properties

- Addition:  $(x_1, \dots, x_N) + (y_1, \dots, y_N) = (x_1 + y_1, \dots, x_N + y_N)$
- Multiplication by a scalar  $\lambda \in \mathbb{R}$ :  $\lambda x = \lambda(x_1, \dots, x_N) = (\lambda x_1, \dots, \lambda x_N)$
- Associative:  $\lambda, \mu \in \mathbb{R}, (\lambda\mu)x = \lambda(\mu x)$
- Additive Identity: There exists a vector  $\bar{0} = (0, \dots, 0)$  such that  $x + \bar{0} = x$
- Additive Inverse:  $\forall x = (x_1, \dots, x_N), \exists \bar{x} = (-x_1, \dots, -x_N)$  such that  $x + \bar{x} = \bar{0}$
- Distributive Property (over vector addition):

$$\lambda((x_1, \dots, x_N) + (y_1, \dots, y_N)) = \lambda(x_1, \dots, x_N) + \lambda(y_1, \dots, y_N)$$

- Distributive Property (over scalar addition):

$$(\lambda + \mu)(x_1, \dots, x_N) = \lambda(x_1, \dots, x_N) + \mu(x_1, \dots, x_N)$$

- Scalar Multiplication Identity:  $1 \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)$
- Zero Scalar Multiplication:  $0 \cdot (x_1, \dots, x_N) = (0, \dots, 0)$

### Norm

The euclidean space in  $\mathbb{R}^N$  is a normal space with an associated norm function.

$$\|\cdot\| : \mathbb{R}^N \rightarrow \mathbb{R}_+ \cup \{0\}$$

$$x = (x_1, \dots, x_N) \rightarrow \|x\| = \sqrt{x_1^2 + \dots + x_N^2}$$

## Properties

The norm satisfies the following properties:

- (a)  $\forall x \in \mathbb{R}^N$ 
  - $\|x\| > 0 \iff x \neq 0$
  - $\|x\| = 0 \iff x = 0$
- (b)  $\|\lambda x\| = |\lambda| \|x\|$
- (c)  $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$ 
  - Triangular inequality.

## Remark: Distance

We can define the distance between two elements in  $\mathbb{R}^N$  as

$$\begin{aligned} \text{dist}(x, y) &= \|x - y\| = \|y - x\| \\ \text{dist}(\cdot, \cdot) &: \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \end{aligned}$$

- $\text{dist}(x, y) = \|x - y\| > 0$  if  $x \neq y$ , and  $\text{dist}(x, y) = 0$  if  $x = y$
- $\text{dist}(x, y) = \|x - y\| = \|-(y - x)\| = \|-1\| \cdot \|y - x\| = \text{dist}(y, x)$
- $\text{dist}(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = \text{dist}(x, z) + \text{dist}(z, y)$

## Remark

For  $\mathbb{R}$  such a distance is the absolute value,  $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$

## 2 Inner or scalar product

Let  $x, y$  be two vectors in  $\mathbb{R}^N$ , then

$$\begin{aligned} x \cdot y &= x_1 y_1 + \cdots + x_N y_N \\ x \cdot y &= \langle x, y \rangle = (x, y) \end{aligned}$$

### 2.1 Properties

The inner product satisfies the following properties:

- $\forall x \in \mathbb{R}^N \quad \langle x, x \rangle \geq 0$ 
  - $\langle x, x \rangle = 0$  if  $x = 0$
- Symmetric:  $\langle x, y \rangle = \langle y, x \rangle$
- Bilinear:  $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

### 2.2 Cauchy-Schwartz inequality

$$|x \cdot y| \leq \|x\| \|y\|$$

**Proof** If  $y = \lambda x$ ,  $|\langle x, \lambda y \rangle| = |\lambda| \|x\|^2 = \|x\| |\lambda| \|x\| = \|x\| \|y\|$

If  $y \neq \lambda x$  ( $x$  and  $y$  are linearly independent).

Assume  $z = \lambda x + y$

$$0 \leq \langle z, z \rangle = \langle \lambda x + y, \lambda x + y \rangle = \langle \lambda x, \lambda x \rangle + \langle \lambda x, y \rangle + \langle y, \lambda x \rangle + \langle y, y \rangle$$

$$\text{Since } \|x\|^2 > 0, \quad = \lambda^2 \|x\|^2 + 2\lambda \langle x, y \rangle + \|y\|^2$$

If we represent it as a parabola in function of  $\lambda$ , it has no roots.

$$\lambda = \frac{-2\langle x, y \rangle \pm \sqrt{4(\langle x, y \rangle)^2 - 4\|x\|^2\|y\|^2}}{2\|x\|^2}$$

So the discriminant  $\leq 0$

$$4(\langle x, y \rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\implies |\langle x, y \rangle| \leq \|x\| \|y\|$$

## 2.3 Theorem

$$\langle x, y \rangle = \|x\| \|y\| \cos \varphi$$

Writing  $x$  and  $y$  in polar coordinates:

$$x_1 = \|x\| \cos \alpha, \quad x_2 = \|x\| \sin \alpha$$

$$y_1 = \|y\| \cos \beta, \quad y_2 = \|y\| \sin \beta$$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 = \|x\| \cos \alpha \|y\| \cos \beta + \|x\| \sin \alpha \|y\| \sin \beta$$

$$= \|x\| \|y\| (\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

$$= \|x\| \|y\| \cos(\alpha - \beta) = \|x\| \|y\| \cos \varphi$$

## Remark

$$\cos \varphi = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$$\text{Then } x \perp y \iff \langle x, y \rangle = 0$$

## Examples:

- $C((a, b)) \cong$  continuous functions in  $(a, b)$

$$f, g \in C((a, b)), \quad \text{then } \langle f, g \rangle = \int_a^b f(t)g(t) dt$$

We can add some weights:

$$\langle f, g \rangle = \int_a^b w(t) f(t) g(t) dt$$

An example:

$$\langle f, g \rangle = \int_a^b e^{-t} f(t) g(t) dt$$

- We might have an orthogonal family (with infinite elements) of functions.

$$\{\cos nx, \sin nx\}_{n \in \mathbb{Z}} \quad \text{in } [0, 2\pi]$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx \, dx$$

$$\int_0^{2\pi} \cos nx \sin nx \, dx = \int_0^{2\pi} \cos nx \cos mx \, dx = 0 \quad \text{for } n \neq m$$

### 3 Vector Product (Only in $\mathbb{R}^3$ )

Take  $x, y \in \mathbb{R}^3$

$$\begin{aligned} x \times y &= \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k \end{aligned}$$

Recall:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

#### 3.1 Triple product and properties

We take the triple product

$$\begin{aligned} a \cdot (b \times c) &= (a_1, a_2, a_3) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

If  $u \in \text{span}\{b, c\}$  then  $a \cdot (b \times c) = 0$  and  $a, b, c$  are coplanar if  $a \cdot (b \times c) = 0$

#### 3.2 Geometric Interpretation

- The magnitude of  $x \times y$  represents the area of the parallelogram formed by  $x$  and  $y$ .
- The direction of  $x \times y$  is perpendicular to the plane spanned by  $x$  and  $y$ , following the right-hand rule.
- The cross product satisfies:  $x \times y = -(y \times x)$ .

## 4 Topology of $\mathbb{R}^n$

Definition of open spaces: we define an open ball in  $\mathbb{R}^n$  centered at  $x_0$  and of radius  $R$ .

$$\text{Denoted by } B_R(x_0) = \{x \in \mathbb{R}^n : \text{dist}(x, x_0) < R\}$$

This set includes all points in  $\mathbb{R}^n$  whose distance from  $x_0$  is less than  $R$ . Open sets are the building blocks of topological spaces, and they help define concepts such as convergence, continuity, and compactness.

### 4.1 Open set

A set  $A \subset \mathbb{R}^n$  is open if  $\forall x \in A, \exists R > 0$  such that  $B_R(x) \subset A$ .

For example:

$$(x, y) \in \mathbb{R}^2, \quad A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 3\}$$

### 4.2 Closed set

A set  $A \subset \mathbb{R}^n$  is closed if its complement is open.

$$A \subset \mathbb{R}^n \text{ is closed if } \mathbb{R}^n \setminus A \text{ is open.}$$

### 4.3 Boundary of a set

The boundary of a set  $A \subset \mathbb{R}^n$  denoted by  $\partial A$ :

$$\partial A = \{x \in \mathbb{R}^n : \forall R > 0, \quad B_R(x) \cap A \neq \emptyset \text{ and } B_R(x) \cap (\mathbb{R}^n \setminus A) \neq \emptyset\}$$

Following the previous example, the boundary of  $A$  is the circle of radius 3 centered at the origin:

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 3\}$$

#### Remark

A set  $A \subset \mathbb{R}^n$  is closed if and only if it contains its boundary.

#### Example:

$$D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1 \text{ and } x > 0\}$$

$$D^C = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 1 \text{ or } x \leq 0\}$$

$$\partial D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1 \text{ and } x > 0\}$$

$D$  is open,  $D^C$  is closed, and  $\partial D$  is the semicircle of radius 1 centered at the origin.

#### Example:

$$S = \{x = 1 \text{ and } 1 < y \leq 2\}$$

$$S^C = \{x \neq 1 \text{ or } y \leq 1 \text{ or } y > 2\}$$

$$\partial S = \{x = 1 \text{ and } y = 1 \text{ or } y = 2\}$$

$S$  is neither open nor closed, as  $S^C$ , and  $\partial S$  is the line segment from  $(1, 1)$  to  $(1, 2)$ .

## 4.4 Compact set

A set  $A \subset \mathbb{R}^n$  is compact if and only if it is closed and bounded.

**Example:**

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$

$A = \partial A$  so  $A$  is closed.  $A$  is also bounded, as all points in  $A$  are contained within the circle of radius 2 centered at the origin.

$\implies A$  is compact.

**Example: (Exercise 11a)**

$$A = xy - \text{plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3\}$$

This set is closed, and  $B_R(x, R) \cap A = \emptyset$  and  $B_R(x, R) \cap (\mathbb{R}^3 \setminus A) = \emptyset$ . Therefore,  $A$  is not compact.

**Example: (Exercise 11b)**

$$A = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$$

For any point  $(x, y) \in A$ , we can find an open ball  $B_R(x, R) \subset B$  that does not intersect the  $x$  or  $y$  axes.

$$R = \min\{x, y\}$$

## 4.5 Ball in $\mathbb{R}^n$

For a ball at any part of radius  $r$  in  $\mathbb{R}^n$ :

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2$$

$$\text{dist}(x - a) = \|x - a\| = r$$

**Example: (Exercise 1a)**

Sphere centered at  $(0, 1, -1)$  with  $r = 4$

$$(x - 0)^2 + (y - 1)^2 + (z + 1)^2 = 16$$

Intersection with the  $x, y, z$ -planes:

$$\text{If } z = -1, \quad x^2 + (y - 1)^2 = 16$$

$$\text{If } y = 1, \quad x^2 + (z + 1)^2 = 16$$

$$\text{If } x = 0, \quad (y - 1)^2 + (z + 1)^2 = 16$$

**Example: (Exercise 1b)**

Sphere going through the origin and centered at  $(1, 2, 3)$ :

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = r^2$$

$$\text{dist}(0, (1, 2, 3)) = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

So the sphere is  $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

**Example: (Exercise 1c)**

Find center and radius of the sphere:

$$x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$$

$$(x + 1)^2 + (y + 4)^2 + (z - 2)^2 = 49$$

So the center is  $(-1, -4, 2)$  and the radius is 7.

**Example: (Exercise 3)**

Check if the vectors are orthogonal:

$$a = (-5, 3, 7), \quad b = (6, -8, 2)$$

$$a \cdot b = -30 - 24 + 14 = -40 \neq 0$$

So  $a$  and  $b$  are not orthogonal.

**Example: (Exercise 4a)**

Let  $P$  be a point not on the line  $L$  that passes through the points  $Q$  and  $R$ . Show that the distance  $d$  from the point  $P$  to the line  $L$  is:

$$d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Let } a = R - Q, \quad b = P - Q$$

$$\text{Then } d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Area of parallelogram} = \|a \times b\| = \|a\| \cdot d$$

**Example: (Exercise 4b)**

Use the formula in part (a) to find the distance from the point  $P(1, 1, 1)$  to the line through  $Q(0, 6, 8)$  and  $R(-1, 4, 7)$ .

$$\mathbf{a} = \overrightarrow{QR} = R - Q = (-1 - 0, 4 - 6, 7 - 8) = (-1, -2, -1)$$

$$\mathbf{b} = \overrightarrow{QP} = P - Q = (1 - 0, 1 - 6, 1 - 8) = (1, -5, -7)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -1 \\ -5 & -7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & -7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -2 \\ 1 & -5 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = (9, -8, 7)$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{9^2 + (-8)^2 + 7^2} = \sqrt{81 + 64 + 49} = \sqrt{194}$$

$$\|\mathbf{a}\| = \sqrt{(-1)^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|} = \frac{\sqrt{194}}{\sqrt{6}} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$$

**Example: (Exercise 5)**

Calculate the volume of the parallelepiped with edges adjacent to  $\overrightarrow{PQ}$ ,  $\overrightarrow{PR}$ , and  $\overrightarrow{PS}$ , where

$$P(1, 1, 1), \quad Q(2, 0, 3), \quad R(4, 1, 7), \quad S(3, -1, -2).$$

The volume of the parallelepiped can be calculated using the scalar triple product of the vectors  $\overrightarrow{PQ}$ ,  $\overrightarrow{PR}$ , and  $\overrightarrow{PS}$ .

$$\overrightarrow{PQ} = Q - P = (2 - 1, 0 - 1, 3 - 1) = (1, -1, 2)$$

$$\overrightarrow{PR} = R - P = (4 - 1, 1 - 1, 7 - 1) = (3, 0, 6)$$

$$\overrightarrow{PS} = S - P = (3 - 1, -1 - 1, -2 - 1) = (2, -2, -3)$$

$$\begin{aligned} \overrightarrow{PR} \times \overrightarrow{PS} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} = \mathbf{i}(0 \cdot (-3) - 6 \cdot (-2)) - \mathbf{j}(3 \cdot (-3) - 6 \cdot 2) + \mathbf{k}(3 \cdot (-2) - 0 \cdot 2) \\ &= \mathbf{i}(0 + 12) - \mathbf{j}(-9 - 12) + \mathbf{k}(-6 - 0) \\ &= 12\mathbf{i} + 21\mathbf{j} - 6\mathbf{k} \\ &= (12, 21, -6) \end{aligned}$$

$$\begin{aligned} \overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) &= (1, -1, 2) \cdot (12, 21, -6) \\ &= 1 \cdot 12 + (-1) \cdot 21 + 2 \cdot (-6) \\ &= 12 - 21 - 12 \\ &= -21 \end{aligned}$$

The volume of the parallelepiped is the absolute value of this scalar triple product:

$$\text{Volume} = |-21| = 21$$

**Example: (Exercise 6)**

Use the scalar product to check if the following vectors are coplanar:  $a = 2i + 3j + k$ ,  $b = i - j$  and  $c = 7i + 3j + 2k$ .

The vectors are coplanar if the scalar triple product of the vectors is zero.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = (-2, -2, 10)$$

$$\begin{aligned} \mathbf{a} \cdot (-2, -2, 10) &= (2 \times -2) + (3 \times -2) + (1 \times 10) \\ &= -4 - 6 + 10 = 0 \end{aligned}$$

Since the scalar triple product is zero, the vectors are coplanar.

**5 Functions of several variables**

A function  $f : A \rightarrow B$  is a correspondence between two sets  $A$  and  $B$  such that each element in  $A$  is associated with exactly one element in  $B$ .



**Example:**

$$\begin{aligned}f(x, y) &= x^2 + y^2, \text{ where} \\f &: \mathbb{R}^2 \rightarrow \mathbb{R} \\(x, y) &\rightarrow f(x, y) = x^2 + y^2\end{aligned}$$

The formula for that function:

$$z = x^2 + y^2$$

The graph results in a paraboloid (surface).

### 5.1 Domain for a function $f$

The domain is the set of points where the function is well defined.

### 5.2 Image of a function $f$

The image is the set of points in  $B$  that are associated with points in  $A$ .

**Example: (Exercise 8)**

- The function

$$x^2 + 2z^2 = 1$$

is a cylinder with an ellipse section.

- The function

$$x^2 - y^2 + z^2 = 1$$

- If  $y = k$ , then  $x^2 + z^2 = 1 + k^2$  is a circle.
- If  $z = 0$ , then  $x^2 - y^2 = 1$  is a hyperbola.
- If  $x = 0$ , then  $z^2 - y^2 = 1$  is a hyperbola.

### 5.3 Types of functions

- Scalar functions:  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f(x_1, \dots, x_n) \in \mathbb{R}$ .
- Vector functions:  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f(x_1, \dots, x_n) \in \mathbb{R}^m$ .  
If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , then  $f$  is a vector field.

**Example:**

Parametric equations for a line in  $\mathbb{R}^3$ :

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

### 5.4 Level curves

The level curves of a function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  are the curves in the domain of  $f$  where  $f(x, y) = k$  for some constant  $k$ .

$$\begin{aligned}(x_1, \dots, x_N) &\rightarrow f(x_1, \dots, x_N) \in \mathbb{R} \\f(x, y) &= c, \quad c \in \mathbb{R}\end{aligned}$$

The graph of a scalar function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a surface in  $\mathbb{R}^3$ .

**Example:**

$$f(x, y) = x^2 + y^2$$
$$x^2 + y^2 = c, \quad c \in \mathbb{R}$$

The level curves are circles centered at the origin. They can be seen as the projection of the surface onto the  $xy$ -plane (from above).

### 5.5 Remark

In  $\mathbb{R}^3$ , the level curves of a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  are the curves in the domain of  $f$  where  $f(x, y, z) = c$  for some constant  $c \in \mathbb{R}$ .

They allow us to visualize a 3D graph of a function in 2D.

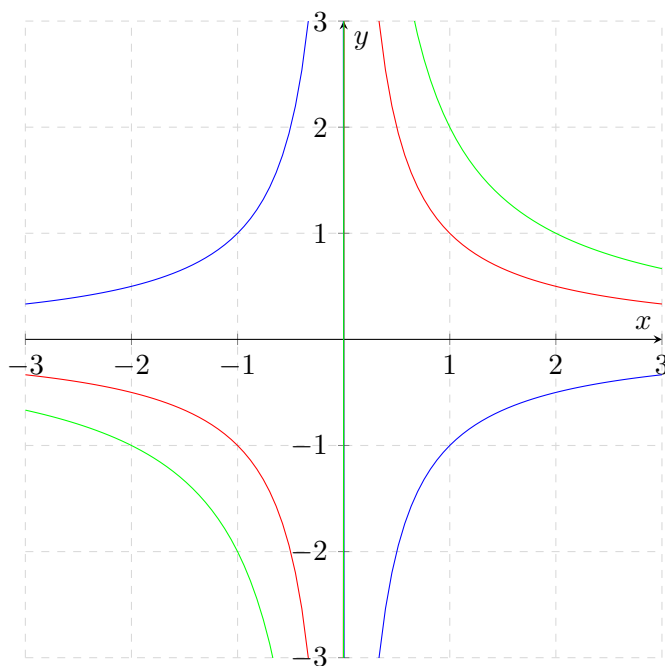
If the level curves are very close to each other, then the function is steep.

Two level curves never touch.

**Example:**

Find the level curves of the function  $f(x, y) = xy$ .

$$xy = c, \quad c = 1, -1, 2.$$



The level curves are a family of hyperbolas.

**Example:**

Find the level curves of the function  $f(x, y) = \log(x - y)$ .

$$\log(x - y) = c, \quad c = 1, -1, 0.$$



The level curves are a family of straight lines.

### Example:

Find the level curves of the functions

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{and} \quad g(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$$

The level curves for  $f$  are spheres centered at the origin.

The level curves for  $g$  are ellipsoids centered at the origin.





## 5.6 Graph of a function

$$\{(x, f(x)), x \in \text{Dom}(f)\}, \quad \text{where } f = 9y^2 + 4z^2 = x^2 + 36$$

Intersection with the  $x, y, z$ -planes:

$$\text{If } z = 0, \quad 9y^2 - x^2 = 36 \rightarrow \text{Hyperbola}$$

$$\text{If } y = 0, \quad 4z^2 - x^2 = 36 \rightarrow \text{Hyperbola}$$

$$\text{If } x = 0, \quad 9y^2 + 4z^2 = 36 \rightarrow \text{Ellipse}$$

### Example:

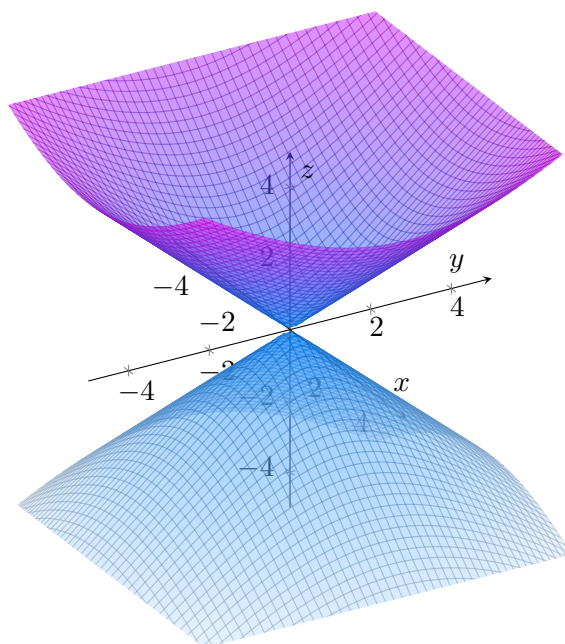
Plot the function  $f = x^2 + 4y^2 = z^2$ .

$$\text{If } z = 0, \quad x^2 + 4y^2 = 0 \rightarrow x = 0, y = 0$$

$$\text{If } y = 0, \quad x^2 = z^2 \rightarrow x = z, x = -z$$

$$\text{If } x = 0, \quad 4y^2 = z^2 \rightarrow y = z/2, y = -z/2$$

$$\text{If } z = k, \quad x^2 + 4y^2 = k^2 \rightarrow \text{Ellipses}$$



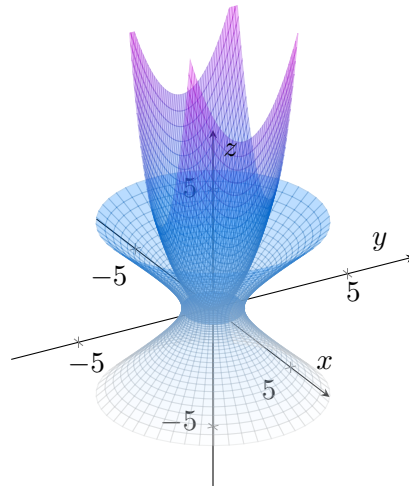
The graph is a cone.

### Problem 8

$$x^2 + y^2 + 9z^2 = 1 \rightarrow \text{Ellipsoid}$$

$$x^2 - y^2 + z^2 = 1 \rightarrow \text{Hyperboloid of one sheet}$$

$$y = 2x^2 + z^2 \rightarrow \text{Paraboloid}$$



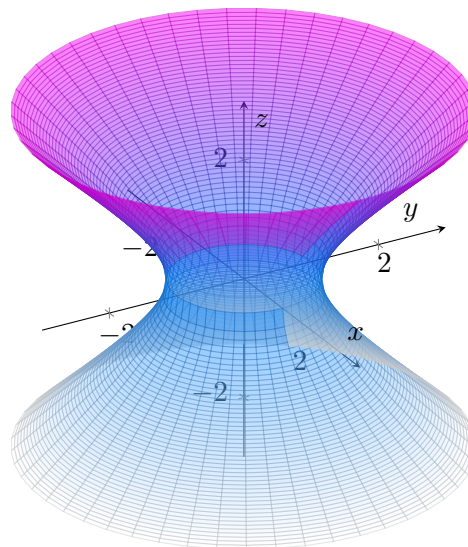
### Example:

$$-x^2 + y^2 - z^2 = 1 \rightarrow \text{Hyperboloid}$$

If  $z = 0$ ,  $-x^2 + y^2 = 1 \rightarrow \text{Hyperbola}$

If  $y = 0$ ,  $-x^2 - z^2 = 1 \rightarrow \text{No solution}$

If  $x = 0$ ,  $y^2 - z^2 = 1 \rightarrow \text{Hyperbola}$



### Example:

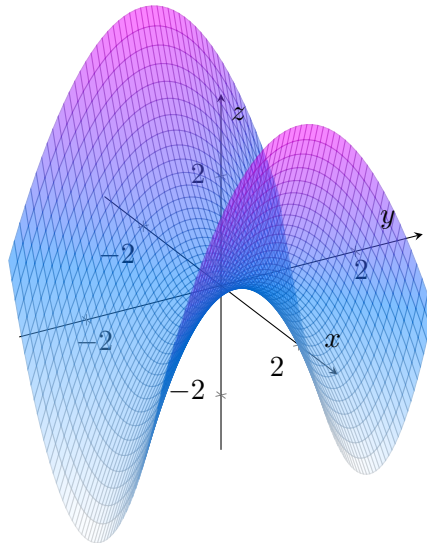
$$z = x^2 - y^2 \rightarrow \text{Paraboloid}$$

If  $z = 0$ ,  $x^2 - y^2 = 0 \rightarrow \text{Hyperbola}$

If  $z = k$ ,  $x^2 = y^2 + k \rightarrow \text{Hyperbola}$

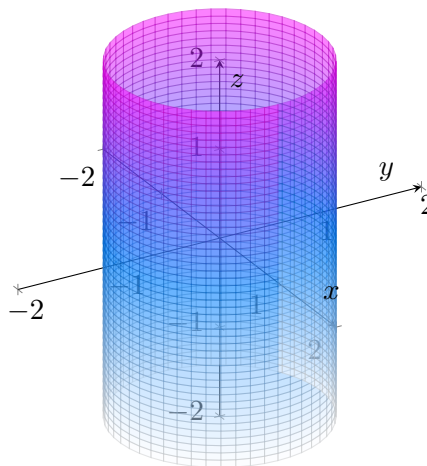
If  $y = 0$ ,  $z = x^2 \rightarrow$  Parabola

If  $x = 0$ ,  $z = -y^2 \rightarrow$  Parabola



**Example:**

Plot the function  $x^2 + y^2 = 1$ .



## 6 Cartesian coordinates in $\mathbb{R}^N$

In  $\mathbb{R}^2$ , the Cartesian coordinates are  $(x, y)$ .

In  $\mathbb{R}^3$ , the Cartesian coordinates are  $(x, y, z)$ .

### 6.1 Polar coordinates in $\mathbb{R}^2$

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

Inverse transformation:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

**Lemma** Let  $A = (0, \infty) \times (0, 2\pi)$ .

The function  $g : A \rightarrow \mathbb{R}^2$  defined by  $g(r, \theta) = (r \cos(\theta), r \sin(\theta))$  is a bijection, continuous in a ball  $B(0, \alpha)$  such that  $\{g(r, \theta), 0 < r < \alpha, 0 \leq \theta < 2\pi\}$  is a subset of  $B(0, \alpha)$ . To see if the function is one-to-one, assume that  $g(r_1, \theta_1) = g(r_2, \theta_2)$  for  $r_1, r_2 \geq 0$  and  $0 \leq \theta_1, \theta_2 < 2\pi$ .

Then  $r_1 \cos(\theta_1) = r_2 \cos(\theta_2)$  and  $r_1 \sin(\theta_1) = r_2 \sin(\theta_2)$ . This implies that  $r_1 = r_2$ , since  $r_1 \geq 0$  and  $0 \leq \theta_1, \theta_2 < 2\pi$ .

As a consequence  $\theta_1 = \theta_2$  so that  $g$  is one-to-one.

Now taking  $(x, y) \in B(0, \alpha)$ , and  $r = \sqrt{x^2 + y^2} > 0$ .

Then, the point  $(\frac{x}{r}, \frac{y}{r})$  is in  $B(0, 1)$ .

Therefore, there exists  $\theta \in [0, 2\pi)$  such that  $\cos(\theta) = \frac{x}{r}$  and  $\sin(\theta) = \frac{y}{r}$ .

Which implies that  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ . So  $g$  is onto.

## 6.2 Cylindrical coordinates in $\mathbb{R}^3$

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z$$



**Example:**

Transform into cartesian coordinates:

$$(3, \frac{\pi}{2}, 1) = (3 \cos(\frac{\pi}{2}), 3 \sin(\frac{\pi}{2}), 1) = (0, 3, 1)$$

**Example:**

Transform into cartesian coordinates:

$$(4, -\frac{\pi}{3}, 5) = (4 \cos(-\frac{\pi}{3}), 4 \sin(-\frac{\pi}{3}), 5) = (2, -2\sqrt{3}, 5)$$

## 6.3 Spherical coordinates in $\mathbb{R}^3$

$$x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad z = \rho \cos(\phi), \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$\rho$  is the distance from the origin,  $\phi$  is the angle from the  $z$ -axis, and  $\theta$  is the angle from the  $x$ -axis.



### Example: (Problem 6)

Transform from spherical coordinates to cartesian coordinates:

$$(1, 0, 0) = (1 \sin(0) \cos(0), 1 \sin(0) \sin(0), 1 \cos(0)) = (0, 0, 1)$$

$$(2, \frac{\pi}{3}, \frac{\pi}{4}) = (2 \sin(\frac{\pi}{4}) \cos(\frac{\pi}{3}), 2 \sin(\frac{\pi}{4}) \sin(\frac{\pi}{3}), 2 \cos(\frac{\pi}{4})) = (\sqrt{3}, 1, \sqrt{2})$$

## 6.4 Parametric equation of a line in $\mathbb{R}^3$

Having two points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$ , the parametric equation of the line passing through  $P_1$  and  $P_2$  is:

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

### Example: (Problem 14)

Find the parametric equation of the line passing through  $P(1, 0, 1)$  and  $Q(2, 3, 1)$ .

$$\begin{cases} x = 1 + t(2 - 1) = 1 + t \\ y = 0 + t(3 - 0) = 3t \\ z = 1 + t(1 - 1) = 1 \end{cases}$$

## 6.5 Parametric equation of a plane in $\mathbb{R}^3$

For a plane in parametric form:

$$\begin{cases} x = x_0 + a_1t + a_2s \\ y = y_0 + b_1t + b_2s \\ z = z_0 + c_1t + c_2s \end{cases}$$

**Remark** A line in  $\mathbb{R}^3$  is a manifold of dimension 1.  
A plane in  $\mathbb{R}^3$  is a manifold of dimension 2.



**Example: (Problem 9)**

Write  $x^2 + y^2 + z^2 = 4$  (Sphere) in parametric form.

$$\begin{cases} x = \lambda \\ y = \mu \\ z = \sqrt{4 - \lambda^2 - \mu^2} \end{cases} \quad \text{where } \lambda^2 + \mu^2 \leq 4$$

Using spherical coordinates:

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \quad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi, \quad \rho = 2$$

$$\rho^2 \cos^2(\theta) \sin^2(\phi) + \rho^2 \sin^2(\theta) \sin^2(\phi) + \rho^2 \cos^2(\phi) = 4$$

$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) = 4$$

$$\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi) = 4$$

$$\rho^2 = 4 \rightarrow \rho = 2$$

**Example: (Problem 11)**

A solid above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ .

Because the solid is inside the cone, we have  $z \geq \sqrt{x^2 + y^2}$

Because the solid is below the sphere, we have  $x^2 + y^2 + z^2 \leq z$

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \quad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$$\sqrt{\rho^2 \cos^2(\theta) \sin^2(\phi) + \rho^2 \sin^2(\theta) \sin^2(\phi)} \leq \rho \cos(\phi)$$

$$\rho \sin(\phi) \leq \rho \cos(\phi) \rightarrow \tan(\phi) \leq 1 \rightarrow \phi \leq \frac{\pi}{4}$$

$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) \leq \rho \cos(\phi)$$

$$\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi) \leq \rho \cos(\phi)$$

$$\rho^2 \leq \rho \cos(\phi) \rightarrow \rho \leq \cos(\phi)$$



Solid in cartesian coordinates:

$$\begin{cases} z \geq \sqrt{x^2 + y^2} \\ x^2 + y^2 + z^2 \leq z \end{cases}$$

In spherical coordinates:

$$\begin{cases} \rho \leq \cos(\phi) \\ \phi \leq \frac{\pi}{4} \\ \theta \in [0, 2\pi) \end{cases}$$

## 6.6 Intersection of two bodies

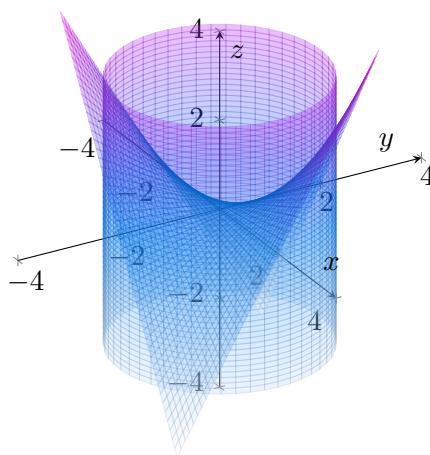
The intersection of two bodies gives (in general) a curve.

### Example: (Problem 12)

Parametrize the intersection (a curve  $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ ) of the two bodies:

$$\begin{cases} \text{Cylinder: } x^2 + y^2 = 4 \\ \text{Surface: } z = xy \end{cases}$$

$$\begin{cases} x = 2 \cos(t) = \gamma_1(t) \\ y = 2 \sin(t) = \gamma_2(t) \\ z = 4 \cos(t) \sin(t) = \gamma_3(t) \end{cases} \quad \text{where } t \in [0, 2\pi)$$



$$\begin{cases} \text{Paraboloid: } z = 4x^2 + y \\ \text{Cylinder: } y = x^2 \end{cases} \implies \begin{cases} x = t \\ y = t^2 \\ z = 4t^2 + t^2 = 5t^2 \end{cases}$$



## 7 Limit of functions

Assume a scalar  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ . We say that the limit of  $f(x)$  as  $x$  approaches  $x_0$  is  $L$  and we denote it by:

$$\lim_{x \rightarrow x_0} f(x) = L \quad \in \mathbb{R}, \quad x, x_0 \in \mathbb{R}^N$$

### 7.1 Definition of the limit

We say that  $\lim_{x \rightarrow x_0} f(x) = L$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that:

$$\forall x \in \mathbb{R}^N, \quad 0 < \|x - x_0\| < \delta \implies |f(x) - L| < \varepsilon$$

**Theorem** If the limit of  $f(x)$  as  $x$  approaches  $x_0$  exists, then the limit is unique.

**Proof** Argue by contradiction. Assume that there are two limits  $L_1$  and  $L_2$  such that  $L_1 \neq L_2$ .

Actually, we can say that:

$$B(L_1, r_1) \cap B(L_2, r_2) = \emptyset$$

Where  $B(L_1, r_1)$  is the ball of radius  $r_1$  centered at  $L_1$  and  $B(L_2, r_2)$  is the ball of radius  $r_2$  centered at  $L_2$ .

$$\lim_{x \rightarrow x_0} f(x) = L_1, \quad \text{for any } \varepsilon_1 > 0, \quad \exists \delta_1 > 0 \text{ such that } |f(x) - L_1| < \varepsilon_1 \text{ for } 0 < \|x - x_0\| < \delta_1$$

$$\lim_{x \rightarrow x_0} f(x) = L_2, \quad \text{for any } \varepsilon_2 > 0, \quad \exists \delta_2 > 0 \text{ such that } |f(x) - L_2| < \varepsilon_2 \text{ for } 0 < \|x - x_0\| < \delta_2$$

Indeed,

$$\begin{cases} |f(x) - L_1| < r_1 = \varepsilon_1 \\ |f(x) - L_2| < r_2 = \varepsilon_2 \end{cases} \implies \text{However, taking } \delta = \min(\delta_1, \delta_2) \text{ so that } \|x - x_0\| < \delta$$

$$\text{such that } \begin{cases} |f(x) - L_1| < \min(r_1, r_2) \\ |f(x) - L_2| < \min(r_1, r_2) \end{cases} \implies L_1 = L_2, \text{ which is a contradiction}$$

## 7.2 Computing limits

Using the definition of the limit, we can compute the limit of a function  $f(x)$  as  $x$  approaches  $x_0$ . To do so we must choose the value of  $L$  towards the function is going. The process is as follows:

$$\text{For example in } \mathbb{R}^2, \quad \begin{cases} f : \mathbb{R}^2 \rightarrow \mathbb{R} \\ (x, y) \rightarrow f(x, y) \end{cases}$$

Then:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Hence, we must find  $\delta \cong \delta(\varepsilon)$ .

### Example:

Prove that

$$\lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) = 0$$

$$\text{Let } \varepsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } \left| \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) \right| < \varepsilon$$

$$\text{Since } \left| \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq 1 \text{ and } \sqrt{x^2 + y^2} \geq 0$$

$$\text{We have } \left| \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq \sqrt{x^2 + y^2} < \varepsilon$$

Therefore, we can choose  $\delta = \varepsilon$

### Example:

Prove that

$$\lim_{(x,y) \rightarrow (a,b)} y = b$$

$$\text{Let } \varepsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } ||f(x) - b|| = |y - b| < \varepsilon$$

$$\text{when } ||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

$$\text{We start from } |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

Therefore, we can choose  $\delta = \varepsilon$

## 7.3 Iterative limits

In 2D,

$$\lim_{x \rightarrow a} \left( \lim_{y \rightarrow b} f(x, y) \right) = \lim_{y \rightarrow b} \left( \lim_{x \rightarrow a} f(x, y) \right)$$

However, this technique only gives us negative answers (non-existence of the limit). Since we are just following one direction.

## 7.4 Approach following families of functions

They might be straight lines, parabolas, etc. around the point of approach.

**Example:**

Compute the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2 + y^2}}$$

Let  $y = \lambda x$ ,  $\lambda \in \mathbb{R}$ . Then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2 + y^2}} = \lim_{x \rightarrow 0} \frac{x}{\sqrt{x^2 + \lambda^2 x^2}} = \lim_{x \rightarrow 0} \frac{x}{|x| \sqrt{1 + \lambda^2}} = \frac{1}{\sqrt{1 + \lambda^2}}$$

This limit depends on  $\lambda$ . Therefore, the limit does not exist.

**Example:**

Compute the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2}$$

Let  $y = \lambda x$ ,  $\lambda \in \mathbb{R}$ . Then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^3}{x^2 + \lambda^2 x^2} = \lim_{x \rightarrow 0} \frac{x}{1 + \lambda^2} = 0$$

Following the family of functions  $y = \lambda x$ , the limit is 0. However, we cannot confirm that the value of the limit is 0 or that the limit exists using this method.

This method is necessary but not sufficient. It is a good way to check if the limit does not exist.

**Problem 1a**

Taking  $y = \lambda x$  and knowing that  $\sin x \approx x$  for  $x \approx 0$ :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 + \sin^2(\lambda x)}{2x^2 + \lambda^2 x^2} = \lim_{x \rightarrow 0} \frac{x^2 + \lambda^2 x^2}{2x^2 + \lambda^2 x^2} = \frac{1 + \lambda^2}{2 + \lambda^2}$$

This limit depends on  $\lambda$ . Therefore, the limit does not exist.

**Problem 1d**

Taking  $y = \lambda x$ :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{x \rightarrow 0} \frac{x \lambda x}{\sqrt{x^2 + \lambda^2 x^2}} = \lim_{x \rightarrow 0} \frac{\lambda x^2}{\sqrt{x^2(1 + \lambda^2)}} = \lim_{x \rightarrow 0} \frac{\lambda x}{\sqrt{1 + \lambda^2}} = 0$$

This limit does not depend on  $\lambda$ . Therefore, the limit exists and is 0.

Using polar coordinates:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0} \frac{r^2 \cos(\theta) \sin(\theta)}{r} = \lim_{r \rightarrow 0} r \cos(\theta) \sin(\theta) = 0$$

**7.5 Polar coordinates**

In  $\mathbb{R}^2$ , the polar coordinates are  $(r, \theta)$ .

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

**Theorem** Let us consider two functions  $f$  and  $g$  such that

$$\lim_{x \rightarrow x_0} f(x) = 0, \quad \text{and } g \text{ is bounded for } \|x - x_0\| < \delta$$

Then

$$\lim_{x \rightarrow x_0} f(x)g(x) = 0$$

### Problem 1e

Taking  $y = \lambda x$ :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x \rightarrow 0} \frac{x^2 \lambda x e^{\lambda x}}{x^4 + 4\lambda^2 x^2} = \lim_{x \rightarrow 0} \frac{\lambda x^3 e^{\lambda x}}{x^2(x^2 + 4\lambda^2)} = \lim_{x \rightarrow 0} \frac{\lambda x e^{\lambda x}}{x^2 + 4\lambda^2} = 0$$

Now taking  $y = x^2$ :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x \rightarrow 0} \frac{x^2 x^2 e^{x^2}}{x^4 + 4x^4} = \lim_{x \rightarrow 0} \frac{x^4 e^{x^2}}{5x^4} = \lim_{x \rightarrow 0} \frac{e^{x^2}}{5} = \frac{1}{5}$$

This limit depends on the direction of approach. Therefore, the limit does not exist.

### Problem 1f

Applying generalized spherical coordinates:

$$\begin{cases} x = r \sin(\phi) \cos(\theta) \\ y = \frac{1}{2} r \sin(\phi) \sin(\theta) \\ z = \frac{1}{3} r \cos(\phi) \end{cases}$$

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{r \rightarrow 0} \frac{\frac{1}{2} r \sin(\phi) \sin(\theta) \frac{1}{3} r \cos(\phi)}{r^2} = \lim_{r \rightarrow 0} \frac{1}{6} \sin(\phi) \cos(\phi) \sin(\theta)$$

This limit depends on  $\phi$  and  $\theta$ . Therefore, the limit does not exist.

## 8 Continuity of functions

A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous at  $x_0 \in \mathbb{R}^N$  if:

1.  $f(x_0)$  is defined.
2.  $\lim_{x \rightarrow x_0} f(x)$  exists.
3.  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

**Theorem** Consider  $A$  a subset in  $\mathbb{R}^N$  and  $f : A \rightarrow \mathbb{R}^M$  a vector function.

$$F(x) = (f_1(x), f_2(x), \dots, f_M(x))$$

If  $f_1, f_2, \dots, f_M$  are continuous at  $x_0 \in A$ , then  $F$  is continuous at  $x_0$ .

**Theorem** Assume that  $f$  and  $g$  are continuous at  $x_0 \in \mathbb{R}^N$  and  $f(x_0)$  respectively. Then, the composition  $g \circ f$  is continuous at  $x_0$ .

$$\text{For example: } f(x) = x^2 y \sin(x + y), \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Since  $f$  is a composite function and  $x^2, y, \sin(x + y)$  are continuous, then  $f$  is continuous.

### Problem 2a

Prove the continuity:

$$f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

If  $(x, y) \neq (0, 0)$ , then  $f(x, y)$  is a composition of continuous functions. So that

$$\text{for any } (x_0, y_0) \neq (0, 0), \quad \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(x_0, y_0) = \frac{x_0^2 y_0^3}{2x_0^2 + y_0^2}$$

At  $(0, 0)$ , the function is well defined:  $f(0, 0) = 1$ , so we must check that the limit exists.

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y^3}{2x^2 + y^2} =$$

Using polar coordinates:

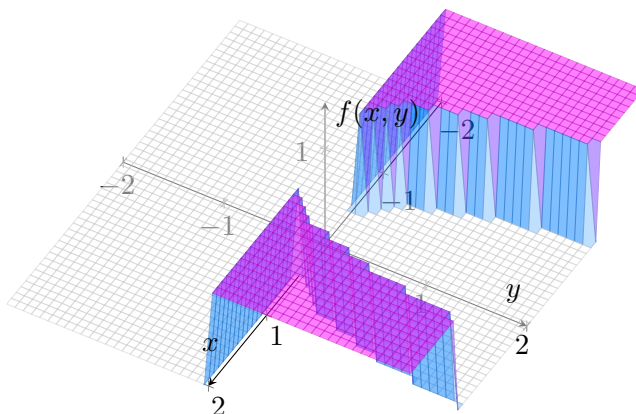
$$\begin{aligned} & \begin{cases} x = \frac{1}{\sqrt{2}} r \cos(\theta) \\ y = r \sin(\theta) \end{cases} \\ & = \lim_{r \rightarrow 0} \frac{\frac{1}{2} r^3 \cos^2(\theta) \sin^3(\theta)}{r^2} = \lim_{r \rightarrow 0} \frac{1}{2} r \cos^2(\theta) \sin^3(\theta) = 0 \end{aligned}$$

The limit exists and is equal to 0. However,  $f(0, 0) = 1 \neq 0$ . Therefore, the function is not continuous at  $(0, 0)$ .

### Problem 3

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

a)  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$  along any path going through  $y = mx^\alpha$ .  $0 < \alpha < 4$



Let  $\alpha = 1$ :  $y = mx$ . Then

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} f(x, mx) = \lim_{x \rightarrow 0} f(x, mx) = 0$$

In this case, the limit exists and is equal to 0.

b) Despite (a), prove that the function is not continuous.

Take any point  $(a, 0)$ ,  $a \in \mathbb{R}$ . Then:

$$\lim_{(x,y) \rightarrow (a,0)} f(x,y) = \begin{cases} 0 & \text{if } y \rightarrow 0^- \\ 1 & \text{if } y \rightarrow 0^+ \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

c)  $f$  discontinuous on two entire curves:  $\begin{cases} y = x^4 \\ y = 0 \end{cases}$

Take any point  $(a, a^4)$  on  $y = x^4$ ,  $a \in \mathbb{R}$ . Then:

$$\lim_{(x,y) \rightarrow (a,a^4)} f(x,y) =$$

$$|| (x,y) - (a,a^4) || < \delta \implies \begin{cases} 0 < |f(x,y)| = 1 < \varepsilon & \text{if } y < x^4 \\ 0 < |f(x,y)| = 0 < \varepsilon & \text{if } y > x^4 \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

### Example:

Consider the function:

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let  $y = \lambda x$ ,  $\lambda \in \mathbb{R}$ . Then

$$a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x \rightarrow 0} \frac{x^3 \lambda x}{x^6 + \lambda^2 x^2} = \lim_{x \rightarrow 0} \frac{\lambda x^2}{x^4 + \lambda^2} = 0$$

Now taking  $y = x^3$ :

$$b) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x \rightarrow 0} \frac{x^3 x^3}{x^6 + x^6} = \lim_{x \rightarrow 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

Therefore the function is not continuous at  $(0,0)$ , since the limit depends on the direction of approach.

## 9 Differentiability of functions

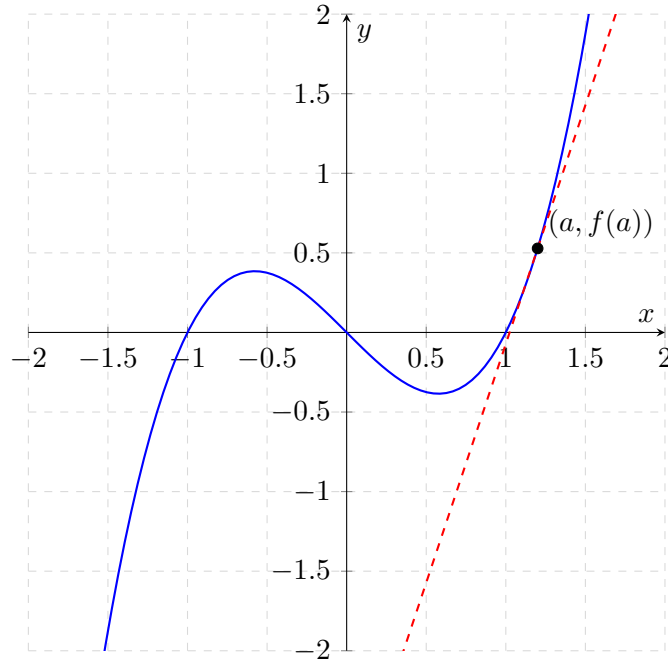
In 1D when we consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the derivative of  $f$  at  $a \in \mathbb{R}$  describes the ratio of the change of the function  $f$  at  $x = a$  and is denoted by:

$$f'(a) = \lim_{t \rightarrow 0} \frac{f(a+t) - f(a)}{t} = \frac{df(a)}{dt}$$

Geometrically, the derivative of a function at a point is the slope of the tangent line to the curve at that point. And this tangent line is defined as:

$$y = f(a) + f'(a)(x - a)$$





A way of seeing this is written by the Taylor polynomial:

$$f(x) = f(a) + M(x - a) + r(|x - a|)$$

Where  $M = f'(a)$  is the slope of the tangent line and  $r(|x - a|)$  is the remainder.

If  $x \rightarrow a$ , then  $r(|x - a|) \rightarrow 0$  but this only guarantees that the function is continuous at  $a$ . However, if

$$\lim_{x \rightarrow a} \frac{r(|x - a|)}{|x - a|} = 0$$

we actually find the existence of the tangent line at  $a$ .

$$\frac{f(x) - f(a)}{x - a} = f'(a) + \frac{r(|x - a|)}{x - a}$$

## 9.1 Generalizing to several variables

$$z = f(x, y)$$

We would like to understand the derivative with respect to each variable.

To do so, we use the knowledge of derivatives in 1 variable.

## 9.2 Partial derivatives

Let  $f : A \subset \mathbb{R}^N \rightarrow \mathbb{R}$  be a scalar function. And let  $x_0 \in A$ . The partial derivative of  $f$  with respect to the variable  $x_i$  at  $x_0$  is defined as:

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

Where  $e_i$  is the unit vector in the direction of  $x_i$ .

Somehow, we are computing the ratio of the change of the function  $f$  in the direction of  $x_i$ , in other words, of the vector  $e_i$ .

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

$$B = \{e_1, e_2, \dots, e_N\} \text{ is the standard basis of } \mathbb{R}^N$$

If  $F : \mathbb{R}^N \rightarrow \mathbb{R}^M$  is a vector function, then the partial derivative of  $F$  with respect to the variable  $x_i$  at the point  $x$  is defined as:

$$\frac{\partial F(x)}{\partial x_i} = \left( \frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \dots, \frac{\partial f_M}{\partial x_i} \right)$$

**Example:**

$$f(x, y) = xy + x - y \quad \text{at } (0, 0)$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0 + te_1) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t \cdot 0 + t - 0}{t} = 1$$

Where  $t$  is the norm of the vector  $te_1$ :

$$t = t||e_1|| = t \cdot 1 = t$$

Now,

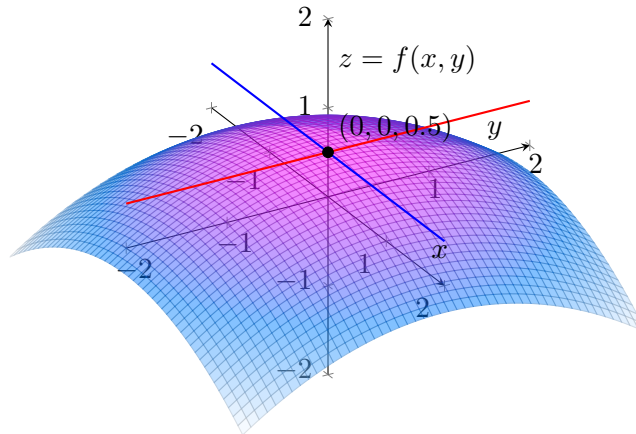
$$\frac{\partial f}{\partial y}(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0 \cdot k + 0 - k}{k} = -1$$

### 9.3 Geometrical interpretation of the partial derivative

The partial derivative of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  at a point  $(x_0, y_0)$  is the slope of the tangent line to the curve  $z = f(x, y)$  at the point  $(x_0, y_0)$  in the direction of the variable  $x_i$ .

$$\frac{\partial f}{\partial x}(x_0, y_0) = \text{slope of the tangent line in the direction of } x$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \text{slope of the tangent line in the direction of } y$$



Obviously, we can apply the rules for derivatives:

$$\frac{\partial f}{\partial x}(x, y) = y + 1, \quad \frac{\partial f}{\partial y}(x, y) = x - 1$$

### 9.4 Directional derivative

Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a scalar function and  $P \in A \subset \mathbb{R}^N$  and  $v \in \mathbb{R}^N$  be a vector. The directional derivative of  $f$  at  $P$  in the direction of  $v$  is defined as:

$$D_v f(P) = \lim_{t \rightarrow 0} \frac{f(P + tv) - f(P)}{t||v||} = \frac{df(P + tv)}{dt}$$

**Example:**

$$f(x, y) = \sqrt{|xy|} \quad \text{at } (0, 0) \quad \text{in the direction of } v = (1, 1)$$

$$D_{(1,1)}f(0,0) = \lim_{t \rightarrow 0} \frac{f(0+t, 0+t) - f(0,0)}{t\|(1,1)\|} = \lim_{t \rightarrow 0} \frac{\sqrt{|t^2|} - 0}{t\|(1,1)\|} = \lim_{t \rightarrow 0} \frac{|t|}{t\sqrt{2}} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } t \rightarrow 0^+ \\ -\frac{1}{\sqrt{2}} & \text{if } t \rightarrow 0^- \end{cases}$$

Therefore, the directional derivative does not exist.

## 9.5 Remark

Existence of all directional derivatives at a point  $P$  does not guarantee the continuity of the function at  $P$ .

$$\text{For example: } f(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ does not exist.}$$

**Example:**

$$f(x, y) = x^{1/3}y^{1/3} \quad \text{at } (0,0)$$

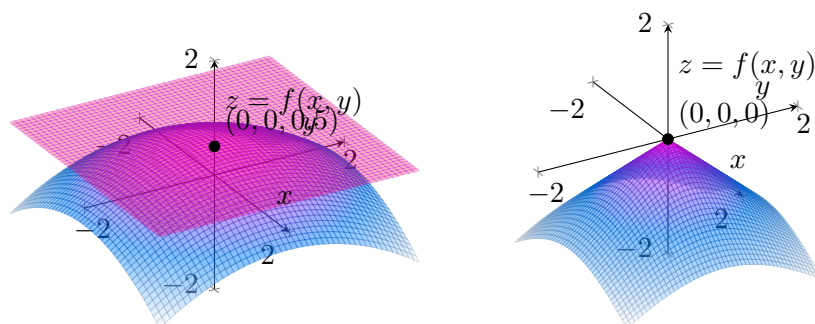
$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{t^{1/3} \cdot 0^{1/3} - 0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \rightarrow 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 \cdot t^{1/3} - 0}{t} = 0$$

Derivatives exist and the function is also continuous at  $(0,0)$ . However, there is no tangent plane at  $(0,0)$ .

## 9.6 Differentiability

Heuristically it means the construction of a tangent plane.



Let  $A \subset \mathbb{R}^2$  be an open subset in  $\mathbb{R}^2$  such that  $(x, y) \in A$ , with  $f : A \rightarrow \mathbb{R}$  a scalar function. We say that  $f$  is differentiable at  $(x_0, y_0)$  if:

1.  $\frac{\partial f}{\partial x}(x_0, y_0)$  and  $\frac{\partial f}{\partial y}(x_0, y_0)$  exist.

$$2. \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x-x_0) - \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

For a function to be differentiable we say that there exists a tangent plane at  $(x_0, y_0)$ :

$$z = f(x_0, y_0) + A(x - x_0) + B(y - y_0)$$

Following similar ideas to the 1D case, we can write:

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + r(\|(x, y) - (x_0, y_0)\|)$$

If  $\|(x, y) - (x_0, y_0)\| \rightarrow 0$  then  $r(\|(x, y) - (x_0, y_0)\|) \rightarrow 0$ , and the function is continuous at  $(x_0, y_0)$ .

However if we have:

$$\frac{f(x, y) - f(x_0, y_0) - A(x - x_0) - B(y - y_0)}{\|(x - x_0) + (y - y_0)\|} = \frac{r(\|(x, y) - (x_0, y_0)\|)}{\|(x - x_0) + (y - y_0)\|} \rightarrow 0$$

Then the function is differentiable at  $(x_0, y_0)$  if:

$$A = \frac{\partial f}{\partial x}(x_0, y_0), \quad B = \frac{\partial f}{\partial y}(x_0, y_0)$$

**Remark** Taylor's polynomial around  $(x_0, y_0)$  of degree 1 is:

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + r(\|(x, y) - (x_0, y_0)\|)$$

Where  $\frac{\partial f}{\partial x}(x_0, y_0)$  and  $\frac{\partial f}{\partial y}(x_0, y_0)$  form  $L((x - x_0), (y - y_0))$ , the linearity function.

$$\|f(x, y) - L((x - x_0), (y - y_0))\| < \varepsilon \text{ as } \|(x, y) - (x_0, y_0)\| < \delta$$

We can write  $L$  in terms of the derivative of the scalar function  $f$ :

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

We define such a derivative as the gradient of  $f$ :

$$D(f(x, y)) = \nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

The gradient provides us with the direction of the maximum increase of the function  $f$  at  $(x, y)$ .

**Problem 4a**

$$f(x, y) = xy, \quad \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

**Problem 4c**

$$f(x, y) = \sqrt{x^2 + y^2}, \quad \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

**Problem 5a**

$$\begin{aligned} f(x, y, z) &= (x + z)e^{x-y}, \quad \text{gradient at } (1, 1, 1) \\ \nabla f(x, y, z) &= \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (e^{x-y} + (x + z)e^{x-y}, -(x + z)e^{x-y}, e^{x-y}) \\ \nabla f(1, 1, 1) &= (e^0 + 2e^0, -2e^0, e^0) = (3, -2, 1) \end{aligned}$$

## 10 Differentiability in $\mathbb{R}^N$

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \rightarrow \mathbb{R}^M$$

We say that  $f$  is differentiable at  $x_0$  if:

1.  $\frac{\partial f_i}{\partial x_j}(x_0)$  exists for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, N$ .
2.  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - Jf(x_0)(x - x_0)}{\|x - x_0\|} = 0$

Where  $Jf(x_0)$  is the Jacobian matrix of  $f$  at  $x_0$ :

$$Jf(x_0) = Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix}$$

### Problem 6a

Compute the jacobian matrix:

$$F(x, y) = (y, x, xy, y^2 - x^2), \quad F : \mathbb{R}^2 \rightarrow \mathbb{R}^4$$

$$JF(x, y) = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \\ \frac{\partial(y^2 - x^2)}{\partial x} & \frac{\partial(y^2 - x^2)}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ y & x \\ -2x & 2y \end{pmatrix}$$

Now at  $(1, 2)$ :

$$JF(1, 2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \\ -2 & 4 \end{pmatrix}$$

### Problem 6c

$$F(x, y, z) = z^2 e^x \cos(y), \quad \text{at } (0, \frac{\pi}{2}, -1) \quad F : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$\begin{aligned} JF(x, y, z) &= \begin{pmatrix} \frac{\partial z^2 e^x \cos(y)}{\partial x} & \frac{\partial z^2 e^x \cos(y)}{\partial y} & \frac{\partial z^2 e^x \cos(y)}{\partial z} \end{pmatrix} = \\ &= (z^2 e^x \cos(y) \quad -z^2 e^x \sin(y) \quad 2z e^x \cos(y)) \end{aligned}$$

$$JF(0, \frac{\pi}{2}, -1) = (0 \quad -1 \quad 0)$$

**Example:**

$$f(x, y) = \begin{cases} \frac{2xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$f$  is not differentiable at  $(0, 0)$ , the partial derivatives exist at  $(0, 0)$ .

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{2t \cdot 0}{t \cdot 0} = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{2 \cdot 0 \cdot t}{t \cdot 0} = 0$$

However, the limit of the difference quotient does not exist:

$$\begin{aligned} \lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)x - \frac{\partial f}{\partial y}(0, 0)y}{\sqrt{x^2 + y^2}} &= \\ &= \lim_{(x, y) \rightarrow (0, 0)} \frac{\frac{2xy}{\sqrt{x^2 + y^2}} - 0 - 0 \cdot x - 0 \cdot y}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \frac{2xy}{x^2 + y^2} \end{aligned}$$

If we use polar coordinates:

$$\begin{aligned} &\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases} \\ &= \lim_{r \rightarrow 0} \frac{2r^2 \cos(\theta) \sin(\theta)}{r^2} = \lim_{r \rightarrow 0} 2 \cos(\theta) \sin(\theta) = 2 \cos(\theta) \sin(\theta) \end{aligned}$$

The limit depends on  $\theta$ . Therefore, the function is not differentiable at  $(0, 0)$ .

### Problem 7

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The partial derivatives exist at  $(0, 0)$ :

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0$$

Take the limit:

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2}{x^2 + y^4} = \lim_{r \rightarrow 0} \frac{r^3 \cos(\theta) \sin^2(\theta)}{r^2 \cos^2(\theta) + r^4 \sin^4(\theta)}$$

The limit does not exist, therefore the function is not continuous at  $(0, 0)$ . The function is not differentiable at  $(0, 0)$ , due to it not being continuous at that point.

Direction of any vector at  $(0, 0)$ :

$$D_{(u, v)} f(0, 0) = \lim_{t \rightarrow 0} \frac{f(tu, tv) - f(0, 0)}{t \|(u, v)\|}$$

Assume  $\|(u, v)\| = 1$ :

$$= \lim_{t \rightarrow 0} \frac{f(tu, tv)}{t} = \lim_{t \rightarrow 0} \frac{t^3 uv^2}{t(t^2 u^2 + t^4 v^4)} = \lim_{t \rightarrow 0} \frac{uv^2}{u^2 + t^2 v^4} = \frac{v^2}{u}$$

Where  $\frac{v^2}{u}$  is the slope of the tangent line in the direction of  $(u, v)$ .

### 10.1 Proposition:

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \rightarrow \mathbb{R}$$

$f$  is a scalar function, differentiable at  $x_0$  and  $v \in \mathbb{R}^N \setminus \{0\}$  a vector. Then:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle$$

#### Remark

$$\langle \nabla f(x_0), \alpha v \rangle = \alpha \langle \nabla f(x_0), v \rangle \neq \langle \nabla f(x_0), v \rangle$$

Using the properties of the scalar product:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle = \|\nabla f(x_0)\| \cdot \|v\| \cdot \cos(\theta) = \|\nabla f(x_0)\| \cdot \cos(\theta)$$

Where  $\theta$  is the angle between  $\nabla f(x_0)$  and  $v$ , and we assume  $\|v\| = 1$ .

The directional derivative is maximum in the direction of  $\nabla f(x_0)$  and minimum in the direction of  $-\nabla f(x_0)$ .

### 10.2 Proposition:

Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a scalar function and let  $(x_0, y_0, z_0) \in \mathbb{R}^3$  be a point on the level surface  $\{f(x, y, z) = k \in \mathbb{R}\} = S$ .

Then,  $\nabla f(x_0, y_0, z_0) \perp v = 0$  where  $v$  is the tangent vector to the trajectory  $c(t)$  at  $t = 0$  over the surface  $S$ .

#### Proof

Since  $c(t) \subset S$ , we have  $f(c(t)) = k$ .

By construction,  $v = c'(0)$ .

Now, applying the chain rule, we get:

$$\left. \frac{d}{dt} f(c(t)) \right|_{t=0} = \nabla f(c(0)) \cdot c'(0) = \nabla f(c(0)) \cdot v = 0$$

This proves the orthogonality between  $\nabla f$  and the tangent vector to the curve (to the level curve).

#### Remark

Take  $P^*(p_1, p_2, p_3)$  as a point and  $(x, y, z)$  as any point on the tangent plane.

Then,

$$n \cdot (x - p_1, y - p_2, z - p_3) = 0$$

where  $n = \nabla f(P^*)$ .

### 10.3 Tangent Plane to a Surface

The tangent plane to  $f(x, y, z) = k$  at  $p_0$  is given by

$$\nabla f(p_0) \cdot \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix} = 0$$

### Example

Tangent plane of  $3xy + z^2 = 4$  at  $(1, 1, 1)$ :

- **Gradient:**

$$\nabla f(x, y, z) = (3x, 3y, 2z)$$

- **Gradient at  $(1, 1, 1)$ :**

$$\nabla f(1, 1, 1) = (3, 3, 2)$$

- **Tangent Plane:**

$$\nabla f(1, 1, 1) \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0$$

- **Equation:**

$$3(x-1) + 3(y-1) + 2(z-1) = 0$$

$$3x + 3y + 2z = 8$$

### Problem 11c

Find the tangent plane of  $z = \frac{x}{y^2}$  at  $(-2, 2, 1)$ :

- **Gradient:**

$$\nabla f(x, y, z) = \left( \frac{1}{y^2}, -\frac{2x}{y^3}, -1 \right)$$

- **Gradient at  $(-2, 2, 1)$ :**

$$\nabla f(-2, 2, 1) = \left( \frac{1}{4}, \frac{1}{2}, -1 \right)$$

- **Tangent Plane:**

$$\nabla f(-2, 2, 1) \cdot \begin{pmatrix} x+2 \\ y-2 \\ z-1 \end{pmatrix} = 0$$

- **Equation:**

$$\frac{1}{4}(x+2) + \frac{1}{2}(y-2) - (z-1) = 0$$

$$\frac{1}{4}x + \frac{1}{2}y - z = 0$$

## 10.4 Theorems

Let  $A \subseteq \mathbb{R}^n$ ,  $x_0 \in A$ , and  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ . If  $f$  is differentiable at  $x_0$ ,  $f$  is continuous at  $x_0$ .  
Let  $A \subseteq \mathbb{R}^n$ ,  $x_0 \in A$ , and  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . For  $x = (x_1, \dots, x_n)$ :

- a)  $\frac{\partial f}{\partial x_i}$  exists for any  $i$ .
- b)  $\frac{\partial f}{\partial x_i}$  is continuous at  $x_0$  for any  $i$ .

Then,  $f$  is differentiable at  $x_0$ .

### Problem 9

Given  $f(x, y) = e^{\left(\frac{1}{x^2+y^2}\right)}$  for  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$ :

- a) Show that  $f$  is differentiable at  $(0, 0)$ .
- b) Prove the existence of the tangent plane at  $(0, 0)$ .

To show  $f$  is differentiable at  $(0, 0)$ , we need to check the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)x - \frac{\partial f}{\partial y}(0, 0)y}{\sqrt{x^2 + y^2}} = 0$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{e^{\left(\frac{-1}{t^2}\right)}}{t} = 0 = \frac{\partial f}{\partial y}(0, 0)$$

Thus,  $f$  is differentiable at  $(0, 0)$ .



## 10.5 Properties

Let  $A, B \subseteq \mathbb{R}^n$ ,  $x_0 \in A$ , and  $f, g : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable functions.

1.  $\lambda f(x)$  is differentiable at  $x_0$ , where  $\lambda \in \mathbb{R}$ . The derivative is given by:

$$D(\lambda f(x_0)) = \lambda D(f(x_0))$$

2.  $f(x) + g(x)$  is differentiable at  $x_0$ . The derivative is given by:

$$D(f(x_0) + g(x_0)) = D(f(x_0)) + D(g(x_0))$$

3.  $f(x) \cdot g(x)$  is differentiable at  $x_0$ . The derivative is given by:

$$D(f(x_0) \cdot g(x_0)) = D(f(x_0)) \cdot g(x_0) + f(x_0) \cdot D(g(x_0))$$

4.  $\frac{f(x)}{g(x)}$  is differentiable at  $x_0$ , provided  $g(x_0) \neq 0$ . The derivative is given by:

$$D\left(\frac{f(x_0)}{g(x_0)}\right) = \frac{D(f(x_0)) \cdot g(x_0) - f(x_0) \cdot D(g(x_0))}{(g(x_0))^2}$$

## 10.6 Chain Rule

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ . If  $x_0 \in \mathbb{R}^m$  and  $f(x_0) \in \mathbb{R}^n$ , then  $(g \circ f)(x) = g(f(x))$  is differentiable at  $x_0$ . The derivative is given by:

$$D(g \circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0)$$

### Example

Given  $g(x, y) = (x^2 + y, y^2)$  and  $F(u, v) = (u + v, u, v^2)$ , find  $D(F \circ g)|_{(1,1)}$ .

$$\begin{aligned} g : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ f : \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \end{aligned}$$

$$g(x, y) = (x^2 + 1, y^2)$$

$$f(x, y) = (u + v, u, v^2)$$

$$(x, y) \rightarrow (g_1(x, y), g_2(x, y)) \rightarrow (f_1(g_1, g_2), f_2(g_1, g_2), f_3(g_1, g_2))$$

$$f(g_1(x, y), g_2(x, y)) = (x^2 + 1 + y^2, x^2 + 1, y^4) \rightarrow Df(x, y) = Jf(x, y)$$

$$Dg(x, y) = Jg(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

$$Df(u, v) = Jf(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$$

$$D(f \circ g)|_{(x,y)} = D(f(g(x,y))) \cdot Dg(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix} \cdot \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

$$D(F \circ g)|_{(1,1)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$D(F \circ g)|_{(1,1)} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$$

### Example

$$f(x,y) = \left( \tan \frac{y}{x} - x + y, \log \frac{y+1}{x} \right), g(s,t) = (t \cdot \cos s, e^t, s - 2t) \text{ and } h(u,v,w) = uv^2 \cdot e^w$$

$$F = h \circ g \circ f$$

$$DF(x,y) = D(h(g(f(x,y)))) \cdot D(g(f(x,y))) \cdot D(f(x,y))$$

$$Df(x,y) = \begin{pmatrix} -\frac{\sec^2 \frac{y}{x} \cdot y}{x^2} - 1 & \frac{\sec^2 \frac{y}{x}}{x} + 1 \\ -\frac{1}{x} & \frac{1}{y+1} \end{pmatrix}$$

$$Dg(s,t) = \begin{pmatrix} -t \cdot \sin s & \cos s \\ 0 & e^t \\ 1 & -2 \end{pmatrix}$$

$$Dh(u,v,w) = (v^2 \cdot e^w \quad 2uv \cdot e^w \quad uv^2 \cdot e^w)$$

### Example

Particle of mass  $m$  following a trajectory  $s(t) \in \mathbb{R}^3$

According to Newton's law, the field force satisfies

$$F = -\nabla V \quad \text{with} \quad V = \text{potential function}$$

Prove that the total energy is constant  $E(t) = E_k + E_p = c$

Following  $s(t)$  we find  $F(s(t)) = -\nabla V(s(t))$

Taking the arbitrary times  $t_1, t_2$

$$\begin{aligned} \int_{t_1}^{t_2} F(s(t)) \cdot d(s(t)) &= - \int_{t_1}^{t_2} \nabla V(s(t)) \cdot s'(t) dt \quad (\text{chain rule}) \\ &= -(V(s(t_2)) - V(s(t_1))) \end{aligned}$$

Thanks to Newton's second law

$$F = m s''(t)$$

$$\begin{aligned} \int_{t_1}^{t_2} F(s(t)) \cdot s'(t) dt &= \int_{t_1}^{t_2} m s''(t) \cdot s'(t) dt = \left[ \frac{m \|s'(t)\|^2}{2} \right]_{t_1}^{t_2} \\ &= \frac{m \|s'(t_2)\|^2}{2} - \frac{m \|s'(t_1)\|^2}{2} \\ E(t) &= \frac{m \|s'(t)\|^2}{2} + V(s(t)) \end{aligned}$$

### Example

$$f(x, y) = \begin{cases} \frac{2xy}{2x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Partial Derivatives:

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = 0 \\ \frac{\partial f(x, y)}{\partial y} &= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = 0 \end{aligned}$$

Differentiability at  $(0, 0)$ :

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\frac{2xy}{2x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \frac{2xy}{\sqrt{x^2 + y^2} (2x^2 + y^2)}$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\lim_{r \rightarrow 0} \frac{2 \cos \theta \sin \theta}{r (2 \cos^2 \theta + \sin^2 \theta)} = \infty \quad f \text{ is not differentiable at } (0, 0)$$

Directional derivative following  $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ :

$$\text{Using } D_v f(0, 0) = \langle \nabla f(0, 0), v \rangle = \left\langle (0, 0), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\rangle = 0$$

$$D_v f(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, 0) + tv - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{f(tv) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{\frac{2t^2}{2} - 0}{\frac{2t^2}{2} + \frac{t^2}{2}} = \frac{2}{3}$$

### Problem 17

Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

$$\mathbf{a)} \quad u = f(x, y), \quad x = x(r, s, t), \quad y = y(r, s, t)$$

$$(r, s, t) \rightarrow (x(r, s, t), y(r, s, t)) \rightarrow f(x(r, s, t), y(r, s, t))$$

$$\mathbb{R}^3 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$D(f \circ g)(r, s, t) = D(f(g(r, s, t))) \cdot Dg(r, s, t)$$

$$D(f \circ g)(r, s, t) = Df(x(r, s, t), y(r, s, t)) \cdot \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$D(f \circ g)(r, s, t) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$D(f \circ g)(r, s, t) = \left( \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \right)$$

$$\begin{aligned} \mathbf{b)} \quad w &= f(x, y, z), \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \\ (u, v) &\rightarrow (x(u, v), y(u, v), z(u, v)) \rightarrow f(x(u, v), y(u, v), z(u, v)) \\ \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \rightarrow \mathbb{R} \end{aligned}$$

$$D(f \circ g)(u, v) = D(f(g(u, v))) \cdot Dg(u, v)$$

$$D(f \circ g)(u, v) = Df(x(u, v), y(u, v), z(u, v)) \cdot \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$D(f \circ g)(u, v) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$D(f \circ g)(u, v) = \begin{pmatrix} \frac{\partial f}{\partial x}|_{g(u, v)} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}|_{g(u, v)} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}|_{g(u, v)} \cdot \frac{\partial z}{\partial u} \\ \frac{\partial f}{\partial x}|_{g(u, v)} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}|_{g(u, v)} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}|_{g(u, v)} \cdot \frac{\partial z}{\partial v} \end{pmatrix}^t$$

## Problem 12

Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be differentiable. Making the transformation to spherical coordinates, compute  $\frac{\partial f}{\partial \rho}$ ,  $\frac{\partial f}{\partial \theta}$ , and  $\frac{\partial f}{\partial \phi}$  in terms of  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ , and  $\frac{\partial f}{\partial z}$ .

The transformation to spherical coordinates is given by:

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases}$$

$$(\rho, \theta, \phi) \rightarrow (x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) \rightarrow f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi))$$

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \rho} = \frac{\partial f}{\partial x} \cdot \sin(\phi) \cos(\theta) + \frac{\partial f}{\partial y} \cdot \sin(\phi) \sin(\theta) + \frac{\partial f}{\partial z} \cdot \cos(\phi)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x} \cdot \rho \sin(\phi) \sin(\theta) + \frac{\partial f}{\partial y} \cdot \rho \sin(\phi) \cos(\theta)$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \phi} = \frac{\partial f}{\partial x} \cdot \rho \cos(\phi) \cos(\theta) + \frac{\partial f}{\partial y} \cdot \rho \cos(\phi) \sin(\theta) - \frac{\partial f}{\partial z} \cdot \rho \sin(\phi)$$

## 11 Linearization

Recall: if  $\exists \frac{\partial f}{\partial x}$ , then  $\exists \frac{\partial f}{\partial y}$

$$\lim_{(h_1, h_2) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}|_{(x_0, y_0)} h_1 - \frac{\partial f}{\partial y}|_{(x_0, y_0)} h_2}{\sqrt{h_1^2 + h_2^2}} = 0$$

With  $h_1 = x - x_0$  and  $h_2 = y - y_0$ , we have:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}|_{(x_0, y_0)}(x - x_0) - \frac{\partial f}{\partial y}|_{(x_0, y_0)}(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $x_0 \in \mathbb{R}^n$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - Df|_{x_0} \cdot h}{\|h\|} = 0$$

With  $h \in \mathbb{R}^n$ ,  $h = t \cdot v$ ,  $t \rightarrow 0 \implies h \rightarrow 0$

$$\lim_{t \rightarrow 0} \frac{f(x_0 + t \cdot v) - f(x_0) - Df|_{x_0} \cdot t \cdot v}{t} = 0$$

With  $\|v\| = 1$  and  $\|h\| = t$

### 11.1 Proposition

If  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $U$  open in  $\mathbb{R}^n$  is differentiable at  $x_0 \in U$ , then  $f$  is continuous at  $x_0$ , then all the directional derivatives exist at  $x_0$ .

$$D_v f(x_0) = Df \cdot v = Jf \cdot v$$

**Proof**

$$v \in \mathbb{R}^n, \quad D_v f = \lim_{t \rightarrow 0} \frac{f(x_0 + t \cdot v) - f(x_0)}{t} = \lim_{t \rightarrow 0} \frac{Df|_{x_0} \cdot t \cdot v}{t} = Df|_{x_0} \cdot v$$

### 11.2 Special cases

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ , differentiable at  $x_0$ , scalar function

$$D_v f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \frac{\partial f}{\partial x_1} \cdot v_1 + \dots + \frac{\partial f}{\partial x_n} \cdot v_n = \langle \nabla f, v \rangle$$

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$D_{(1,0)} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial f}{\partial x}$$

$$D_{(0,1)} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial f}{\partial y}$$

$$D_{v_1, v_2} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}^t \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\partial f}{\partial x} \cdot v_1 + \frac{\partial f}{\partial y} \cdot v_2$$

$$v_1 + v_2 = 1$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$D_{(1,0,\dots,0)} f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}^t \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}$$

$$D_{(0,1,\dots,0)} f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}^t \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \vdots \\ \frac{\partial f_m}{\partial x_2} \end{pmatrix}$$

### 11.3 Mean Value Theorem

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

In vectorial calculus, the mean value theorem is only for scalar functions.

**Theorem** Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}$  be differentiable, and  $[a, b] \subset U$ . Then,  $\forall x \in (a, b)$ ,  $\exists c \in (a, b)$  such that

$$f(b) - f(a) = Df|_c \cdot (b - a) = \nabla f|_c \cdot (b - a)$$

**Proof** We need to use the chain rule:

$$(1 - t) \cdot a + t \cdot b, \quad t \in [0, 1], \quad [0, 1] \rightarrow U$$

$$g(t) = f((1 - t) \cdot a + t \cdot b)$$

We apply the mean value theorem for one variable in  $[0, 1]$ :

$$g(1) - g(0) = g'(t_0) \cdot (1 - 0)$$

$$g'(c) = Df|_{(1-t_0) \cdot a + t_0 \cdot b} \cdot (b - a)$$

Let  $c = (1 - t_0) \cdot a + t_0 \cdot b$ , then:

$$f(b) - f(a) = Df|_c \cdot (b - a) = \nabla f|_c \cdot (b - a)$$

### 11.4 Theorem

If  $U \subset \mathbb{R}^n$  is open,  $f : U \rightarrow \mathbb{R}^m$  and all the partial derivatives exist and are continuous on a point  $a$ , then  $f$  is differentiable at  $a$  and  $Df|_a = Jf|_a$ .

### Example

$$f(x, y) = \begin{cases} (x^4 + y^4) \cdot \sin\left(\frac{1}{x^4 + y^4}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Is  $f$  continuous at  $(0, 0)$ ?

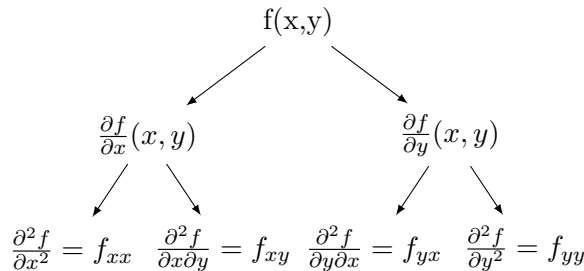
$$\begin{aligned} & \left| \sin\left(\frac{1}{x^4 + y^4}\right) \right| \leq 1, \quad \text{is bounded} \\ \implies & \exists \lim_{(x, y) \rightarrow (0, 0)} (x^4 + y^4) \cdot \sin\left(\frac{1}{x^4 + y^4}\right) = 0 \\ \frac{\partial f}{\partial x} \Big|_{(0, 0)} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^4 \cdot \sin\left(\frac{1}{h^4}\right)}{h} = 0 \\ \frac{\partial f}{\partial y} \Big|_{(0, 0)} &= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{k^4 \cdot \sin\left(\frac{1}{k^4}\right)}{k} = 0 \\ \implies & f \text{ is continuous at } (0, 0) \end{aligned}$$

Is  $f$  differentiable at  $(0, 0)$ ?

$$\begin{aligned} & \lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - \frac{\partial f}{\partial x} \Big|_{(0, 0)} x - \frac{\partial f}{\partial y} \Big|_{(0, 0)} y}{\sqrt{x^2 + y^2}} = \\ &= \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^4 + y^4) \cdot \sin\left(\frac{1}{x^4 + y^4}\right) - 0 - 0 \cdot x - 0 \cdot y}{\sqrt{x^2 + y^2}}, \quad \text{where } \sin\left(\frac{1}{x^4 + y^4}\right) \text{ is bounded} \\ & \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^4 + y^4)}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4 + y^4}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4}{\sqrt{x^2 + y^2}} + \lim_{(x, y) \rightarrow (0, 0)} \frac{y^4}{\sqrt{x^2 + y^2}} \\ & \frac{x^4}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}} \cdot x^3 = 0, \quad \text{because } \left| \frac{x}{\sqrt{x^2 + y^2}} \right| \leq 1 \\ & \frac{y^4}{\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}} \cdot y^3 = 0, \quad \text{because } \left| \frac{y}{\sqrt{x^2 + y^2}} \right| \leq 1 \\ \implies & \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^4 + y^4)}{\sqrt{x^2 + y^2}} = 0 \implies f \text{ is differentiable at } (0, 0) \end{aligned}$$

## 11.5 Higher Order Partial Derivatives

Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , then:



$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

**Hessian Matrix**

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

## 11.6 Partial Differential Equations (PDEs)

### Problem 2

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \quad (x, y, z) \neq (0, 0, 0)$$

Where  $u$  satisfies the PDE:

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad \text{which is } \Delta u = 0, \quad \textbf{Laplace's equation}$$

$$u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$u_x = -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x$$

$$u_{xx} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2x^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$u_{yy} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2y^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$u_{zz} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2z^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$u_{xx} + u_{yy} + u_{zz} = \frac{6}{4} \left( 3(x^2 + y^2 + z^2)^{-5} - (x^2 + y^2 + z^2)^{-\frac{3}{2}} \right) = 0$$

### Problem 1

$$u_t = \alpha^2 u_{xx}, \quad \text{where } \alpha^2 \text{ is known as the diffusion coefficient in the } \textbf{heat equation}$$

$$u(x, t) = e^{-\alpha k^2 t} \cdot \sin(kx)$$

$$u_t = -\alpha k^2 e^{-\alpha k^2 t} \cdot \sin(kx)$$

$$u_x = e^{-\alpha k^2 t} \cdot k \cdot \cos(kx)$$

$$u_{xx} = -e^{-\alpha k^2 t} \cdot k^2 \cdot \sin(kx)$$

### Problem 4c

$$u(x, t) = (x - at)^6 + (x + at)^6$$

$$u_{tt} = a^2 u_{xx} \quad \text{is known as the } \textbf{Wave equation}$$

## 11.7 Cross Derivatives

$$\text{When it is true that } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$



**Theorem: Equality of Crossed Partial Derivatives**

Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be defined on an open set  $U$  and such that the partial derivatives  $\frac{\partial f}{\partial x_i}$  and  $\frac{\partial f}{\partial x_j}$  exist and are differentiable at  $a \in U$ . Then,  $\forall x_i, x_j \in U$ , it is true that:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \quad \text{at } a$$

**Example**

$$f(x, y) = \begin{cases} xy \cdot \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f \right) \Big|_{(0,0)} \neq \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f \right) \Big|_{(0,0)}$$

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = 0 & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = 0 & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} \Big|_{(0,0)} = \lim_{t \rightarrow 0} \frac{\frac{\partial f}{\partial x}(t, 0) - \frac{\partial f}{\partial x}(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{\frac{h^5}{h^4} - 0}{h} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x} \Big|_{(0,0)} = \lim_{t \rightarrow 0} \frac{\frac{\partial f}{\partial y}(0, t) - \frac{\partial f}{\partial y}(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{-\frac{h^5}{h^4} - 0}{h} = -1$$

**11.8 Theorem**

Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be defined on an open set  $U$  and such that all the partial derivatives exist and are continuous at  $a \in U$ . Then,  $f$  is differentiable at  $a$  and its differential is given by the Jacobian matrix  $Jf|_a$ .

**Proof**

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a) - Jf|_a \cdot h}{\|h\|} = 0$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$a + h = \begin{pmatrix} a_1 + h_1 \\ a_2 + h_2 \\ \vdots \\ a_n + h_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 + h_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \dots - \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n + h_n \end{pmatrix}$$

$$f(a+h) = f \begin{pmatrix} a_1+h_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} = f \begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + \dots - f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$f(a+h) - f(a) = f \begin{pmatrix} a_1+h_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f \begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f \begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - \dots - f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$f(a+h) - f(a) = D_{v_1}f \cdot h_1 + D_{v_2}f \cdot h_2 + \dots + D_{v_n}f \cdot h_n$$

$$\text{Where } v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ is the } i\text{th unit vector}$$

$$f(a+h) - f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix} \cdot h_1 + \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \vdots \\ \frac{\partial f_m}{\partial x_2} \end{pmatrix} \cdot h_2 + \dots + \begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{pmatrix} \cdot h_n$$

So:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \big|_{x_1} & \frac{\partial f_1}{\partial x_2} \big|_{x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \big|_{x_n} \\ \frac{\partial f_2}{\partial x_1} \big|_{x_1} & \frac{\partial f_2}{\partial x_2} \big|_{x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \big|_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} \big|_{x_1} & \frac{\partial f_m}{\partial x_2} \big|_{x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \big|_{x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

Finally:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - Jf|_a \cdot h}{\|h\|} = \frac{Jfh - Jf|_a h}{\|h\|} = \frac{(Jf - Jf|_a) \cdot h}{\|h\|} = 0$$

$$Jf|_h \rightarrow Jf|_a \text{ as } h \rightarrow 0$$

## 11.9 Proof of the Chain Rule

Let  $U \subseteq \mathbb{R}^m, V \subseteq \mathbb{R}^n$  be open sets. Let  $g : U \rightarrow V$  and  $f : V \rightarrow \mathbb{R}^p$  be mappings, and let  $a$  be a point in  $U$ . If  $g$  is differentiable at  $a$  and  $f$  is differentiable at  $g(a)$ , then  $f \circ g$  is differentiable at  $a$  and

$$D(f \circ g)(a) = Df(g(a)) \cdot Dg(a).$$

We define two "remainder" functions,  $r$  and  $s$ :

- $r$ : difference between the increment to  $g$  and its linear approximation at  $a$ ,
- $s$ : difference between the increment to  $f$  and its linear approximation at  $g(a)$ .

We know that  $g$  is differentiable at  $a$ :

$$g(a+h) = g(a) + Dg|_a h + r(h),$$

where

$$r(h) = g(a+h) - g(a) - Dg|_a h.$$

This represents the increase to  $g$  relative to its linear approximation at  $a$ . Then,

$$\lim_{h \rightarrow 0} \frac{r(h)}{\|h\|} = 0.$$

We know that  $f$  is differentiable at  $g(a)$ :

$$f(g(a)+k) = f(g(a)) + Df|_{g(a)} k + s(k),$$

where

$$s(k) = f(g(a)+k) - f(g(a)) - Df|_{g(a)} k.$$

This represents the increase to  $f$  relative to its linear approximation at  $g(a)$ . Then,

$$\lim_{k \rightarrow 0} \frac{s(k)}{\|k\|} = 0,$$

we have:

$$f(g(a+h)) = f(g(a) + Dg|_a h + r(h)),$$

where we treat  $Dg|_a h + r(h)$  as  $k$ .

Expanding further:

$$\begin{aligned} f(g(a+h)) &= f(g(a)) + Df|_{g(a)}[Dg|_a h + r(h)] + s(Dg|_a h + r(h)), \\ &= f(g(a)) + Df|_{g(a)} Dg|_a h + [Df|_{g(a)} r(h) + s(Dg|_a h + r(h))]. \end{aligned}$$

Subtract  $f(g(a))$ :

$$f(g(a+h)) - f(g(a)) = Df|_{g(a)} Dg|_a h + \text{remainder}.$$

Thus, the linear approximation of the composition is:

$$Df|_{g(a)} Dg|_a h,$$

with the remainder terms:

$$Df|_{g(a)} r(h) + s(Dg|_a h + r(h)).$$

Now, we need to prove that:

$$\lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a)) - Df|_{g(a)} Dg|_a h}{\|h\|} = 0,$$

since:

$$\lim_{h \rightarrow 0} \frac{Df|_{g(a)} r(h) + s(Dg|_a h + r(h))}{\|h\|} = 0.$$

For the first term:

$$\|Df|_{g(a)} r(h)\| \leq \|Df|_{g(a)}\| \|r(h)\|,$$

and:

$$\lim_{h \rightarrow 0} \frac{\|Df|_{g(a)} r(h)\|}{\|h\|} \leq \|Df|_{g(a)}\| \lim_{h \rightarrow 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

For the second part:

$$\lim_{h \rightarrow 0} \frac{s(Dg|_a h + r(h))}{\|h\|} = 0.$$

There exists  $\delta > 0$  such that  $\|r(h)\| \leq \|h\|$  when  $\|h\| \leq \delta$ . Thus:

$$\|Dg|_a h + r(h)\| \leq \|Dg|_a h\| + \|r(h)\| \leq (\|Dg|_a\| + 1)\|h\|.$$

Also, for any  $\varepsilon > 0$ , there exists  $0 < \delta' \leq \delta$  such that when  $\|k\| \leq \delta'$ :

$$\frac{\|s(h)\|}{\|k\|} \leq \varepsilon.$$

$$\|Dg|_a h + r(h)\| \leq \delta'$$

Substitute for  $\|k\|$  in the equation:

$$\|s(k)\| \leq \varepsilon \|k\|$$

$$\|s(Dg|_a h + r(h))\| \leq \varepsilon \|Dg|_a h + r(h)\|$$

where  $\|s(Dg|_a h + r(h))\| = s(k)$

$$s(k) \leq \varepsilon (\|Dg|_a\| + 1) \|h\|$$

Divide by  $\|h\|$ :

$$\frac{\|s(Dg|_a h + r(h))\|}{\|h\|} \leq \varepsilon (\|Dg|_a\| + 1)$$

since this is true for all  $\varepsilon > 0$ , we have:

$$\lim_{h \rightarrow 0} \frac{\|s(Dg|_a h + r(h))\|}{\|h\|} = 0$$

$$\lim_{h \rightarrow 0} \frac{Df|_{g(a)} r(h) + s(Dg|_a h + r(h))}{\|h\|} = 0$$

$$\lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a)) - Df|_{g(a)} Dg|_a h}{\|h\|} = 0$$

$$Df|_{g(a)} Dg|_a = D(f \circ g)|_a$$

## 11.10 Induced Norm

A matrix  $A$  is a linear operator, because

$$A(x + \alpha y) = A(x) + \alpha A(y)$$

The induced norm of a matrix  $A$  is defined as:

$$\|A\| = \sup_{\|x\|=1} \|A(x)\|$$

$$\|x\| = \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \implies \|A\| = \sqrt{\rho(AA^t)}$$

where  $\rho$  is the spectral radius of the matrix  $AA^t$ . The spectral radius is defined as the maximum eigenvalue of the matrix  $AA^t$ .

### 11.11 Inverse Function Theorem for One Variable

Let  $f : [a, b] \rightarrow [c, d]$  be a continuous function.  $f(a) = c$  and  $f(b) = d$ . If  $f$  is strictly increasing or decreasing in  $[a, b]$ , then  $f$  is invertible and there exists  $g : [c, d] \rightarrow [a, b]$  such that  $g(f(x)) = x$  and  $f(g(y)) = y$ .

If  $f$  is differentiable at  $x_0 \in (a, b)$  and  $f'(x_0) \neq 0$ , then  $f$  is invertible in a neighborhood of  $x_0$  and the inverse function  $f^{-1}$  is differentiable at  $f(x_0)$ :

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x)$$

### 11.12 Inverse Function Theorem for Higher Dimensions

#### 11.12.1 Preliminary Definitions

**Lipschitz Condition** A function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  differentiable in  $U$  satisfies the Lipschitz condition in a subset  $V \subset U$  if there exists a constant known as the Lipschitz ratio  $M > 0$  for all  $x, y \in V$ :

$$\forall x, y \in V, \quad \exists M > 0 \text{ such that } \|Df(x) - Df(y)\| \leq M\|x - y\|$$

#### Example

$$f(x, y) = (x - y^2 \quad x^2 + y)$$

$$Df = \begin{pmatrix} 1 & -2y \\ 2x & 1 \end{pmatrix}$$

$$Df \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2y_1 \\ 2x_1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2y_2 \\ 2x_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2(y_1 - y_2) \\ 2(x_1 - x_2) & 0 \end{pmatrix}$$

$$A_{ij} \Big|_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}, \quad |A| = \left( \sum_{i=1}^n \sum_{j=1}^m A_{ij}^2 \right)^{\frac{1}{2}}$$

$$\left| Df \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right| = 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 2x - y$$

#### Example

$$f(x, y) = \begin{pmatrix} x - y^3 \\ x^3 - y \end{pmatrix}$$

$$Df = \begin{pmatrix} 1 & -3y^2 \\ 3x^2 & -1 \end{pmatrix}$$

$$Df \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -3y_1^2 \\ 3x_1^2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -3y_2^2 \\ 3x_2^2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -3(y_1^2 - y_2^2) \\ 3(x_1^2 - x_2^2) & 0 \end{pmatrix}$$

$$\left| Df \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right| = 3\sqrt{(x_1^2 - x_2^2)^2 + (y_1^2 - y_2^2)^2}$$

If  $(y_1 + y_2)^2 \leq A$  and  $(x_1 + x_2)^2 \leq A$ , then:  $M = 3A$ .

**Best Constant for the Lipschitz Condition** The best constant for the Lipschitz condition  $M^*$  is defined as:

$$M^* = \inf M$$

If we pick  $x \in U \subset V, y \in U \subset V \implies x, y \in V$ .

$$\|Df(x) - Df(y)\| \leq M_V \|x - y\| \implies M_U < M_V$$

$$B_{R_1}(x_0, x_1) \subset B_{R_2}(x_0, x_1) \implies M_{R_1} < M_{R_2}$$

### 11.12.2 Inverse Function Theorem for Higher Dimensions

Given a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  and an image  $y$ , we need to obtain an  $x$  such that  $f(x) = y$ . If  $y$  is fixed, then we say  $x$  is a root of the function  $f_y(x) = f(x) - y = 0$ .

**Theorem** If  $f$  is continuously differentiable in an open set  $U \subset \mathbb{R}^n$  and  $Df(x)$  is invertible in  $x_0 \in U$ , then  $f$  is locally invertible, with differentiable inverse in some neighborhood of  $f(x_0)$ .

$$Df|_{x_0}, \quad \exists (Df|_{x_0})^{-1}$$

$$D(f^{-1}) = (Df)^{-1}$$

#### Example

$$f(x, y) = (\sin(x + y) \quad x^2 - y^2)$$

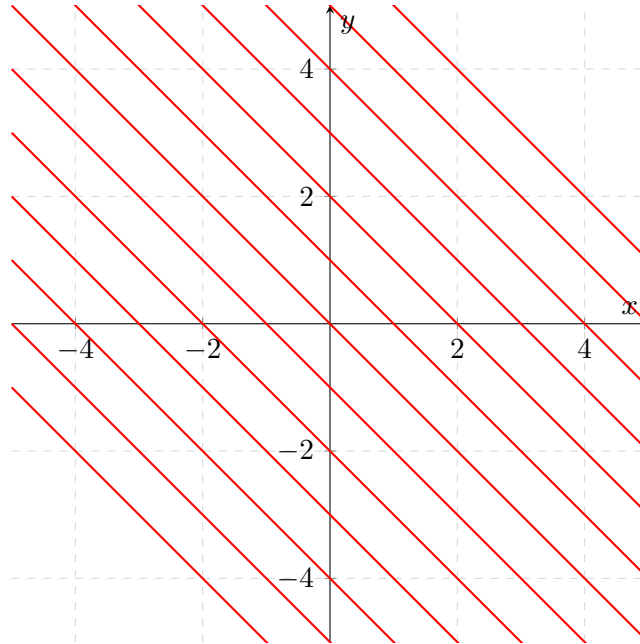
$$Df = \begin{pmatrix} \cos(x + y) & \cos(x + y) \\ 2x & -2y \end{pmatrix}$$

If  $\det(Df)|_{(x_0, y_0)} \neq 0$ , then  $f$  is invertible at  $(x_0, y_0)$  and there exists  $(Df|_{(x_0, y_0)})^{-1}$

$$\det(Df|_{(x_0, y_0)}) = \cos(x_0 + y_0) \cdot (-2y_0) - 2x_0 \cdot \cos(x_0 + y_0) = -2\cos(x_0 + y_0)(x_0 + y_0) \neq 0$$

$$\implies x_0 + y_0 \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$x_0 + y_0 \neq 0$$



The function  $f$  is invertible in the region  $x + y \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}, x + y \neq 0$ , anywhere outside the red lines.

### 11.13 Newton Method

$$x_{n+1} = x_n - (Df|_{x_n})^{-1}f(x_n)$$

$$x_n \rightarrow \alpha, \quad f(\alpha) = 0, \quad \text{where } \alpha \text{ is the solution of the application}$$

### 11.14 Theorem (Kontorovich, Nobel Prize Economics 1975)

Let  $a_0$  be a point in  $\mathbb{R}^n$ ,  $\tilde{U}$  be an open neighborhood of  $a_0$ , and  $f : \tilde{U} \rightarrow \mathbb{R}^n$  be a continuously differentiable function with its derivative  $Df|_{a_0}$  invertible. Define

$$h_0 = -(Df|_{a_0})^{-1}f(a_0), \quad a_1 = a_0 + h_0$$

Let  $U_0 = B(a_1)$ . If  $U_0 \subset U$ , and the derivative  $Df$  satisfies the Lipschitz condition:

$$\|Df(u_1) - Df(u_2)\| \leq M\|u_1 - u_2\|, \quad \forall u_1, u_2 \in U$$

and

$$\|f(a_0)\| \cdot \|(Df|_{a_0})^{-1}\|^2 \cdot M \leq \frac{1}{2}$$

is satisfied, the equation  $f(x) = 0$  has a unique solution in  $U$ .

### 11.15 Inverse Function Theorem

Let  $W \subset \mathbb{R}^n$  be an open neighborhood of  $x_0$  and  $f : W \rightarrow \mathbb{R}^n$  be a continuously differentiable function. If  $Df(x_0)$  is invertible, then  $f$  is invertible, with a continuously differentiable inverse in a neighborhood of  $f(x_0)$ .

To clarify this statement, we will specify the radius  $R$  of a ball  $V$  centered at  $x_0$ , in which the inverse of the function  $f$  is defined.

Setting  $L = Df|_{x_0}$ , now we will define the following conditions:

1. The ball  $W_0$  of radius  $2R\|L^{-1}\|$  centered at  $x_0$  is contained in  $W$ .
2. In the ball  $W_0$ , the derivative of  $f$  satisfies the Lipschitz condition:

$$\|Df(u) - Df(v)\| \leq \frac{1}{2R\|L^{-1}\|^2}\|u - v\|, \quad \forall u, v \in W_0$$

Then,

1. There exists a unique continuous differentiable function  $g : V \rightarrow W$  such that  $f(g(y)) = y$  for all  $y \in V$ . By the chain rule,

$$Df(g(y)) \cdot Dg(y) = I \implies Dg(y) = (Df(g(y)))^{-1}$$

2. The image of  $g$  contains the ball of radius  $R_1$  centered at  $x_0$ :

$$R_1 = 2R\|L^{-1}\|^2 \left( \sqrt{\|L\|^2 + \frac{1}{\|L^{-1}\|^2}} - \|L\| \right)$$

### Example

$$f(x) = 2x + \sin x, \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f : [-k\pi, k\pi] \rightarrow [-2k\pi, 2k\pi]$$

$$f(-k\pi) = -2k\pi, \quad f(k\pi) = 2k\pi$$

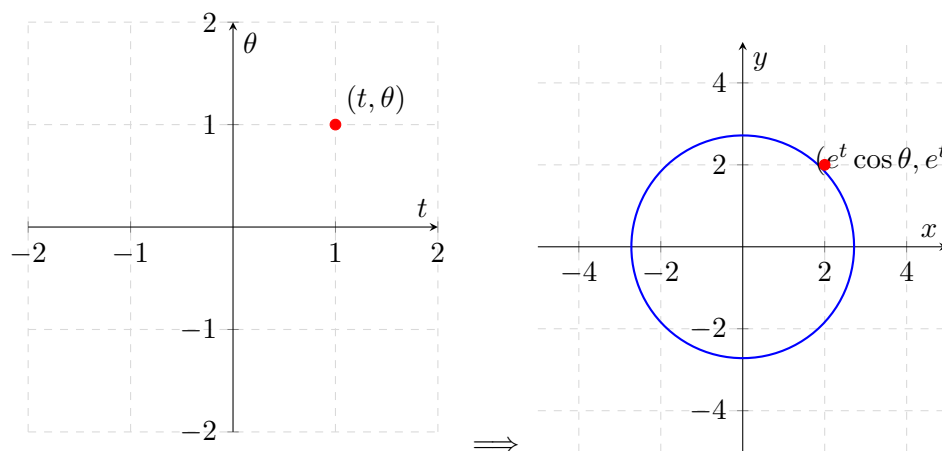
$$f'(x) = 2 + \cos x \neq 0, \quad \forall x \in \mathbb{R}$$

Then,  $f$  is invertible in  $\mathbb{R}$  and  $f^{-1}$  is differentiable, with derivative:

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{2 + \cos x}$$

## Example

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (t, \theta) \rightarrow (e^t \cos \theta, e^t \sin \theta)$$



$$Df = \begin{pmatrix} e^t \cos \theta & -e^t \sin \theta \\ e^t \sin \theta & e^t \cos \theta \end{pmatrix}$$

As  $\det(Df) = e^{2t} \neq 0$ ,  $f$  does not have a global inverse, but it does have a local inverse.

## Example

$$f(x) = x^3, \quad \text{at } x_0 = 0$$

$$f'(x) = 3x^2, \quad f'(0) = 0$$

$$f^{-1}(y) = \sqrt[3]{y}$$

Is  $f^{-1}$  differentiable at  $y_0 = 0$ ?

$$(f^{-1})'(y) = \frac{1}{3}y^{-\frac{2}{3}}, \quad (f^{-1})'(0) = \infty \text{ (does not exist)}$$

## 11.16 Kontorovich Theorem

$$h_0 = -(Df|_{a_0})^{-1}f(a_0), \quad a_1 = a_0 + h_0$$

$$\|f(a_0)\| \cdot \|(Df|_{a_0})^{-1}\|^2 \cdot M \leq \frac{1}{2}$$

$$Df|_{a_0} \text{ is invertible and } \|Df|_{a_0}\| = L$$

$$M < \frac{1}{2} \cdot \frac{1}{\|f(a_0)\| \cdot \|L^2\|}$$

$(Df)^{-1}$  exists in  $x_0$  and  $\exists x \in U_0$  such that  $f(x) = 0$ .



### 11.16.1 Proof

Given  $y \in V$ , where  $V = V_R(f(x_0))$ , we want to find  $x \in U$  such that  $f(x) = y$  once  $y$  is fixed.

$$f_y(x) = f(x) - y, \quad x \text{ is a root of } f_y(x) = 0$$

We apply Kontorovich's theorem to  $f_y(x)$ :

$$f_y(x) = f(x) - y = y_0 - y$$

$$\|f_y(x_0)\| = \|y_0 - y\| < R$$

$$Df_y(x_0) = Df|_{x_0}, \quad \text{take care, } f_y(x) \text{ depends on } x.$$

$$\|f_y(x_0)\| \cdot \|(Df_y(x_0))^{-1}\|^2 \cdot M < \frac{1}{2}$$

$$\|f_y(x_0)\| \cdot \|(Df_y(x_0))^{-1}\|^2 \cdot M = \|y_0 - y\| \cdot L^2 \cdot M < \frac{1}{2}$$

$$M < \frac{1}{2RL^2}$$

$$f_y(x) = f(x) - y, \quad Df_y = Df$$

Lipschitz constant for  $f_y(x)$  and  $f(x)$ :

$$\|Df_y(u) - Df_y(v)\| = \|Df(u) - Df(v)\| \leq M\|u - v\|$$

Let  $M_R$  be the least Lipschitz constant

$$\text{If } M < M_R, \text{ then } M < \frac{1}{2RL^2}$$

$$h_0 = -(Df|_{x_0})^{-1}f(x_0)$$

$$f_y(x) = f(x) - y = 0$$

We have found  $x$  such that  $f(x) = y$ .

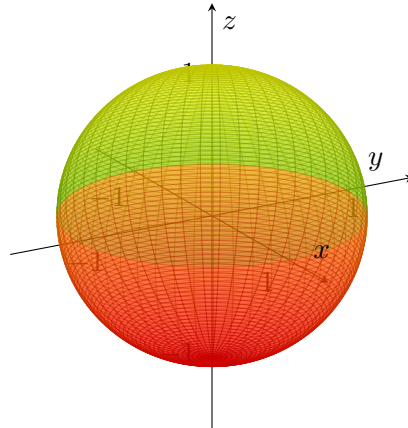
$$g(y) = x \implies f(g(y)) = y$$

### 11.17 Implicit Function Theorem

Let  $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ , in this case for example:

$$(x, y, z) \rightarrow x^2 + y^2 + z^2 - 1$$

$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$



We look for a zero of  $f$ :

$$f(x_0, y_0, z_0) = 0, \quad x_0^2 + y_0^2 + z_0^2 - 1 = 0$$

$$z_0 = \pm \sqrt{1 - x_0^2 - y_0^2}$$

$$(x_c, y_c) \in \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

$$z_c = \sqrt{1 - x_c^2 - y_c^2}$$

$$f : \mathbb{R}^{2+1} \rightarrow \mathbb{R}^1, \quad n = 1, \quad m = 2$$

Where  $z_c$  is the passive variable and  $(x_c, y_c)$  are the active variables. In general we will have  $n$  passive variables and  $m$  active variables.

$$\begin{cases} F_1(x_1, \dots, x_n, \dots, x_{n+m}) = 0 \\ F_2(x_1, \dots, x_n, \dots, x_{n+m}) = 0 \\ \vdots \\ F_n(x_1, \dots, x_n, \dots, x_{n+m}) = 0 \end{cases} \quad \text{Non-linear system with } n \text{ equations and } n + m \text{ variables}$$

### 11.17.1 Implicit Function Theorem (short version)

Let  $U \subset \mathbb{R}^{n+m}$  be an open set and  $F : U \rightarrow \mathbb{R}^n$  be a continuously differentiable function such that  $F(c) = 0$  and  $DF(c)$  is onto for some  $c \in U$ . Then, the system of linear equations

$$DF(c) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

has  $n$  passive variables and  $m$  active variables. There exists a neighborhood of  $c$  in which  $F = 0$  implicitly defines  $n$  of the active variables as functions of the passive variables.

### 11.17.2 Implicit Function Theorem Continued

Let  $W$  be an open neighborhood of  $\mathbb{R}^{n+m}$  and  $F : W \rightarrow \mathbb{R}^n$  be a continuously differentiable function such that  $F(c) = 0$ . If  $DF(c)$  is onto, then it has  $n$  passive columns (arranging the variables so that the passive variables are first), that is  $c = \begin{pmatrix} a \\ b \end{pmatrix}$  when the entries of the corresponding to then  $n$  passive variables are  $a$  and the entries corresponding to the  $m$  active variables are  $b$ .

Then, there exists a unique continuously differentiable function  $g$  from a neighborhood of  $b$  to a neighborhood of  $a$  that expresses the first  $n$  passive variables as functions of the last  $m$  active variables.

To quantify this statement, we can define the radius  $R$  of a ball  $V$  centered at  $b$  in which the  $g$  is defined. First note that the  $n \times n$  matrix

$$(D_1F(c) \quad D_2F(c) \quad \dots \quad D_nF(c))$$

representing the first  $n$  columns of  $DF(c)$  is invertible.

$$L = \left( \begin{array}{cccc|ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} & \cdots & \frac{\partial f_1}{\partial x_{n+m}} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial x_{n+1}} & \cdots & \frac{\partial f_2}{\partial x_{n+m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} & \cdots & \frac{\partial f_n}{\partial x_{n+m}} \\ \hline 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array} \right)$$

$$DF(c) \text{ is onto} \implies DF(c) : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$$

Given  $w \in \mathbb{R}^n$ , there exists  $v \in \mathbb{R}^{n+m}$  such that  $DF(c)v = w$ .

This is satisfied if and only if  $DF(c)$  has  $n$  linearly independent columns.

By convention,  $D_1F(c)$ ,  $D_2F(c)$ ,  $\dots$ ,  $D_nF(c)$  are the columns of  $DF(c)$  corresponding to the passive variables and are linearly independent.

We only have to prove that there are  $x_1, x_2, \dots, x_n$  passive variables because we have changed the order of the columns.

Let

$$c = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{where } a \text{ and } b \text{ are the passive and active variables, respectively}$$

and let

$\tilde{F}$  be an extension of  $F$  in the image

$$\tilde{F} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}, \quad \text{to apply the inverse function theorem}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\tilde{F} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x, y) \\ y \end{pmatrix}$$

$$D\tilde{F} = \left( \begin{array}{cccc|cccc} D_1F & D_2F & \cdots & D_nF & D_{n+1}F & D_{n+2}F & \cdots & D_{n+m}F \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array} \right)$$

We need  $D\tilde{F}$  to be Lipschitz. If  $DF$  is Lipschitz, then  $D\tilde{F}$  is Lipschitz.

$$\|D\tilde{F}(u) - D\tilde{F}(v)\| \leq M\|u - v\|$$

$$\left( \begin{array}{ccc|ccc} D_1F(u) - D_1F(v) & \dots & D_nF(u) - D_nF(v) & D_{n+1}F(u) - D_{n+1}F(v) & \dots & D_{n+m}F(u) - D_{n+m}F(v) \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{array} \right)$$

Now we find a number  $R > 0$  satisfying the following hypotheses:

1. The ball  $W_0$  of radius  $2R\|L^{-1}\|$  centered at  $c$  is contained in  $W$ .
2. In the ball  $W_0$ , the derivative of  $\tilde{F}$  satisfies the Lipschitz condition:

$$\|F(u) - F(v)\| \leq \frac{1}{2R\|L^{-1}\|^2} \|u - v\|$$

Then, there exists a unique continuously differentiable function  $g$  from a neighborhood of  $b$  to a neighborhood of  $a$  that expresses the first  $n$  passive variables as functions of the last  $m$  active variables.

$$g : B_R(b) \rightarrow B_R(a)$$

such that  $g(b) = a$  and  $F \begin{pmatrix} g(y) \\ y \end{pmatrix} = 0$  for all  $y \in B_R(b)$ .

By the chain rule, the derivative of this implicit function  $g$  at a point  $b$  is given by:

$$Dg(b) = - [D_1F(c) \ D_2F(c) \ \dots \ D_nF(c)]^{-1} [D_{n+1}F(c) \ D_{n+2}F(c) \ \dots \ D_{n+m}F(c)]$$

Taking into account all the previous results,

$$\tilde{F} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x, y) \\ y \end{pmatrix}, \quad \tilde{F}(c) = \tilde{F} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} F(a, b) \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\implies \exists (D\tilde{F})^{-1} \implies \text{we can apply the inverse function theorem}$$

$$\tilde{F} \left( \tilde{G} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x \\ y \end{pmatrix}$$

We restrict  $\tilde{G}$  to  $\begin{pmatrix} 0 \\ y \end{pmatrix}$

$$\tilde{F} \begin{pmatrix} g(y) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \tilde{F} \begin{pmatrix} g(y) \\ y \end{pmatrix} = \begin{pmatrix} F(g(y), y) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$\tilde{F} \left( \tilde{G} \begin{pmatrix} 0 \\ y \end{pmatrix} \right) = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

**Example: Unit Circle**

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2 - 1$$

$$F(x, y) = x^2 + y^2 - 1 = 0$$

$$DF = (2x \ 2y)$$

We need to find a  $c = (x_0, y_0)$  such that  $F(c) = 0$  and  $DF(c)$  is onto.

$$y = \pm\sqrt{1-x^2}$$

In this case we can obtain  $g(x)$  explicitly.

$x$  will be the active variable and  $y$  will be the passive variable.

$$DF \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (2x_0 \quad 2y_0)$$

$$\begin{cases} 2x_0 \neq 0 & \text{if } y > 0 \\ 2y_0 \neq 0 & \text{if } x > 0 \end{cases}$$

Is  $DF$  onto?

$$(2x \quad 2y) \begin{pmatrix} x \\ y \end{pmatrix} = w, \quad w \in \mathbb{R}$$

Does there exist a solution  $\begin{pmatrix} x \\ y \end{pmatrix}$  such that  $DF \begin{pmatrix} x \\ y \end{pmatrix} = w$ ?

$$2x_0x + 2y_0y = w$$

**Example**

$$\begin{cases} x^2 - y = a \\ y^2 - z = b \\ z^2 - x = 0 \end{cases}$$

$$DF = \begin{pmatrix} 2x & -1 & 0 & -1 & 0 \\ 0 & 2y & -1 & 0 & -1 \\ -1 & 0 & 2z & 0 & 0 \end{pmatrix}$$

$$c = (0, 0, 0, 0, 0), \quad DF(c) = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \left( \begin{array}{ccc|cc} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left\| DF \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - D\tilde{F} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| =$$

$$= \left\| \left( \begin{array}{ccc|cc} 2x_1 & -1 & 0 & -1 & 0 \\ 0 & 2y_1 & -1 & 0 & -1 \\ -1 & 0 & 2z_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) - \left( \begin{array}{ccc|cc} 2x_2 & -1 & 0 & -1 & 0 \\ 0 & 2y_2 & -1 & 0 & -1 \\ -1 & 0 & 2z_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \right\| =$$

$$= \left\| \left( \begin{array}{ccc|cc} 2(x_1 - x_2) & 0 & 0 & 0 & 0 \\ 0 & 2(y_1 - y_2) & 0 & 0 & 0 \\ 0 & 0 & 2(z_1 - z_2) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\|$$

$$\left\| DF \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - D\tilde{F} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| = 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = 2 \left\| \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| \implies M = 2$$

### Problem 5.20

Are the following functions locally invertible with differentiable inverse?

- (a)  $F(x, y) = (x^2y, -2x)$  at  $(1, 1)$
- (b)  $F(x, y, z) = (xyz, x^2, z^2)$  at  $(0, 0, 0)$
- (c)  $F(x, y, z) = (xyz, x^2, z^2)$  at  $(1, 1, 1)$

For (a), we have  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $F(x, y) = (x^2y, -2x)$ . We need to calculate  $M$  and  $L$ .

$$DF = \begin{pmatrix} 2xy & x^2 \\ -2 & 0 \end{pmatrix}, \quad DF|_{(1,1)} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}, \quad \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} = 2 \neq 0$$

So  $F$  is differentiable at  $(1, 1)$  and  $DF$  is invertible:

$$(DF|_{(1,1)})^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

By the inverse function theorem, we take  $G = F^{-1}$  and  $DG = (DF|_{(1,1)})^{-1}$ .

$$DG|_{F(1,1)} = DG|_{(1,-2)} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

as  $F(1, 1) = (1, -2)$ .

$$R < \frac{1}{2\|L^{-1}\|M}, \quad \text{where } L = DF|_{(1,1)}$$

We calculate  $M$  by taking all the second derivatives of  $F$ :

$$2xy \begin{cases} 2y = \frac{\partial}{\partial x}(2xy) \\ 2x = \frac{\partial}{\partial y}(2xy) \end{cases}, \quad x^2 \begin{cases} 2x = \frac{\partial}{\partial x}(x^2) \\ 0 = \frac{\partial}{\partial y}(x^2) \end{cases}, \quad -2 \begin{cases} 0 = \frac{\partial}{\partial x}(-2) \\ 0 = \frac{\partial}{\partial y}(-2) \end{cases}, \quad 0 \begin{cases} 0 = \frac{\partial}{\partial x}(0) \\ 0 = \frac{\partial}{\partial y}(0) \end{cases}$$

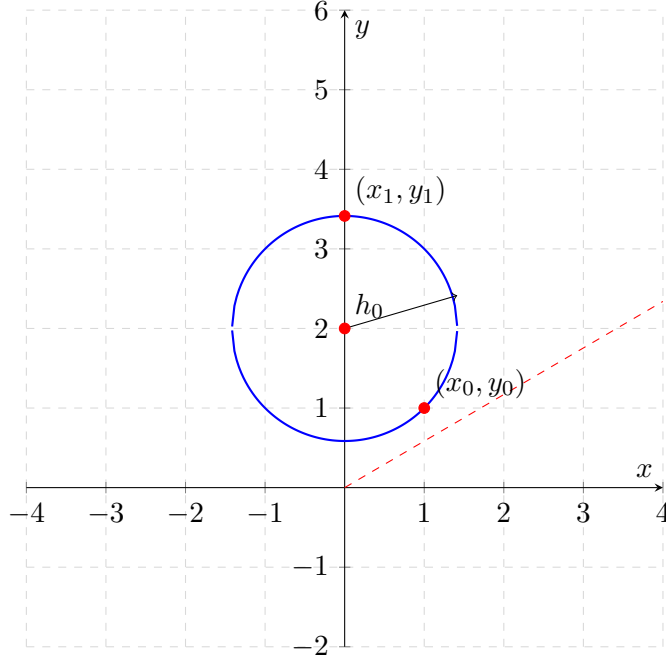
$$(2y)^2 + (2x)^2 + 2x^2 \leq 8(x^2 + y^2)$$

$$h_0 = - (DF|_{(1,1)})^{-1} F(1, 1) = -\frac{1}{2} \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(x_1, y_1) = (x_0, y_0) + (-1, 1) = (1, 1) + (-1, 1) = (0, 2)$$

so we obtain  $M$ :

$$M = \sqrt{8(x^2 + y^2)} = \sqrt{8(0^2 + 2^2)} = \sqrt{8(2 + \sqrt{2})} = 4\sqrt{2} + 4$$



For (c), we have  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $F(x, y, z) = (xyz, x^2, z^2)$ . We need to calculate  $M$  and  $L$ .

$$DF = \begin{pmatrix} yz & xz & xy \\ 2x & 0 & 0 \\ 0 & 0 & 2z \end{pmatrix}, \quad DF|_{(1,1,1)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -4 \neq 0$$

So  $F$  is differentiable at  $(1, 1, 1)$  and  $DF$  is invertible:

$$(DF|_{(1,1,1)})^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

By the inverse function theorem, we take  $G = F^{-1}$  and  $DG = (DF|_{(1,1,1)})^{-1}$ .

$$DG|_{F(1,1,1)} = DG|_{(1,1,1)} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

as  $F(1, 1, 1) = (1, 1, 1)$ .

$$R < \frac{1}{2\|L^{-1}\|M}, \quad \text{where } L = DF|_{(1,1,1)}$$

We calculate  $M$  by taking all the second derivatives of  $F$ :

$$\begin{aligned} yz & \begin{cases} z = \frac{\partial}{\partial x}(yz) \\ y = \frac{\partial}{\partial y}(yz) \\ 0 = \frac{\partial}{\partial z}(yz) \end{cases}, & xz & \begin{cases} z = \frac{\partial}{\partial x}(xz) \\ 0 = \frac{\partial}{\partial y}(xz) \\ x = \frac{\partial}{\partial z}(xz) \end{cases}, & xy & \begin{cases} y = \frac{\partial}{\partial x}(xy) \\ x = \frac{\partial}{\partial y}(xy) \\ 0 = \frac{\partial}{\partial z}(xy) \end{cases} \\ 2x & \begin{cases} 2 = \frac{\partial}{\partial x}(x^2) \\ 0 = \frac{\partial}{\partial y}(x^2) \\ 0 = \frac{\partial}{\partial z}(x^2) \end{cases}, & 0 & \begin{cases} 0 = \frac{\partial}{\partial x}(0) \\ 0 = \frac{\partial}{\partial y}(0) \\ 0 = \frac{\partial}{\partial z}(0) \end{cases}, & 2z & \begin{cases} 0 = \frac{\partial}{\partial x}(z^2) \\ 0 = \frac{\partial}{\partial y}(z^2) \\ 2z = \frac{\partial}{\partial z}(z^2) \end{cases} \end{aligned}$$

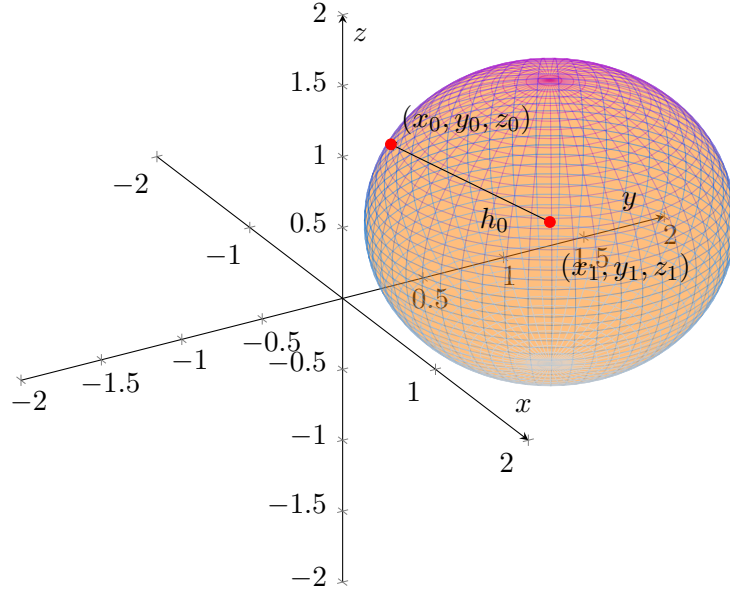
$$(yz)^2 + (xz)^2 + (xy)^2 + 2x^2 + 2z^2 \leq 8(x^2 + y^2 + z^2)$$

$$h_0 = -\left(DF|_{(1,1,1)}\right)^{-1} F(1,1,1) = -\frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$(x_1, y_1, z_1) = (1, 1, 1) + \left(-\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$$

so we obtain  $M$ :

$$M = \sqrt{8(x^2 + y^2 + z^2)} = \sqrt{8(0^2 + 2^2 + 1^2)} = \sqrt{8}(3) = 6\sqrt{2}$$



### 11.18 Curl and Divergence

Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a continuously differentiable function of class  $C'$  (all first derivatives are continuous). The curl of  $F$  is defined as:

$$\begin{aligned} \text{curl}(F) = \nabla \times F &= \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x, y, z) & F_2(x, y, z) & F_3(x, y, z) \end{pmatrix} = \\ &= \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix} \end{aligned}$$

where  $\nabla = \left( \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)$  is the gradient operator.

The divergence of  $F$  is defined as:

$$\text{div}(F) = \nabla \cdot F = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{pmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



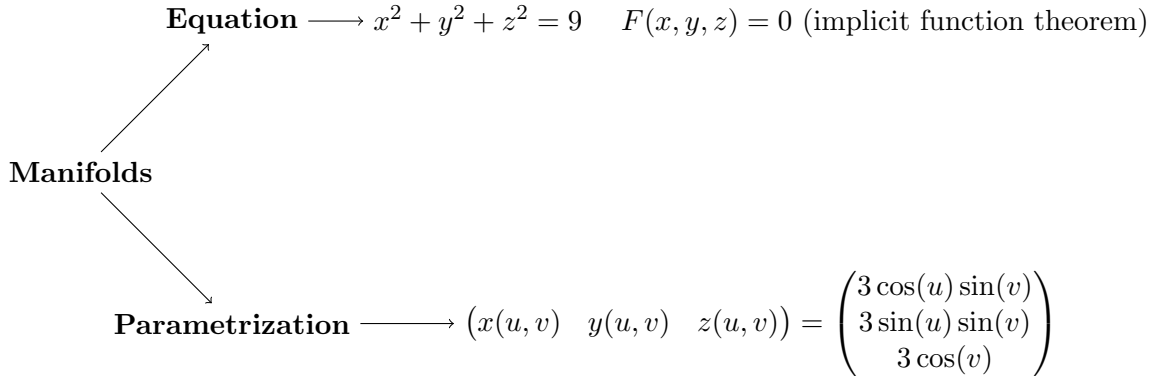
## 12 Manifolds

A curve is a manifold of dimension 1. A surface is a manifold of dimension 2. A manifold of dimension  $n$  is a set that locally looks like  $\mathbb{R}^n$ .

$$M \subset \mathbb{R}^n, \quad M \text{ locally is a graph of } f : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$$

The graph is defined as:

$$\Gamma(f) = \{(x, f(x)) : x \in \mathbb{R}^k, f(x) \in \mathbb{R}^{n-k}\} \subset \mathbb{R}^n$$



### 12.1 Smooth Manifolds in $\mathbb{R}^n$

The graph  $\Gamma(f)$  of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the set of pairs  $(x, y)$  such that  $y = f(x)$ , where  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$  such that  $y = f(x)$ .

For example:

- (A)  $(x, x^2)$  is a smooth graphic.
- (B)  $(x, |x|)$  is not smooth at  $(0, 0)$ .
- (C)  $(x, x^{\frac{1}{3}})$  is a smooth manifold.

**Example:**

$$x^2 + y^2 = 1, \quad \text{Is it locally a manifold?}$$

$$(\cos \theta, \sin \theta) \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

### 12.2 Surface

A smooth surface  $S \subset \mathbb{R}^3$  is a manifold of dimension 2 in  $\mathbb{R}^3$  if it is locally a graph of a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  expressing one variable as a function of the other two.

**Example:**

$$(a, b, c) \in S$$

$$I = (a - \varepsilon, a + \varepsilon) \quad J = (b - \delta, b + \delta) \quad K = (c - \gamma, c + \gamma)$$

$$U = I \times J \times K \text{ is open in } \mathbb{R}^3$$

$$f : I \times J \rightarrow K, \quad (x, y, f(x, y))$$

$$g : I \times K \rightarrow J, \quad (x, g(x, z), z)$$

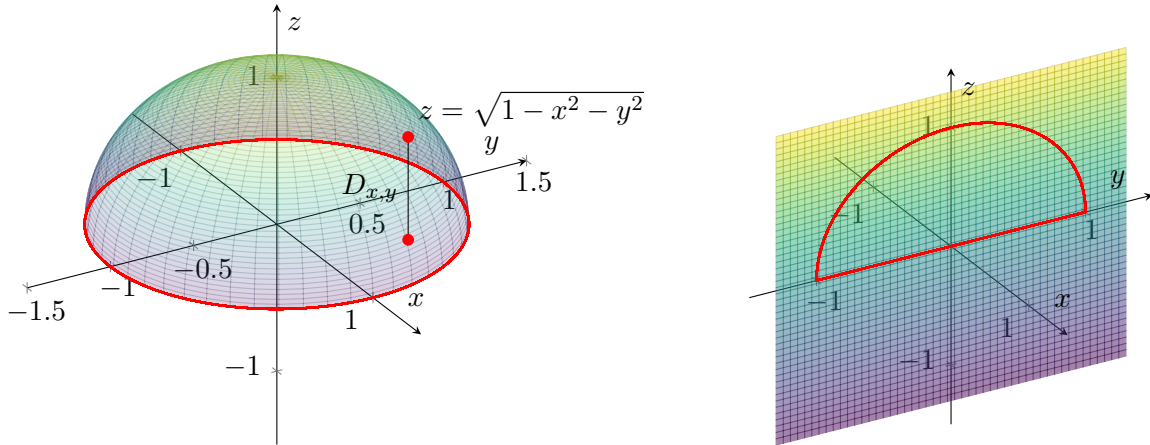
$$h : J \times K \rightarrow I, \quad (g(y, z), y, z)$$

The functions  $f$ ,  $g$ , and  $h$  are obtained by the implicit function theorem.

**Example:**

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$D_{x,y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$



It is a graph of a function  $f : D_{x,y} \rightarrow \mathbb{R}_z^+$ , where  $f(x, y) = \sqrt{1 - x^2 - y^2}$ .

$$(x, y, f(x, y)) \in S^2 \cap (D_{x,y} \times \mathbb{R}_z^+) \subset \mathbb{R}^3$$

### 12.3 Using the Implicit Function Theorem to identify smooth manifolds

Is the locus defined by  $x^8 + 2x^3 + y + y^5 = 1$  a smooth manifold?

**Theorem:** Let  $M \subset \mathbb{R}^n$  be a subset,  $U \subset \mathbb{R}^n$  an open set, and  $F : U \rightarrow \mathbb{R}^{n-k}$  a continuously differentiable function such that  $M \cap U$  is the set of solutions of  $F(z) = 0$ . If  $DF(c)$  is onto for every  $z \in M \cap U$ , then  $M$  is a smooth manifold of dimension  $k$  in  $\mathbb{R}^n$ . Conversely, if  $M$  is a smooth manifold of dimension  $k$  in  $\mathbb{R}^n$ , then every point  $z \in M$  has a neighborhood  $U$  such that there exists a continuously differentiable function  $F : U \rightarrow \mathbb{R}^{n-k}$  with  $DF(z)$  onto and  $M \cap U$  is the set of solutions of  $F(z) = 0$ .

Following the theorem, we apply the implicit function theorem to the function  $F(x, y) = x^8 + 2x^3 + y + y^5 - 1$ .

#### Exercise 6.1b

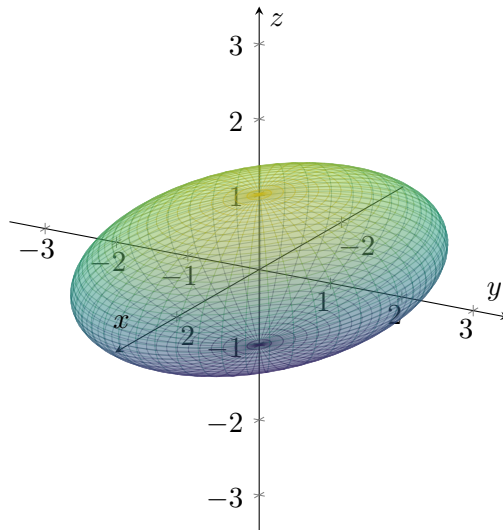
$$x = 3 \cos \theta \sin \phi, \quad y = 3 \sin \theta \sin \phi, \quad z = 3 \cos \phi$$

$$\frac{x}{3} = \cos \theta \sin \phi, \quad \frac{y}{3} = \sin \theta \sin \phi, \quad z = \cos \phi$$

So we have:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1$$

which is the equation of an ellipsoid.

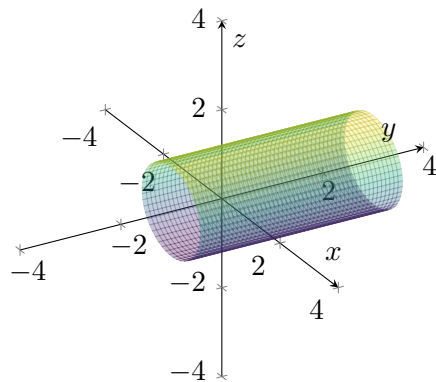
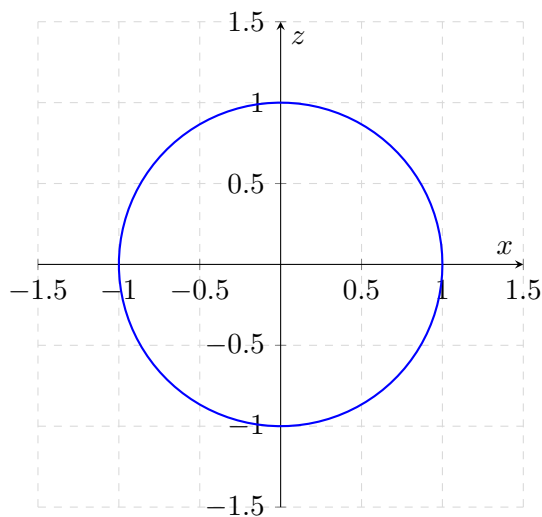


### Exercise 6.1c

$$x = \sin v, \quad y = u, \quad z = \cos v$$

where  $-1 \leq u \leq 3$  and  $0 \leq v \leq 2\pi$ .

$$x^2 + z^2 = 1$$

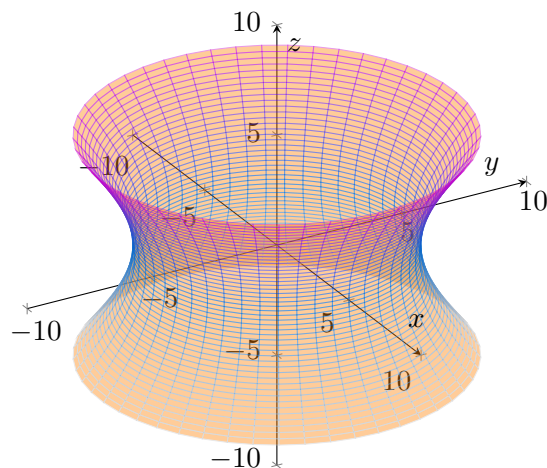


### Example

$$x^2 + y^2 - z^2 = 25 \quad \text{is a hyperbola.}$$

For  $z = 0$  we have  $x^2 + y^2 = 25$  which is a circle.

For  $z \neq 0$  we have  $x^2 + y^2 = 25 + z^2$



$$F(x, y, z) = (x^2 + y^2 - z^2 - 25), \quad F : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$F(x, y, z) = 0, \quad DF = (2x \quad 2y \quad -2z), \quad DF|_{(x_0, y_0, z_0)} = (2x_0 \quad 2y_0 \quad -2z_0)$$

The only way for  $DF$  to not be onto is if  $x_0 = 0$ ,  $y_0 = 0$ , and  $z_0 = 0$ . In this case, we have  $F(0, 0, 0) = -25$  and  $DF(0, 0, 0) = (0, 0, 0)$  which is not onto. So the only point where  $F$  is not a smooth manifold is at the origin.

$$(x^2 + y^2) = 25 \cosh^2(\theta) \quad \text{and} \quad z = 5 \sinh(\theta)$$

$$x^2 + y^2 - z^2 = 25 \cosh^2(\theta) - 25 \sinh^2(\theta) = 25(\cosh^2(\theta) - \sinh^2(\theta)) = 25$$

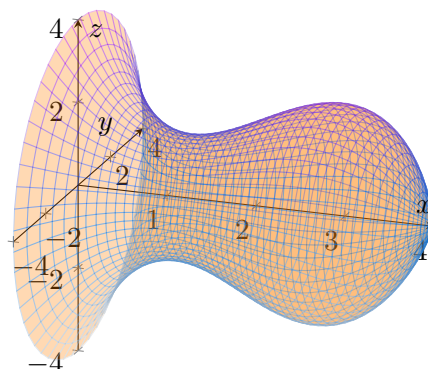
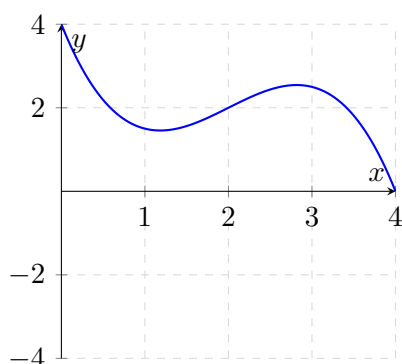
$$\begin{cases} x = 5 \cosh(\theta) \cos(\phi) \\ y = 5 \cosh(\theta) \sin(\phi) \\ z = 5 \sinh(\theta) \end{cases}$$

### Exercise 6.5

Parametrization of a donut (torus):

$$x(\theta, \phi) = (R + \cos(\phi)) \cos(\theta), \quad y(\theta, \phi) = (R + \cos(\phi)) \sin(\theta), \quad z(\theta, \phi) = \sin(\phi)$$

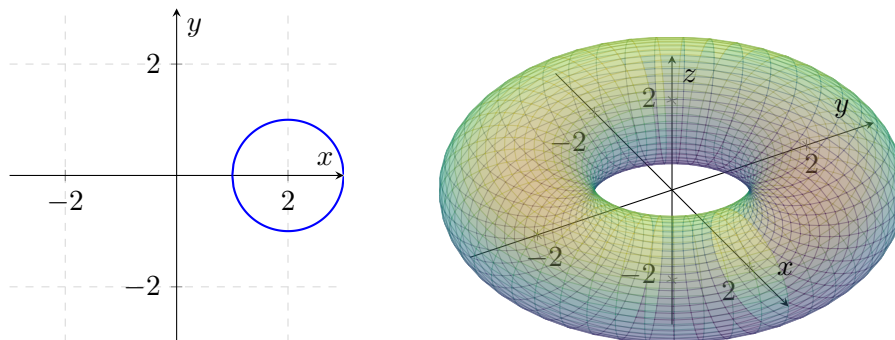
**Surface of revolution:** A surface of revolution is a surface generated by rotating a curve around an axis. The curve is called the generating curve and the axis is called the axis of revolution. The surface of revolution can be described by a parametrization that depends on the angle of rotation and the distance from the axis of revolution.



With  $z = \sin \phi$ ,  $R + \cos \phi = R \pm \sqrt{1 - z^2}$ , we have:

$$\begin{cases} x = (R \pm \sqrt{1 - z^2}) \cos \theta \\ y = (R \pm \sqrt{1 - z^2}) \sin \theta \\ z = z, \quad \text{where } -1 < z < 1 \end{cases}$$

If  $\theta = 0$  we have  $(x - R)^2 + z^2 = 1$  which is a circle.



### Exercise 6.10

$F_1 : x^2 + y^3 + z = a^2$  is a smooth surface.

and also  $F_2 : x + y + z = b$

$$DF_1 = (2x \quad 3y^2 \quad 1), \quad DF_2 = (1 \quad 1 \quad 1)$$

Is the intersection of the two surfaces a smooth curve?

$$F_n = F_1 \cap F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$DF_n = \begin{pmatrix} 2x & 3y^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad DF_n|_{(x_0, y_0, z_0)} = \begin{pmatrix} 2x_0 & 3y_0^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We observe that the rank of  $DF_n$  is 2, so we have two passive variables and one active variable. The intersection of the two surfaces is a smooth curve.

For  $3y^2 = 1$  and  $2x = 1$ , we have:

$$DF_n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{which is not onto.}$$

Then we look at the points where  $x = \frac{1}{2}, y = \frac{1}{\sqrt{3}}$  and  $x = \frac{1}{2}, y = -\frac{1}{\sqrt{3}}$ .

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^3 + z = a^2 \quad \text{and} \quad \frac{1}{2} + \frac{1}{\sqrt{3}} + z = b$$

In this points:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{3}} \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{3}} \\ z \end{pmatrix}$$

$x + y + z = b$  is the plane tangent to  $x^2 + y^3 + z = a^2$ , which is a surface. The intersection of the two surfaces is a curve.

## 13 Tangent Space

Let  $M \subset \mathbb{R}^n$  be a smooth manifold of dimension  $k$  so that near  $z \in M$ ,  $M$  is a graph of a function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ , where  $f$  is continuously differentiable. This function expresses  $n - k$  variables as a function of  $k$  variables.

Then, the tangent space to the manifold at  $a \in M$ , denoted by  $T_a M$ , is the graph of  $DF|_a$ .

### Example

Consider a function

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}, \quad c = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a \in \mathbb{R}^{n-k}, b \in \mathbb{R}^k$$

By the implicit function theorem, we have:

$$F(c) = 0, \quad DF(c) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{n-k}}{\partial x_1} & \cdots & \frac{\partial F_{n-k}}{\partial x_n} \end{pmatrix} \text{ is onto}$$

$$\implies \exists g \text{ such that } F \begin{pmatrix} g(y) \\ y \end{pmatrix} = 0 \implies x - a = Dg|_b \cdot (y - b)$$

### Example

$$\begin{aligned} (a, f(a)) & \quad y = f(a) + f'(a)(x - a) \\ (g(t), t) & \quad x = g(b) + g'(b)(y - b) \end{aligned}$$

### Example

$$S = \{(x, y, f(x, y)) : (x, y) \in U \subset \mathbb{R}^2\}$$

$$z = f(x, y) \quad Df = \nabla f = z - z_0 = \nabla f|_{(x_0, y_0)} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$z = z_0 + \frac{\partial f}{\partial x}|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0, y_0)}(y - y_0)$$

### 13.1 Theorem

If  $f(z) = 0$  describes a manifold  $M$  and if  $DF(c)$  is onto for every  $c \in M$ , then the tangent space  $T_c M$  is the set of solutions of the linear system  $DF(c) \cdot (x - c) = 0$ , i.e. the tangent space is the kernel of the differential  $DF(c)$  at the point  $c$ .

$$T_c M = \ker DF(c)$$

### Example

$$DF(c) \cdot \begin{bmatrix} x - c_1 \\ x_2 - c_2 \\ \vdots \\ x_{n+m} - c_{n+m} \end{bmatrix} =$$

$$DF, \quad \text{if } F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m, \quad DF = \begin{bmatrix} D_1F & D_2F & \cdots & D_{n+m}F \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ x_2 - c_2 \\ \vdots \\ x_{n+m} - c_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

### Example

The curve is a locus defined at  $(1, 1)$  by the equation  $x^9 + 2x^3 + y + y^5 = 5$ .

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad DF = (9x^8 + 6x^2 \quad 1 + 5y^4)$$

$$\text{As } 1 + 5y^4 \neq 0 \implies y(x)$$

The tangent space is the set of solutions of  $DF|_{(1,1)} \cdot \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = 0$

$$DF|_{(1,1)} = (15 \quad 6) \implies 15(x-1) + 6(y-1) = 0 \implies y = -\frac{5}{2}(x-1) + 1$$

### Example

The paraboloid  $z = x^2 + y^2$

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad F(x, y, z) = z - x^2 - y^2$$

$$F(x_0, y_0, z_0) = 0 \quad \text{and} \quad DF = (-2x \quad -2y \quad 1)$$

$$DF|_{(x_0, y_0, z_0)} = (-2x_0 \quad -2y_0 \quad 1) \quad z_0 = x_0^2 + y_0^2$$

$$-2x_0(x - x_0) - 2y_0(y - y_0) + (z - z_0) = 0$$

## 14 Taylor Polynomial

Let  $U \subset \mathbb{R}^n$  be an open subset and  $f : U \rightarrow \mathbb{R}$  a continuously differentiable function. The Taylor polynomial of degree 2 at the point  $(x_0, y_0)$  is given by:

$$P(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + \left\langle \nabla f|_{(x_0, y_0)}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle + \frac{1}{2} \left\langle H|_{(x_0, y_0)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle$$

where  $H$  is the Hessian matrix of  $f$  at the point  $(x_0, y_0)$ , and  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^2$ .

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \quad \text{is the Hessian matrix of } f$$

$$P(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0, y_0)} h_1 + \frac{\partial f}{\partial y}|_{(x_0, y_0)} h_2 +$$

$$+ \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2}|_{(x_0, y_0)} h_1^2 + \frac{\partial^2 f}{\partial x \partial y}|_{(x_0, y_0)} h_1 h_2 + \frac{\partial^2 f}{\partial y \partial x}|_{(x_0, y_0)} h_1 h_2 + \frac{\partial^2 f}{\partial y^2}|_{(x_0, y_0)} h_2^2 \right)$$

### 14.1 Theorem

Let  $U \subset \mathbb{R}^2$ ,  $(x_0, y_0) \in U$ , and  $f : U \rightarrow \mathbb{R}$  be a continuously differentiable function. Then:

1. The Taylor polynomial is the unique polynomial of degree 2 with the same partial derivatives up to order 2 at the point  $(x_0, y_0)$  of the function  $f$ .
2. The Taylor polynomial is the best approximation of the function  $f$  at the point  $(x_0, y_0)$  in the sense that:

$$\lim_{(h_1, h_2) \rightarrow (0,0)} \frac{|f(x_0 + h_1, y_0 + h_2) - P(x_0 + h_1, y_0 + h_2)|}{h_1^2 + h_2^2} = 0$$

### 14.2 Maximum and Minimum

The maximum and minimum of a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  are the points  $x_0 \in U$  such that:

1.  $f(x_0) \geq f(x)$  for all  $x \in B_\varepsilon(x_0)$  (maximum)
2.  $f(x_0) \leq f(x)$  for all  $x \in B_\varepsilon(x_0)$  (minimum)

We build a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g(t) = f(x_0 + t \cdot e_i)$ , where  $e_i$  is the vector with 1 in the  $i$ -th position and 0 elsewhere. The function  $g(t)$  has a maximum or minimum at  $t = 0$  because  $f$  has a maximum or minimum at  $x_0$ , and  $g'(0) = 0$ .

$$g'(t) = \langle \nabla f, e_i \rangle = 0 \implies \nabla f|_{x_0} = 0$$

Now we have to check the second derivative: Given a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , we consider  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , where  $f \in \mathcal{C}^2$  in  $U$ . For  $(x_0, y_0) \in U$ ,  $f$  has a maximum or minimum at  $(x_0, y_0)$ :

$$f(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + \left\langle \nabla f|_{(x_0, y_0)}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle + \frac{1}{2} \left\langle H|_{(x_0, y_0)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle + r(h_1^2 + h_2^2)$$

Knowing that  $\nabla f|_{(x_0, y_0)} = (0, 0)$ , we have:

$$f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) \simeq \frac{1}{2} \left\langle H|_{(x_0, y_0)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle < 0$$

where  $H|_{(x_0, y_0)}$  is negative definite.

### 14.3 Classifying Critical Points

**Extremum:** An extremum of a function is a point where the function achieves a maximum or a minimum.

#### 14.3.1 Theorem

Let  $U \subset \mathbb{R}^n$  be an open subset and  $f : U \rightarrow \mathbb{R}$  a continuously differentiable function. If  $x_0 \in U$  is an extremum of  $f$ , then  $Df|_{x_0} = \vec{0}$ .

**Critical point:** Let  $U \subset \mathbb{R}^n$  be an open subset and  $f : U \rightarrow \mathbb{R}$  a continuously differentiable function. A point  $x_0 \in U$  is called a critical point of  $f$  if the derivative vanishes at that point. The value of the function at the critical point is called a critical value.



### 14.3.2 Theorem: Existence of minima and maxima

Let  $W \subset \mathbb{R}^n$  be a compact set and  $f : W \rightarrow \mathbb{R}$  a continuous function. Then, there exists  $a \in W$  such that  $f(a) \leq f(x)$  for all  $x \in W$  (minimum) and there exists  $b \in W$  such that  $f(b) \geq f(x)$  for all  $x \in W$  (maximum).

*"A continuous function on a compact set attains its maximum and minimum values."*

**Note:** A closed set contains all its limit points, and a bounded set is a set that can be contained within a ball of finite radius. In  $\mathbb{R}^n$ , a set is compact if and only if it is closed and bounded.

### 14.3.3 Theorem

Let  $U \subset \mathbb{R}^n$  be an open set,  $f : U \rightarrow \mathbb{R}$  a function in  $\mathcal{C}^2$ , and  $x_0 \in U$  a critical point of  $f$ . Then,  $x_0$  is a local minimum of  $f$  if and only if the Hessian matrix  $H|_{x_0}$  is positive definite. Similarly,  $x_0$  is a local maximum of  $f$  if and only if the Hessian matrix  $H|_{x_0}$  is negative definite.

**Note:** Given a matrix  $A \in \mathbb{R}^{2 \times 2}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix  $A$  is positive definite if and only if  $a > 0$  and  $\det(A) > 0$ . The matrix  $A$  is negative definite if and only if  $a < 0$  and  $\det(A) > 0$ .

If applied to the Hessian matrix:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

**Saddle point:** A saddle point is a critical point of a function that is neither a local maximum nor a local minimum. It is a point where the function changes direction, and  $\det(H) < 0$ .

### 14.3.4 Theorem: Behaviour of functions near saddle points

Let  $U \subset \mathbb{R}^n$  be an open set,  $f : U \rightarrow \mathbb{R}$  a function in  $\mathcal{C}^2$ . If  $f$  has a saddle point at  $a \in U$ , then in every neighborhood of  $a$ , there are points where  $f(b) > f(a)$  and points where  $f(c) < f(a)$  for  $b \in U$  and  $c \in U$ .

**Proof:** There exists a subspace  $W$  in which  $H$  is positive definite and a subspace  $Z$  in which  $H$  is negative definite.

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in W$$

$$\frac{f(a+th) - f(a)}{t^2} = \frac{t^2 \langle h, Hh \rangle}{2t^2} + \frac{r(t^2 \|h\|^2)}{t^2} \implies f(a+th) > f(a)$$

$$f(a+th) = f(a) + \langle \nabla f|_a, ht \rangle - \frac{t^2}{2} \langle h, Hh \rangle + r(ht)$$

$$\frac{f(a+th) - f(a)}{t^2} \simeq \frac{1}{2} \langle h, Hh \rangle \begin{cases} > 0 & \text{if } h \in W \\ < 0 & \text{if } h \in Z \end{cases}$$

## 14.4 Conclusion

1. Let  $(x_0, y_0)$  be a local minimum of  $f$ . Then:

- $\nabla f|_{(x_0, y_0)} = (0, 0)$
- $H|_{(x_0, y_0)}$  is positive definite

2. Let  $(x_0, y_0)$  be a local maximum of  $f$ . Then:

- $\nabla f|_{(x_0, y_0)} = (0, 0)$
- $H|_{(x_0, y_0)}$  is negative definite

3. Let  $(x_0, y_0)$  be a saddle point of  $f$ . Then:

- $\nabla f|_{(x_0, y_0)} = (0, 0)$
- $\det(H|_{(x_0, y_0)}) < 0$

### Exercise

Given the function  $f(x, y) = x^2 + y^2 - 2x - 6y + 14$ , find the critical points and classify them.

$$2x - 2 = 0 \implies x = 1$$

$$2y - 6 = 0 \implies y = 3$$

The unique critical point is  $(1, 3)$ , and we have:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ is positive definite.}$$

so  $(1, 3)$  is a local minimum.

$$f(x, y) = x^2 - 2x + 1 + y^2 - 6y + 9 + 4 = (x - 1)^2 + (y - 3)^2 + 4$$

$$f(x, y) \geq 4 = f(1, 3) \implies f(x, y) \geq f(1, 3) \implies (1, 3) \text{ is a global minimum.}$$

### Exercise

Find the critical points of the function  $f(x, y) = x^4 + y^4 - 4xy + 1$  and classify them.

$$\nabla f = \begin{pmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{pmatrix}, \quad \begin{cases} 4x^3 - 4y = 0 \implies x^3 = y \\ 4y^3 - 4x = 0 \implies y^3 = x \end{cases}$$

$$\text{As } y = x^3, x^9 = x \implies x^9 - x = 0 \implies x(x^8 - 1) = 0.$$

$$x(x^4 - 1)(x^4 + 1) = 0 \implies x = 0, \pm 1, (\pm i)$$

The critical points are  $(0, 0)$ ,  $(1, 1)$  and  $(-1, -1)$ .

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}, \quad H|_{(0,0)} = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{ is indefinite.}$$

$$H|_{(1,1)} = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \text{ is positive definite.} \quad H|_{(-1,-1)} = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \text{ is positive definite.}$$

So  $(1, 1)$  and  $(-1, -1)$  are local minima because  $H$  is positive definite. The critical point  $(0, 0)$  is a saddle point because  $H$  is indefinite.

**Exercise 8.2c**

Classify the critical points of the function  $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$ .

$$\frac{\partial f}{\partial x} = 3x^2 - 6x - 9, \quad \frac{\partial f}{\partial y} = 3y^2 - 6y$$

$$H = \begin{pmatrix} 6x - 6 & 0 \\ 0 & 6y - 6 \end{pmatrix}, \quad \det(H) = (6x - 6)(6y - 6) \quad \text{is positive definite.}$$

$$\frac{\partial f}{\partial x} = 0 \implies 3x^2 - 6x - 9 = 0 \implies x^2 - 2x - 3 = 0 \implies (x - 3)(x + 1) = 0 \implies x = 3, -1$$

$$\frac{\partial f}{\partial y} = 0 \implies 3y^2 - 6y = 0 \implies y(y - 2) = 0 \implies y = 0, 2$$

So we have the critical points  $(3, 0)$ ,  $(3, 2)$ ,  $(-1, 0)$  and  $(-1, 2)$ .

$$H|_{(3,0)} = \begin{pmatrix} 12 & 0 \\ 0 & -6 \end{pmatrix} \quad H|_{(3,2)} = \begin{pmatrix} 12 & 0 \\ 0 & 6 \end{pmatrix}$$

$$H|_{(-1,0)} = \begin{pmatrix} -12 & 0 \\ 0 & -6 \end{pmatrix} \quad H|_{(-1,2)} = \begin{pmatrix} -12 & 0 \\ 0 & 6 \end{pmatrix}$$

$\det(H|_{(3,0)}) < 0 \implies (3, 0)$  is a saddle point.

$\det(H|_{(3,2)}) > 0 \implies f(3, 2) = -31$  is a local minimum.

$\det(H|_{(-1,0)}) > 0 \implies (-1, 0) = 5$  is a local maximum.

$\det(H|_{(-1,2)}) < 0 \implies (-1, 2)$  is a saddle point.

**Exercise 8.3b**

Find the critical points of the function  $f(x, y) = x^2 + y^2 + x^2y + 4$  and classify them.

$$D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$$

$f$  is continuous on  $D$  and differentiable on  $D$ . As  $\partial D \subset D$ ,  $D$  is closed. There exists  $R$  such that  $D \subset B_R(0)$ .  $D$  is bounded and closed, so  $D$  is compact.

Given  $f$  continuous on  $D$ , we know that  $f$  attains its maximum and minimum values on  $D$ .

$$\frac{\partial f}{\partial x} = 2x + 2xy, \quad \frac{\partial f}{\partial y} = 2y + x^2$$

$$\frac{\partial f}{\partial x} = 0 \implies 2x + 2xy = 0 \implies x(1 + y) = 0 \implies x = 0, y = -1$$

$$\frac{\partial f}{\partial y} = 0 \implies 2y + x^2 = 0 \implies y = -\frac{x^2}{2}$$

$$x = 0 \implies y = 0 \implies (0, 0) \text{ is a critical point.}$$

$$y = -1 \implies x^2 = 2 \implies x = \pm\sqrt{2} \implies (-\sqrt{2}, -1) \text{ and } (\sqrt{2}, -1) \text{ are critical points.}$$

We only keep  $(0, 0)$ :

$$H|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{is positive definite.}$$

so  $(0, 0)$  is a local minimum.

## Exercise 7.2

Determine the second-order Taylor formula for the given functions about the point  $(x_0, y_0)$ :

$$g(x, y) = \sin(x + y^2) \quad \text{at } (0, 0)$$

We know that when  $x \simeq 0$  and  $y \simeq 0$ , we have:

$$g(x \simeq 0, y \simeq 0) = x + y^2 \simeq 0$$

$$\sin(x + y^2) = x + y^2 - \frac{(x + y^2)^3}{6} + r(x, y) \implies g(x, y) = x + y^2 - \frac{(x + y^2)^3}{6} + r(x, y)$$

We expand  $(x + y^2)^3$  and rule out terms of degree greater than needed:

$$(x + y^2)^3 = x^3 + 3x^2y^2 + 3xy^4 + y^6$$

So the Taylor polynomial of degree 2 is:

$$\sin(x + y^2) \simeq x + y^2 \quad \text{at } (0, 0)$$

The Taylor polynomial of degree 3 is:

$$\sin(x + y^2) \simeq x + y^2 + x^3 \quad \text{at } (0, 0)$$

The Taylor polynomial of degree 4 is:

$$\sin(x + y^2) \simeq x + y^2 + x^3 + 3x^2y^2 \quad \text{at } (0, 0)$$

## Exercise

Given  $F(x, y) = x^3 + yx + y^3 - 3$ , prove that  $x = g(y)$  near  $(1, 1)$  and find the Taylor polynomial of degree 2 of  $g$ .

To use the implicit function theorem, we need to check that  $DF|_{(1,1)}$  is onto. We have:

$$DF = (3x^2 + y \quad 3y^2 + x), \quad DF|_{(1,1)} = (4 \quad 4)$$

which is onto. So we take  $x$  as the passive variable and  $y$  as the active variable. There exists a function  $g$  such that  $F(g(y), y) = 0$ .

$$g(1) = 1 \quad \text{and} \quad g(1 + h) = 1g'(1)h + \frac{1}{2}g''(1)h^2 + r(h^2)$$

To calculate  $g'(1)$ , we differentiate the equation  $F(g(y), y) = 0$  with respect to  $y$ :

$$DF|_{(g(1),1)} \cdot \begin{pmatrix} g'(y) \\ 1 \end{pmatrix} = 0 \implies g'(y) = -\frac{DF_2}{DF_1}|_{(g(1),1)} \implies g'(1) = -\frac{4}{4} = -1$$

$$g(1 + h) = 1 - h + Ah^2$$

With  $F(1, 1) = 0$ , we have:

$$\begin{aligned} F(g(1 + h), 1 + h) &= (g(1 + h))^3 + (1 + h)g(1 + h) + (1 + h)^3 - 3 = \\ &= (1 - h + Ah^2)^3 + (1 + h)(1 - h + Ah^2) + (1 + h)^3 - 3 \\ &\implies (4A + 5)h^2 = 0 \implies A = -\frac{5}{4} \implies g(1 + h) = 1 - h - \frac{5}{4}h^2 \end{aligned}$$

### Exercise 7.5

Show that the equation  $x^2 + y^3 + xyz^3 - 3 = 0$  determines  $z$  as a function of  $g(x, y)$  near the point  $(1, 1, 1)$ . Find the Taylor polynomial of degree 2 of  $g$  at the point  $(1, 1)$ .

$$F(x, y, z) = x^2 + y^3 + xyz^3 - 3 \quad \text{at } (1, 1, 1)$$

$$DF = (2x + yz^2 \quad 3y^2 + yz^3 \quad xy(3z^2)), \quad DF|_{(1,1,1)} = (3 \quad 4 \quad 3)$$

$$DF|_{(1,1,1)} \text{ is onto, so there exists a function } g \text{ such that } F(x, y, g(x, y)) = 0$$

$$g(1+h_1, 1+h_2) = g(1, 1) + \frac{\partial g}{\partial x}|_{(1,1)}h_1 + \frac{\partial g}{\partial y}|_{(1,1)}h_2 + \frac{1}{2} \left( \frac{\partial^2 g}{\partial x^2}|_{(1,1)}h_1^2 + \frac{\partial^2 g}{\partial y^2}|_{(1,1)}h_2^2 + 2\frac{\partial^2 g}{\partial x \partial y}|_{(1,1)}h_1h_2 \right)$$

The implicit function theorem says that  $g$  is differentiable:

$$\left( \frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \quad \frac{\partial^2 g}{\partial g(x, y)} \right) |_{(1,1,1)} \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x}|_{(1,1)} \end{pmatrix} = 0$$

$$\Rightarrow (3 \quad 4 \quad 3) \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x}|_{(1,1)} \end{pmatrix} = 0 \Rightarrow \frac{\partial g}{\partial x}|_{(1,1)} = -1$$

$$\frac{\partial g}{\partial y}|_{(1,1)} = -\frac{DF_2}{DF_1}|_{(1,1,1)} = -\frac{4}{3} \Rightarrow \frac{\partial g}{\partial y}|_{(1,1)} = -\frac{4}{3}$$

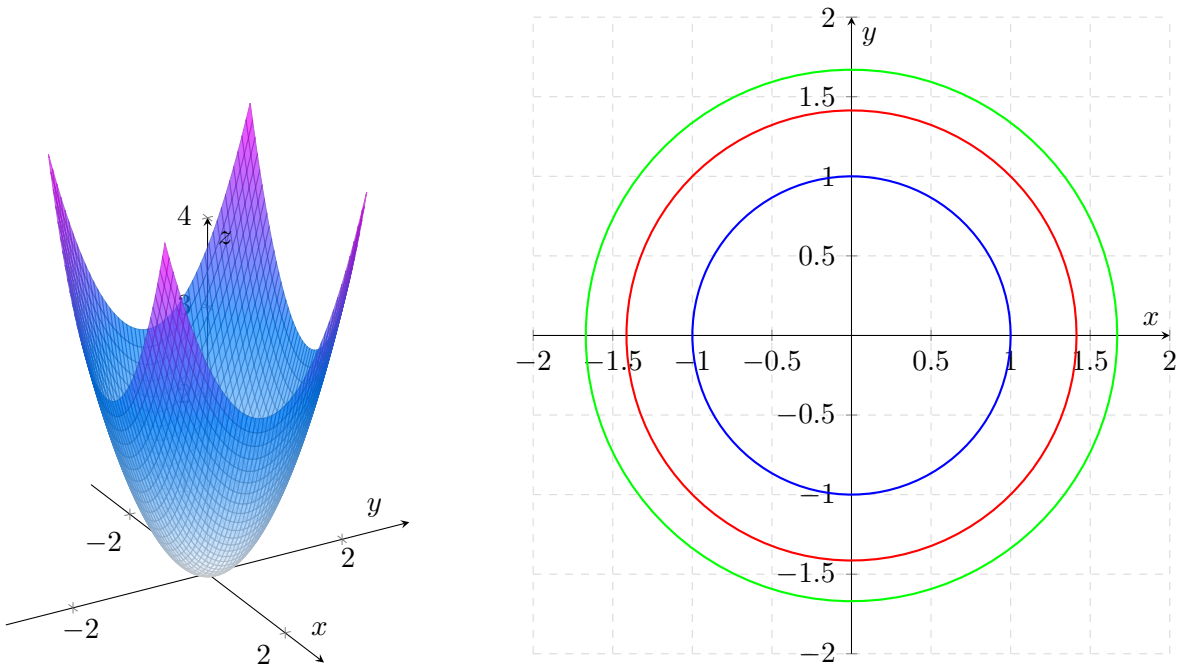
$$(1+h_1)^2 + (1+h_2)^3 + (1+h_1)(1+h_2)(1-h_1-\frac{4}{3}h_2)^3 + A\frac{h_1^2}{2} + B\frac{h_2^2}{2} + Ch_1h_2 = 0$$

So  $A = -\frac{2}{3}$ ,  $B = -\frac{26}{9}$ , and  $C = -\frac{2}{3}$ . The Taylor polynomial of degree 2 is:

$$g(x, y) \simeq 1 - (x-1) - \frac{4}{3}(y-1) - \frac{1}{3}(x-1)^2 - \frac{13}{9}(y-1)^2 - \frac{2}{3}(x-1)(y-1) \quad \text{at } (1, 1)$$

## 15 Lagrange Multipliers

We have to remember level curves and level surfaces, and the fact that the gradient is perpendicular to the level curves and level surfaces. The gradient is a vector that points in the direction of the steepest ascent of a function.



## 15.1 Constrained Optimization

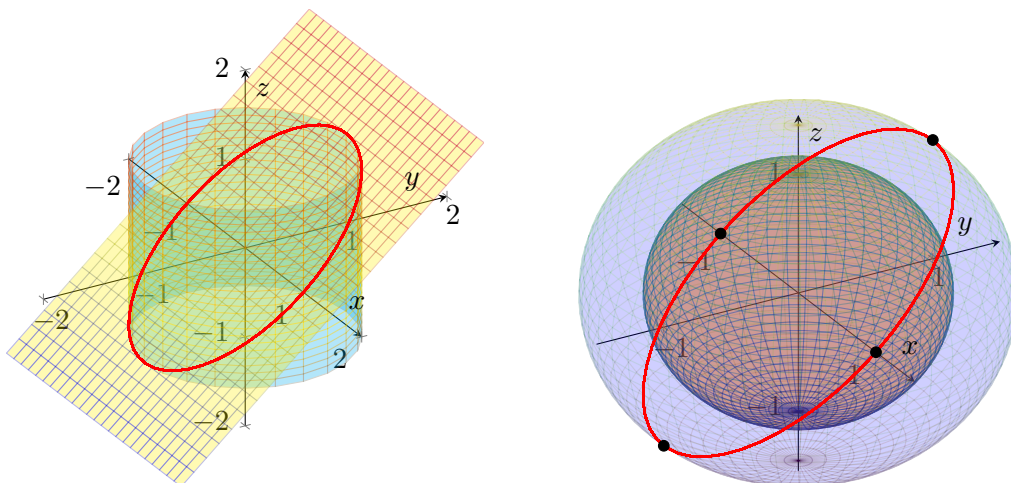
Graphically, restriction is to be on the unit circle of  $F(x, y) = x^2 + 2y^2$ .

We have seen that if  $f(x, y)$  is a continuous function on a compact set  $D \subset \mathbb{R}^2$ , then  $f$  attains its maximum and minimum values on  $D$ .

$$D \begin{cases} \partial D & \begin{cases} \text{Lagrange multipliers} \\ \text{Restriction constrained} \end{cases} \\ \text{Interior of } D & \begin{cases} \text{Hessian} \\ \text{Unconstrained optimization} \end{cases} \end{cases}$$

### Example

Obtain, graphically, the maximum and minimum of the function  $f(x, y, z) = x^2 + y^2 + z^2$ , which is the distance from the origin squared, in the curve intersection of the cylinder  $x^2 + y^2 = 1$  and  $z = y$ .



So we have the following points:

- $(1, 0, 0)$
- $(-1, 0, 0)$
- $(0, 1, 1)$
- $(0, -1, -1)$

The maximums are  $(0, 1, 1)$  and  $(0, -1, -1)$ , and the minimums are  $(1, 0, 0)$  and  $(-1, 0, 0)$ .

## 15.2 Tangent Space

Let  $F(x) = 0$  be given by an equation. Then the tangent space at the point  $x_0$  is given by the equation:

$$T_{M_{x_0}} = \ker(DF|_{x_0}) = \{x \in \mathbb{R}^n \mid F(x) = 0\} \quad \text{is the tangent space at } x_0$$

$T_{M_{x_0}}$  is the tangent space at  $x_0$  and is a manifold.

## 15.3 Constrained Optimization

We have to find the maximum and minimum of  $f(x, y)$  subject to the constraint  $F(x) = 0$ . We need to optimize the function  $f$  on the manifold  $M$  defined by the constraint  $F(x) = 0$ .

$$\langle v, \nabla f \rangle = 0, \quad (\nabla f) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = 0$$

where  $v \in T_{M_{x_0}} = \ker(DF|_{x_0})$ .

## 15.4 Lemma

Let  $A = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}$  be linear transformations. Then,

$$\ker(A) \subset \ker(\beta) \iff \exists \lambda_1, \dots, \lambda_m \in \mathbb{R} \text{ such that } \beta = \sum_{i=1}^m \lambda_i \alpha_i$$

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \dots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} \quad \text{and } v \in \ker(A) \implies v \in \mathbb{R}^n$$

$$\beta = \nabla f = \left( \frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right) \quad v \in \ker(\beta)$$

## 15.5 Theorem (Lagrange multipliers)

Let  $M$  be a manifold known by a vector function  $F(x) = 0$ , where  $F = F(x_1, \dots, x_n)$  goes from an open subset  $U \subset \mathbb{R}^n$  to  $\mathbb{R}^m$ . And  $DF(x)$  is onto for all  $x \in M$ . Then, the numbers  $\lambda_1, \dots, \lambda_m$  are called Lagrange multipliers of  $f$  at  $x_0$ .

If  $a \in x$  is a critical point of  $f$  restricted to  $M$ , then there exists  $\lambda_1, \dots, \lambda_m$  such that the derivative of  $f$  at  $a$  is a linear combination of the derivatives of the constrained functions  $F$  at  $a$ :

$$Df|_a = \lambda_1 DF_1|_a + \dots + \lambda_m DF_m|_a$$

### Example

Find the maximum and minimum of the function  $f(x, y) = x^2 + 2y^2$  subject to the constraint  $g(x, y) = x^2 + y^2 = 1$ .

$F(x) = 0$  defines the manifold  $M$  and  $F(x, y) = x^2 + y^2 - 1 = 0$

$$DF = (2x \ 2y), \quad \nabla f = (2x \ 4y)$$

$$\begin{cases} \nabla f = \lambda \nabla F \\ F(x, y) = 0 \end{cases} \implies \begin{cases} (2x \ 4y) = \lambda (2x \ 2y) \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \implies \text{three equations, three unknowns.}$$

From the first equation, we have two cases:

- $x = 0 \implies y^2 = 1 \implies y = \pm 1$
- $x \neq 0 \implies \lambda = 1$  and  $y = 2x$ .

The critical points are  $(0, 1)$  and  $(0, -1)$ .

### Exercise

Find the points of the sphere  $x^2 + y^2 + z^2 = 4$  that are closest and furthest to the point  $(3, 1, -1)$ . We have to maximize and minimize the function  $f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$  because it achieves its maximum and minimum values on the same point.

$F(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$  defines the manifold  $M$ . We have:

$$\nabla f = (2(x - 3) \ 2(y - 1) \ 2(z + 1)), \quad \nabla F = (2x \ 2y \ 2z)$$

$$\begin{cases} \nabla f = \lambda \nabla F \\ F(x, y, z) = 0 \end{cases} \implies \begin{cases} (2(x - 3) \ 2(y - 1) \ 2(z + 1)) = \lambda (2x \ 2y \ 2z) \\ x^2 + y^2 + z^2 - 4 = 0 \end{cases}$$

$$2x - 6 = \lambda 2x \implies 2x(1 - \lambda) = 6 \implies \begin{cases} 2x(1 - \lambda) = 6 \\ 2y(1 - \lambda) = 2 \\ 2z(1 - \lambda) = -2 \end{cases}$$

so  $\lambda \neq 1$  and we have:

$$\begin{cases} x = \frac{3}{1 - \lambda} \\ y = \frac{1}{1 - \lambda} \\ z = \frac{-1}{1 - \lambda} \end{cases} \implies \frac{9}{(1 - \lambda)^2} + \frac{1}{(1 - \lambda)^2} + \frac{1}{(1 - \lambda)^2} = 4$$
$$\implies \frac{11}{(1 - \lambda)^2} = 4 \implies (1 - \lambda)^2 = \frac{11}{4} \implies 1 - \lambda = \pm \frac{\sqrt{11}}{2}$$



### Exercise

Find the maximum value of  $f(x, y, z) = x + 2y + 3z$  in the curve intersection of the plane  $x - y + z = 1$  and the cylinder  $x^2 + y^2 = 1$ .

$$F(x, y, z) = 0 \text{ defines the manifold } M \text{ and } F(x, y, z) = \begin{cases} x - y + z - 1 \\ x^2 + y^2 - 1 \end{cases} = 0$$

$$\nabla f = (1 \ 2 \ 3), \quad \nabla F = \begin{pmatrix} 1 & -1 & 1 \\ 2x & 2y & 0 \end{pmatrix}$$

$$\nabla f = \lambda \nabla F_1 + \mu \nabla F_2 \implies (1 \ 2 \ 3) = \lambda (1 \ -1 \ 1) + \mu (2x \ 2y \ 0) \implies \begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \end{cases}$$

### Exercise

Find, graphically, the closest point  $P(x_0, y_0, z_0)$  to an arbitrary sphere.

Our restriction is  $F(x, y, z) = 0$ , and we have the distance function:

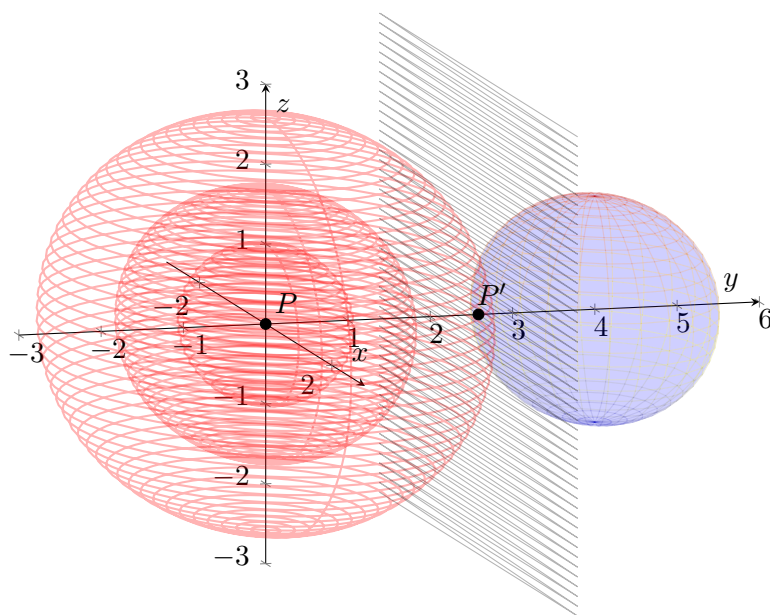
$$f(x, y, z) = (d(P, (x, y, z)))^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

The level surfaces of  $f$  are:

$$\{(x, y, z) \mid f(x, y, z) = c\} = \{(x, y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c\}$$

We need to find the level surface of  $f$  that is tangent to the level surface of  $F$ . The level surface of  $F$  is given by the equation:

$$\{(x, y, z) \mid F(x, y, z) = c\} = \{(x, y, z) \mid x^2 + y^2 + z^2 - c = 0\}$$



### Exercise

Find the minimum value of  $f(x, y, z) = x^2 + 2y^2 + 3z^2$  subject to the constraint  $x + 2y + 3z = 10$ . The constraint is not bounded, so we cannot say that the function attains its maximum value, but we can say that it attains its minimum value. We have:

$$\begin{aligned} F(x, y, z) = 0 \text{ defines the manifold } M \text{ and } F(x, y, z) = x + 2y + 3z - 10 = 0 \\ \nabla f = (2x \quad 4y \quad 6z), \quad \nabla F = (1 \quad 2 \quad 3) \\ \nabla f = \lambda \nabla F \implies (2x \quad 4y \quad 6z) = \lambda (1 \quad 2 \quad 3) \implies \begin{cases} 2x = \lambda \\ 4y = 2\lambda \\ 6z = 3\lambda \end{cases} \implies \begin{cases} x = \frac{\lambda}{2} \\ y = \frac{\lambda}{2} \\ z = \frac{\lambda}{2} \end{cases} \\ \implies \frac{\lambda}{2} + 2\frac{\lambda}{2} + 3\frac{\lambda}{2} = 10 \implies \frac{6\lambda}{2} = 10 \implies \lambda = \frac{10}{3} \\ \implies x = \frac{5}{3}, y = \frac{5}{6}, z = \frac{5}{9} \implies f(x, y, z) = \left(\frac{5}{3}\right)^2 + 2\left(\frac{5}{6}\right)^2 + 3\left(\frac{5}{9}\right)^2 = \frac{25}{9} + \frac{50}{36} + \frac{75}{81} \end{aligned}$$

### Exercise

Find the extreme values of the function  $f$  on the region described by the inequality:

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5 \quad \text{and} \quad x^2 + y^2 \leq 16$$

$$D = \{(x, y) \mid x^2 + y^2 \leq 16\} \text{ is the disk of radius 4}$$

$D$  is a compact set, so  $f$  attains its maximum and minimum values on  $D$ . We have:

$$\nabla f = \begin{pmatrix} 4x - 4 \\ 6y \end{pmatrix} \quad \nabla f = 0 \implies \begin{cases} 4x - 4 = 0 \\ 6y = 0 \end{cases} \implies \begin{cases} x = 1 \\ y = 0 \end{cases} \implies f(1, 0) = -1$$

We have to check the Hessian:

$$H = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \quad \text{is positive definite.}$$

so  $(1, 0)$  is a local minimum.

## 16 Curvature of a curve (In $\mathbb{R}^3$ or in $\mathbb{R}^2$ )

We need the "best coordinate system" to describe the curve. We have:

$$Y = G(x), \quad Y = \frac{A_2}{2}X^2 + \text{remainder}$$

### 16.1 Curvature of a curve in $\mathbb{R}^2$

Let a curve in  $\mathbb{R}^2$  be locally the graph of a function  $G(x)$ , with Taylor polynomial:

$$G(x) = \frac{A_2}{2}x^2 + \frac{A_3}{6}x^3 + \cdots + \frac{A_n}{n!}x^n + r(x) \quad \text{at } (0, 0)$$

Then, the curvature of the curve at the point 0, i.e. the point of the curve, is given by:

$$\kappa = |A_2| \quad \text{at } (0, 0)$$

### Example

Consider the unit circle at  $(1, 0)$ :

$$\begin{aligned} Y &= y - 1, \quad X = x \\ y &= \sqrt{1 - x^2} \implies Y = \sqrt{1 - X^2} - 1 \end{aligned}$$

With  $Y = G(x)$ , we have:

$$\begin{aligned} G(0) &= 0, \quad G'(x) = -\frac{X}{\sqrt{1 - X^2}} \implies G'(0) = 0 \\ G''(x) &= -\frac{1}{\sqrt{1 - X^2}} + \frac{X^2}{(1 - X^2)^{3/2}} \implies G''(0) = -1 \\ G(x) &= G(0) + G'(0)x + \frac{G''(0)}{2}x^2 \implies \kappa = |-1| = 1 \end{aligned}$$

## 16.2 Computing the curvature of a plane curve known as a graph $(x, f(x))$

In this case, the curvature of  $f$  at the point  $(x_0, f(x_0))$  is given by:

$$\kappa = \frac{|f''(x_0)|}{(1 + (f'(x_0))^2)^{3/2}} \quad \text{at } (x_0, f(x_0))$$

### Example

Calculate the curvature of the parabola  $y = x^2$  at the point  $(a, f(a))$ .

$$\begin{aligned} f(x) &= x^2 \implies f'(x) = 2x, \quad f''(x) = 2 \\ \kappa &= \frac{|f''(a)|}{(1 + (f'(a))^2)^{3/2}} = \frac{2}{(1 + (2a)^2)^{3/2}} \quad \text{at } (a, a^2) \end{aligned}$$

### Example: Curvature of a curve given by the implicit function theorem

With  $x = g(y)$ , calculate the curvature of the curve

$$\begin{aligned} F(x, y) &= x^3 + xy + y^3 - 3 \quad \text{at } (1, 1) \\ DF &= (3x^2 + y \quad x + 3y^2), \quad DF|_{(1,1)} = (4 \quad 4) \\ DF|_{(1,1)} &\text{ is onto, so there exists a function } g \text{ such that } F(g(y), y) = 0 \\ g(1) &= 1 \implies F(g(1), 1) = 0 \implies g'(y) = -\frac{DF_2}{DF_1}|_{(g(1),1)} \implies g'(1) = -\frac{4}{4} = -1 \\ g(1+h) &= g(1) + g'(1)h + \frac{g''(1)}{2}h^2 + \dots \\ F(g(1+h), 1+h) &= (g(1+h))^3 + (1+h)(g(1+h)) + (1+h)^3 - 3 \\ &= (1-h+Ah^2)^3 + (1+h)(1-h+Ah^2) + (1+h)^3 - 3 \\ &= 1 - 3h + 3Ah^2 + (1+h)(1-h+Ah^2) + 1 + 3h + 3h^2 - 3 = 0 \\ A &= -\frac{5}{4} \implies \frac{g''(1)}{2} = -\frac{5}{4} \implies g''(1) = -\frac{5}{2} \\ \kappa &= \frac{|g''(1)|}{(1 + (g'(1))^2)^{3/2}} = \frac{|-5/2|}{(1 + (-1)^2)^{3/2}} = \frac{5/2}{2\sqrt{2}} = \frac{5}{4\sqrt{2}} \end{aligned}$$