Vector Calculus

1 Euclidean space

We define the euclidean space in $\mathbb{R}^N, N \geq 1$ using cartesian coordinates.
Any element $x \in \mathbb{R}^N, \quad x = (x_1, x_2, \dots, x_N), \quad x \in \mathbb{R}$

Standard basis

$$e_1 = (1, 0, \dots, 0), \dots, e_N = (0, 0, \dots, 1)$$

$$\text{Then, } x = \sum_{j=1}^{N} x_j \cdot e_j$$

In particular, $B_{\mathbb{R}^3} = \{i, j, k\}$, the canonical basis.

Properties

- Addition: $\quad (x_1, \dots, x_N) + (y_1, \dots, y_N) = (x_1 + y_1, \dots, x_N + y_N)$
- Multiplication by a scalar $\lambda \in \mathbb{R}$: $\quad \lambda x = \lambda(x_1, \dots, x_N) = (\lambda x_1, \dots, \lambda x_N)$
- Associative: $\quad \lambda, \mu \in \mathbb{R}, \quad (\lambda\mu)x = \lambda(\mu x)$
- Additive Identity: $\quad$ There exists a vector $\overline{0} = (0, \dots, 0)$ such that $x + \overline{0} = x$
- Additive Inverse: $\quad \forall x = (x_1, \dots, x_N), \exists\ \overline{x} = (-x_1, \dots, -x_N)$ such that $x + \overline{x} = \overline{0}$
- Distributive Property (over vector addition):

$$\lambda\left((x_1, \dots, x_N) + (y_1, \dots, y_N)\right) = \lambda(x_1, \dots, x_N) + \lambda(y_1, \dots, y_N)$$

- Distributive Property (over scalar addition):

$$(\lambda + \mu)(x_1, \dots, x_N) = \lambda(x_1, \dots, x_N) + \mu(x_1, \dots, x_N)$$

- Scalar Multiplication Identity: $\quad 1 \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)$
- Zero Scalar Multiplication: $\quad 0 \cdot (x_1, \dots, x_N) = (0, \dots, 0)$

Norm

The euclidean space in $\mathbb{R}^N$ is a normal space with an associated norm function.

$$\| \cdot \| : \mathbb{R}^N \to \mathbb{R}_+ \cup \{0\}$$

$$x = (x_1, \dots, x_N) \to \|x\| = \sqrt{x_1^2 + \cdots + x_N^2}$$

Properties

The norm satisfies the following properties:

(a) $\forall x \in \mathbb{R}^N$

- $\|x\| > 0 \iff x \neq 0$
- $\|x\| = 0 \iff x = 0$

(b) $\|\lambda x\| = |\lambda| \, \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$

- Triangular inequality.

Remark: Distance

We can define the distance between two elements in $\mathbb{R}^N$ as

$$dist(x, y) = \|x - y\| = \|y - x\|$$
$$dist(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

- $dist(x, y) = \|x - y\| > 0$ if $x \neq y$, and $dist(x, y) = 0$ if $x = y$
- $dist(x, y) = \|x - y\| = \| - (y - x)\| = \| - 1\| \cdot \|y - x\| = dist(y, x)$
- $dist(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = dist(x, z) + dist(z, y)$

Remark

For $\mathbb{R}$ such a distance is the absolute value, $| \cdot | : \mathbb{R} \to \mathbb{R}$

2 Inner or scalar product

Let x, y be two vectors in $\mathbb{R}^N$, then

$$x \cdot y = x_1 y_1 + \cdots + x_N y_N$$
$$x \cdot y = \langle x, y \rangle = (x, y)$$

2.1 Properties

The inner product satisfies the following properties:

- $\forall x \in \mathbb{R}^N \ \langle x, x \rangle \geq 0$

  $\langle x, x \rangle = 0$ if $x = 0$
- Symmetric: $\langle x, y \rangle = \langle y, x \rangle$
- Bilinear: $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

2.2 Cauchy-Schwartz inequality

$$|x \cdot y| \leq \|x\| \|y\|$$

Proof If $y = \lambda x, |\langle x, \lambda y\rangle| = |\lambda|\|x\|^2 = \|x\||\lambda|\|x\| = \|x\|\|y\|$
If $y \neq \lambda x$ (x and y are linearly independent).
Assume $z = \lambda x + y$

$$0 \leq \langle z, z\rangle = \langle \lambda x + y, \lambda x + y\rangle = \langle \lambda x, \lambda x\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + \langle y, y\rangle$$
$$\text{Since } \|x\|^2 > 0, \quad = \lambda^2\|x\|^2 + 2\lambda \langle x, y\rangle + \|y\|^2$$

If we represent it as a parabola in function of λ, it has no roots.

$$\lambda = \frac{-2\langle x, y\rangle \pm \sqrt{4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2}}{2\|x\|^2}$$

So the discriminant ≤ 0

$$4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\implies |\langle x, y\rangle| \leq \|x\|\|y\|$$

2.3 Theorem

$$\langle x, y\rangle = \|x\|\|y\| \cos\varphi$$

Writing x and y in polar coordinates:

$$x_1 = \|x\| \cos\alpha, \quad x_2 = \|x\| \sin\alpha$$
$$y_1 = \|y\| \cos\beta, \quad y_2 = \|y\| \sin\beta$$

$$\langle x, y\rangle = x_1y_1 + x_2y_2 = \|x\| \cos\alpha\|y\| \cos\beta + \|x\| \sin\alpha\|y\| \sin\beta$$
$$= \|x\|\|y\|(\cos\alpha \cos\beta + \sin\alpha \sin\beta)$$
$$= \|x\|\|y\| \cos(\alpha - \beta) = \|x\|\|y\| \cos\varphi$$

Remark

$$\cos\varphi = \frac{\langle x, y\rangle}{\|x\|\|y\|}$$
$$\text{Then } x \perp y \iff \langle x, y\rangle = 0$$

Examples:

- $C((a, b)) \cong$ continuous functions in (a, b)

$$f, g \in C((a, b)), \quad then\, \langle f, g\rangle = \int_a^b f(t)g(t)\, dt$$

We can add some weights:

$$\langle f, g\rangle = \int_a^b w(t)f(t)g(t)\, dt$$

An example:

$$\langle f, g\rangle = \int_a^b e^{-t}f(t)g(t)\, dt$$

- We might have an orthogonal family (with infinite elements) of functions.

$$\{\cos nx, \sin nx\}_{n\in\mathbb{Z}} \quad \text{in } [0, 2\pi]$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx\, dx$$

$$\int_0^{2\pi} \cos nx \sin nx\, dx = \int_0^{2\pi} \cos nx \cos mx\, dx = 0 \quad \text{for } n \neq m$$

3 Vector Product (Only in $\mathbb{R}^3$)

Take $x, y \in \mathbb{R}^3$

$$x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

Recall:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3.1 Triple product and properties

We take the triple product

$$a \cdot (b \times c) = (a_1, a_2, a_3) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $u \in \text{span}\{b, c\}$ then $a \cdot (b \times c) = 0$ and a, b, c are coplanar if $a \cdot (b \times c) = 0$

3.2 Geometric Interpretation

- The magnitude of $x \times y$ represents the area of the parallelogram formed by x and y.
- The direction of $x \times y$ is perpendicular to the plane spanned by x and y, following the right-hand rule.
- The cross product satisfies: $x \times y = -(y \times x)$.

4 Topology of $\mathbb{R}^n$

Definition of open spaces: we define an open ball in $\mathbb{R}^n$ centered at x_0 and of radius R.

$$\text{Denoted by} \quad B_R(x_0) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, x_0) < R\}$$

This set includes all points in $\mathbb{R}^n$ whose distance from x_0 is less than R. Open sets are the building blocks of topological spaces, and they help define concepts such as convergence, continuity, and compactness.

4.1 Open set

A set $A \subset \mathbb{R}^n$ is open if $\forall x \in A, \quad \exists R > 0$ such that $B_R(x) \subset A$.
For example:

$$(x, y) \in \mathbb{R}^2, \quad A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 3\}$$

4.2 Closed set

A set $A \subset \mathbb{R}^n$ is closed if its complement is open.

$$A \subset \mathbb{R}^n \text{ is closed if } \mathbb{R}^n \setminus A \text{ is open.}$$

4.3 Boundary of a set

The boundary of a set $A \subset \mathbb{R}^n$ denoted by ∂A:

$$\partial A = \{x \in \mathbb{R}^n : \forall R > 0, \quad B_R(x) \cap A \neq \emptyset \text{ and } B_R(x) \cap (\mathbb{R}^n \setminus A) \neq \emptyset\}$$

Following the previous example, the boundary of A is the circle of radius 3 centered at the origin:

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 3\}$$

Remark

A set $A \subset \mathbb{R}^n$ is closed if and only if it contains its boundary.

Example:

$$D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1 \text{ and } x > 0\}$$

$$D^C = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 1 \text{ or } x \leq 0\}$$

$$\partial D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1 \text{ and } x > 0\}$$

D is open, D^C is closed, and ∂D is the semicircle of radius 1 centered at the origin.

Example:

$$S = \{x = 1 \text{ and } 1 < y \leq 2\}$$

$$S^C = \{x \neq 1 \text{ or } y \leq 1 \text{ or } y > 2\}$$

$$\partial S = \{x = 1 \text{ and } y = 1 \text{ or } y = 2\}$$

S is neither open nor closed, as S^C, and ∂S is the line segment from $(1, 1)$ to $(1, 2)$.

4.4 Compact set

A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Example:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$

$A = \partial A$ so A is closed. A is also bounded, as all points in A are contained within the circle of radius 2 centered at the origin.
$\implies A$ is compact.

Example: (Exercise 11a)

$$A = xy - \text{plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3\}$$

This set is closed, and $B_R(x, R) \cap A = \emptyset$ and $B_R(x, R) \cap (\mathbb{R}^3 \setminus A) = \emptyset$.
Therefore, A is will not be compact.

Example: (Exercise 11b)

$$A = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$$

For any point $(x, y) \in A$, we can find an open ball $B_R(x, R) \subset B$ that does not intersect the x or y axes.

$$R = \min\{x, y\}$$

4.5 Ball in $\mathbb{R}^n$

For a ball at any part of radius r in $\mathbb{R}^n$:

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2$$

$$\text{dist}(x - a) = \|x - a\| = r$$

Example: (Exercise 1a)

Sphere centered at (0,1,-1) with $r = 4$

$$(x - 0)^2 + (y - 1)^2 + (z + 1)^2 = 16$$

Intersection with the x, y, z-planes:

$$\text{If } z = -1, \quad x^2 + (y - 1)^2 = 16$$

$$\text{If } y = 1, \quad x^2 + (z + 1)^2 = 16$$
$$\text{If } x = 0, \quad (y - 1)^2 + (z + 1)^2 = 16$$

Example: (Exercise 1b)

Sphere going through the origin and centered at $(1, 2, 3)$:

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = r^2$$

$$\text{dist}(0, (1, 2, 3)) = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

So the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

Example: (Exercise 1c)

Find center and radius of the sphere:

$$x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$$

$$(x + 1)^2 + (y + 4)^2 + (z - 2)^2 = 49$$

So the center is $(-1, -4, 2)$ and the radius is 7.

Example: (Exercise 3)

Check if the vectors are orthogonal:

$$a = (-5, 3, 7), \quad b = (6, -8, 2)$$

$$a \cdot b = -30 - 24 + 14 = -40! = 0$$

So a and b are not orthogonal.

Example: (Exercise 4a)

Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is:

$$d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Let } a = R - Q, \quad b = P - Q$$

$$\text{Then } d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Area of parallelogram} = \|a \times b\| = \|a\| \cdot d$$

Example: (Exercise 4b)

Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

$$\mathbf{a} = \overrightarrow{QR} = R - Q = (-1 - 0, 4 - 6, 7 - 8) = (-1, -2, -1)$$

$$\mathbf{b} = \overrightarrow{QP} = P - Q = (1 - 0, 1 - 6, 1 - 8) = (1, -5, -7)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -1 \\ -5 & -7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & -7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -2 \\ 1 & -5 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = (9, -8, 7)$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{9^2 + (-8)^2 + 7^2} = \sqrt{81 + 64 + 49} = \sqrt{194}$$

$$\|\mathbf{a}\| = \sqrt{(-1)^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|} = \frac{\sqrt{194}}{\sqrt{6}} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$$

Example: (Exercise 5)

Calculate the volume of the parallelepiped with edges adjacent to $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$, where

$$P(1,1,1), \quad Q(2,0,3), \quad R(4,1,7), \quad S(3,-1,-2).$$

The volume of the parallelepiped can be calculated using the scalar triple product of the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2-1, 0-1, 3-1) = (1,-1,2) \\
\overrightarrow{PR} &= R - P = (4-1, 1-1, 7-1) = (3,0,6) \\
\overrightarrow{PS} &= S - P = (3-1, -1-1, -2-1) = (2,-2,-3)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PR} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} &= \mathbf{i}(0 \cdot (-3) - 6 \cdot (-2)) - \mathbf{j}(3 \cdot (-3) - 6 \cdot 2) + \mathbf{k}(3 \cdot (-2) - 0 \cdot 2) \\
&= \mathbf{i}(0+12) - \mathbf{j}(-9-12) + \mathbf{k}(-6-0) \\
&= 12\mathbf{i} + 21\mathbf{j} - 6\mathbf{k} \\
&= (12, 21, -6)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) &= (1,-1,2) \cdot (12,21,-6) \\
&= 1 \cdot 12 + (-1) \cdot 21 + 2 \cdot (-6) \\
&= 12 - 21 - 12 \\
&= -21
\end{aligned}$$

The volume of the parallelepiped is the absolute value of this scalar triple product:

$$\text{Volume} = |-21| = 21$$

Example: (Exercise 6)

Use the scalar product to check if the following vectors are coplanar: $a = 2i + 3j + k, b = i - j$ and $c = 7i + 3j + 2k$.
The vectors are coplanar if the scalar triple product of the vectors is zero.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} \quad = (-2, -2, 10)$$

$$\begin{aligned}
\mathbf{a} \cdot (-2,-2,10) &= (2 \times -2) + (3 \times -2) + (1 \times 10) \\
&= -4 - 6 + 10 = 0
\end{aligned}$$

Since the scalar triple product is zero, the vectors are coplanar.

5 Functions of several variables

A function $f : A \to B$ is a correspondence between two sets A and B such that each element in A is associated with exactly one element in B.

Example:

$$f(x,y) = x^2 + y^2 \quad \text{, where}$$
$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \to f(x,y) = x^2 + y^2$$

The formula for that function:

$$z = x^2 + y^2$$

The graph results in a paraboloid (surface).

5.1 Domain for a function f

The domain is the set of points where the function is well defined.

5.2 Image of a function f

The image is the set of points in B that are associated with points in A.

Example: (Exercise 8)

- The function
$$x^2 + 2z^2 = 1$$
is a cylinder with an elipse section.
- The function
$$x^2 - y^2 + z^2 = 1$$
  - If $y = k$, then $x^2 + z^2 = 1 + k^2$ is a circle.
  - If $z = 0$, then $x^2 - y^2 = 1$ is a hyperbola.
  - If $x = 0$, then $z^2 - y^2 = 1$ is a hyperbola.

5.3 Types of functions

- Scalar functions: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, \ldots, x_n) \in \mathbb{R}$.
- Vector functions: $f : \mathbb{R}^n \to \mathbb{R}^m, \quad f(x_1, \ldots, x_n) \in \mathbb{R}^m$.
If $f : \mathbb{R}^n \to \mathbb{R}^n$, then f is a vector field.

Example:

Paramatric equations for a line in $\mathbb{R}^3$:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

5.4 Level curves

The level curves of a function $f : \mathbb{R}^N \to \mathbb{R}$ are the curves in the domain of f where $f(x,y) = k$ for some constant k.

$$(x_1, \ldots, x_N) \to f(x_1, \ldots, x_N) \in \mathbb{R}$$
$$f(x,y) = c, \quad c \in \mathbb{R}$$

The graph of a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$.

Example:

$$f(x,y) = x^2 + y^2$$
$$x^2 + y^2 = c, \quad c \in \mathbb{R}$$

The level curves are circles centered at the origin. They can be seen as the projection of the surface onto the xy-plane (from above).

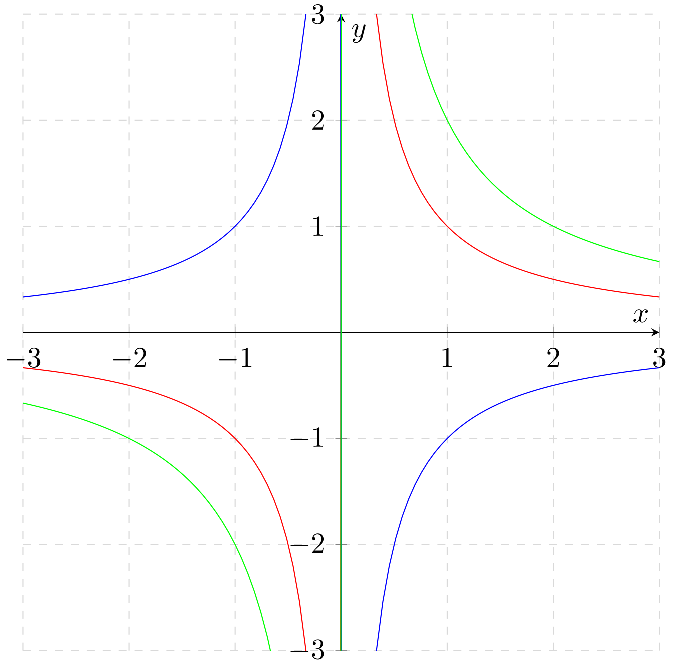
5.5 Remark

In $\mathbb{R}^3$, the level curves of a function $f : \mathbb{R}^3 \to \mathbb{R}$ are the curves in the domain of f where $f(x,y,z) = c$ for some constant $c \in \mathbb{R}$.
They allow us to visualize a 3D graph of a function in 2D.
If the level curves are very close to each other, then the function is steep.
Two level curves never touch.

Example:

Find the level curves of the function $f(x,y) = xy$.

$$xy = c, \quad c = 1, -1, 2.$$

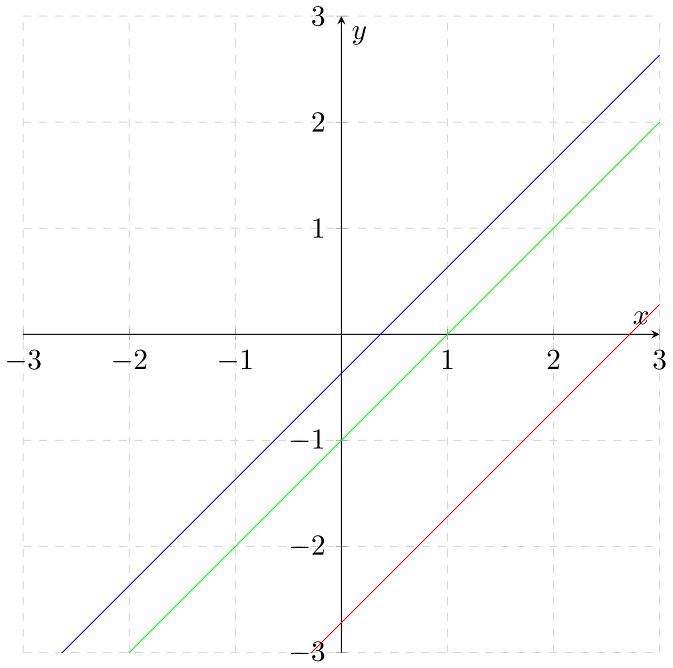


The level curves are a family of hyperbolas.

Example:

Find the level curves of the function $f(x,y) = log(x - y)$.

$$log(x-y) = c, \quad c = 1, -1, 0.$$



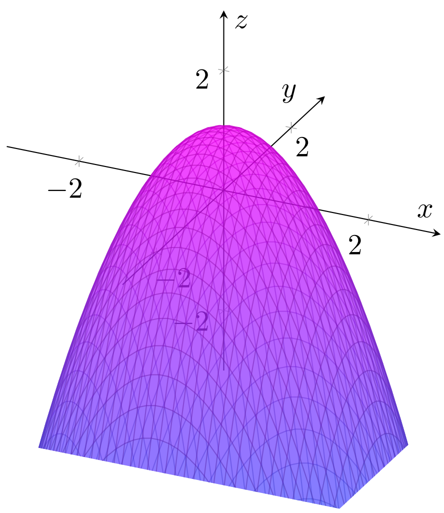
The level curves are a family of straight lines.

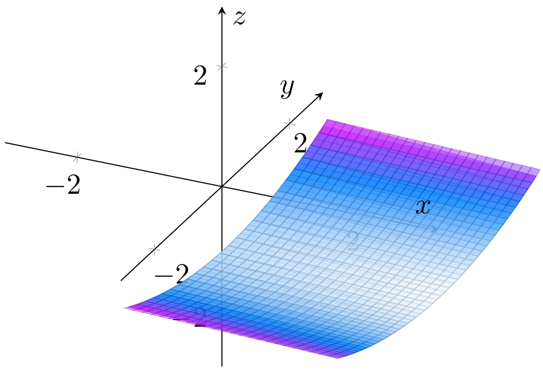
Example:

Find the level curves of the functions

$$f(x,y,z) = x^2 + y^2 + z^2 \quad \text{and} \quad g(x,y,z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$$

The level curves for f are spheres centered at the origin.
The level curves for g are ellipsoids centered at the origin.





5.6 Graph of a function

$$\{(x, f(x)), x \in Dom(f)\}, \quad \text{where } f = 9y^2 + 4z^2 = x^2 + 36$$

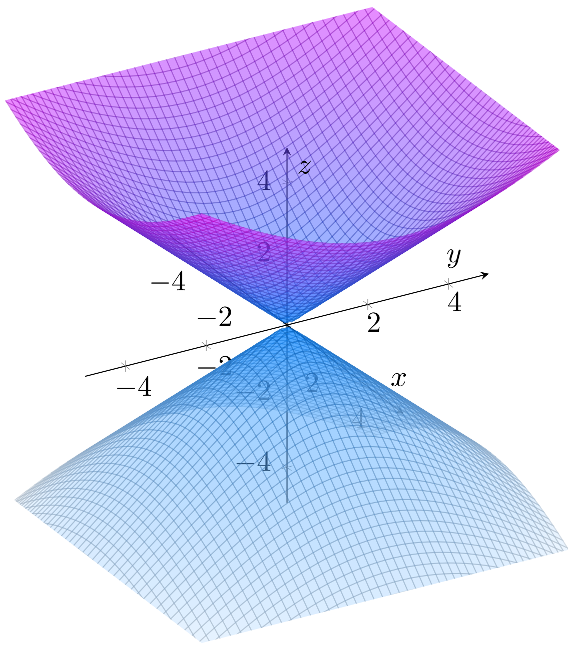
Intersection with the x, y, z-planes:

$$\text{If } z = 0, \quad 9y^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad 4z^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } x = 0, \quad 9y^2 + 4z^2 = 36 \to \text{Ellipse}$$

Example:

Plot the function $f = x^2 + 4y^2 = z^2$.

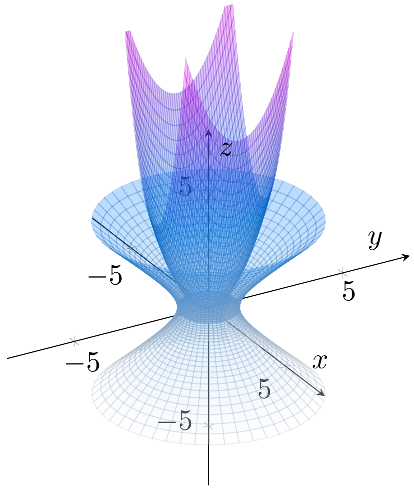
$$\text{If } z = 0, \quad x^2 + 4y^2 = 0 \to x = 0, y = 0$$
$$\text{If } y = 0, \quad x^2 = z^2 \to x = z, x = -z$$
$$\text{If } x = 0, \quad 4y^2 = z^2 \to y = z/2, y = -z/2$$
$$\text{If } z = k, \quad x^2 + 4y^2 = k^2 \to \text{Ellipses}$$



The graph is a cone.

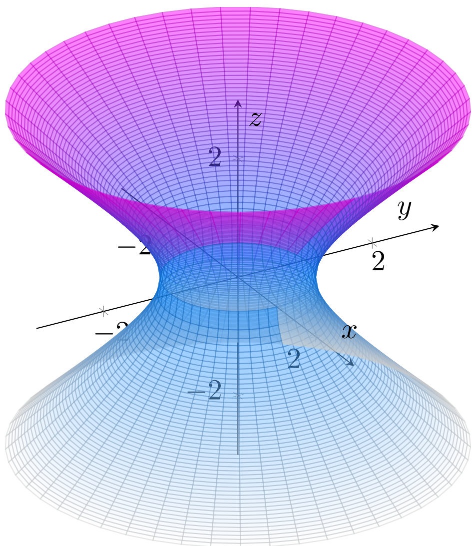
Problem 8

$$x^2 + y^2 + 9z^2 = 1 \to \text{Ellipsoid}$$
$$x^2 - y^2 + z^2 = 1 \to \text{Hyperboloid of one sheet}$$
$$y = 2x^2 + z^2 \to \text{Paraboloid}$$



Example:

$$-x^2 + y^2 - z^2 = 1 \to \text{Hyperboloid}$$
$$\text{If } z = 0, \quad -x^2 + y^2 = 1 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad -x^2 - z^2 = 1 \to \text{No solution}$$
$$\text{If } x = 0, \quad y^2 - z^2 = 1 \to \text{Hyperbola}$$

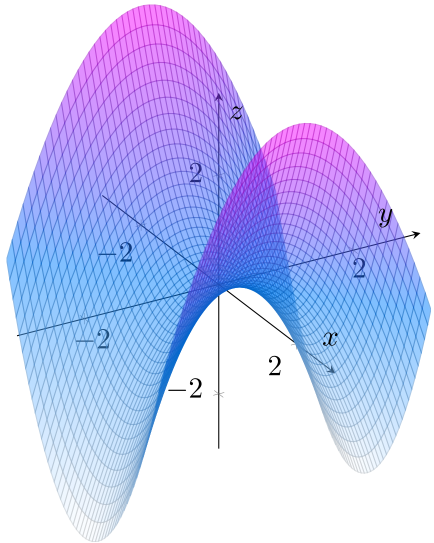


Example:

$$z = x^2 - y^2 \to \text{Paraboloid}$$

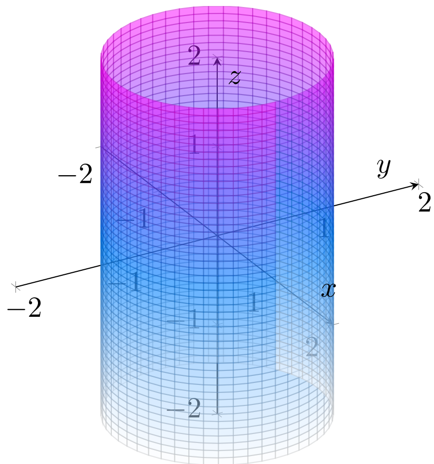
$$\text{If } z = 0, \quad x^2 - y^2 = 0 \to \text{Hyperbola}$$
$$\text{If } z = k, \quad x^2 = y^2 + k \to \text{Hyperbola}$$

$$\text{If } y = 0, \quad z = x^2 \rightarrow \text{Parabola}$$
$$\text{If } x = 0, \quad z = -y^2 \rightarrow \text{Parabola}$$



Example:

Plot the function $x^2 + y^2 = 1$.



6 Cartesian coordinates in $\mathbb{R}^N$

In $\mathbb{R}^2$, the Cartesian coordinates are (x, y).
In $\mathbb{R}^3$, the Cartesian coordinates are (x, y, z).

6.1 Polar coordinates in $\mathbb{R}^2$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

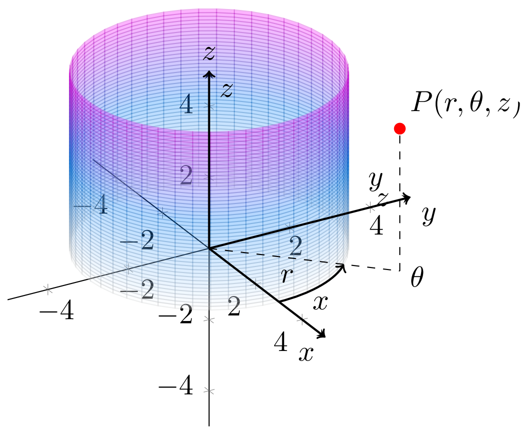
Inverse transformation:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Lemma Let $A = (0, \inf) \times (0, 2\pi)$.
The function $g : A \to \mathbb{R}^2$ defined by $g(r, \theta) = (r\cos(\theta), r\sin(\theta))$ is a bijection, continuous in a ball $B(0, \alpha)$ such that $\{g(r, \theta), 0 < r < \alpha, 0 - leq\theta < 2\pi\}$ is a subset of $B(0, \alpha)$.
To see if the function is one-to-one, assume that $g(r_1, \theta_1) = f(r_2, \theta_2)$ for $r_1, r_2 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
Then $r_1 \cos(\theta_1) = r_2 \cos(\theta_2)$ and $r_1 \sin(\theta_1) = r_2 \sin(\theta_2)$. This implies that $r_1 = r_2$, since $r_1 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
As a consequence $\theta_1 = \theta_2$ so that g is one-to-one.
Now taking $(x, y) \in B(0, \alpha)$, and $r = \sqrt{x^2 + y^2} > 0$.
Then, the part $(\frac{x}{2}, \frac{y}{2})$ is in $B(0, 1)$.
Therefore, there exists $\theta \in [0, 2\pi)$ such that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$.
Which implies that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. So g is onto.

6.2 Cylindrical coordinates in $\mathbb{R}^3$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$



Example:

Transform into cartesian coordinates:

$$(3, \frac{\pi}{2}, 1) = (3\cos(\frac{\pi}{2}), 3\sin(\frac{\pi}{2}), 1) = (0, 3, 1)$$

Example:

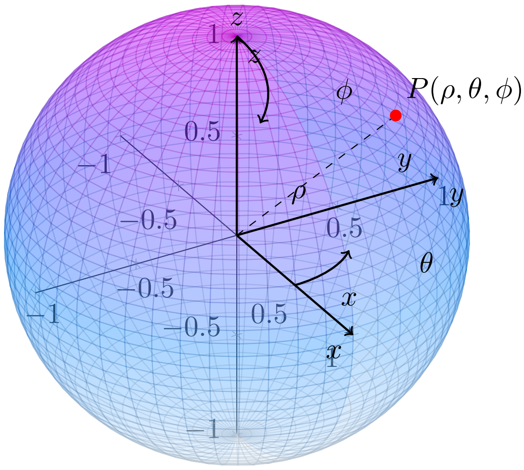
Transform into cartesian coordinates:

$$(4, -\frac{\pi}{3}, 5) = (4\cos(-\frac{\pi}{3}), 4\sin(-\frac{\pi}{3}), 5) = (2, -2\sqrt{3}, 5)$$

6.3 Spherical coordinates in $\mathbb{R}^3$

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi), \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

ρ is the distance from the origin, ϕ is the angle from the z-axis, and θ is the angle from the x-axis.



Example: (Problem 6)

Transform into spherical coordinates:

$$(1, 0, 0) = (1\sin(\phi)\cos(\theta), 1\sin(\phi)\sin(\theta), 1\cos(\phi)) = (0, 0, 1)$$

$$(2, \frac{\pi}{3}, \frac{\pi}{4}) = (2\sin(\frac{\pi}{4})\cos(\frac{\pi}{3}), 2\sin(\frac{\pi}{4})\sin(\frac{\pi}{3}), 2\cos(\frac{\pi}{4})) = (\sqrt{3}, 1, \sqrt{2})$$

6.4 Parametric equation of a line in $\mathbb{R}^3$

Having two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the parametric equation of the line passing through P_1 and P_2 is:

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

Example: (Problem 14)

Find the parametric equation of the line passing through $P(1, 0, 1)$ and $Q(2, 3, 1)$.

$$\begin{cases} x = 1 + t(2 - 1) = 1 + t \\ y = 0 + t(3 - 0) = 3t \\ z = 1 + t(1 - 1) = 1 \end{cases}$$

6.5 Parametric equation of a plane in $\mathbb{R}^3$

For a plane in parametric form:

$$\begin{cases} x = x_0 + a_1 t + a_2 s \\ y = y_0 + b_1 t + b_2 s \\ z = z_0 + c_1 t + c_2 s \end{cases}$$

Remark A line in $\mathbb{R}^3$ is a manifold of dimension 1.
A plane in $\mathbb{R}^3$ is a manifold of dimension 2.

Example: (Problem 9)

Write $x^2 + y^2 + z^2 = 4$ (Sphere) in parametric form.

$$\begin{cases} x = \lambda \\ y = \mu \\ z = \sqrt{4 - \lambda^2 - \mu^2} \end{cases} \qquad \text{where } \lambda^2 + \mu^2 \leq 4$$

Using spherical coordinates:

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi, \quad \rho = 2$$

$$\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi) + \rho^2\cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi) + \rho^2\cos^2(\phi) = 4$$
$$\rho^2 = 4 \to \rho = 2$$

Example: (Problem 11)

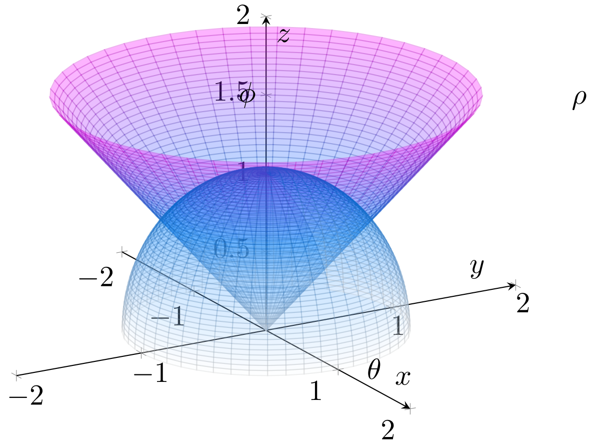
A solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Becase the solid is inside the cone, we have $z \geq \sqrt{x^2 + y^2}$
Because the solid is below the sphere, we have $x^2 + y^2 + z^2 \leq z$

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$$\sqrt{\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi)} \leq \rho\cos(\phi)$$
$$\rho\sin(\phi) \leq \rho\cos(\phi) \to \tan(\phi) \leq 1 \to \phi \leq \frac{\pi}{4}$$

$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2\cos^2(\phi) \leq \rho\cos(\phi)$$
$$\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi) \leq \rho\cos(\phi)$$
$$\rho^2 \leq \rho\cos(\phi) \to \rho \leq \cos(\phi)$$



Solid in cartesian coordinates:

$$\begin{cases} z \geq \sqrt{x^2+y^2} \\ x^2+y^2+z^2 \leq z \end{cases}$$

In spherical coordinates:

$$\begin{cases} \rho \leq \cos(\phi) \\ \phi \leq \frac{\pi}{4} \\ \theta \in [0, 2\pi) \end{cases}$$

6.6 Intersection of two bodies

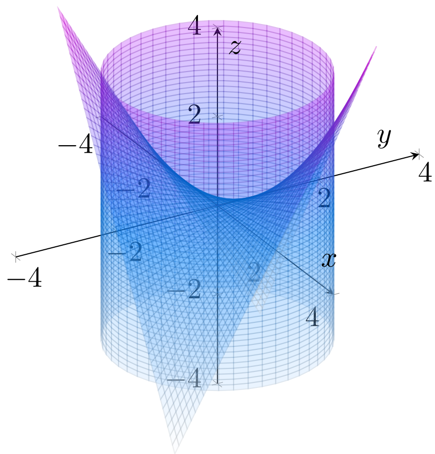
The intersection of two bodies gives (in general) a curve.

Example: (Problem 12)

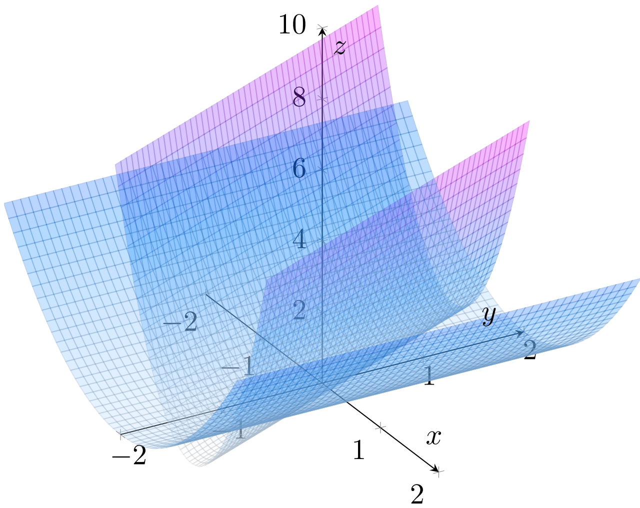
Parametrize the intersection (a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$) of the two bodies:

$$\begin{cases} \text{Cylinder: } x^2+y^2=4 \\ \text{Surface: } z=xy \end{cases}$$

$$\begin{cases} x = 2\cos(t) = \gamma_1(t) \\ y = 2\sin(t) = \gamma_2(t) \\ z = 2\cos(t)\sin(t) = \gamma_3(t) \end{cases} \qquad \text{where } t \in [0, 2\pi)$$



$$\begin{cases} \text{Paraboloid: } z = 4x^2+y \\ \text{Cylinder: } y = x^2 \end{cases} \implies \begin{cases} x = t \\ y = t^2 \\ z = 4t^2+t^4 \end{cases}$$



7 Limit of functions

Assume a scalar $f : \mathbb{R}^N \to \mathbb{R}$. We say that the limit of $f(x)$ as x approaches x_0 is L and we denote it by:

$$\lim_{x \to x_0} f(x) = L \quad \in \mathbb{R}, \quad x, x_0 \in \mathbb{R}^N$$

7.1 Definition of the limit

We say that $\lim_{x \to x_0} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\forall x \in \mathbb{R}^N, \quad 0 < ||x - x_0|| < \delta \implies |f(x) - L| < \epsilon$$

Theorem If the limit of $f(x)$ as x approaches x_0 exists, then the limit is unique.

Proof Argue by contradiction. Assume that there are two limits L_1 and L_2 such that $L_1 \neq L_2$.

Actually, we can say that:

$$B(L_1, r_1) \cap B(L_2, r_2) = \emptyset$$

Where $B(L_1, r_1)$ is the ball of radius r_1 centered at L_1 and $B(L_2, r_2)$ is the ball of radius r_2 centered at L_2.

$$\lim_{x \to x_0} f(x) = L_1, \quad \text{for any } \epsilon_1 > 0, \quad \exists \delta_1 > 0 \text{ such that } |f(x) - L_1| < \epsilon_1 \text{ for } 0 < ||x - x_0|| < \delta_1$$

$$\lim_{x \to x_0} f(x) = L_2, \quad \text{for any } \epsilon_2 > 0, \quad \exists \delta_2 > 0 \text{ such that } |f(x) - L_2| < \epsilon_2 \text{ for } 0 < ||x - x_0|| < \delta_2$$

Indeed,

$$\begin{cases} |f(x) - L_1| < r_1 = \epsilon_1 \\ |f(x) - L_2| < r_2 = \epsilon_2 \end{cases} \implies \text{However, taking } \delta = \min(\delta_1, \delta_2) \text{ so that } ||x - x_0|| < \delta$$

$$\text{such that } \begin{cases} |f(x) - L_1| < \min(r_1, r_2) \\ |f(x) - L_2| < \min(r_1, r_2) \end{cases} \implies L_1 = L_2, \text{ which is a contradiction}$$

7.2 Computing limits

Using the definition of the limit, we can compute the limit of a function $f(x)$ as x approaches x_0. To do so we must choose the value of L towards the function is going. The process is as follows:

$$\text{For example in } \mathbb{R}^2, \quad \begin{cases} f : \mathbb{R}^2 \to \mathbb{R} \\ (x, y) \to f(x, y) \end{cases}$$

Then:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \epsilon$$

Hence, we must find $\delta \cong \delta(\epsilon)$.

Example:

Prove that

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) = 0$$

$$\text{Let } \epsilon > 0, \quad \text{we must find } \delta > 0 \text{ such taht } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| < \epsilon$$

$$\text{Since } \left|\sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq 1 \text{ and } \sqrt{x^2 + y^2} \geq 0$$

$$\text{We have } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq \sqrt{x^2 + y^2} < \epsilon$$

$$\text{Therefore, we can choose } \delta = \epsilon$$

Example:

Prove that

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\text{Let } \epsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } ||f(x) - b|| = |y - b| < \epsilon$$
$$\text{when } ||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

$$\text{We start from } |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta$$
$$\text{Therefore, we can choose } \delta = \epsilon$$

7.3 Iterative limits

In 2D,

$$\lim_{x\to a} \left(\lim_{y\to b} f(x, y)\right) = \lim_{y\to b} \left(\lim_{x\to a} f(x, y)\right)$$

However, this technique only gives us negative answers (non-existence of the limit). Since we are just following one direction.

7.4 Approach following families of functions

They might be straight lines, parabolas, etc. around the point of approach.

Example:

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

Let y $= \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{x}{|x|\sqrt{1+\lambda^2}} = \frac{1}{\sqrt{1+\lambda^2}}$$

This limit depends on λ. Therefore, the limit does not exist.

Example:

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

Let $y = \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3}{x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x}{1+\lambda^2} = 0$$

Following the family of functions $y = \lambda x$, the limit is 0. However, we cannot confirm that the value of the limit is 0 or that the limit exists using this method.

This method is necessary but not sufficient. It is a good way to check if the limit does not exist.

Problem 1a

Taking $y = \lambda x$ and knowing that $\sin x \approx x$ for $x \approx 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2+\sin^2(\lambda x)}{2x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x^2+\lambda^2 x^2}{2x^2+\lambda^2 x^2} = \frac{1+\lambda^2}{2+\lambda^2}$$

This limit depends on λ. Therefore, the limit does not exist.

Problem 1d

Taking $y = \lambda x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x\lambda x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{\lambda x^2}{\sqrt{x^2(1+\lambda^2)}} = \lim_{x\to 0} \frac{\lambda x}{\sqrt{1+\lambda^2}} = 0$$

This limit does not depend on λ. Therefore, the limit exists and is 0.
Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{r} = \lim_{r\to 0} r\cos(\theta)\sin(\theta) = 0$$

7.5 Polar coordinates

In $\mathbb{R}^2$, the polar coordinates are (r, θ).

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

Theorem Let us consider two functions f and g such that

$$\lim_{x \to x_0} f(x) = 0, \quad \text{and } g \text{ is bounded for } ||x - x_0|| < \delta$$

Then

$$\lim_{x \to x_0} f(x)g(x) = 0$$

Problem 1e

Taking $y = \lambda x$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x\to 0} \frac{x^2 \lambda x e^{\lambda x}}{x^4 + 4\lambda^2 x^2} = \lim_{x\to 0} \frac{\lambda x^3 e^{\lambda x}}{x^2(x^2 + 4\lambda^2)} = \lim_{x\to 0} \frac{\lambda x e^{\lambda x}}{x^2 + 4\lambda^2} = 0$$

Now taking $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x\to 0} \frac{x^2 x^2 e^{x^2}}{x^4 + 4x^4} = \lim_{x\to 0} \frac{x^4 e^{x^2}}{5x^4} = \lim_{x\to 0} \frac{e^{x^2}}{5} = \frac{1}{5}$$

This limit depends on the direction of approach. Therefore, the limit does not exist.

Problem 1f

Applying generalized spherical coordinates:

$$\begin{cases} x = r\sin(\phi)\cos(\theta) \\ y = \frac{1}{2} r \sin(\phi)\sin(\theta) \\ z = \frac{1}{3} r\cos(\phi) \end{cases}$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{r\to 0} \frac{\frac{1}{2} r\sin(\phi)\sin(\theta)\frac{1}{3} r\cos(\phi)}{r^2} = \lim_{r\to 0} \frac{1}{6}\sin(\phi)\cos(\phi)\sin(\theta)$$

This limit depends on ϕ and θ. Therefore, the limit does not exist.

8 Continuity of functions

A function $f : \mathbb{R}^N \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}^N$ if:

1. $f(x_0)$ is defined.
2. $\lim_{x\to x_0} f(x)$ exists.
3. $\lim_{x\to x_0} f(x) = f(x_0)$

Theorem Consider A a subset in $\mathbb{R}^N$ and $f : A \to \mathbb{R}^M$ a vector function.

$$F(x) = (f_1(x), f_2(x), \ldots, f_M(x))$$

If $f_1, f_2, \ldots, f_M$ are continuous at $x_0 \in A$, then F is continuous at x_0.

Theorem Assume that f and g are continuous at $x_0 \in \mathbb{R}^N$ and $f(x_0)$ respectively. Then, the composition $g \circ f$ is continuous at x_0.

$$\text{For example: } f(x) = x^2 y \sin(x+y), \quad f : \mathbb{R}^2 \to \mathbb{R}$$

Since f is a composite function and $x^2, y, \sin(x+y)$ are continuous, then f is continuous.

Problem 2a

Prove the continuity:

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

If $(x,y) \neq (0,0)$, then $f(x,y)$ is a composition of continuous functions. So that

$$\text{for any } (x_0,y_0) \neq (0,0), \qquad \lim_{(x,y)\to(0,0)} f(x,y) = f(x_0,y_0) = \frac{x_0^2y_0^3}{2x_0^2+y_0^2}$$

At $(0,0)$, the function is well defined: $f(0,0) = 1$, so we must check that the limit exists.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} =$$

Using polar coordinates:

$$\begin{cases} x = \dfrac{1}{\sqrt{2}} r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

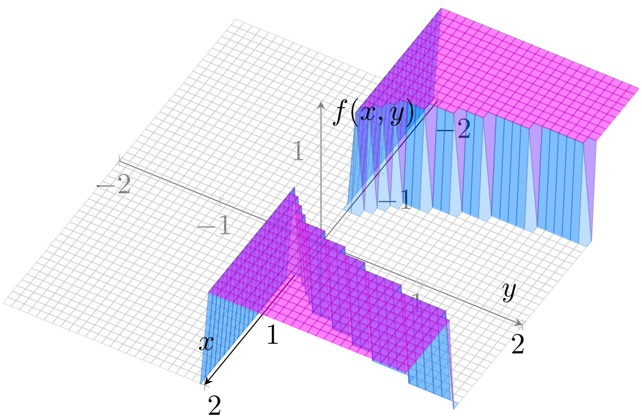
$$= \lim_{r\to 0} \frac{\frac{1}{2}r^3\cos^2(\theta)\sin^3(\theta)}{r^2} = \lim_{r\to 0} \frac{1}{2} r\cos^2(\theta)\sin^3(\theta) = 0$$

The limit exists and is equal to 0. However, $f(0,0) = 1 \neq 0$. Therefore, the function is not continuous at $(0,0)$.

Problem 3

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

a) $lim_{(x,y)\to(0,0)} f(x,y) = 0$ along any path going through $y = mx^{\alpha}$. $\quad 0 < \alpha < 4$



Let $\alpha = 1$: $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} f(x,mx) = 0$$

In this case, the limit exists and is equal to 0.

b) Despite (a), prove that the function is not continuous.

Take any point $(a, 0), \quad a \in \mathbb{R}$. Then:

$$\lim_{(x,y)\to(a,0)} f(x,y) = \begin{cases} 0 & \text{if } y \to 0^- \\ 1 & \text{if } y \to 0^+ \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

c) f discontinuous on two entire curves: $\begin{cases} y = x^4 \\ y = 0 \end{cases}$

Take any point (a, a^4) on $y = x^4, \quad a \in \mathbb{R}$. Then:

$$lim_{(x,y)\to(a,a^4)} f(x,y) =$$

$$||(x,y) - (a, a^4)|| < \delta \implies \begin{cases} 0 < |f(x,y)| = 1 < \epsilon & \text{if } y < x^4 \\ 0 < |f(x,y)| = 0 < \epsilon & \text{if } y > x^4 \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

Example:

Consider the function:

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let $y = \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$a) \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3 \lambda x}{x^6 + \lambda^2 x^2} = \lim_{x\to 0} \frac{\lambda x^2}{x^4 + \lambda^2} = 0$$

Now taking $y = x^3$:

$$b) \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3 x^3}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

Therefore the function is not continuous at $(0,0)$, since the limit depends on the direction of approach.

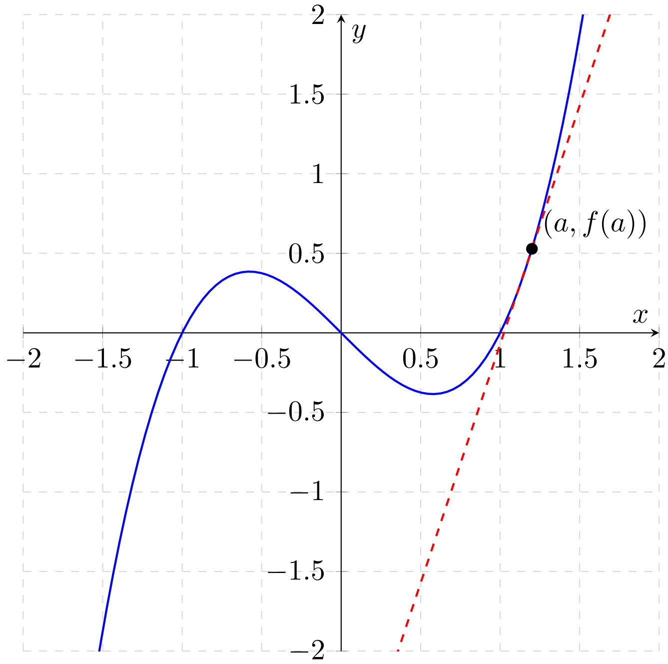
9 Differentiability of functions

In 1D when we consider a function $f : \mathbb{R} \to \mathbb{R}$, the derivative of f at $a \in \mathbb{R}$ describes the ratio of the change of the function f at $x = a$ and is denoted by:

$$f'(a) = \lim_{t\to 0} \frac{f(a+t) - f(a)}{t} = \frac{df(a)}{dt}$$

Geometrically, the derivative of a function at a point is the slope of the tangent line to the curve at that point. And this tangent line is defined as:

$$y = f(a) + f'(a)(x - a)$$



A way of seeing this is written by the Taylor polynomial:

$$f(x) = f(a) + M(x-a) + r(|x-a|)$$

Where $M = f'(a)$ is the slope of the tangent line and $r(|x-a|)$ is the remainder.
If $x \to a$, then $r(|x-a|) \to 0$ but this only guarantees that the function is continuous at a.
However, if

$$\lim_{x \to a} \frac{r(|x-a|)}{|x-a|} = 0$$

we actually find the existence of the tangent line at a.

$$\frac{f(x) - f(a)}{x-a} = f'(a) + \frac{r(|x-a|)}{x-a}$$

9.1 Generalizing to several variables

$$z = f(x, y)$$

We would like to understand the derivative with respect to each variable.
To do so, we use the knowledge of derivatives in 1 variable.

9.2 Partial derivatives

Let $f : A \subset \mathbb{R}^N \to \mathbb{R}$ be a scalar function. And let $x_0 \in A$. The partial derivative of f with respect to the variable x_i at x_0 is defined as:

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

Where e_i is the unit vector in the direction of x_i.
Somehow, we are computing the ratio of the change of the function f in the direction of x_i, in other words, of the vector e_i.

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$B = \{e_1, e_2, \ldots, e_N\} \quad \text{is the standard basis of } \mathbb{R}^N$$

If $F : \mathbb{R}^N \to \mathbb{R}^M$ is a vector function, then the partial derivative of F with respect to the variable x_i at the point x is defined as:

$$\frac{\partial F(x)}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \dots, \frac{\partial f_M}{\partial x_i}\right)$$

Example:

$$f(x,y) = xy + x - y \quad \text{at } (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+te_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{t\cdot 0 + t - 0}{t} = 1$$

Where t is the norm of the vector te_1:

$$t = t||e_1|| = t \cdot 1 = t$$

Now,

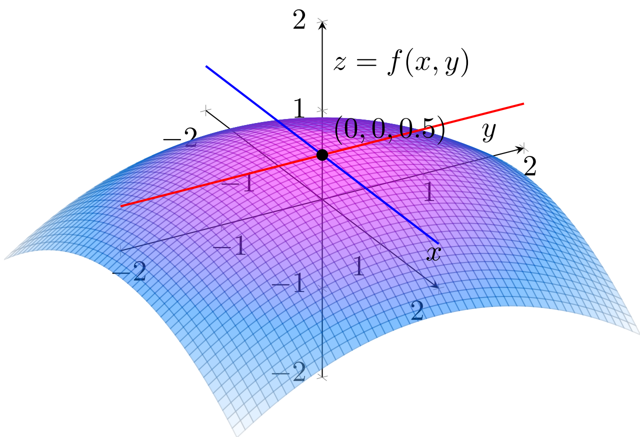
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{t} = \lim_{k\to 0} \frac{0\cdot k + 0 - k}{k} = -1$$

9.3 Geometrical interpretation of the partial derivative

The partial derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point (x_0, y_0) is the slope of the tangent line to the curve $z = f(x,y)$ at the point (x_0, y_0) in the direction of the variable x_i.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \text{slope of the tangent line in the direction of } x$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \text{slope of the tangent line in the direction of } y$$



Obviously, we can apply the rules for derivatives:

$$\frac{\partial f}{\partial x}(x,y) = y + 1, \quad \frac{\partial f}{\partial y}(x,y) = x - 1$$

9.4 Directional derivative

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a scalar function and $P \in A \subset \mathbb{R}^N$ and $v \in \mathbb{R}^N$ be a vector. The directional derivative of f at P in the direction of v is defined as:

$$D_v f(P) = \lim_{t\to 0} \frac{f(P+tv) - f(P)}{t||v||} = \frac{df(P+tv)}{dt}$$

Example:

$$f(x,y) = \sqrt{|xy|} \quad \text{at } (0,0) \quad \text{in the direction of } v = (1,1)$$

$$D_{(1,1)}f(0,0) = \lim_{t\to 0} \frac{f(0+t,0+t) - f(0,0)}{t||(1,1)||} = \lim_{t\to 0} \frac{\sqrt{|t^2|} - 0}{t||(1,1)||} = \lim_{t\to 0} \frac{|t|}{t\sqrt{2}} = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{if } t \to 0^+ \\ -\dfrac{1}{\sqrt{2}} & \text{if } t \to 0^- \end{cases}$$

Therefore, the directional derivative does not exist.

9.5 Remark

Existence of all directional derivatives at a point P does not guarantee the continuity of the function at P.

$$\text{For example: } f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

Example:

$$f(x,y) = x^{1/3}y^{1/3} \quad \text{at } (0,0)$$

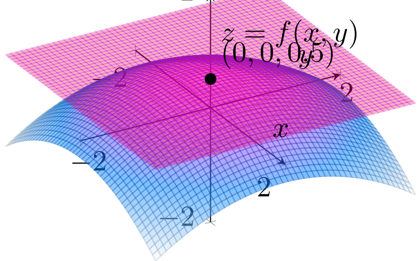
$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^{1/3}\cdot 0^{1/3} - 0}{t} = 0$$

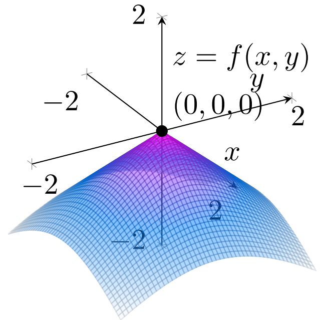
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0 \cdot t^{1/3} - 0}{t} = 0$$

Derivatives exist and the function is also continuous at $(0,0)$. However, there is no tangent plane at $(0,0)$.

9.6 Differentiability

Heuristically it means the construction of a tangent plane.





Let $A \subset \mathbb{R}^2$ be an open subset in $\mathbb{R}^2$ such that $(x,y) \in A$, with $f : A \to \mathbb{R}$ a scalar function. We say that f is differentiable at (x_0, y_0) if:

1. $\dfrac{\partial f}{\partial x}(x_0,y_0)$ and $\dfrac{\partial f}{\partial y}(x_0,y_0)$ exist.

2. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y) - f(x_0,y_0) - \dfrac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

To be differentiable we say that there exists a tangent plane at (x_0, y_0):

$$z = f(x_0,y_0) + A(x-x_0) + B(y-y_0)$$

Following similar ideas to the 1D case, we can write:

$$f(x,y) = f(x_0,y_0) + A(x-x_0) + B(y-y_0) + r(\|(x,y)-(x_0,y_0)\|)$$

If $\|(x,y)-(x_0,y_0)\| \to 0$ then $r(\|(x,y)-(x_0,y_0)\|) \to 0$, and the function is continuous at (x_0, y_0).
However if we have:

$$\frac{f(x,y) - f(x_0,y_0) - A(x-x_0) - B(y-y_0)}{\|(x-x_0)+(y-y_0)}\| = \frac{r(\|(x,y)-(x_0,y_0)\|)}{\|(x-x_0)+(y-y_0)\|} \to 0$$

Then the function is differentiable at (x_0, y_0) if:

$$A = \frac{\partial f}{\partial x}(x_0,y_0), \quad B = \frac{\partial f}{\partial y}(x_0,y_0)$$

Remark Taylor's polynomial around (x_0, y_0) of degree 1 is:

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + r(\|(x,y)-(x_0,y_0)\|)$$

Where $\dfrac{\partial f}{\partial x}(x_0,y_0)$ and $\dfrac{\partial f}{\partial y}(x_0,y_0)$ form $L\left((x-x_0),(y-y_0)\right)$, the linearity function.

$$\|f(x,y) - L\left((x-x_0),(y-y_0)\right)\| < \varepsilon \text{ as } \|(x,y)-(x_0,y_0)\| < \delta$$

We can write L in terms of the dericative of the scalar function f:

$$f : \mathbb{R}^2 \to \mathbb{R}$$

We define such a derivative as the gradient of f:

$$D(f(x,y)) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

The gradient provides us with the direction of the maximum increase of the function f at (x, y).

Problem 4a

$$f(x,y) = xy, \quad \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

Problem 4c

$$f(x,y) = \sqrt{^2 + y^2}, \quad \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Problem 5a

$$f(x,y,z) = (x+z)e^{x-y}, \quad \text{gradient at } (1,1,1)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(e^{x-y} + (x+z)e^{x-y}, -(x+z)e^{x-y}, e^{x-y}\right)$$

$$\nabla f(1,1,1) = (e^0 + 2e^0, -2e^0, e^0) = (3, -2, 1)$$

10 Differentiability in $\mathbb{R}^N$

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \to \mathbb{R}^M$$

We say that f is differentiable at x_0 if:

1. $\dfrac{\partial f_i}{\partial x_j}(x_0)$ exists for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$.

2. $\lim_{x \to x_0} \dfrac{f(x) - f(x_0) - Jf(x_0)(x - x_0)}{||x - x_0||} = 0$

Where $Jf(x_0)$ is the Jacobian matrix of f at x_0:

$$Jf(x_0) = Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix}$$

Problem 6a

Compute the jacobian matrix:

$$F(x, y) = (y, x, xy, y^2 - x^2), \quad F : \mathbb{R}^2 \to \mathbb{R}^4$$

$$JF(x, y) = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial xy}{\partial x} & \frac{\partial xy}{\partial y} \\ \frac{\partial y^2 - x^2}{\partial x} & \frac{\partial y^2 - x^2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ y & x \\ -2x & 2y \end{pmatrix}$$

Now at $(1, 2)$:

$$JF(1, 2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \\ -2 & 4 \end{pmatrix}$$

Problem 6c

$$F(x, y, z) = z^2 e^x \cos(y), \text{ at } (0, \frac{\pi}{2}, -1) \quad F : \mathbb{R}^3 \to \mathbb{R}$$

$$JF(x, y, z) = \begin{pmatrix} \frac{\partial z^2 e^x \cos(y)}{\partial x} & \frac{\partial z^2 e^x \cos(y)}{\partial y} & \frac{\partial z^2 e^x \cos(y)}{\partial z} \end{pmatrix} =$$
$$= \begin{pmatrix} z^2 e^x \cos(y) & -z^2 e^x \sin(y) & 2ze^x \cos(y) \end{pmatrix}$$

$$JF(0, \frac{\pi}{2}, -1) = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$$

Example:

$$f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f is not differentiable at $(0,0)$, the partial derivatives exist at $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{2t\cdot 0}{t\cdot 0} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{2\cdot 0\cdot t}{t\cdot 0} = 0$$

However, the limit of the difference quotient does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{2xy}{\sqrt{x^2+y^2}}-0-0\cdot x-0\cdot y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

If we use polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$$= \lim_{r\to 0} \frac{2r^2\cos(\theta)\sin(\theta)}{r^2} = \lim_{r\to 0} 2\cos(\theta)\sin(\theta) = 2\cos(\theta)\sin(\theta)$$

The limit depends on θ. Therefore, the function is not differentiable at $(0,0)$.

Problem 7

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The partial derivatives exist at $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Take he limit:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = lim_{r\to 0}\frac{r^3\cos(\theta)\sin^2(\theta)}{r^2\cos^2(\theta)+r^4\sin^4(\theta)}$$

The limit does not exist, therefore the function is not continuous at $(0,0)$. The function is not differentiable at $(0,0)$, due to it not being continuous at that point.
Direction of any vector at $(0,0)$:

$$D_(u,v)f(0,0) = \lim_{t\to 0} \frac{f(tu,tv)-f(0,0)}{t\|(u,v)\|}$$

Assume $\|(u,v)\| = 1$:

$$= \lim_{t\to 0} \frac{f(tu,tv)}{t} = \lim_{t\to 0} \frac{t^3uv^2}{t(t^2u^2+t^4v^4)} = \lim_{t\to 0} \frac{uv^2}{u^2+t^2v^4} = \frac{v^2}{u}$$

Where $\dfrac{v^2}{u}$ is the slope of the tangent line in the direction of (u,v).

10.1 Proposition:

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \to \mathbb{R}$$

f is a scalar function, differentiable at x_0 and $v \in \mathbb{R}^N \backslash 0$ a vector. Then:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle$$

Remark

$$\langle \nabla f(x_0), \alpha v \rangle = \alpha \langle \nabla f(x_0), v \rangle \neq \langle \nabla f(x_0), v \rangle$$

Using the properties of the scalar product:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle = ||\nabla f(x_0)|| \cdot ||v|| \cdot \cos(\theta) = ||\nabla f(x_0)|| \cdot \cos(\theta)$$

Where θ is the angle between $\nabla f(x_0)$ and v, and we assume $\|v\| = 1$.
The directional derivative is maximum in the direction of $\nabla f(x_0)$ and minimum in the direction of $-\nabla f(x_0)$.