Vector Calculus

1 Euclidean space

We define the euclidean space in $\mathbb{R}^N, N \geq 1$ using cartesian coordinates.
Any element $x \in \mathbb{R}^N, \quad x = (x_1, x_2, \dots, x_N), \quad x \in \mathbb{R}$

Standard basis

$$e_1 = (1, 0, \dots, 0), \dots, e_N = (0, 0, \dots, 1)$$

$$\text{Then, } x = \sum_{j=1}^{N} x_j \cdot e_j$$

In particular, $B_{\mathbb{R}^3} = \{i, j, k\}$, the canonical basis.

Properties

- Addition: $\quad (x_1, \dots, x_N) + (y_1, \dots, y_N) = (x_1 + y_1, \dots, x_N + y_N)$
- Multiplication by a scalar $\lambda \in \mathbb{R}$: $\quad \lambda x = \lambda(x_1, \dots, x_N) = (\lambda x_1, \dots, \lambda x_N)$
- Associative: $\quad \lambda, \mu \in \mathbb{R}, \quad (\lambda\mu)x = \lambda(\mu x)$
- Additive Identity: $\quad$ There exists a vector $\overline{0} = (0, \dots, 0)$ such that $x + \overline{0} = x$
- Additive Inverse: $\quad \forall x = (x_1, \dots, x_N), \exists\ \overline{x} = (-x_1, \dots, -x_N)$ such that $x + \overline{x} = \overline{0}$
- Distributive Property (over vector addition):

$$\lambda\left((x_1, \dots, x_N) + (y_1, \dots, y_N)\right) = \lambda(x_1, \dots, x_N) + \lambda(y_1, \dots, y_N)$$

- Distributive Property (over scalar addition):

$$(\lambda + \mu)(x_1, \dots, x_N) = \lambda(x_1, \dots, x_N) + \mu(x_1, \dots, x_N)$$

- Scalar Multiplication Identity: $\quad 1 \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)$
- Zero Scalar Multiplication: $\quad 0 \cdot (x_1, \dots, x_N) = (0, \dots, 0)$

Norm

The euclidean space in $\mathbb{R}^N$ is a normal space with an associated norm function.

$$\| \cdot \| : \mathbb{R}^N \to \mathbb{R}_+ \cup \{0\}$$

$$x = (x_1, \dots, x_N) \to \|x\| = \sqrt{x_1^2 + \cdots + x_N^2}$$

Properties

The norm satisfies the following properties:

(a) $\forall x \in \mathbb{R}^N$

- $\|x\| > 0 \iff x \neq 0$
- $\|x\| = 0 \iff x = 0$

(b) $\|\lambda x\| = |\lambda| \, \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$

- Triangular inequality.

Remark: Distance

We can define the distance between two elements in $\mathbb{R}^N$ as

$$dist(x, y) = \|x - y\| = \|y - x\|$$
$$dist(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

- $dist(x, y) = \|x - y\| > 0$ if $x \neq y$, and $dist(x, y) = 0$ if $x = y$
- $dist(x, y) = \|x - y\| = \| - (y - x)\| = \| - 1\| \cdot \|y - x\| = dist(y, x)$
- $dist(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = dist(x, z) + dist(z, y)$

Remark

For $\mathbb{R}$ such a distance is the absolute value, $|\cdot| : \mathbb{R} \to \mathbb{R}$

2 Inner or scalar product

Let x, y be two vectors in $\mathbb{R}^N$, then

$$x \cdot y = x_1 y_1 + \cdots + x_N y_N$$
$$x \cdot y = \langle x, y \rangle = (x, y)$$

2.1 Properties

The inner product satisfies the following properties:

- $\forall x \in \mathbb{R}^N \ \langle x, x \rangle \geq 0$

  $\langle x, x \rangle = 0$ if $x = 0$
- Symmetric: $\langle x, y \rangle = \langle y, x \rangle$
- Bilinear: $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

2.2 Cauchy-Schwartz inequality

$$|x \cdot y| \leq \|x\| \|y\|$$

Proof If $y = \lambda x$, $|\langle x, \lambda y\rangle| = |\lambda| \|x\|^2 = \|x\| |\lambda| \|x\| = \|x\| \|y\|$
If $y \neq \lambda x$ (x and y are linearly independent).
Assume $z = \lambda x + y$

$$0 \leq \langle z, z\rangle = \langle \lambda x + y, \lambda x + y\rangle = \langle \lambda x, \lambda x\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + \langle y, y\rangle$$
$$\text{Since } \|x\|^2 > 0, \quad = \lambda^2 \|x\|^2 + 2\lambda \langle x, y\rangle + \|y\|^2$$

If we represent it as a parabola in function of λ, it has no roots.

$$\lambda = \frac{-2\langle x, y\rangle \pm \sqrt{4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2}}{2\|x\|^2}$$

So the discriminant ≤ 0

$$4(\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\implies |\langle x, y\rangle| \leq \|x\| \|y\|$$

2.3 Theorem

$$\langle x, y\rangle = \|x\| \|y\| \cos\varphi$$

Writing x and y in polar coordinates:

$$x_1 = \|x\| \cos\alpha, \quad x_2 = \|x\| \sin\alpha$$
$$y_1 = \|y\| \cos\beta, \quad y_2 = \|y\| \sin\beta$$

$$\begin{aligned}\langle x, y\rangle = x_1 y_1 + x_2 y_2 &= \|x\| \cos\alpha \|y\| \cos\beta + \|x\| \sin\alpha \|y\| \sin\beta \\ &= \|x\| \|y\| (\cos\alpha \cos\beta + \sin\alpha \sin\beta) \\ &= \|x\| \|y\| \cos(\alpha - \beta) = \|x\| \|y\| \cos\varphi\end{aligned}$$

Remark

$$\cos\varphi = \frac{\langle x, y\rangle}{\|x\| \|y\|}$$
$$\text{Then } x \perp y \iff \langle x, y\rangle = 0$$

Examples:

- $C((a, b)) \cong$ continuous functions in (a, b)

  $$f, g \in C((a, b)), \quad then\, \langle f, g\rangle = \int_a^b f(t) g(t)\, dt$$

  We can add some weights:

  $$\langle f, g\rangle = \int_a^b w(t) f(t) g(t)\, dt$$

  An example:

  $$\langle f, g\rangle = \int_a^b e^{-t} f(t) g(t)\, dt$$

- We might have an orthogonal family (with infinite elements) of functions.

$$\{\cos nx, \sin nx\}_{n\in\mathbb{Z}} \quad \text{in } [0, 2\pi]$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx\, dx$$

$$\int_0^{2\pi} \cos nx \sin nx\, dx = \int_0^{2\pi} \cos nx \cos mx\, dx = 0 \quad \text{for } n \neq m$$

3 Vector Product (Only in $\mathbb{R}^3$)

Take $x, y \in \mathbb{R}^3$

$$x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

Recall:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3.1 Triple product and properties

We take the triple product

$$a \cdot (b \times c) = (a_1, a_2, a_3) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $u \in \operatorname{span}\{b, c\}$ then $a \cdot (b \times c) = 0$ and a, b, c are coplanar if $a \cdot (b \times c) = 0$

3.2 Geometric Interpretation

- The magnitude of $x \times y$ represents the area of the parallelogram formed by x and y.
- The direction of $x \times y$ is perpendicular to the plane spanned by x and y, following the right-hand rule.
- The cross product satisfies: $x \times y = -(y \times x)$.

4 Topology of $\mathbb{R}^n$

Definition of open spaces: we define an open ball in $\mathbb{R}^n$ centered at x_0 and of radius R.

$$\text{Denoted by} \quad B_R(x_0) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, x_0) < R\}$$

This set includes all points in $\mathbb{R}^n$ whose distance from x_0 is less than R. Open sets are the building blocks of topological spaces, and they help define concepts such as convergence, continuity, and compactness.

4.1 Open set

A set $A \subset \mathbb{R}^n$ is open if $\forall x \in A, \quad \exists R > 0$ such that $B_R(x) \subset A$.
For example:

$$(x, y) \in \mathbb{R}^2, \quad A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 3\}$$

4.2 Closed set

A set $A \subset \mathbb{R}^n$ is closed if its complement is open.

$$A \subset \mathbb{R}^n \text{ is closed if } \mathbb{R}^n \setminus A \text{ is open.}$$

4.3 Boundary of a set

The boundary of a set $A \subset \mathbb{R}^n$ denoted by ∂A:

$$\partial A = \{x \in \mathbb{R}^n : \forall R > 0, \quad B_R(x) \cap A \neq \emptyset \text{ and } B_R(x) \cap (\mathbb{R}^n \setminus A) \neq \emptyset\}$$

Following the previous example, the boundary of A is the circle of radius 3 centered at the origin:

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 3\}$$

Remark

A set $A \subset \mathbb{R}^n$ is closed if and only if it contains its boundary.

Example:

$$D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1 \text{ and } x > 0\}$$

$$D^C = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq 1 \text{ or } x \leq 0\}$$

$$\partial D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1 \text{ and } x > 0\}$$

D is open, D^C is closed, and ∂D is the semicircle of radius 1 centered at the origin.

Example:

$$S = \{x = 1 \text{ and } 1 < y \leq 2\}$$

$$S^C = \{x \neq 1 \text{ or } y \leq 1 \text{ or } y > 2\}$$

$$\partial S = \{x = 1 \text{ and } y = 1 \text{ or } y = 2\}$$

S is neither open nor closed, as S^C, and ∂S is the line segment from $(1, 1)$ to $(1, 2)$.

4.4 Compact set

A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Example:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$

$A = \partial A$ so A is closed. A is also bounded, as all points in A are contained within the circle of radius 2 centered at the origin.
$\implies$ A is compact.

Example: (Exercise 11a)

$$A = xy - \text{plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3\}$$

This set is closed, and $B_R(x, R) \cap A = \emptyset$ and $B_R(x, R) \cap (\mathbb{R}^3 \setminus A) = \emptyset$.
Therefore, A is not compact.

Example: (Exercise 11b)

$$A = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$$

For any point $(x, y) \in A$, we can find an open ball $B_R(x, R) \subset B$ that does not intersect the x or y axes.

$$R = \min\{x, y\}$$

4.5 Ball in $\mathbb{R}^n$

For a ball at any part of radius r in $\mathbb{R}^n$:

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2$$

$$\text{dist}(x - a) = \|x - a\| = r$$

Example: (Exercise 1a)

Sphere centered at (0,1,-1) with $r = 4$

$$(x - 0)^2 + (y - 1)^2 + (z + 1)^2 = 16$$

Intersection with the x, y, z-planes:

$$\text{If } z = -1, \quad x^2 + (y - 1)^2 = 16$$

$$\text{If } y = 1, \quad x^2 + (z + 1)^2 = 16$$

$$\text{If } x = 0, \quad (y - 1)^2 + (z + 1)^2 = 16$$

Example: (Exercise 1b)

Sphere going through the origin and centered at $(1, 2, 3)$:

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = r^2$$

$$\text{dist}(0, (1, 2, 3)) = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

So the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

Example: (Exercise 1c)

Find center and radius of the sphere:

$$x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$$

$$(x+1)^2 + (y+4)^2 + (z-2)^2 = 49$$

So the center is $(-1, -4, 2)$ and the radius is 7.

Example: (Exercise 3)

Check if the vectors are orthogonal:

$$a = (-5, 3, 7), \quad b = (6, -8, 2)$$

$$a \cdot b = -30 - 24 + 14 = -40 \neq 0$$

So a and b are not orthogonal.

Example: (Exercise 4a)

Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is:

$$d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Let } a = R - Q, \quad b = P - Q$$

$$\text{Then } d = \frac{\|a \times b\|}{\|a\|}$$

$$\text{Area of parallelogram} = \|a \times b\| = \|a\| \cdot d$$

Example: (Exercise 4b)

Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

$$\mathbf{a} = \overrightarrow{QR} = R - Q = (-1-0, 4-6, 7-8) = (-1, -2, -1)$$

$$\mathbf{b} = \overrightarrow{QP} = P - Q = (1-0, 1-6, 1-8) = (1, -5, -7)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -1 \\ -5 & -7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & -7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -2 \\ 1 & -5 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = (9, -8, 7)$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{9^2 + (-8)^2 + 7^2} = \sqrt{81 + 64 + 49} = \sqrt{194}$$

$$\|\mathbf{a}\| = \sqrt{(-1)^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|} = \frac{\sqrt{194}}{\sqrt{6}} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$$

Example: (Exercise 5)

Calculate the volume of the parallelepiped with edges adjacent to $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$, where

$$P(1,1,1), \quad Q(2,0,3), \quad R(4,1,7), \quad S(3,-1,-2).$$

The volume of the parallelepiped can be calculated using the scalar triple product of the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2-1, 0-1, 3-1) = (1,-1,2) \\
\overrightarrow{PR} &= R - P = (4-1, 1-1, 7-1) = (3,0,6) \\
\overrightarrow{PS} &= S - P = (3-1, -1-1, -2-1) = (2,-2,-3)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PR} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} &= \mathbf{i}(0 \cdot (-3) - 6 \cdot (-2)) - \mathbf{j}(3 \cdot (-3) - 6 \cdot 2) + \mathbf{k}(3 \cdot (-2) - 0 \cdot 2) \\
&= \mathbf{i}(0+12) - \mathbf{j}(-9-12) + \mathbf{k}(-6-0) \\
&= 12\mathbf{i} + 21\mathbf{j} - 6\mathbf{k} \\
&= (12, 21, -6)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) &= (1,-1,2) \cdot (12,21,-6) \\
&= 1 \cdot 12 + (-1) \cdot 21 + 2 \cdot (-6) \\
&= 12 - 21 - 12 \\
&= -21
\end{aligned}$$

The volume of the parallelepiped is the absolute value of this scalar triple product:

$$\text{Volume} = |-21| = 21$$

Example: (Exercise 6)

Use the scalar product to check if the following vectors are coplanar: $a = 2i + 3j + k, b = i - j$ and $c = 7i + 3j + 2k$.
The vectors are coplanar if the scalar triple product of the vectors is zero.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} \quad = (-2,-2,10)$$

$$\begin{aligned}
\mathbf{a} \cdot (-2,-2,10) &= (2 \times -2) + (3 \times -2) + (1 \times 10) \\
&= -4 - 6 + 10 = 0
\end{aligned}$$

Since the scalar triple product is zero, the vectors are coplanar.

5 Functions of several variables

A function $f : A \to B$ is a correspondence between two sets A and B such that each element in A is associated with exactly one element in B.

Example:

$$f(x,y) = x^2 + y^2 \quad \text{, where}$$
$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \to f(x,y) = x^2 + y^2$$

The formula for that function:

$$z = x^2 + y^2$$

The graph results in a paraboloid (surface).

5.1 Domain for a function f

The domain is the set of points where the function is well defined.

5.2 Image of a function f

The image is the set of points in B that are associated with points in A.

Example: (Exercise 8)

- The function
$$x^2 + 2z^2 = 1$$
is a cylinder with an elipse section.
- The function
$$x^2 - y^2 + z^2 = 1$$
  - If $y = k$, then $x^2 + z^2 = 1 + k^2$ is a circle.
  - If $z = 0$, then $x^2 - y^2 = 1$ is a hyperbola.
  - If $x = 0$, then $z^2 - y^2 = 1$ is a hyperbola.

5.3 Types of functions

- Scalar functions: $f : \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, \ldots, x_n) \in \mathbb{R}$.
- Vector functions: $f : \mathbb{R}^n \to \mathbb{R}^m, \quad f(x_1, \ldots, x_n) \in \mathbb{R}^m$.
If $f : \mathbb{R}^n \to \mathbb{R}^n$, then f is a vector field.

Example:

Paramatric equations for a line in $\mathbb{R}^3$:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

5.4 Level curves

The level curves of a function $f : \mathbb{R}^N \to \mathbb{R}$ are the curves in the domain of f where $f(x,y) = k$ for some constant k.

$$(x_1, \ldots, x_N) \to f(x_1, \ldots, x_N) \in \mathbb{R}$$
$$f(x,y) = c, \quad c \in \mathbb{R}$$

The graph of a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$.

Example:

$$f(x, y) = x^2 + y^2$$
$$x^2 + y^2 = c, \quad c \in \mathbb{R}$$

The level curves are circles centered at the origin. They can be seen as the projection of the surface onto the xy-plane (from above).

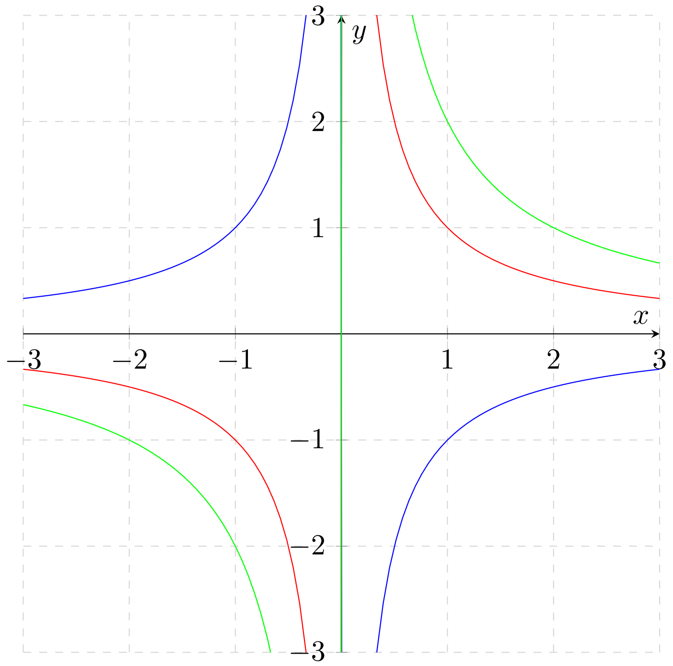
5.5 Remark

In $\mathbb{R}^3$, the level curves of a function $f : \mathbb{R}^3 \to \mathbb{R}$ are the curves in the domain of f where $f(x, y, z) = c$ for some constant $c \in \mathbb{R}$.
They allow us to visualize a 3D graph of a function in 2D.
If the level curves are very close to each other, then the function is steep.
Two level curves never touch.

Example:

Find the level curves of the function $f(x, y) = xy$.

$$xy = c, \quad c = 1, -1, 2.$$

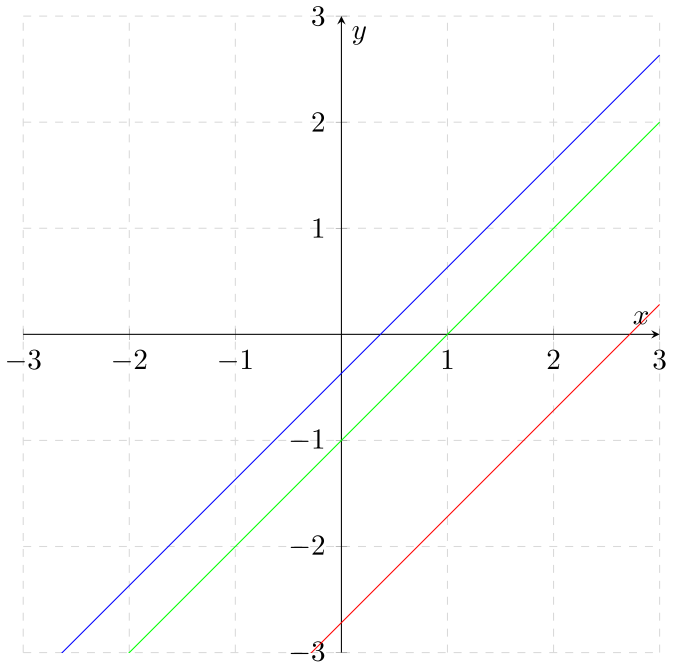


The level curves are a family of hyperbolas.

Example:

Find the level curves of the function $f(x, y) = log(x - y)$.

$$log(x - y) = c, \quad c = 1, -1, 0.$$



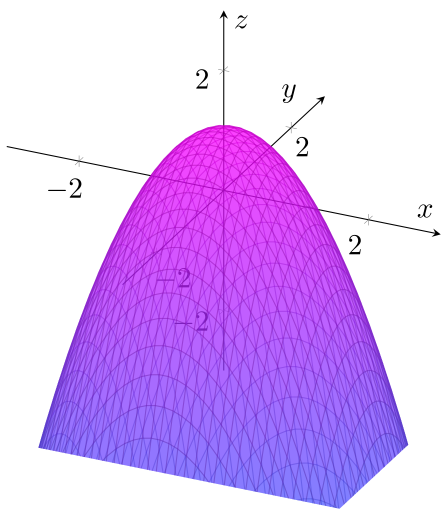
The level curves are a family of straight lines.

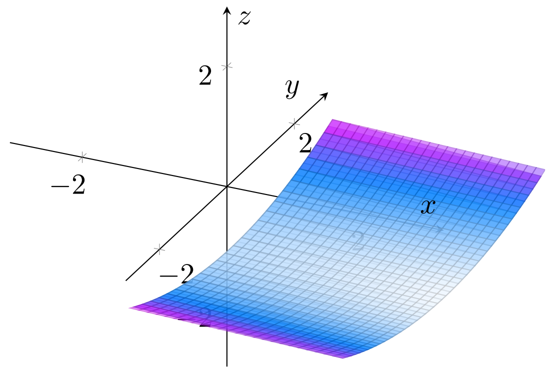
Example:

Find the level curves of the functions

$$f(x,y,z) = x^2 + y^2 + z^2 \quad \text{and} \quad g(x,y,z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$$

The level curves for f are spheres centered at the origin.
The level curves for g are ellipsoids centered at the origin.





5.6 Graph of a function

$$\{(x, f(x)), x \in Dom(f)\}, \quad \text{where } f = 9y^2 + 4z^2 = x^2 + 36$$

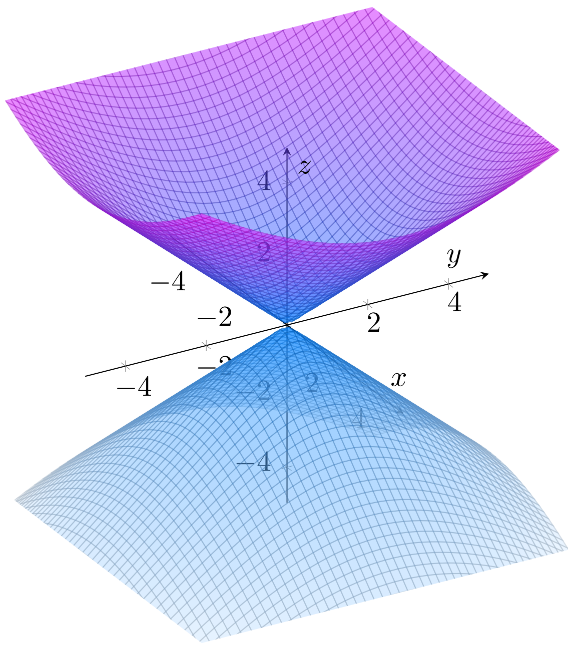
Intersection with the x, y, z-planes:

$$\text{If } z = 0, \quad 9y^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad 4z^2 - x^2 = 36 \to \text{Hyperbola}$$
$$\text{If } x = 0, \quad 9y^2 + 4z^2 = 36 \to \text{Ellipse}$$

Example:

Plot the function $f = x^2 + 4y^2 = z^2$.

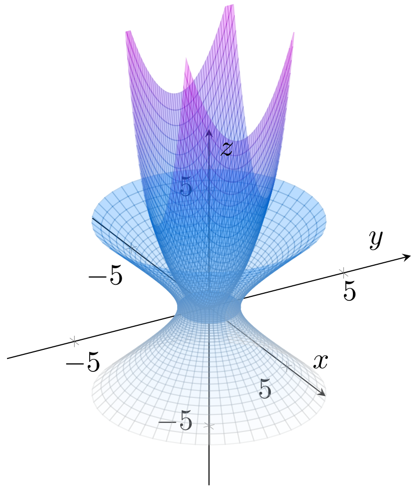
$$\text{If } z = 0, \quad x^2 + 4y^2 = 0 \to x = 0, y = 0$$
$$\text{If } y = 0, \quad x^2 = z^2 \to x = z, x = -z$$
$$\text{If } x = 0, \quad 4y^2 = z^2 \to y = z/2, y = -z/2$$
$$\text{If } z = k, \quad x^2 + 4y^2 = k^2 \to \text{Ellipses}$$



The graph is a cone.

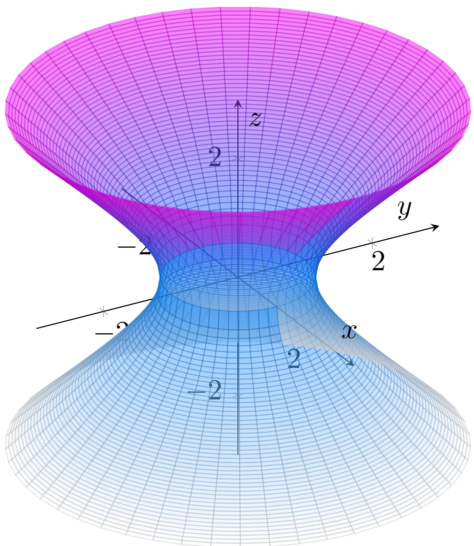
Problem 8

$$x^2 + y^2 + 9z^2 = 1 \to \text{Ellipsoid}$$
$$x^2 - y^2 + z^2 = 1 \to \text{Hyperboloid of one sheet}$$
$$y = 2x^2 + z^2 \to \text{Paraboloid}$$



Example:

$$-x^2 + y^2 - z^2 = 1 \to \text{Hyperboloid}$$
$$\text{If } z = 0, \quad -x^2 + y^2 = 1 \to \text{Hyperbola}$$
$$\text{If } y = 0, \quad -x^2 - z^2 = 1 \to \text{No solution}$$
$$\text{If } x = 0, \quad y^2 - z^2 = 1 \to \text{Hyperbola}$$

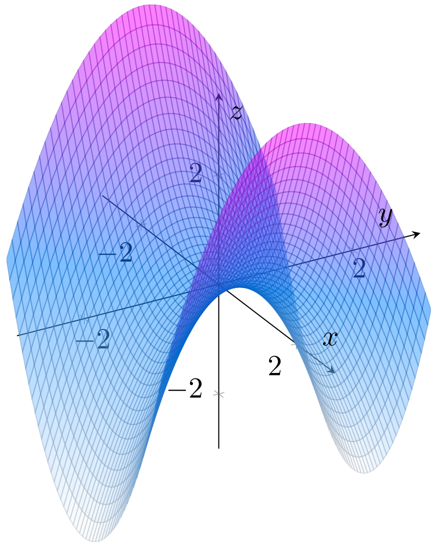


Example:

$$z = x^2 - y^2 \to \text{Paraboloid}$$

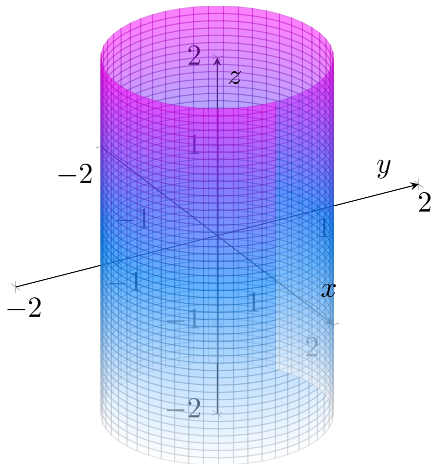
$$\text{If } z = 0, \quad x^2 - y^2 = 0 \to \text{Hyperbola}$$
$$\text{If } z = k, \quad x^2 = y^2 + k \to \text{Hyperbola}$$

$$\text{If } y = 0, \quad z = x^2 \rightarrow \text{Parabola}$$
$$\text{If } x = 0, \quad z = -y^2 \rightarrow \text{Parabola}$$



Example:

Plot the function $x^2 + y^2 = 1$.



6 Cartesian coordinates in $\mathbb{R}^N$

In $\mathbb{R}^2$, the Cartesian coordinates are (x, y).
In $\mathbb{R}^3$, the Cartesian coordinates are (x, y, z).

6.1 Polar coordinates in $\mathbb{R}^2$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

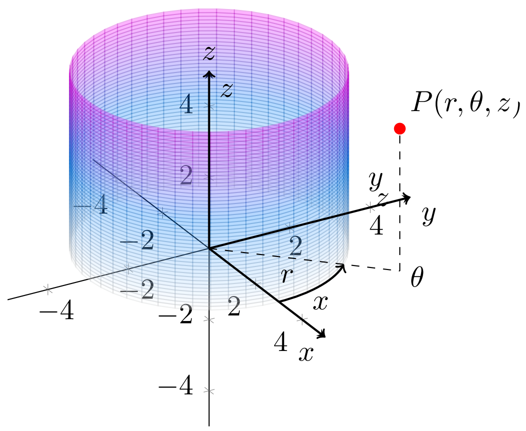
Inverse transformation:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Lemma Let $A = (0, \infty) \times (0, 2\pi)$.
The function $g : A \to \mathbb{R}^2$ defined by $g(r, \theta) = (r\cos(\theta), r\sin(\theta))$ is a bijection, continuous in a ball $B(0, \alpha)$ such that $\{g(r, \theta), 0 < r < \alpha, 0 - leq\theta < 2\pi\}$ is a subset of $B(0, \alpha)$.
To see if the function is one-to-one, assume that $g(r_1, \theta_1) = f(r_2, \theta_2)$ for $r_1, r_2 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
Then $r_1 \cos(\theta_1) = r_2 \cos(\theta_2)$ and $r_1 \sin(\theta_1) = r_2 \sin(\theta_2)$. This implies that $r_1 = r_2$, since $r_1 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.
As a consequence $\theta_1 = \theta_2$ so that g is one-to-one.
Now taking $(x, y) \in B(0, \alpha)$, and $r = \sqrt{x^2 + y^2} > 0$.
Then, the part $(\frac{x}{2}, \frac{y}{2})$ is in $B(0, 1)$.
Therefore, there exists $\theta \in [0, 2\pi)$ such that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$.
Which implies that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. So g is onto.

6.2 Cylindrical coordinates in $\mathbb{R}^3$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$



Example:

Transform into cartesian coordinates:

$$(3, \frac{\pi}{2}, 1) = (3\cos(\frac{\pi}{2}), 3\sin(\frac{\pi}{2}), 1) = (0, 3, 1)$$

Example:

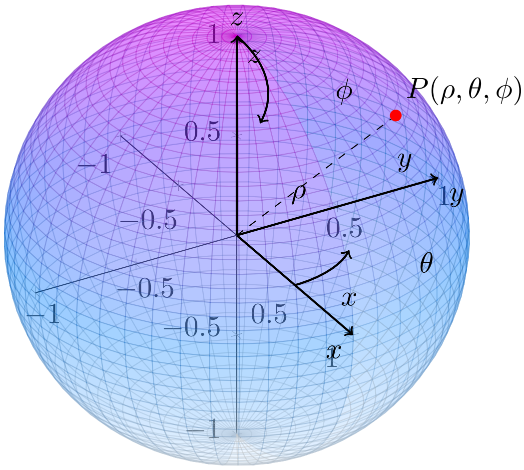
Transform into cartesian coordinates:

$$(4, -\frac{\pi}{3}, 5) = (4\cos(-\frac{\pi}{3}), 4\sin(-\frac{\pi}{3}), 5) = (2, -2\sqrt{3}, 5)$$

6.3 Spherical coordinates in $\mathbb{R}^3$

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi), \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

ρ is the distance from the origin, ϕ is the angle from the z-axis, and θ is the angle from the x-axis.



Example: (Problem 6)

Transform from spherical coordinates to cartesian coordinates:

$$(1, 0, 0) = (1\sin(0)\cos(0), 1\sin(0)\sin(0), 1\cos(0)) = (0, 0, 1)$$

$$(2, \frac{\pi}{3}, \frac{\pi}{4}) = (2\sin(\frac{\pi}{4})\cos(\frac{\pi}{3}), 2\sin(\frac{\pi}{4})\sin(\frac{\pi}{3}), 2\cos(\frac{\pi}{4})) = (\sqrt{3}, 1, \sqrt{2})$$

6.4 Parametric equation of a line in $\mathbb{R}^3$

Having two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the parametric equation of the line passing through P_1 and P_2 is:

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

Example: (Problem 14)

Find the parametric equation of the line passing through $P(1, 0, 1)$ and $Q(2, 3, 1)$.

$$\begin{cases} x = 1 + t(2 - 1) = 1 + t \\ y = 0 + t(3 - 0) = 3t \\ z = 1 + t(1 - 1) = 1 \end{cases}$$

6.5 Parametric equation of a plane in $\mathbb{R}^3$

For a plane in parametric form:

$$\begin{cases} x = x_0 + a_1 t + a_2 s \\ y = y_0 + b_1 t + b_2 s \\ z = z_0 + c_1 t + c_2 s \end{cases}$$

Remark A line in $\mathbb{R}^3$ is a manifold of dimension 1.
A plane in $\mathbb{R}^3$ is a manifold of dimension 2.

Example: (Problem 9)

Write $x^2 + y^2 + z^2 = 4$ (Sphere) in parametric form.

$$\begin{cases} x = \lambda \\ y = \mu \\ z = \sqrt{4 - \lambda^2 - \mu^2} \end{cases} \qquad \text{where } \lambda^2 + \mu^2 \leq 4$$

Using spherical coordinates:

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi, \quad \rho = 2$$

$$\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi) + \rho^2\cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) = 4$$
$$\rho^2 \sin^2(\phi) + \rho^2\cos^2(\phi) = 4$$
$$\rho^2 = 4 \to \rho = 2$$

Example: (Problem 11)

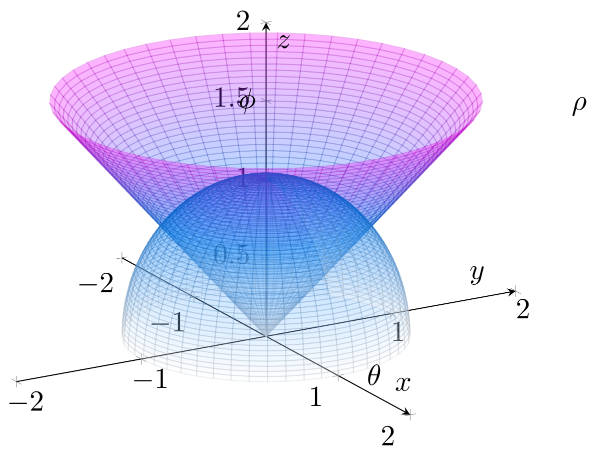
A solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Becase the solid is inside the cone, we have $z \geq \sqrt{x^2 + y^2}$
Because the solid is below the sphere, we have $x^2 + y^2 + z^2 \leq z$

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho \cos(\phi) \end{cases} \qquad \text{where } 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

$$\sqrt{\rho^2 \cos^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi)} \leq \rho\cos(\phi)$$
$$\rho \sin(\phi) \leq \rho \cos(\phi) \to \tan(\phi) \leq 1 \to \phi \leq \frac{\pi}{4}$$

$$\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) \leq \rho\cos(\phi)$$
$$\rho^2 \sin^2(\phi) + \rho^2\cos^2(\phi) \leq \rho\cos(\phi)$$
$$\rho^2 \leq \rho\cos(\phi) \to \rho \leq \cos(\phi)$$



Solid in cartesian coordinates:

$$\begin{cases} z \geq \sqrt{x^2+y^2} \\ x^2+y^2+z^2 \leq z \end{cases}$$

In spherical coordinates:

$$\begin{cases} \rho \leq \cos(\phi) \\ \phi \leq \frac{\pi}{4} \\ \theta \in [0, 2\pi) \end{cases}$$

6.6 Intersection of two bodies

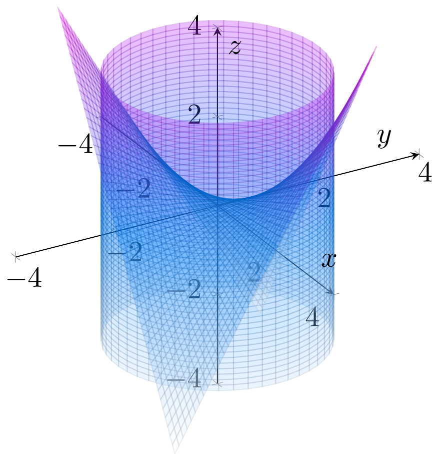
The intersection of two bodies gives (in general) a curve.

Example: (Problem 12)

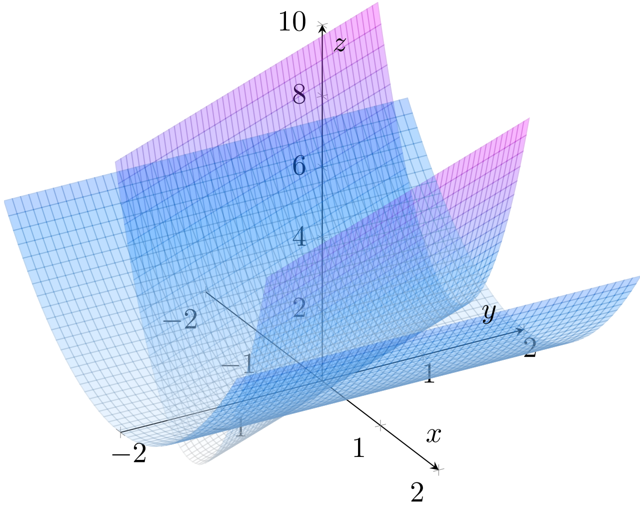
Parametrize the intersection (a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$) of the two bodies:

$$\begin{cases} \text{Cylinder: } x^2+y^2=4 \\ \text{Surface: } z=xy \end{cases}$$

$$\begin{cases} x = 2\cos(t) = \gamma_1(t) \\ y = 2\sin(t) = \gamma_2(t) \\ z = 4\cos(t)\sin(t) = \gamma_3(t) \end{cases} \qquad \text{where } t \in [0, 2\pi)$$



$$\begin{cases} \text{Paraboloid: } z = 4x^2+y \\ \text{Cylinder: } y = x^2 \end{cases} \implies \begin{cases} x = t \\ y = t^2 \\ z = 4t^2+t^2 = 5t^2 \end{cases}$$



7 Limit of functions

Assume a scalar $f : \mathbb{R}^N \to \mathbb{R}$. We say that the limit of $f(x)$ as x approaches x_0 is L and we denote it by:

$$\lim_{x \to x_0} f(x) = L \quad \in \mathbb{R}, \quad x, x_0 \in \mathbb{R}^N$$

7.1 Definition of the limit

We say that $\lim_{x \to x_0} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\forall x \in \mathbb{R}^N, \quad 0 < ||x - x_0|| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem If the limit of $f(x)$ as x approaches x_0 exists, then the limit is unique.

Proof Argue by contradiction. Assume that there are two limits L_1 and L_2 such that $L_1 \neq L_2$.

Actually, we can say that:

$$B(L_1, r_1) \cap B(L_2, r_2) = \emptyset$$

Where $B(L_1, r_1)$ is the ball of radius r_1 centered at L_1 and $B(L_2, r_2)$ is the ball of radius r_2 centered at L_2.

$$\lim_{x \to x_0} f(x) = L_1, \quad \text{for any } \varepsilon_1 > 0, \quad \exists \delta_1 > 0 \text{ such that } |f(x) - L_1| < \varepsilon_1 \text{ for } 0 < ||x - x_0|| < \delta_1$$

$$\lim_{x \to x_0} f(x) = L_2, \quad \text{for any } \varepsilon_2 > 0, \quad \exists \delta_2 > 0 \text{ such that } |f(x) - L_2| < \varepsilon_2 \text{ for } 0 < ||x - x_0|| < \delta_2$$

Indeed,

$$\begin{cases} |f(x) - L_1| < r_1 = \varepsilon_1 \\ |f(x) - L_2| < r_2 = \varepsilon_2 \end{cases} \implies \text{However, taking } \delta = \min(\delta_1, \delta_2) \text{ so that } ||x - x_0|| < \delta$$

$$\text{such that } \begin{cases} |f(x) - L_1| < \min(r_1, r_2) \\ |f(x) - L_2| < \min(r_1, r_2) \end{cases} \implies L_1 = L_2, \text{ which is a contradiction}$$

7.2 Computing limits

Using the definition of the limit, we can compute the limit of a function $f(x)$ as x approaches x_0. To do so we must choose the value of L towards the function is going. The process is as follows:

$$\text{For example in } \mathbb{R}^2, \quad \begin{cases} f : \mathbb{R}^2 \to \mathbb{R} \\ (x, y) \to f(x, y) \end{cases}$$

Then:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Hence, we must find $\delta \cong \delta(\varepsilon)$.

Example:

Prove that

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right) = 0$$

$$\text{Let } \varepsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| < \varepsilon$$

$$\text{Since } \left|\sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq 1 \text{ and } \sqrt{x^2 + y^2} \geq 0$$

$$\text{We have } \left|\sqrt{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)\right| \leq \sqrt{x^2 + y^2} < \varepsilon$$

$$\text{Therefore, we can choose } \delta = \varepsilon$$

Example:

Prove that

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\text{Let } \varepsilon > 0, \quad \text{we must find } \delta > 0 \text{ such that } ||f(x) - b|| = |y - b| < \varepsilon$$
$$\text{when } ||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

$$\text{We start from } |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta$$
$$\text{Therefore, we can choose } \delta = \varepsilon$$

7.3 Iterative limits

In 2D,

$$\lim_{x\to a}\left(\lim_{y\to b} f(x, y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x, y)\right)$$

However, this technique only gives us negative answers (non-existence of the limit). Since we are just following one direction.

7.4 Approach following families of functions

They might be straight lines, parabolas, etc. around the point of approach.

Example:

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

Let y $= \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{x}{|x|\sqrt{1+\lambda^2}} = \frac{1}{\sqrt{1+\lambda^2}}$$

This limit depends on λ. Therefore, the limit does not exist.

Example:

Compute the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

Let $y = \lambda x, \quad \lambda \in \mathbb{R}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3}{x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x}{1+\lambda^2} = 0$$

Following the family of functions $y = \lambda x$, the limit is 0. However, we cannot confirm that the value of the limit is 0 or that the limit exists using this method.

This method is necessary but not sufficient. It is a good way to check if the limit does not exist.

Problem 1a

Taking $y = \lambda x$ and knowing that $\sin x \approx x$ for $x \approx 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2+\sin^2(\lambda x)}{2x^2+\lambda^2 x^2} = \lim_{x\to 0} \frac{x^2+\lambda^2 x^2}{2x^2+\lambda^2 x^2} = \frac{1+\lambda^2}{2+\lambda^2}$$

This limit depends on λ. Therefore, the limit does not exist.

Problem 1d

Taking $y = \lambda x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x\lambda x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{\lambda x^2}{\sqrt{x^2(1+\lambda^2)}} = \lim_{x\to 0} \frac{\lambda x}{\sqrt{1+\lambda^2}} = 0$$

This limit does not depend on λ. Therefore, the limit exists and is 0.
Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{r} = \lim_{r\to 0} r\cos(\theta)\sin(\theta) = 0$$

7.5 Polar coordinates

In $\mathbb{R}^2$, the polar coordinates are (r,θ).

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

Theorem Let us consider two functions f and g such that

$$\lim_{x\to x_0} f(x) = 0, \quad \text{and } g \text{ is bounded for } ||x - x_0|| < \delta$$

Then

$$\lim_{x\to x_0} f(x)g(x) = 0$$

Problem 1e

Taking $y = \lambda x$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x\to 0} \frac{x^2 \lambda x e^{\lambda x}}{x^4 + 4\lambda^2 x^2} = \lim_{x\to 0} \frac{\lambda x^3 e^{\lambda x}}{x^2(x^2 + 4\lambda^2)} = \lim_{x\to 0} \frac{\lambda x e^{\lambda x}}{x^2 + 4\lambda^2} = 0$$

Now taking $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} = \lim_{x\to 0} \frac{x^2 x^2 e^{x^2}}{x^4 + 4x^4} = \lim_{x\to 0} \frac{x^4 e^{x^2}}{5x^4} = \lim_{x\to 0} \frac{e^{x^2}}{5} = \frac{1}{5}$$

This limit depends on the direction of approach. Therefore, the limit does not exist.

Problem 1f

Applying generalized spherical coordinates:

$$\begin{cases} x = r\sin(\phi)\cos(\theta) \\ y = \frac{1}{2} r \sin(\phi)\sin(\theta) \\ z = \frac{1}{3} r\cos(\phi) \end{cases}$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{r\to 0} \frac{\frac{1}{2} r\sin(\phi)\sin(\theta)\frac{1}{3} r\cos(\phi)}{r^2} = \lim_{r\to 0} \frac{1}{6}\sin(\phi)\cos(\phi)\sin(\theta)$$

This limit depends on ϕ and θ. Therefore, the limit does not exist.

8 Continuity of functions

A function $f : \mathbb{R}^N \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}^N$ if:

1. $f(x_0)$ is defined.
2. $\lim_{x\to x_0} f(x)$ exists.
3. $\lim_{x\to x_0} f(x) = f(x_0)$

Theorem Consider A a subset in $\mathbb{R}^N$ and $f : A \to \mathbb{R}^M$ a vector function.

$$F(x) = (f_1(x), f_2(x), \ldots, f_M(x))$$

If $f_1, f_2, \ldots, f_M$ are continuous at $x_0 \in A$, then F is continuous at x_0.

Theorem Assume that f and g are continuous at $x_0 \in \mathbb{R}^N$ and $f(x_0)$ respectively. Then, the composition $g \circ f$ is continuous at x_0.

$$\text{For example: } f(x) = x^2 y \sin(x+y), \quad f : \mathbb{R}^2 \to \mathbb{R}$$

Since f is a composite function and $x^2, y, \sin(x+y)$ are continuous, then f is continuous.

Problem 2a

Prove the continuity:

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

If $(x,y) \neq (0,0)$, then $f(x,y)$ is a composition of continuous functions. So that

$$\text{for any } (x_0,y_0) \neq (0,0), \qquad \lim_{(x,y)\to(0,0)} f(x,y) = f(x_0,y_0) = \frac{x_0^2y_0^3}{2x_0^2+y_0^2}$$

At $(0,0)$, the function is well defined: $f(0,0) = 1$, so we must check that the limit exists.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} =$$

Using polar coordinates:

$$\begin{cases} x = \dfrac{1}{\sqrt{2}} r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

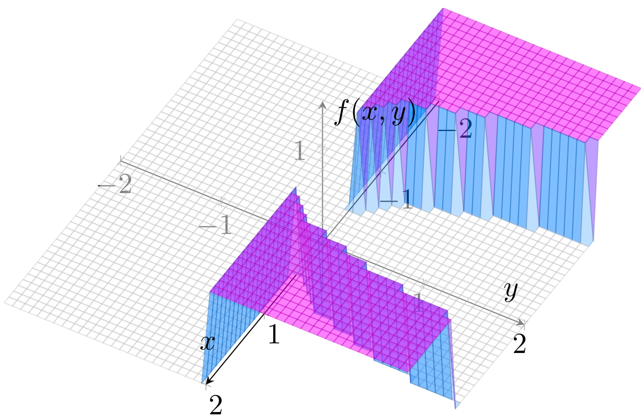
$$= \lim_{r\to 0} \frac{\frac{1}{2}r^3\cos^2(\theta)\sin^3(\theta)}{r^2} = \lim_{r\to 0} \frac{1}{2} r\cos^2(\theta)\sin^3(\theta) = 0$$

The limit exists and is equal to 0. However, $f(0,0) = 1 \neq 0$. Therefore, the function is not continuous at $(0,0)$.

Problem 3

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

a) $lim_{(x,y)\to(0,0)} f(x,y) = 0$ along any path going through $y = mx^\alpha$. $\quad 0 < \alpha < 4$



Let $\alpha = 1$: $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} f(x,mx) = 0$$

In this case, the limit exists and is equal to 0.

b) Despite (a), prove that the function is not continuous.

Take any point $(a, 0)$, $\quad a \in \mathbb{R}$. Then:

$$\lim_{(x,y)\to(a,0)} f(x,y) = \begin{cases} 0 & \text{if } y \to 0^- \\ 1 & \text{if } y \to 0^+ \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

c) f discontinuous on two entire curves: $\begin{cases} y = x^4 \\ y = 0 \end{cases}$

Take any point (a, a^4) on $y = x^4$, $\quad a \in \mathbb{R}$. Then:

$$lim_{(x,y)\to(a,a^4)} f(x,y) =$$

$$||(x,y) - (a,a^4)|| < \delta \implies \begin{cases} 0 < |f(x,y)| = 1 < \varepsilon & \text{if } y < x^4 \\ 0 < |f(x,y)| = 0 < \varepsilon & \text{if } y > x^4 \end{cases}$$

The limit depends on the direction of approach. Therefore, the function is not continuous.

Example:

Consider the function:

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let $y = \lambda x$, $\quad \lambda \in \mathbb{R}$. Then

$$a) \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3 \lambda x}{x^6 + \lambda^2 x^2} = \lim_{x\to 0} \frac{\lambda x^2}{x^4 + \lambda^2} = 0$$

Now taking $y = x^3$:

$$b) \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3 x^3}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

Therefore the function is not continuous at $(0,0)$, since the limit depends on the direction of approach.

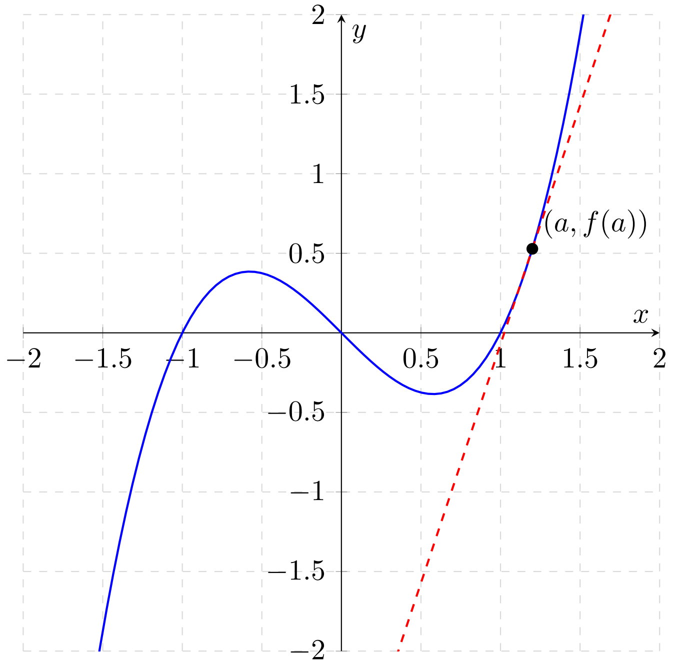
9 Differentiability of functions

In 1D when we consider a function $f : \mathbb{R} \to \mathbb{R}$, the derivative of f at $a \in \mathbb{R}$ describes the ratio of the change of the function f at $x = a$ and is denoted by:

$$f'(a) = \lim_{t\to 0} \frac{f(a+t) - f(a)}{t} = \frac{df(a)}{dt}$$

Geometrically, the derivative of a function at a point is the slope of the tangent line to the curve at that point. And this tangent line is defined as:

$$y = f(a) + f'(a)(x - a)$$



A way of seeing this is written by the Taylor polynomial:

$$f(x) = f(a) + M(x-a) + r(|x-a|)$$

Where $M = f'(a)$ is the slope of the tangent line and $r(|x-a|)$ is the remainder.
If $x \to a$, then $r(|x-a|) \to 0$ but this only guarantees that the function is continuous at a.
However, if

$$\lim_{x\to a} \frac{r(|x-a|)}{|x-a|} = 0$$

we actually find the existence of the tangent line at a.

$$\frac{f(x)-f(a)}{x-a} = f'(a) + \frac{r(|x-a|)}{x-a}$$

9.1 Generalizing to several variables

$$z = f(x,y)$$

We would like to understand the derivative with respect to each variable.
To do so, we use the knowledge of derivatives in 1 variable.

9.2 Partial derivatives

Let $f : A \subset \mathbb{R}^N \to \mathbb{R}$ be a scalar function. And let $x_0 \in A$. The partial derivative of f with respect to the variable x_i at x_0 is defined as:

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t\to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

Where e_i is the unit vector in the direction of x_i.
Somehow, we are computing the ratio of the change of the function f in the direction of x_i, in other words, of the vector e_i.

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

$$B = \{e_1, e_2, \dots, e_N\} \quad \text{is the standard basis of } \mathbb{R}^N$$

If $F : \mathbb{R}^N \to \mathbb{R}^M$ is a vector function, then the partial derivative of F with respect to the variable x_i at the point x is defined as:

$$\frac{\partial F(x)}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_M}{\partial x_i}\right)$$

Example:

$$f(x, y) = xy + x - y \quad \text{at } (0, 0)$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t\to 0} \frac{f(0 + te_1) - f(0, 0)}{t} = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t\to 0} \frac{t \cdot 0 + t - 0}{t} = 1$$

Where t is the norm of the vector te_1:

$$t = t||e_1|| = t \cdot 1 = t$$

Now,

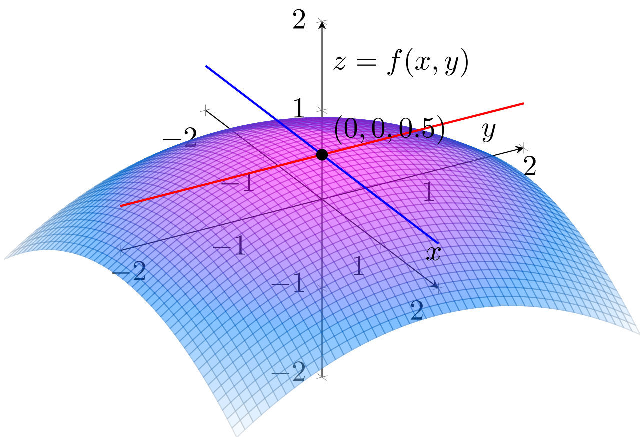
$$\frac{\partial f}{\partial y}(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{t} = \lim_{k\to 0} \frac{0 \cdot k + 0 - k}{k} = -1$$

9.3 Geometrical interpretation of the partial derivative

The partial derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point (x_0, y_0) is the slope of the tangent line to the curve $z = f(x, y)$ at the point (x_0, y_0) in the direction of the variable x_i.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \text{slope of the tangent line in the direction of } x$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \text{slope of the tangent line in the direction of } y$$



Obviously, we can apply the rules for derivatives:

$$\frac{\partial f}{\partial x}(x, y) = y + 1, \quad \frac{\partial f}{\partial y}(x, y) = x - 1$$

9.4 Directional derivative

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a scalar function and $P \in A \subset \mathbb{R}^N$ and $v \in \mathbb{R}^N$ be a vector. The directional derivative of f at P in the direction of v is defined as:

$$D_v f(P) = \lim_{t\to 0} \frac{f(P + tv) - f(P)}{t||v||} = \frac{df(P + tv)}{dt}$$

Example:

$$f(x,y)=\sqrt{|xy|} \quad \text{at } (0,0) \quad \text{in the direction of } v=(1,1)$$

$$D_{(1,1)}f(0,0)=\lim_{t\to 0}\frac{f(0+t,0+t)-f(0,0)}{t||(1,1)||}=\lim_{t\to 0}\frac{\sqrt{|t^2|}-0}{t||(1,1)||}=\lim_{t\to 0}\frac{|t|}{t\sqrt{2}}=\begin{cases}\dfrac{1}{\sqrt{2}} & \text{if } t\to 0^+\\ -\dfrac{1}{\sqrt{2}} & \text{if } t\to 0^-\end{cases}$$

Therefore, the directional derivative does not exist.

9.5 Remark

Existence of all directional derivatives at a point P does not guarantee the continuity of the function at P.

$$\text{For example: } f(x,y)=\begin{cases}1 & \text{if } x=0 \text{ or } y=0\\ 0 & \text{otherwise}\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=0=\frac{\partial f}{\partial y}(0,0)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

Example:

$$f(x,y)=x^{1/3}y^{1/3} \quad \text{at } (0,0)$$

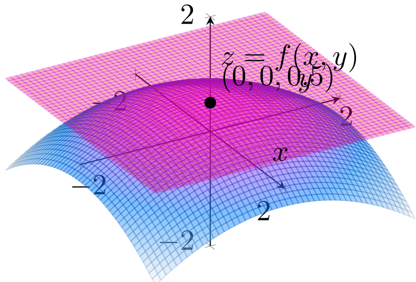
$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{t^{1/3}\cdot 0^{1/3}-0}{t}=0$$

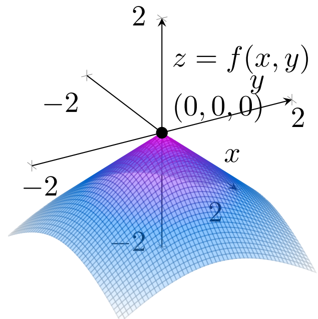
$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to 0}\frac{0\cdot t^{1/3}-0}{t}=0$$

Derivatives exist and the function is also continuous at $(0,0)$. However, there is no tangent plane at $(0,0)$.

9.6 Differentiability

Heuristically it means the construction of a tangent plane.





Let $A\subset\mathbb{R}^2$ be an open subset in $\mathbb{R}^2$ such that $(x,y)\in A$, with $f:A\to\mathbb{R}$ a scalar function. We say that f is differentiable at (x_0,y_0) if:

1. $\dfrac{\partial f}{\partial x}(x_0,y_0)$ and $\dfrac{\partial f}{\partial y}(x_0,y_0)$ exist.

2. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)-f(x_0,y_0)-\frac{\partial f}{\partial x}\big|_{(x_0,y_0)}(x-x_0)-\frac{\partial f}{\partial y}\big|_{(x_0,y_0)}(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$

For a function to be differentiable we say that there exists a tangent plane at (x_0, y_0):

$$z = f(x_0, y_0) + A(x-x_0) + B(y-y_0)$$

Following similar ideas to the 1D case, we can write:

$$f(x,y) = f(x_0,y_0) + A(x-x_0) + B(y-y_0) + r(\|(x,y)-(x_0,y_0)\|)$$

If $\|(x,y)-(x_0,y_0)\| \to 0$ then $r(\|(x,y)-(x_0,y_0)\|) \to 0$, and the function is continuous at (x_0, y_0).
However if we have:

$$\frac{f(x,y)-f(x_0,y_0)-A(x-x_0)-B(y-y_0)}{\|(x-x_0)+(y-y_0)}\| = \frac{r(\|(x,y)-(x_0,y_0)\|)}{\|(x-x_0)+(y-y_0)\|} \to 0$$

Then the function is differentiable at (x_0, y_0) if:

$$A = \frac{\partial f}{\partial x}(x_0,y_0), \quad B = \frac{\partial f}{\partial y}(x_0,y_0)$$

Remark Taylor's polynomial around (x_0, y_0) of degree 1 is:

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + r(\|(x,y)-(x_0,y_0)\|)$$

Where $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ form $L\left((x-x_0),(y-y_0)\right)$, the linearity function.

$$\|f(x,y) - L\left((x-x_0),(y-y_0)\right)\| < \varepsilon \text{ as } \|(x,y)-(x_0,y_0)\| < \delta$$

We can write L in terms of the derivative of the scalar function f:

$$f : \mathbb{R}^2 \to \mathbb{R}$$

We define such a derivative as the gradient of f:

$$D(f(x,y)) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

The gradient provides us with the direction of the maximum increase of the function f at (x, y).

Problem 4a

$$f(x,y) = xy, \quad \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

Problem 4c

$$f(x,y) = \sqrt{x^2+y^2}, \quad \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Problem 5a

$$f(x,y,z) = (x+z)e^{x-y}, \quad \text{gradient at } (1,1,1)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(e^{x-y} + (x+z)e^{x-y}, -(x+z)e^{x-y}, e^{x-y}\right)$$

$$\nabla f(1,1,1) = (e^0 + 2e^0, -2e^0, e^0) = (3, -2, 1)$$

10 Differentiability in $\mathbb{R}^N$

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \to \mathbb{R}^M$$

We say that f is differentiable at x_0 if:

1. $\dfrac{\partial f_i}{\partial x_j}(x_0)$ exists for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$.

2. $\lim_{x \to x_0} \dfrac{f(x) - f(x_0) - Jf(x_0)(x - x_0)}{||x - x_0||} = 0$

Where $Jf(x_0)$ is the Jacobian matrix of f at x_0:

$$Jf(x_0) = Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix}$$

Problem 6a

Compute the jacobian matrix:

$$F(x, y) = (y, x, xy, y^2 - x^2), \quad F : \mathbb{R}^2 \to \mathbb{R}^4$$

$$JF(x, y) = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial (xy)}{\partial x} & \frac{\partial (xy)}{\partial y} \\ \frac{\partial (y^2 - x^2)}{\partial x} & \frac{\partial (y^2 - x^2)}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ y & x \\ -2x & 2y \end{pmatrix}$$

Now at $(1, 2)$:

$$JF(1, 2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \\ -2 & 4 \end{pmatrix}$$

Problem 6c

$$F(x, y, z) = z^2 e^x \cos(y), \text{ at } (0, \frac{\pi}{2}, -1) \quad F : \mathbb{R}^3 \to \mathbb{R}$$

$$JF(x, y, z) = \begin{pmatrix} \frac{\partial z^2 e^x \cos(y)}{\partial x} & \frac{\partial z^2 e^x \cos(y)}{\partial y} & \frac{\partial z^2 e^x \cos(y)}{\partial z} \end{pmatrix} =$$

$$= \begin{pmatrix} z^2 e^x \cos(y) & -z^2 e^x \sin(y) & 2z e^x \cos(y) \end{pmatrix}$$

$$JF(0, \frac{\pi}{2}, -1) = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$$

Example:

$$f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f is not differentiable at $(0,0)$, the partial derivatives exist at $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{2t\cdot 0}{t\cdot 0} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{2\cdot 0\cdot t}{t\cdot 0} = 0$$

However, the limit of the difference quotient does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)x - \frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\dfrac{2xy}{\sqrt{x^2+y^2}} - 0 - 0\cdot x - 0\cdot y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

If we use polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$$= \lim_{r\to 0} \frac{2r^2\cos(\theta)\sin(\theta)}{r^2} = \lim_{r\to 0} 2\cos(\theta)\sin(\theta) = 2\cos(\theta)\sin(\theta)$$

The limit depends on θ. Therefore, the function is not differentiable at $(0,0)$.

Problem 7

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The partial derivatives exist at $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Take the limit:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = lim_{r\to 0} \frac{r^3\cos(\theta)\sin^2(\theta)}{r^2\cos^2(\theta)+r^4\sin^4(\theta)}$$

The limit does not exist, therefore the function is not continuous at $(0,0)$. The function is not differentiable at $(0,0)$, due to it not being continuous at that point.
Direction of any vector at $(0,0)$:

$$D_{(u,v)}f(0,0) = \lim_{t\to 0} \frac{f(tu,tv)-f(0,0)}{t\|(u,v)\|}$$

Assume $\|(u,v)\| = 1$:

$$= \lim_{t\to 0} \frac{f(tu,tv)}{t} = \lim_{t\to 0} \frac{t^3uv^2}{t(t^2u^2+t^4v^4)} = \lim_{t\to 0} \frac{uv^2}{u^2+t^2v^4} = \frac{v^2}{u}$$

Where $\dfrac{v^2}{u}$ is the slope of the tangent line in the direction of (u,v).

10.1 Proposition:

$$A \subset \mathbb{R}^N, \quad x_0 \in A, \quad f : A \to \mathbb{R}$$

f is a scalar function, differentiable at x_0 and $v \in \mathbb{R}^N \backslash 0$ a vector. Then:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle$$

Remark

$$\langle \nabla f(x_0), \alpha v \rangle = \alpha \langle \nabla f(x_0), v \rangle \neq \langle \nabla f(x_0), v \rangle$$

Using the properties of the scalar product:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle = ||\nabla f(x_0)|| \cdot ||v|| \cdot \cos(\theta) = ||\nabla f(x_0)|| \cdot \cos(\theta)$$

Where θ is the angle between $\nabla f(x_0)$ and v, and we assume $\|v\| = 1$.
The directional derivative is maximum in the direction of $\nabla f(x_0)$ and minimum in the direction of $-\nabla f(x_0)$.

10.2 Proposition:

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function and let $(x_0, y_0, z_0) \in \mathbb{R}^3$ be a point on the level surface $\{f(x, y, z) = k \in \mathbb{R}\} = S$.
Then, $\nabla f(x_0, y_0, z_0) \perp v = 0$ where v is the tangent vector to the trajectory $c(t)$ at $t = 0$ over the surface S.

Proof

Since $c(t) \subset S$, we have $f(c(t)) = k$.
By construction, $v = c'(0)$.
Now, applying the chain rule, we get:

$$\left. \frac{d}{dt} f(c(t)) \right|_{t=0} = \nabla f(c(0)) \cdot c'(0) = \nabla f(c(0)) \cdot v = 0$$

This proves the orthogonality between ∇f and the tangent vector to the curve (to the level curve).

Remark

Take $P^*(p_1, p_2, p_3)$ as a point and (x, y, z) as any point on the tangent plane.
Then,

$$n \cdot (x - p_1, y - p_2, z - p_3) = 0$$

where $n = \nabla f(P^*)$.

10.3 Tangent Plane to a Surface

The tangent plane to $f(x, y, z) = k$ at p_0 is given by

$$\nabla f(p_0) \cdot \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix} = 0$$

Example

Tangent plane of $3xy + z^2 = 4$ at $(1, 1, 1)$:

- **Gradient:**
  $\nabla f(x, y, z) = (3x, 3y, 2z)$
- **Gradient at (1, 1, 1):**
  $\nabla f(1, 1, 1) = (3, 3, 2)$
- **Tangent Plane:**
  $\nabla f(1,1,1) \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0$
- **Equation:**
  $3(x-1) + 3(y-1) + 2(z-1) = 0$
  $3x + 3y + 2z = 8$

Problem 11c

Find the tangent plane of $z = \dfrac{x}{y^2}$ at $(-2, 2, 1)$:

- **Gradient:**
  $\nabla f(x, y, z) = \left(\dfrac{1}{y^2}, -\dfrac{2x}{y^3}, -1\right)$
- **Gradient at (-2, 2, 1):**
  $\nabla f(-2, 2, 1) = \left(\dfrac{1}{4}, \dfrac{1}{2}, -1\right)$
- **Tangent Plane:**
  $\nabla f(-2,2,1) \cdot \begin{pmatrix} x+2 \\ y-2 \\ z-1 \end{pmatrix} = 0$
- **Equation:**
  $\dfrac{1}{4}(x+2) + \dfrac{1}{2}(y-2) - (z-1) = 0$
  $\dfrac{1}{4}x + \dfrac{1}{2}y - z = 0$

10.4 Theorems

Let $A \subseteq \mathbb{R}^n$, $x_0 \in A$, and $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. If f is differentiable at x_0, f is continuous at x_0.
Let $A \subseteq \mathbb{R}^n$, $x_0 \in A$, and $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. For $x = (x_1, \ldots, x_n)$:

a) $\dfrac{\partial f}{\partial x_i}$ exists for any i.

b) $\dfrac{\partial f}{\partial x_i}$ is continuous at x_0 for any i.

Then, f is differentiable at x_0.

Problem 9

Given $f(x, y) = e^{\left(\frac{1}{x^2+y^2}\right)}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$:

a) Show that f is differentiable at $(0, 0)$.

b) Prove the existence of the tangent plane at $(0, 0)$.

To show f is differentiable at $(0, 0)$, we need to check the limit:

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)x - \frac{\partial f}{\partial y}(0, 0)y}{\sqrt{x^2 + y^2}} = 0$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t\to 0} \frac{e^{\left(\frac{-1}{t^2}\right)}}{t} = 0 = \frac{\partial f}{\partial y}(0, 0)$$

Thus, f is differentiable at $(0, 0)$.

10.5 Properties

Let $A, B \subseteq \mathbb{R}^n$, $x_0 \in A$, and $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable functions.

1. $\lambda f(x)$ is differentiable at x_0, where $\lambda \in \mathbb{R}$. The derivative is given by:

$$D(\lambda f(x_0)) = \lambda D(f(x_0))$$

2. $f(x) + g(x)$ is differentiable at x_0. The derivative is given by:

$$D(f(x_0) + g(x_0)) = D(f(x_0)) + D(g(x_0))$$

3. $f(x) \cdot g(x)$ is differentiable at x_0. The derivative is given by:

$$D(f(x_0) \cdot g(x_0)) = D(f(x_0)) \cdot g(x_0) + f(x_0) \cdot D(g(x_0))$$

4. $\dfrac{f(x)}{g(x)}$ is differentiable at x_0, provided $g(x_0) \neq 0$. The derivative is given by:

$$D\left(\frac{f(x_0)}{g(x_0)}\right) = \frac{D(f(x_0)) \cdot g(x_0) - f(x_0) \cdot D(g(x_0))}{(g(x_0))^2}$$

10.6 Chain Rule

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^k$. If $x_0 \in \mathbb{R}^m$ and $f(x_0) \in \mathbb{R}^n$, then $(g \circ f)(x) = g(f(x))$ is differentiable at x_0. The derivative is given by:

$$D(g \circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0)$$

Example

Given $g(x, y) = (x^2 + y, y^2)$ and $F(u, v) = (u + v, u, v^2)$, find $D(F \circ g)\big|_{(1,1)}$.

$$\begin{aligned} g :&\mathbb{R}^2 \to \mathbb{R}^2 \\ f :&\mathbb{R}^2 \to \mathbb{R}^3 \end{aligned}$$

$$\begin{aligned} g(x, y) &= (x^2 + 1, y^2) \\ f(x, y) &= (u + v, u, v^2) \end{aligned}$$

$$(x, y) \to (g_1(x, y), g_2(x, y)) \to (f_1(g_1, g_2), f_2(g_1, g_2), f_3(g_1, g_2))$$

$$f(g_1(x, y), g_2(x, y)) = (x^2 + 1 + y^2, x^2 + 1, y^4) \to Df(x, y) = Jf(x, y)$$

$$Dg(x, y) = Jg(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

$$Df(u, v) = Jf(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$$

$$D(f\circ g)\big|_{(x,y)} = D(f(g(x,y)))\cdot Dg(x,y) = \begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 2v\end{pmatrix}\cdot\begin{pmatrix}2x & 0\\ 0 & 2y\end{pmatrix}$$

$$D(F\circ g)\big|_{(1,1)} = \begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 2\end{pmatrix}\cdot\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

$$D(F\circ g)\big|_{(1,1)} = \begin{pmatrix}2 & 2\\ 2 & 0\\ 0 & 4\end{pmatrix}$$

Example

$f(x,y) = \left(\tan\dfrac{y}{x} - x + y, \log\dfrac{y+1}{x}\right)$, $g(s,t) = \left(t\cdot\cos s, e^t, s-2t\right)$ and $h(u,v,w) = uv^2\cdot e^w$

$$F = h\circ g\circ f$$
$$DF(x,y) = D(h(g(f(x,y))))\cdot D(g(f(x,y)))\cdot D(f(x,y))$$

$$Df(x,y) = \begin{pmatrix} -\dfrac{sec^2\dfrac{y}{x}\cdot y}{x^2} - 1 & \dfrac{sec^2\dfrac{y}{x}}{x} + 1 \\ -\dfrac{1}{x} & \dfrac{1}{y+1}\end{pmatrix}$$

$$Dg(s,t) = \begin{pmatrix} -t\cdot\sin s & \cos s\\ 0 & e^t\\ 1 & -2\end{pmatrix}$$

$$Dh(u,v,w) = \begin{pmatrix} v^2\cdot e^w & 2uv\cdot e^w & uv^2\cdot e^w\end{pmatrix}$$

Example

Particle of mass m following a trajectory $s(t)\in\mathbb{R}^3$
According to Newton's law, the field force satisfies

$$F = -\nabla V \quad \text{with} \quad V = \text{potential function}$$

Prove that the total energy is constant $E(t) = E_k + E_p = c$
Following $s(t)$ we find $F(s(t)) = -\nabla V(s(t))$
Taking the arbitrary times t_1, t_2

$$\int_{t_1}^{t_2} F(s(t))\cdot d(s(t)) = -\int_{t_1}^{t_2}\nabla V(s(t))\cdot s'(t)\,dt \quad \text{(chain rule)}$$

$$-\left(V(s(t_2)) - V(s(t_1))\right)$$

Thanks to Newton's second law

$$F = m\,s''(t)$$

$$\int_{t_1}^{t_2} F(s(t))\cdot s'(t)\,dt = \int_{t_1}^{t_2} m\,s''(t)\cdot s'(t)\,dt = \left[\frac{m\|s'(t)\|^2}{2}\right]_{t_1}^{t_2}$$
$$= \frac{m\|s'(t_2)\|^2}{2} - \frac{m\|s'(t_1)\|^2}{2}$$

$$E(t) = \frac{m\|s'(t)\|^2}{2} + V(s(t))$$

Example

$$f(x,y)=\begin{cases}\dfrac{2xy}{2x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Partial Derivatives:

$$\frac{\partial f(x,y)}{\partial x}=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=0$$
$$\frac{\partial f(x,y)}{\partial y}=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=0$$

Differentiability at (0,0):

$$\lim_{(x,y)\to(0,0)}\frac{\dfrac{2xy}{2x^2+y^2}}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{2xy}{\sqrt{x^2+y^2}\,(2x^2+y^2)}$$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

$$\lim_{r\to 0}\frac{2\cos\theta\sin\theta}{r\left(2\cos^2\theta+\sin^2\theta\right)}=\infty \quad f \text{ is not differentiable at } (0,0)$$

Directional derivative following $v=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$:

$$\text{Using } D_vf(0,0)=\langle\nabla f(0,0),v\rangle=\left\langle(0,0)\,,\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle=0$$

$$D_vf(0,0)=\lim_{t\to 0}\frac{f(0,0)+tv)-f(0,0)}{t}=\lim_{t\to 0}\frac{f(tv)-f(0,0)}{t}=\lim_{t\to 0}\frac{\frac{2t^2}{2}}{\frac{2t^2}{2}+\frac{t^2}{2}}=\frac{2}{3}$$

Problem 17

Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

a) $u=f(x,y),\quad x=x(r,s,t),\quad y=y(r,s,t)$

$$(r,s,t)\to(x(r,s,t),y(r,s,t))\to f(x(r,s,t),y(r,s,t))$$
$$\mathbb{R}^3\to\mathbb{R}^2\to\mathbb{R}$$

$$D(f\circ g)(r,s,t)=D(f(g(r,s,t)))\cdot Dg(r,s,t)$$

$$D(f\circ g)(r,s,t)=Df(x(r,s,t),y(r,s,t))\cdot\begin{pmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial s}&\frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial s}&\frac{\partial y}{\partial t}\end{pmatrix}$$

$$D(f\circ g)(r,s,t)=\begin{pmatrix}\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial s}&\frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial s}&\frac{\partial y}{\partial t}\end{pmatrix}$$

$$D(f\circ g)(r,s,t)=\begin{pmatrix}\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}&\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}&\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\end{pmatrix}$$

$$\textbf{b)}\quad w = f(x,y,z), \quad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

$$(u,v) \to (x(u,v), y(u,v), z(u,v)) \to f(x(u,v), y(u,v), z(u,v))$$

$$\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}$$

$$D(f \circ g)(u,v) = D(f(g(u,v))) \cdot Dg(u,v)$$

$$D(f \circ g)(u,v) = Df(x(u,v), y(u,v), z(u,v)) \cdot \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

$$D(f \circ g)(u,v) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

$$D(f \circ g)(u,v) = \begin{pmatrix} \dfrac{\partial f}{\partial x}\Big|_{g(u,v)} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\Big|_{g(u,v)} \cdot \dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\Big|_{g(u,v)} \cdot \dfrac{\partial z}{\partial u} \\ \dfrac{\partial f}{\partial x}\Big|_{g(u,v)} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\Big|_{g(u,v)} \cdot \dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\Big|_{g(u,v)} \cdot \dfrac{\partial z}{\partial v} \end{pmatrix}^t$$

Problem 12

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Making the transformation to spherical coordinates, compute $\frac{\partial f}{\partial \rho}$, $\frac{\partial f}{\partial \theta}$, and $\frac{\partial f}{\partial \phi}$ in terms of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
The transformation to spherical coordinates is given by:

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

$$(\rho,\theta,\phi) \to (x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)) \to f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))$$

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \rho} = \frac{\partial f}{\partial x}\cdot\sin(\phi)\cos(\theta) + \frac{\partial f}{\partial y}\cdot\sin(\phi)\sin(\theta) + \frac{\partial f}{\partial z}\cdot\cos(\phi)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x}\cdot\rho\sin(\phi)\sin(\theta) + \frac{\partial f}{\partial y}\cdot\rho\sin(\phi)\cos(\theta)$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \phi} = \frac{\partial f}{\partial x}\cdot\rho\cos(\phi)\cos(\theta) + \frac{\partial f}{\partial y}\cdot\rho\cos(\phi)\sin(\theta) - \frac{\partial f}{\partial z}\cdot\rho\sin(\phi)$$

11 Linearization

$$\text{Recall: if } \exists \ \frac{\partial f}{\partial x}, \quad \text{then } \exists \ \frac{\partial f}{\partial y}$$

$$\lim_{(h_1,h_2)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}\big|_{(x_0,y_0)} h_1 - \frac{\partial f}{\partial y}\big|_{(x_0,y_0)} h_2}{\sqrt{h_1^2 + h_2^2}} = 0$$

With $h_1 = x - x_0$ and $h_2 = y - y_0$, we have:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}\big|_{(x_0,y_0)}(x - x_0) - \frac{\partial f}{\partial y}\big|_{(x_0,y_0)}(y - y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad \text{is differentiable at } x_0 \in \mathbb{R}^n$$

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - Df\big|_{x_0} \cdot h}{\|h\|} = 0$$

With $h \in \mathbb{R}^n$, $\quad h = t \cdot v,\ t \to 0 \implies h \to 0$

$$\lim_{t\to 0} \frac{f(x_0 + t\cdot v) - f(x_0) - Df\big|_{x_0} \cdot t \cdot v}{t} = 0$$

With $\|v\| = 1$ and $\|h\| = t$

11.1 Proposition

If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, U open in $\mathbb{R}^n$ is differentiable at $x_0 \in U$, then f is continuous at x_0, then all the directional derivatives exist at x_0.

$$D_v f(x_0) = Df \cdot v = Jf \cdot v$$

Proof

$$v \in \mathbb{R}^n, \quad D_v f = \lim_{t\to 0} \frac{f(x_0 + t\cdot v) - f(x_0)}{t} = \lim_{t\to 0} \frac{Df\big|_{x_0} \cdot t \cdot v}{t} = Df\big|_{x_0} \cdot v$$

11.2 Special cases

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \text{differentiable at } x_0, \text{ scalar function}$$

$$D_v f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}^t \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \frac{\partial f}{\partial x_1} \cdot v_1 + \ldots + \frac{\partial f}{\partial x_n} \cdot v_n = \langle \nabla f, v \rangle$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$D_{(1,0)} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}^t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial f}{\partial x}$$

$$D_{(0,1)} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}^t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial f}{\partial y}$$

$$D_{v_1,v_2}f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}^t \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\partial f}{\partial x} \cdot v_1 + \frac{\partial f}{\partial y} \cdot v_2$$

$$v_1 + v_2 = 1$$

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$D_{(1,0,\ldots,0)}f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}^t \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}$$

$$D_{(0,1,\ldots,0)}f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}^t \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \vdots \\ \frac{\partial f_m}{\partial x_2} \end{pmatrix}$$

11.3 Mean Value Theorem

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

In vectorial calculus, the mean value theorem is only for scalar functions.

Theorem Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be differentiable, and $[a, b] \subset U$. Then, $\forall x \in (a, b)$, $\exists c \in (a, b)$ such that

$$f(b) - f(a) = Df\big|_c \cdot (b - a) = \nabla f\big|_c \cdot (b - a)$$

Proof We need to use the chain rule:

$$(1 - t) \cdot a + t \cdot b, \quad t \in [0, 1], \quad [0, 1] \to U$$

$$g(t) = f((1 - t) \cdot a + t \cdot b)$$

We apply the mean value theorem for one variable in $[0, 1]$:

$$g(1) - g(0) = g'(t_0) \cdot (1 - 0)$$

$$g'(c) = Df\big|_{(1-t_0)\cdot a+t_0\cdot b} \cdot (b - a)$$

Let $c = (1 - t_0) \cdot a + t_0 \cdot b$, then:

$$f(b) - f(a) = Df\big|_c \cdot (b - a) = \nabla f\big|_c \cdot (b - a)$$

11.4 Theorem

If $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^m$ and all the partial derivatives exist and are continuous on a point a, then f is differentiable at a and $Df\big|_a = Jf\big|_a$.

Example

$$f(x,y) = \begin{cases} (x^4+y^4)\cdot\sin\left(\dfrac{1}{x^4+y^4}\right) & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Is f continuous at $(0,0)$?

$$\left|sin\left(\frac{1}{x^4+y^4}\right)\right| \leq 1, \quad \text{is bounded}$$

$$\implies \exists \lim_{(x,y)\to(0,0)} (x^4+y^4)\cdot\sin\left(\frac{1}{x^4+y^4}\right) = 0$$

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^4\cdot\sin\left(\frac{1}{h^4}\right)}{h} = 0$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{k^4\cdot\sin\left(\frac{1}{k^4}\right)}{k} = 0$$

$$\implies f \text{ is continuous at } (0,0)$$

Is f differentiable at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}\big|_{(0,0)}x - \frac{\partial f}{\partial y}\big|_{(0,0)}y}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^4+y^4)\cdot\sin\left(\frac{1}{x^4+y^4}\right) - 0 - 0\cdot x - 0\cdot y}{\sqrt{x^2+y^2}}, \quad \text{where } \sin\left(\frac{1}{x^4+y^4}\right) \text{ is bounded}$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^4+y^4)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^4}{\sqrt{x^2+y^2}} + \lim_{(x,y)\to(0,0)} \frac{y^4}{\sqrt{x^2+y^2}}$$

$$\frac{x^4}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}\cdot x^3 = 0, \quad \text{because } \left|\frac{x}{\sqrt{x^2+y^2}}\right| \leq 1$$

$$\frac{y^4}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}\cdot y^3 = 0, \quad \text{because } \left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq 1$$

$$\implies \lim_{(x,y)\to(0,0)} \frac{(x^4+y^4)}{\sqrt{x^2+y^2}} = 0 \implies f \text{ is differentiable at } (0,0)$$

11.5 Higher Order Partial Derivatives

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, then:

$$\begin{tikzcd}
 & & & \text{f(x,y)} \arrow[dll] \arrow[drr] & & & \\
 & \frac{\partial f}{\partial x}(x,y) \arrow[dl] \arrow[dr] & & & & \frac{\partial f}{\partial y}(x,y) \arrow[dl] \arrow[dr] & \\
\frac{\partial^2 f}{\partial x^2} = f_{xx} & & \frac{\partial^2 f}{\partial x \partial y} = f_{xy} & & \frac{\partial^2 f}{\partial y \partial x} = f_{yx} & & \frac{\partial^2 f}{\partial y^2} = f_{yy}
\end{tikzcd}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

Hessian Matrix

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

11.6 Partial Differential Equations (PDEs)

Problem 2

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \quad (x, y, z) \neq (0, 0, 0)$$

Where u satisfies the PDE:

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad \text{which is } \Delta u = 0, \quad \textbf{Laplace's equation}$$

$$u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$
$$u_x = -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x$$
$$u_{xx} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2x^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
$$u_{yy} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2y^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
$$u_{zz} = \frac{3}{4} \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2z^2 - \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$u_{xx} + u_{yy} + u_{zz} = \frac{6}{4} \left(3(x^2 + y^2 + z^2)^{-5} - (x^2 + y^2 + z^2)^{-\frac{3}{2}} \right) = 0$$

Problem 1

$$u_t = \alpha^2 u_{xx}, \quad \text{where } \alpha^2 \text{ is known as the diffusion coefficient in the } \textbf{heat equation}$$

$$u(x, t) = e^{-\alpha k^2 t} \cdot \sin(kx)$$
$$u_t = -\alpha k^2 e^{-\alpha k^2 t} \cdot \sin(kx)$$
$$u_x = e^{-\alpha k^2 t} \cdot k \cdot \cos(kx)$$
$$u_{xx} = -e^{-\alpha k^2 t} \cdot k^2 \cdot \sin(kx)$$

Problem 4c

$$u(x, t) = (x - at)^6 + (x + at)^6$$
$$u_{tt} = a^2 u_{xx} \quad \text{is known as the } \textbf{Wave equation}$$

11.7 Cross Derivatives

$$\text{When it is true that } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem: Equality of Crossed Partial Derivatives
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be defined on an open set U and such that the partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_j}$ exist and are differentiable at $a \in U$. Then, $\forall x_i, x_j \in U$, it is true that:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \quad \text{at } a$$

Example

$$f(x,y) = \begin{cases} xy \cdot \dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)\Big|_{(0,0)} \neq \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)\Big|_{(0,0)}$$

$$\frac{\partial f}{\partial x} = \begin{cases} \dfrac{4x^2y^3 + x^4y - y^5}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ \lim_{t\to 0} \dfrac{f(t,0) - f(0,0)}{t} = 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ \lim_{t\to 0} \dfrac{f(0,t) - f(0,0)}{t} = 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)} = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(t,0) - \frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t\to 0} \frac{\frac{h^5}{h^4} - 0}{h} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}\Big|_{(0,0)} = \lim_{t\to 0} \frac{\frac{\partial f}{\partial y}(0,t) - \frac{\partial f}{\partial y}(0,0)}{t} = \lim_{t\to 0} \frac{-\frac{h^5}{h^4} - 0}{h} = -1$$

11.8 Theorem

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be defined on an open set U and such that all the partial derivatives exist and are continuous at $a \in U$. Then, f is differentiable at a and its differential is given by the Jacobian matrix $Jf\big|_a$.

Proof

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Jf\big|_a \cdot h}{\|h\|} = 0$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$a + h = \begin{pmatrix} a_1 + h_1 \\ a_2 + h_2 \\ \vdots \\ a_n + h_n \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 + h_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 + h_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \ldots - \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n + h_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n + h_n \end{pmatrix}$$

$$f(a+h) = f\begin{pmatrix} a_1+h_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + \ldots - f\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix}$$

$$f(a+h)-f(a) = f\begin{pmatrix} a_1+h_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + \ldots - f\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} + f\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$f(a+h) - f(a) = D_{v_1}f \cdot h_1 + D_{v_2}f \cdot h_2 + \ldots + D_{v_n}f \cdot h_n$$

$$\text{Where } v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ is the } i\text{th unit vector}$$

$$f(a+h) - f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix} \cdot h_1 + \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \vdots \\ \frac{\partial f_m}{\partial x_2} \end{pmatrix} \cdot h_2 + \ldots + \begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{pmatrix} \cdot h_n$$

So:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{x_1} & \frac{\partial f_1}{\partial x_2}\Big|_{x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{x_n} \\ \frac{\partial f_2}{\partial x_1}\Big|_{x_1} & \frac{\partial f_2}{\partial x_2}\Big|_{x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\Big|_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_{x_1} & \frac{\partial f_m}{\partial x_2}\Big|_{x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_{x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

Finally:

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Jf\big|_a \cdot h}{\|h\|} = \frac{Jfh - Jf\big|_a h}{\|h\|} = \frac{(Jf - Jf\big|_a) \cdot h}{\|h\|} = 0$$

$$Jf\big| \to Jf\big|_a \quad \text{as } h \to 0$$

11.9 Proof of the Chain Rule

Let $U \subseteq \mathbb{R}^m, V \subseteq \mathbb{R}^n$ be open sets. Let $g : U \to V$ and $f : V \to \mathbb{R}^p$ be mappings, and let a be a point in U. If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and

$$D(f \circ g)(a) = Df(g(a)) \cdot Dg(a).$$

We define two "remainder" functions, r and s:

- r: difference between the increment to g and its linear approximation at a,
- s: difference between the increment to f and its linear approximation at $g(a)$.

We know that g is differentiable at a:

$$g(a+h) = g(a) + Dg|_a\, h + r(h),$$

where

$$r(h) = g(a+h) - g(a) - Dg|_a\, h.$$

This represents the increase to g relative to its linear approximation at a. Then,

$$\lim_{h\to 0} \frac{r(h)}{\|h\|} = 0.$$

We know that f is differentiable at $g(a)$:

$$f(g(a)+k) = f(g(a)) + Df|_{g(a)}\, k + s(k),$$

where

$$s(k) = f(g(a)+k) - f(g(a)) - Df|_{g(a)}\, k.$$

This represents the increase to f relative to its linear approximation at $g(a)$. Then,

$$\lim_{k\to 0} \frac{s(k)}{\|k\|} = 0,$$

we have:

$$f(g(a+h)) = f(g(a) + Dg|_a h + r(h)),$$

where we treat $Dg|_a h + r(h)$ as k.
Expanding further:

$$f(g(a+h)) = f(g(a)) + Df|_{g(a)}[Dg|_a h + r(h)] + s(Dg|_a h + r(h)),$$

$$= f(g(a)) + Df|_{g(a)} Dg|_a h + [Df|_{g(a)} r(h) + s(Dg|_a h + r(h))].$$

Subtract $f(g(a))$:

$$f(g(a+h)) - f(g(a)) = Df|_{g(a)} Dg|_a h + \text{remainder}.$$

Thus, the linear approximation of the composition is:

$$Df|_{g(a)} Dg|_a h,$$

with the remainder terms:

$$Df|_{g(a)} r(h) + s(Dg|_a h + r(h)).$$

Now, we need to prove that:

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a)) - Df|_{g(a)} Dg|_a h}{\|h\|} = 0,$$

since:

$$\lim_{h\to 0} \frac{Df|_{g(a)} r(h) + s(Dg|_a h + r(h))}{\|h\|} = 0.$$

For the first term:

$$\|Df|_{g(a)} r(h)\| \le \|Df|_{g(a)}\| \|r(h)\|,$$

and:

$$\lim_{h\to 0} \frac{\|Df|_{g(a)} r(h)\|}{\|h\|} \le \|Df|_{g(a)}\| \lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

For the second part:

$$\lim_{h\to 0}\frac{s(Dg|_a h + r(h))}{\|h\|} = 0.$$

There exists $\delta > 0$ such that $\|r(h)\| \le \|h\|$ when $\|h\| \le \delta$. Thus:

$$\|Dg|_a h + r(h)\| \le \|Dg|_a h\| + \|r(h)\| \le (\|Dg|_a\| + 1)\|h\|.$$

Also, for any $\varepsilon > 0$, there exists $0 < \delta' \le \delta$ such that when $\|k\| \le \delta'$:

$$\frac{\|s(h)\|}{\|k\|} \le \varepsilon.$$

$$\|Dg|_a h + r(h)\| \le \delta'$$

Substitute for $\|k\|$ in the equation:

$$\|s(k)\| \le \varepsilon\|k\|$$

$$\|s(Dg|_a h + r(h))\| \le \varepsilon\|Dg|_a h + r(h)\|$$

where $\|s(Dg|_a h + r(h))\| = s(k)$

$$s(k) \le \varepsilon\left(\|Dg|_a\| + 1\right)\|h\|$$

Divide by $\|h\|$:

$$\frac{\|s(Dg|_a h + r(h))\|}{\|h\|} \le \varepsilon\left(\|Dg|_a\| + 1\right)$$

since this is true for all $\varepsilon > 0$, we have:

$$\lim_{h\to 0}\frac{\|s(Dg|_a h + r(h))\|}{\|h\|} = 0$$

$$\lim_{h\to 0}\frac{Df|_{g(a)}r(h) + s(Dg|_a h + r(h))}{\|h\|} = 0$$

$$\lim_{h\to 0}\frac{f(g(a+h)) - f(g(a)) - Df|_{g(a)}Dg|_a h}{\|h\|} = 0$$

$$Df|_{g(a)}Dg|_a = D(f\circ g)|_a$$

11.10 Induced Norm

A matrix A is a linear operator, because

$$A(x + \alpha y) = A(x) + \alpha A(y)$$

The induced norm of a matrix A is defined as:

$$\|A\| = \sup_{\|x\|=1}\|A(x)\|$$

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \implies \|A\| = \sqrt{\rho(AA^t)}$$

where ρ is the spectral radius of the matrix AA^t. The spectral radius is defined as the maximum eigenvalue of the matrix AA^t.

11.11 Inverse Function Theorem for One Variable

Let $f : [a, b] \to [c, d]$ be a continuous function. $f(a) = c$ and $f(b) = d$. If f is strictly increasing or decreasing in $[a, b]$, then f is invertible and there exists g : [c,d] → [a,b] such that $g(f(x)) = x$ and $f(g(y)) = y$.
If f is differentiable at $x_0 \in (a, b)$ and $f'(x_0) \neq 0$, then f is invertible in a neighborhood of x_0 and the inverse function f^{-1} is differentiable at $f(x_0)$:

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x)$$

11.12 Inverse Function Theorem for Higher Dimensions

11.12.1 Preliminary Definitions

Lipschitz Condition A function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ differentiable in U satisfies the Lipschitz condition in a subset $V \subset U$ if there exists a constant known as the Lipschitz ratio $M > 0$ for all $x, y \in V$:

$$\forall x, y \in V, \quad \exists M > 0 \text{ such that } \|Df(x) - Df(y)\| \leq M\|x - y\|$$

Example

$$f(x, y) = \begin{pmatrix} x - y^2 & x^2 + y \end{pmatrix}$$

$$Df = \begin{pmatrix} 1 & -2y \\ 2x & 1 \end{pmatrix}$$

$$Df\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2y_1 \\ 2x_1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2y_2 \\ 2x_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2(y_1 - y_2) \\ 2(x_1 - x_2) & 0 \end{pmatrix}$$

$$A_{ij}\Big|_{1\leq j\leq m}^{1\leq i\leq n}, \quad |A| = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}^2\right)^{\frac{1}{2}}$$

$$\left|Df\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| = 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 2x - y$$

Example

$$f(x, y) = \begin{pmatrix} x - y^3 \\ x^3 - y \end{pmatrix}$$

$$Df = \begin{pmatrix} 1 & -3y^2 \\ 3x^2 & -1 \end{pmatrix}$$

$$Df\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -3y_1^2 \\ 3x_1^2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -3y_2^2 \\ 3x_2^2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -3(y_1^2 - y_2^2) \\ 3(x_1^2 - x_2^2) & 0 \end{pmatrix}$$

$$\left|Df\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - Df\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| = 3\sqrt{(x_1^2 - x_2^2)^2 + (y_1^2 - y_2^2)^2}$$

If $(y_1 + y_2)^2 \leq A$ and $(x_1 + x_2)^2 \leq A$, then: $M = 3A$.

Best Constant for the Lipschitz Condition The best constant for the Lipschitz condition M^* is defined as:

$$M^* = \inf M$$

If we pick $x \in U \subset V$, $y \in U \subset V \implies x, y \in V$.

$$\|Df(x) - Df(y)\| \leq M_V \|x - y\| \implies M_U < M_V$$

$$B_{R_1}(x_0, x_1) \subset B_{R_2}(x_0, x_1) \implies M_{R_1} < M_{R_2}$$

11.12.2 Inverse Function Theorem for Higher Dimensions

Given a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and an image y, we need to obtain an x such that $f(x) = y$. If y is fixed, then we say x is a root of the function $f_y(x) = f(x) - y = 0$.

Theorem If f is continuously differentiable in an open set $U \subset \mathbb{R}^n$ and $Df(x)$ is invertible in $x_0 \in U$, then f is locally invertible, with differentiable inverse in some neighborhood of $f(x_0)$.

$$Df\big|_{x_0}, \quad \exists (Df\big|_{x_0})^{-1}$$
$$D(f^{-1}) = (Df)^{-1}$$

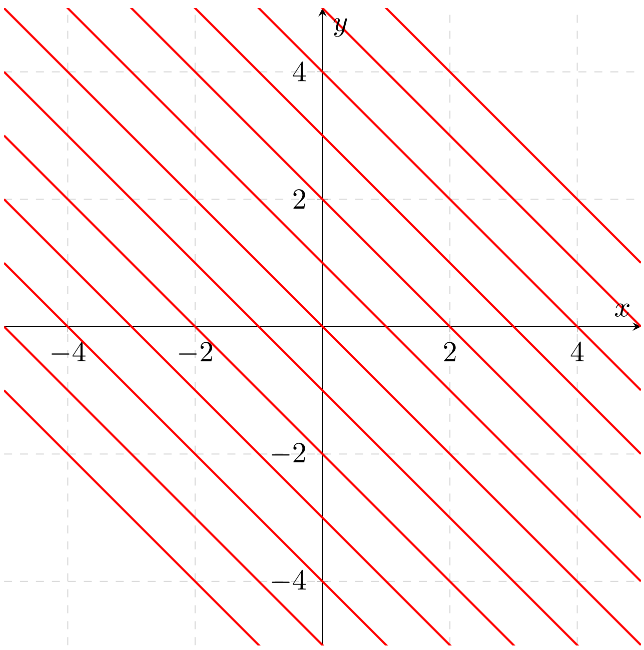
Example

$$f(x, y) = \begin{pmatrix} \sin(x + y) & x^2 - y^2 \end{pmatrix}$$
$$Df = \begin{pmatrix} \cos(x + y) & \cos(x + y) \\ 2x & -2y \end{pmatrix}$$

If $det(Df)\big|_{(x_0,y_0)} \neq 0$, then f is invertible at (x_0, y_0) and there exists $(Df\big|_{(x_0,y_0)})^{-1}$

$$det(Df\big|_{(x_0,y_0)}) = \cos(x_0 + y_0) \cdot (-2y_0) - 2x_0 \cdot \cos(x_0 + y_0) = -2\cos(x_0 + y_0)(x_0 + y_0) \neq 0$$

$$\implies x_0 + y_0 \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$
$$x_0 + y_0 \neq 0$$



The function f is invertible in the region $x + y \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, $x + y \neq 0$, anywhere outside the red lines.

11.13 Newton Method

$$x_{n+1} = x_n - (Df\big|_{x_n})^{-1} f(x_n)$$
$$x_n \to \alpha, \quad f(\alpha) = 0, \quad \text{where } \alpha \text{ is the solution of the application}$$

11.14 Theorem (Kontorovich, Nobel Prize Economics 1975)

Let a_0 be a point in $\mathbb{R}^n$, $\tilde{U}$ be an open neighborhood of a_0, and $f : \tilde{U} \to \mathbb{R}^n$ be a continuously differentiable function with its derivative $Df\big|_{a_0}$ invertible. Define

$$h_0 = -(Df\big|_{a_0})^{-1} f(a_0), \quad a_1 = a_0 + h_0$$

Let $U_0 = B(a_1)$. If $U_0 \subset U$, and the derivative Df satisfies the Lipschitz condition:

$$\|Df(u_1) - Df(u_2)\| \leq M\|u_1 - u_2\|, \quad \forall u_1, u_2 \in U$$

and

$$\|f(a_0)\| \cdot \|(Df\big|_{a_0})^{-1}\|^2 \cdot M \leq \frac{1}{2}$$

is satisfied, the equation $f(x) = 0$ has a unique solution in U.

11.15 Inverse Function Theorem

Let $W \subset \mathbb{R}^n$ be an open neighborhood of x_0 and $f : W \to \mathbb{R}^n$ be a continuously differentiable function. If $Df(x_0)$ is invertible, then f is invertible, with a continuously differentiable inverse in a neighborhood of $f(x_0)$.
To clarify this statement, we will specify the radius R of a ball V centered at x_0, in which the inverse of the function f is defined.
Setting $L = Df\big|_{x_0}$, now we will define the following conditions:

1. The ball W_0 of radius $2R\|L^{-1}\|$ centered at x_0 is contained in W.
2. In the ball W_0, the derivstive of f satisfies the Lipschitz condition:

$$\|Df(u) - Df(v)\| \leq \frac{1}{2R\|L^{-1}\|^2}\|u - v\|, \quad \forall u, v \in W_0$$

Then,

1. There exists a unique continuous differentiable function $g : V \to W$ such that $f(g(y)) = y$ for all $y \in V$. By the chain rule,

$$Df(g(y)) \cdot Dg(y) = I \implies Dg(y) = (Df(g(y)))^{-1}$$

2. The image of g contains the ball of radius R_1 centered at x_0:

$$R_1 = 2R\|L^{-1}\|^2 \left(\sqrt{\|L\|^2 + \frac{1}{\|L^{-1}\|^2}} - \|L\| \right)$$

Example

$$f(x) = 2x + \sin x, \quad f : \mathbb{R} \to \mathbb{R}$$
$$f : [-k\pi, k\pi] \to [-2k\pi, 2k\pi]$$

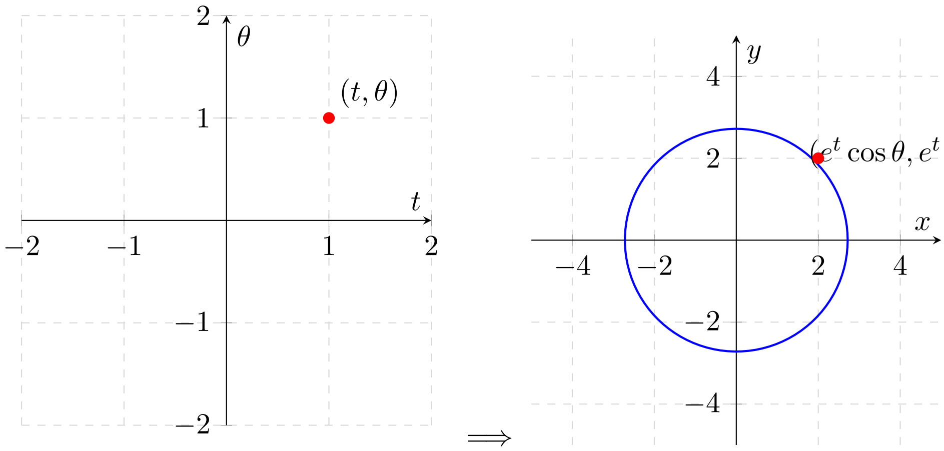
$$f(-k\pi) = -2k\pi, \quad f(k\pi) = 2k\pi$$
$$f'(x) = 2 + \cos x \neq 0, \quad \forall x \in \mathbb{R}$$

Then, f is invertible in $\mathbb{R}$ and f^{-1} is differentiable, with derivative:

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{2 + \cos x}$$

Example

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (t, \theta) \to (e^t \cos\theta, e^t \sin\theta)$$



$$Df = \begin{pmatrix} e^t \cos\theta & -e^t \sin\theta \\ e^t \sin\theta & e^t \cos\theta \end{pmatrix}$$

As $det(Df) = e^{2t} \neq 0$, f does not have a global inverse, but it does have a local inverse.

Example

$$f(x) = x^3, \quad \text{at } x_0 = 0$$
$$f'(x) = 3x^2, \quad f'(0) = 0$$
$$f^{-1}(y) = \sqrt[3]{y}$$

Is f^{-1} differentiable at $y_0 = 0$?

$$(f^{-1})'(y) = \frac{1}{3} y^{-\frac{2}{3}}, \quad (f^{-1})'(0) = \infty \text{ (does not exist)}$$

11.16 Kontorovich Theorem

$$h_0 = -(Df\big|_{a_0})^{-1} f(a_0), \quad a_1 = a_0 + h_0$$
$$\|f(a_0)\| \cdot \|(Df\big|_{a_0})^{-1}\|^2 \cdot M \leq \frac{1}{2}$$
$$Df\big|_{a_0} \text{ is invertible and } = L$$
$$M < \frac{1}{2} \cdot \frac{1}{\|f(a_0)\| \cdot \|L^2\|}$$

$(Df)^{-1}$ exists in x_0 and $\exists\, x \in U_0$ such that $f(x) = 0$.

11.16.1 Proof

Given $y \in V$, where $V = V_R(f(x_0))$, we want to find $x \in U$ such that $f(x) = y$ once y is fixed.

$$f_y(x) = f(x) - y, \quad x \text{ is a root of } f_y(x) = 0$$

We apply Kontorovich's theorem to $f_y(x)$:

$$f_y(x) = f(x) - y = y_0 - y$$

$$\|f_y(x_0)\| = \|y_0 - y\| < R$$

$$Df_y(x_0) = Df\big|_{x_0}, \quad \text{take care, } f_y(x) \text{ depends on } x.$$

$$\|f_y(x_0)\| \cdot \|(Df_y(x_0))^{-1}\|^2 \cdot M < \frac{1}{2}$$

$$\|f_y(x_0)\| \cdot \|(Df_y(x_0))^{-1}\|^2 \cdot M = \|y_0 - y\| \cdot L^2 \cdot M < \frac{1}{2}$$

$$M < \frac{1}{2RL^2}$$

$$f_y(x) = f(x) - y, \quad Df_y = Df$$

Lipschitz constant for $f_y(x)$ and $f(x)$:

$$\|Df_y(u) - Df_y(v)\| = \|Df(u) - Df(v)\| \leq M\|u - v\|$$

Let M_R be the least Lipschitz constant

$$\text{If } M < M_R, \text{ then } M < \frac{1}{2RL^2}$$

$$h_0 = -(Df\big|_{x_0})^{-1} f(x_0)$$

$$f_y(x) = f(x) - y = 0$$

We have found x such that $f(x) = y$.

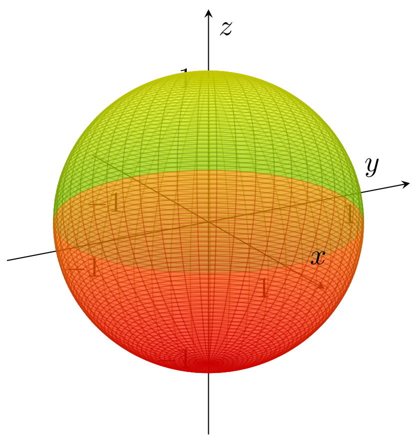
$$g(y) = x \implies f(g(y)) = y$$

11.17 Implicit Function Theorem

Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, in this case for example:

$$(x, y, z) \to x^2 + y^2 + z^2 - 1$$

$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$



We look for a zero of f:

$$f(x_0, y_0, z_0) = 0, \quad x_0^2 + y_0^2 + z_0^2 - 1 = 0$$

$$z_0 = \pm\sqrt{1 - x_0^2 - y_0^2}$$

$$(x_c, y_c) \in \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

$$z_c = \sqrt{1 - x_c^2 - y_c^2}$$

$$f : \mathbb{R}^{2+1} \to \mathbb{R}^1, \quad n = 1, \quad m = 2$$

Where z_c is the passive variable and (x_c, y_c) are the active variables. In general we will have n passive variables and m active variables.

$$\begin{cases} F_1(x_1, \ldots, x_n, \ldots, x_{n+m}) = 0 \\ F_2(x_1, \ldots, x_n, \ldots, x_{n+m}) = 0 \\ \vdots \\ F_n(x_1, \ldots, x_n, \ldots, x_{n+m}) = 0 \end{cases} \qquad \text{Non-linear system with } n \text{ equations and } n+m \text{ variables}$$

11.17.1 Implicit Function Theorem (short version)

Let $U \subset \mathbb{R}^{n+m}$ be an open set and $F : U \to \mathbb{R}^n$ be a continuously differentiable function such that $F(c) = 0$ and $DF(c)$ is onto for some $c \in U$. Then, the system of linear equations

$$DF(c) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

has n passive variables and m active variables. There exists a neighborhood of c in which $F = 0$ implicitly defines n of the active variables as functions of the passive variables.

11.17.2 Implicit Function Theorem Continued

Let W be an open neighborhood of $\mathbb{R}^{n+m}$ and $F : W \to \mathbb{R}^n$ be a continuously differentiable function such that $F(c) = 0$. If $DF(c)$ is onto, then it has n passive columns (arranging the variables so that the passive variables are first), that is $c = \begin{pmatrix} a \\ b \end{pmatrix}$ when the entries of the corresponding to then n passive variables are a and the entries corresponding to the m active variables are b.
Then, there exists a unique continuously differentiable function g from a neighborhood of b to a neighborhood of a that expresses the first n passive variables as functions of the last m active variables.
To quantify this statement, we can define the radius R of a ball V centered at b in which the g is defined. First note that the $n \times n$ matrix

$$\begin{pmatrix} D_1F(c) & D_2F(c) & \ldots & D_nF(c) \end{pmatrix}$$

representing the first n columns of $DF(c)$ is invertible.

$$L = \left(\begin{array}{cccc|cccc}
\dfrac{\partial f_1}{x_1} & \dfrac{\partial f_1}{x_2} & \cdots & \dfrac{\partial f_1}{x_n} & \dfrac{\partial f_1}{x_{n+1}} & \cdots & \dfrac{\partial f_1}{x_{n+m}} & \\
\dfrac{\partial f_2}{x_1} & \dfrac{\partial f_2}{x_2} & \cdots & \dfrac{\partial f_2}{x_n} & \dfrac{\partial f_2}{x_{n+1}} & \cdots & \dfrac{\partial f_2}{x_{n+m}} & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
\dfrac{\partial f_n}{x_1} & \dfrac{\partial f_n}{x_2} & \cdots & \dfrac{\partial f_n}{x_n} & \dfrac{\partial f_n}{x_{n+1}} & \cdots & \dfrac{\partial f_n}{x_{n+m}} & \\
\hline
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right)$$

$$DF(c) \text{ is onto } \implies DF(c) : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

Given $w \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^{n+m}$ such that $DF(c)v = w$.
This is satisified if and only if $DF(c)$ has n linearly independent columns.
By convention, $D_1F(c)$, $D_2F(c)$, …, $D_nF(c)$ are the columns of $DF(c)$ corresponding to the passive variables and are linearly independent.
We only have to prove that there are x_1, x_2, …, x_n passive variables because we have changed the order of the columns.
Let

$$c = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{where } a \text{ and } b \text{ are the passive and active variables, respectively}$$

and let

$$\tilde{F} \text{ be an extension of } F \text{ in the image}$$

$$\tilde{F} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}, \quad \text{to apply the inverse function theorem}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\tilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x,y) \\ y \end{pmatrix}$$

$$D\tilde{F} = \left(\begin{array}{cccc|cccc}
D_1F & D_2F & \cdots & D_nF & D_{n+1}F & D_{n+2}F & \cdots & D_{n+m}F \\
\hline
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right)$$

We need $D\tilde{F}$ to be Lipschitz. If DF is Lipschitz, then $D\tilde{F}$ is Lipschitz.

$$\|D\tilde{F}(u) - D\tilde{F}(v)\| \leq M\|u - v\|$$

$$\left(\begin{array}{ccc|ccc} D_1F(u)-D_1F(v) & \dots & D_nF(u)-D_nF(v) & D_{n+1}F(u)-D_{n+1}F(v) & \dots & D_{n+m}F(u)-D_{n+m}F(v) \\ \hline 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{array}\right)$$

Now we find a number $R > 0$ satisfying the following hypotheses:

1. The ball W_0 of radius $2R\|L^{-1}\|$ centered at c is contained in W.
2. In the ball W_0, the derivative of $\tilde{F}$ satisfies the Lipschitz condition:

$$\|F(u) - F(v)\| \leq \frac{1}{2R\|L^{-1}\|^2}\|u - v\|$$

Then, there exists a unique continuously differentiable function g from a neighborhood of b to a neighborhood of a that expresses the first n passive variables as functions of the last m active variables.

$$g : B_R(b) \to B_R(a)$$

such that $g(b) = a$ and $F\begin{pmatrix} g(y) \\ y \end{pmatrix} = 0$ for all $y \in B_R(b)$.

By the chain rule, the derivative of this implicit function g at a point b is given by:

$$Dg(b) = -\begin{bmatrix} D_1F(c) & D_2F(c) & \dots & D_nF(c) \end{bmatrix}^{-1}\begin{bmatrix} D_{n+1}F(c) & D_{n+2}F(c) & \dots & D_{n+m}F(c) \end{bmatrix}$$

Taking into account all the previous results,

$$\tilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x,y) \\ y \end{pmatrix}, \quad \tilde{F}(c) = \tilde{F}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} F(a,b) \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\implies \exists (D\tilde{F})^{-1} \implies \text{we can apply the inverse function theorem}$$

$$\tilde{F}\left(\tilde{G}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$$

We restrict $\tilde{G}$ to $\begin{pmatrix} 0 \\ y \end{pmatrix}$

$$\tilde{F}\begin{pmatrix} g(y) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \tilde{F}\begin{pmatrix} g(y) \\ y \end{pmatrix} = \begin{pmatrix} F(g(y),y) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$\tilde{F}\left(\tilde{G}\begin{pmatrix} 0 \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

Example: Unit Circle

$$F : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \to x^2 + y^2 - 1$$
$$F(x,y) = x^2 + y^2 - 1 = 0$$

$$DF = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

We need to find a $c = (x_0, y_0)$ such that $F(c) = 0$ and $DF(c)$ is onto.

$$y = \pm\sqrt{1 - x^2}$$

In this case we can obtain $g(x)$ explicitly.
x will be the active variable and y will be the passive variable.

$$DF\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2x_0 & 2y_0 \end{pmatrix}$$

$$\begin{cases} 2x_0 \neq 0 & \text{if } y > 0 \\ 2y_0 \neq 0 & \text{if } x > 0 \end{cases}$$

Is DF onto?

$$\begin{pmatrix} 2x & 2y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = w, \quad w \in \mathbb{R}$$

Does there exist a solution $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $DF\begin{pmatrix} x \\ y \end{pmatrix} = w$?

$$2x_0x + 2y_0y = w$$

Example

$$\begin{cases} x^2 - y = a \\ y^2 - z = b \\ z^2 - x = 0 \end{cases}$$

$$DF = \begin{pmatrix} 2x & -1 & 0 & -1 & 0 \\ 0 & 2y & -1 & 0 & -1 \\ -1 & 0 & 2z & 0 & 0 \end{pmatrix}$$

$$c = (0,0,0,0,0), \quad DF(c) = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \left(\begin{array}{ccc|cc} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left\| DF\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - D\tilde{F}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| =$$

$$= \left\| \left(\begin{array}{ccc|cc} 2x_1 & -1 & 0 & -1 & 0 \\ 0 & 2y_1 & -1 & 0 & -1 \\ -1 & 0 & 2z_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) - \left(\begin{array}{ccc|cc} 2x_2 & -1 & 0 & -1 & 0 \\ 0 & 2y_2 & -1 & 0 & -1 \\ -1 & 0 & 2z_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \right\| =$$

$$= \left\| \left(\begin{array}{ccc|cc} 2(x_1 - x_2) & 0 & 0 & 0 & 0 \\ 0 & 2(y_1 - y_2) & 0 & 0 & 0 \\ 0 & 0 & 2(z_1 - z_2) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \right\|$$

$$\left\| DF\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - D\tilde{F}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| = 2\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} = 2\left\| \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\| \implies M = 2$$

Problem 5.20

Are the following functions locally invertible with differentiable inverse?

(a) $F(x,y) = (x^2y, -2x) \quad$ at $(1,1)$

(b) $F(x,y,z) = (xyz, x^2, z^2) \quad$ at $(0,0,0)$

(c) $F(x,y,z) = (xyz, x^2, z^2) \quad$ at $(1,1,1)$

For (a), we have $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $F(x,y) = (x^2y, -2x)$. We need to calculate M and L.

$$DF = \begin{pmatrix} 2xy & x^2 \\ -2 & 0 \end{pmatrix}, \quad DF\big|_{(1,1)} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}, \quad \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} = 2 \neq 0$$

So F is differentiable at $(1,1)$ and DF is invertible:

$$\left(DF\big|_{(1,1)}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

By the inverse function theorem, we take $G = F^{-1}$ and $DG = \left(DF\big|_{(1,1)}\right)^{-1}$.

$$DG\big|_{F(1,1)} = DG\big|_{(1,-2)} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

as $F(1,1) = (1,-2)$.

$$R < \frac{1}{2\|L^{-1}\|M}, \quad \text{where } L = DF\big|_{(1,1)}$$

We calculate M by taking all the second derivatives of F:

$$2xy\begin{cases} 2y = \dfrac{\partial}{dx}(2xy) \\ 2x = \dfrac{\partial}{dy}(2xy) \end{cases}, \quad x^2\begin{cases} 2x = \dfrac{\partial}{dx}(x^2) \\ 0 = \dfrac{\partial}{dy}(x^2) \end{cases}, \quad -2\begin{cases} 0 = \dfrac{\partial}{dx}(-2) \\ 0 = \dfrac{\partial}{dy}(-2) \end{cases}, \quad 0\begin{cases} 0 = \dfrac{\partial}{dx}(0) \\ 0 = \dfrac{\partial}{dy}(0) \end{cases}$$

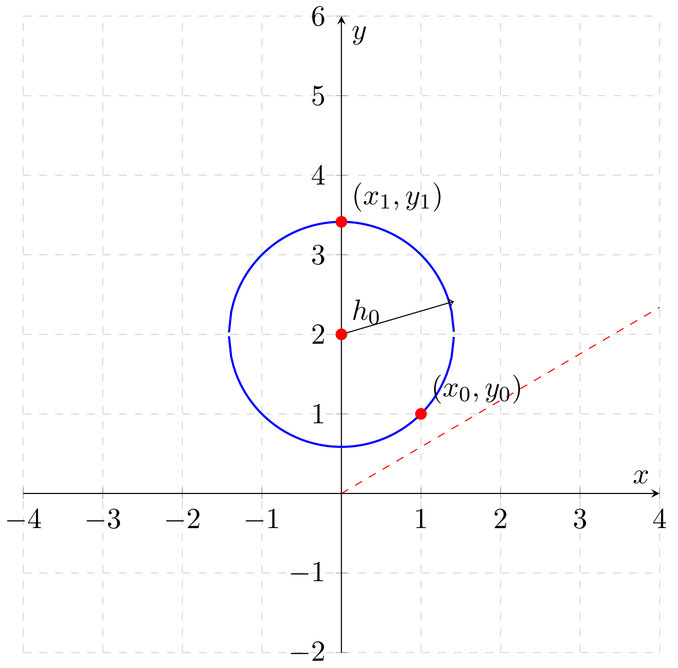
$$(2y)^2 + (2x)^2 + 2x^2 \leq 8(x^2+y^2)$$

$$h_0 = -\left(DF\big|_{(1,1)}\right)^{-1} F(1,1) = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(x_1, y_1) = (x_0, y_0) + (-1, 1) = (1,1) + (-1,1) = (0,2)$$

so we obtain M:

$$M = \sqrt{8(x^2+y^2)} = \sqrt{8(0^2+2^2)} = \sqrt{8}(2+\sqrt{2}) = 4\sqrt{2}+4$$



For (c), we have $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $F(x, y, z) = (xyz, x^2, z^2)$. We need to calculate M and L.

$$DF = \begin{pmatrix} yz & xz & xy \\ 2x & 0 & 0 \\ 0 & 0 & 2z \end{pmatrix}, \quad DF\big|_{(1,1,1)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -4 \neq 0$$

So F is differentiable at $(1, 1, 1)$ and DF is invertible:

$$\left(DF\big|_{(1,1,1)}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

By the inverse function theorem, we take $G = F^{-1}$ and $DG = \left(DF\big|_{(1,1,1)}\right)^{-1}$.

$$DG\big|_{F(1,1,1)} = DG\big|_{(1,1,1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

as $F(1, 1, 1) = (1, 1, 1)$.

$$R < \frac{1}{2\|L^{-1}\|M}, \quad \text{where } L = DF\big|_{(1,1,1)}$$

We calculate M by taking all the second derivatives of F:

$$yz \begin{cases} z = \dfrac{\partial}{dx}(yz) \\ y = \dfrac{\partial}{dy}(yz) \\ 0 = \dfrac{\partial}{dz}(yz) \end{cases}, \quad xz \begin{cases} z = \dfrac{\partial}{dx}(xz) \\ 0 = \dfrac{\partial}{dy}(xz) \\ x = \dfrac{\partial}{dz}(xz) \end{cases}, \quad xy \begin{cases} y = \dfrac{\partial}{dx}(xy) \\ x = \dfrac{\partial}{dy}(xy) \\ 0 = \dfrac{\partial}{dz}(xy) \end{cases}$$

$$2x \begin{cases} 2 = \dfrac{\partial}{dx}(x^2) \\ 0 = \dfrac{\partial}{dy}(x^2) \\ 0 = \dfrac{\partial}{dz}(x^2) \end{cases}, \quad 0 \begin{cases} 0 = \dfrac{\partial}{dx}(0) \\ 0 = \dfrac{\partial}{dy}(0) \\ 0 = \dfrac{\partial}{dz}(0) \end{cases}, \quad 2z \begin{cases} 0 = \dfrac{\partial}{dx}(z^2) \\ 0 = \dfrac{\partial}{dy}(z^2) \\ 2z = \dfrac{\partial}{dz}(z^2) \end{cases}$$

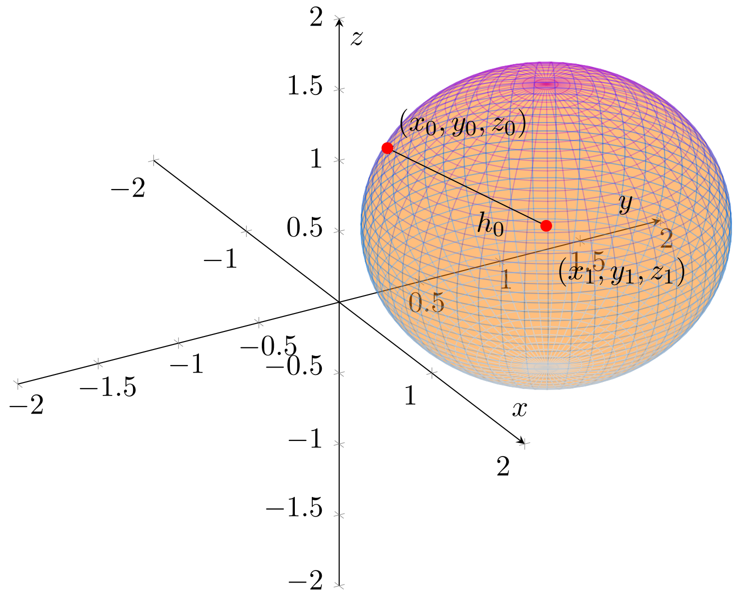
$$(yz)^2 + (xz)^2 + (xy)^2 + 2x^2 + 2z^2 \leq 8(x^2 + y^2 + z^2)$$

$$h_0 = -\left(DF\big|_{(1,1,1)}\right)^{-1} F(1,1,1) = -\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$(x_1, y_1, z_1) = (1,1,1) + (-\frac{1}{2}, 0, -\frac{1}{2}) = (\frac{1}{2}, 1, \frac{1}{2})$$

so we obtain M:

$$M = \sqrt{8(x^2 + y^2 + z^2)} = \sqrt{8(0^2 + 2^2 + 1^2)} = \sqrt{8}(3) = 6\sqrt{2}$$



11.18 Curl and Divergence

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable function of class $\mathcal{C}'$ (all first derivatives arer continuous). The curl of F is defined as:

$$\operatorname{curl}(F) = \nabla \times F = \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1(x,y,z) & F_2(x,y,z) & F_3(x,y,z) \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{pmatrix}$$

where $\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{pmatrix}$ is the gradient operator.
The divergence of F is defined as:

$$\operatorname{div}(F) = \nabla \cdot F = \begin{pmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

12 Manifolds

A curve is a manifold of dimension 1. A surface is a manifold of dimension 2. A manifold of dimension n is a set that locally looks like $\mathbb{R}^n$.

$$M \subset \mathbb{R}^n, \quad M \text{ locally is a graph of } f : \mathbb{R}^k \to \mathbb{R}^{n-k}$$

The graph is defined as:

$$\Gamma(f) = \{(x, f(x)) : x \in \mathbb{R}^k, f(x) \in \mathbb{R}^{n-k}\} \subset \mathbb{R}^n$$

$$\begin{tikzcd}
 & \textbf{Equation} \arrow[r] & x^2+y^2+z^2=9 \quad F(x,y,z)=0 \text{ (implicit function theorem)} \\
\textbf{Manifolds} \arrow[ur] \arrow[dr] & & \\
 & \textbf{Parametrization} \arrow[r] & \begin{pmatrix} x(u,v) & y(u,v) & z(u,v) \end{pmatrix} = \begin{pmatrix} 3\cos(u)\sin(v) \\ 3\sin(u)\sin(v) \\ 3\cos(v) \end{pmatrix}
\end{tikzcd}$$

12.1 Smooth Manifolds in $\mathbb{R}^n$

The graph $\Gamma(f)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the set of pairs (x, y) such that $y = f(x)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that $y = f(x)$.
For example:

(A) (x, x^2) is a smooth graphic.

(B) $(x, |x|)$ is not smooth at $(0, 0)$.

(C) $(x, x^{\frac{1}{3}})$ is a smooth manifold.

Example:

$$x^2 + y^2 = 1, \quad \text{Is it locally a manifold?}$$
$$(\cos\theta, \sin\theta) \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

12.2 Surface

A smooth surface $S \subset \mathbb{R}^3$ is a manifold of dimension 2 in $\mathbb{R}^3$ if it is locally a graph of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^3$ expressing one variable as a function of the other two.

Example:

$$(a, b, c) \in S$$
$$I = (a - \varepsilon, a + \epsilon) \quad J = (b - \delta, b + \delta) \quad K = (c - \gamma, c + \gamma)$$

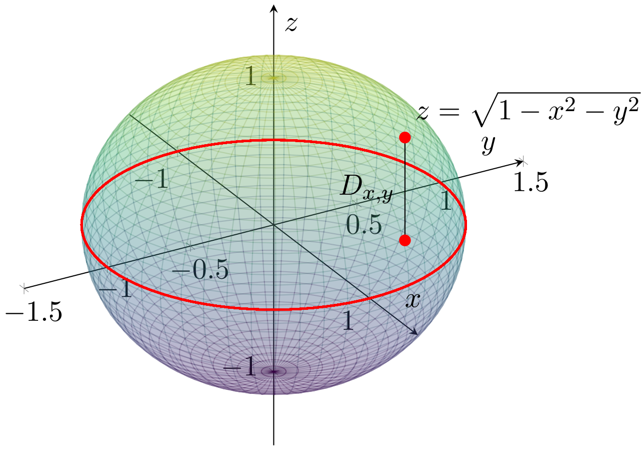
$$U = I \times J \times K \text{ is open in } \mathbb{R}^3$$
$$f : I \times J \to K, \quad (x, y, f(x, y))$$

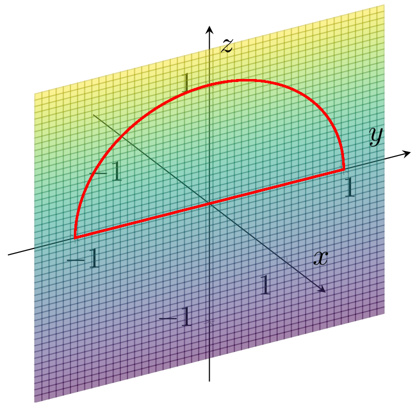
$$g : I \times K \to J, \quad (x, g(x,z), z)$$
$$h : J \times K \to I, \quad (g(y,z), y, z)$$

The functions f, g, and h are obtained by the implicit function theorem.

Example:

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$
$$D_{x,y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$





It is a graph of a function $f : D_{x,y} \to \mathbb{R}_z^+$, where $f(x,y) = \sqrt{1 - x^2 - y^2}$.

$$(x, y, f(x,y)) \in S^2 \cap (D_{x,y} \times \mathbb{R}_z^+) \subset \mathbb{R}^3$$

12.3 Using the Implicit Function Theorem to identify smooth manifolds

Is the locus defined by $x^8 + 2x^3 + y + y^5 = 1$ a smooth manifold?

Theorem: Let $M \subset \mathbb{R}^n$ be a subset, $U \subset \mathbb{R}^n$ an open set, and $F : U \to \mathbb{R}^{n-k}$ a continuously differentiable function such that $M \cap U$ is the set of solutions of $F(z) = 0$. If $DF(c)$ is onto for every $z \in M \cap U$, then M is a smooth manifold of dimension k in $\mathbb{R}^n$. Conversely, if M is a smooth manifold of dimension k in $\mathbb{R}^n$, then every point $z \in M$ has a neighborhood U such that there exists a continuously differentiable function $F : U \to \mathbb{R}^{n-k}$ with $DF(z)$ onto and $M \cap U$ is the set of solutions of $F(z) = 0$.
Following the theorem, we apply the implicit function theorem to the function $F(x,y) = x^8 + 2x^3 + y + y^5 - 1$.

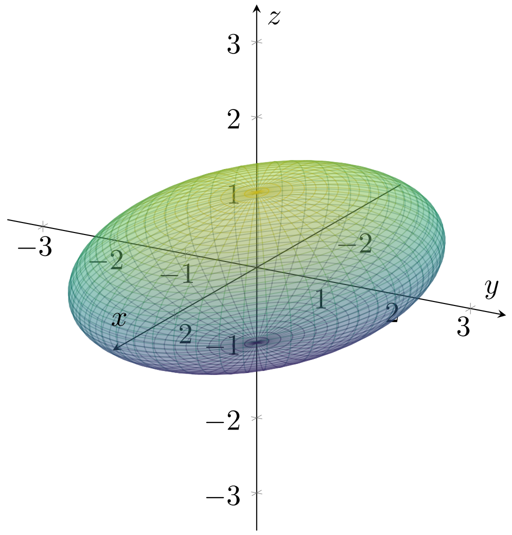
Exercise 6.1b

$$x = 3\cos\theta\sin\phi, \quad y = 3\sin\theta\sin\phi, \quad z = 3\cos\phi$$
$$\frac{x}{3} = \cos\theta\sin\phi, \quad \frac{y}{2} = \sin\theta\sin\phi, \quad z = \cos\phi$$

So we have:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1$$

which is the equation of an ellipsoid.

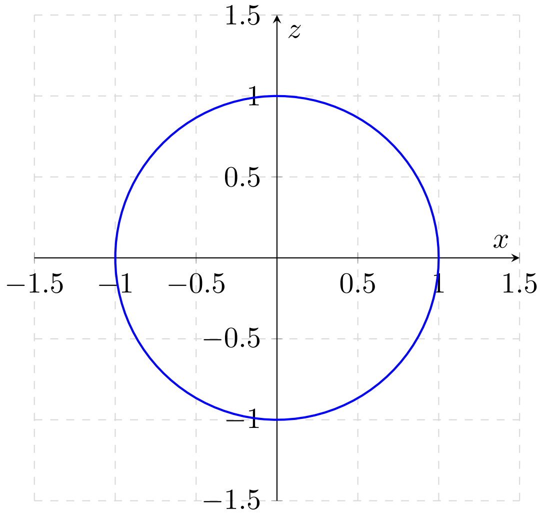


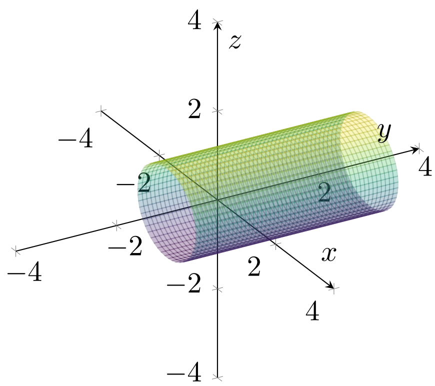
Exercise 6.1c

$$x = \sin v, \quad y = u, \quad z = \cos v$$

where $-1 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.

$$x^2 + z^2 = 1$$



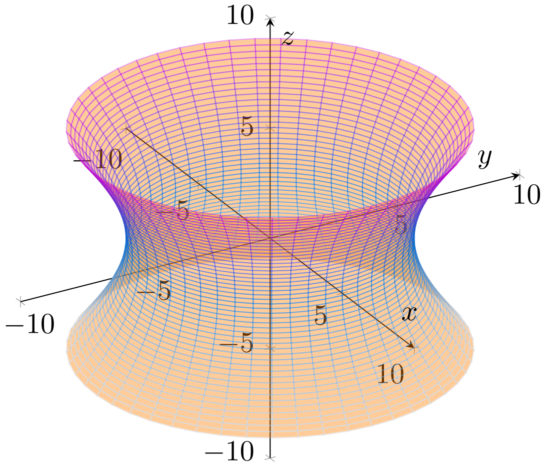


Example

$$x^2 + y^2 - z^2 = 25 \quad \text{is a hyperbola.}$$

For $z = 0$ we have $x^2 + y^2 = 25$ which is a circle.

For $z \neq 0$ we have $x^2 + y^2 = 25 + z^2$



$$F(x,y,z) = (x^2+y^2-z^2-25), \quad F:\mathbb{R}^3 \to \mathbb{R}$$

$$F(x,y,z)=0, \quad DF = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix}, \quad DF\big|_{(x_0,y_0,z_0)} = \begin{pmatrix} 2x_0 & 2y_0 & -2z_0 \end{pmatrix}$$

The only way for DF to not be onto is if $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$. In this case, we have $F(0,0,0) = -25$ and $DF(0,0,0) = (0,0,0)$ which is not onto. So the only point where F is not a smooth manifold is at the origin.

$$(x^2+y^2) = 25\cosh^2(\theta) \quad \text{and } z = 5\sinh(\theta)$$

$$x^2+y^2-z^2 = 25\cosh^2(\theta) - 25\sinh^2(\theta) = 25(\cosh^2(\theta)-\sinh^2(\theta)) = 25$$

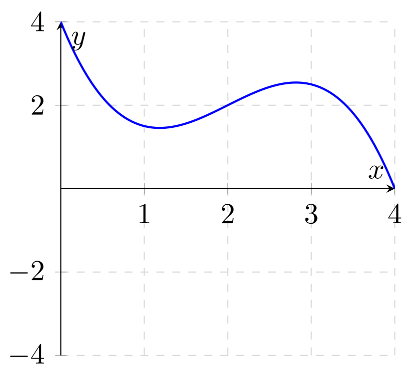
$$\begin{cases} x = 5\cosh(\theta)\cos(\phi) \\ y = 5\cosh(\theta)\sin(\phi) \\ z = 5\sinh(\theta) \end{cases}$$

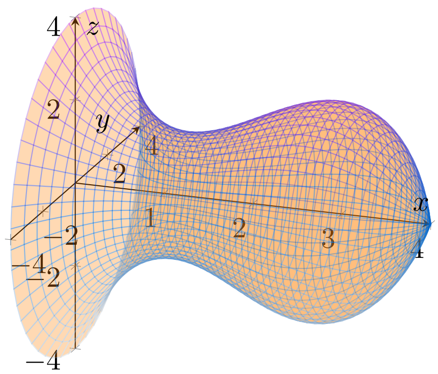
Exercise 6.5

Parametrization of a donut (torus):

$$x(\theta,\phi) = (R+\cos(\phi))\cos(\theta), \quad y(\theta,\phi) = (R+\cos(\phi))\sin(\theta), \quad z(\theta,\phi) = \sin(\phi)$$

Surface of revolution: A surface of revolution is a surface generated by rotating a curve around an axis. The curve is called the generating curve and the axis is called the axis of revolution. The surface of revolution can be described by a parametrization that depends on the angle of rotation and the distance from the axis of revolution.

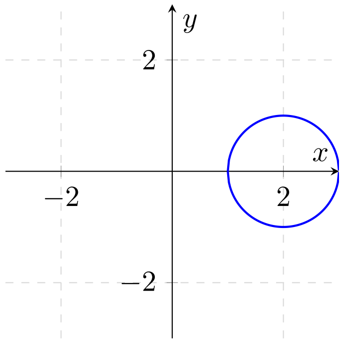


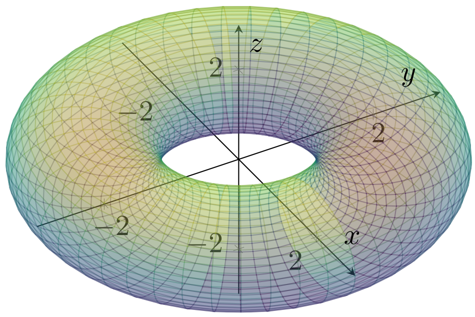


With $z = \sin\phi$, $\quad R + \cos\phi = R \pm \sqrt{1 - z^2}$, we have:

$$\begin{cases} x = (R \pm \sqrt{1-z^2}) \cos\theta \\ y = (R \pm \sqrt{1-z^2}) \sin\theta \\ z = z, \quad \text{where } -1 < z < 1 \end{cases}$$

If $\theta = 0$ we have $(x - R)^2 + z^2 = 1$ which is a circle.





Exercise 6.10

$$F_1 : x^2 + y^3 + z = a^2 \quad \text{is a smooth surface.}$$

$$\text{and also } F_2 : x + y + z = b$$

$$DF_2 = \begin{pmatrix} 2x & 3y^2 & 1 \end{pmatrix}, \quad DF_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

Is the intersection of the two surfaces a smooth curve?

$$F_n = F_1 \cap F_2 : \mathbb{R}^3 \to \mathbb{R}^2$$

$$DF_n = \begin{pmatrix} 2x & 3y^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad DF_n\big|_{(x_0,y_0,z_0)} = \begin{pmatrix} 2x_0 & 3y_0^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We observe that the rank of DF_n is 2, so we have two passive variables and one active variable. The intersection of the two surfaces is a smooth curve.
For $3y^2 = 1$ and $2x = 1$, we have:

$$DF_n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{which is not onto.}$$

Then we look at the points where $x = \frac{1}{2}, y = \frac{1}{\sqrt{3}}$ and $x = \frac{1}{2}, y = -\frac{1}{\sqrt{3}}$.

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^3 + z = a^2 \quad \text{and} \quad \frac{1}{2} + \frac{1}{\sqrt{3}} + z = b$$

In this points:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{3}} \\ z \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{3}} \\ z \end{pmatrix}$$

$x + y + z = b$ is the plane tangent to $x^2 + y^3 + z = a^2$, which is a surface. The intersection of the two surfaces is a curve.

13 Tangent Space

Let $M \subset \mathbb{R}^n$ be a smooth manifold of dimension k so that near $z \in M$, M is a graph of a function $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$, where f is continuously differentiable. This function expresses $n-k$ variables as a function of k variables.
Then, the tangent space to the manifold at $a \in M$, denoted by T_aM, is the graph of $DF\big|_a$.

Example

Consider a function

$$F : \mathbb{R}^n \to \mathbb{R}^{n-k}, \quad c = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a \in \mathbb{R}^{n-k}, b \in \mathbb{R}^k$$

By the implicit function theorem, we have:

$$F(c) = 0, \quad DF(c) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{n-k}}{\partial x_1} & \cdots & \frac{\partial F_{n-k}}{\partial x_n} \end{pmatrix} \text{ is onto}$$

$$\implies \exists\, g \text{ such that } F\begin{pmatrix} g(y) \\ y \end{pmatrix} = 0 \implies x - a = Dg\big|_b \cdot (y - b)$$

Example

$$\begin{aligned} (a, f(a)) \qquad & y = f(a) + f'(a)(x - a) \\ (g(t), t) \qquad & x = g(b) + g'(b)(y - b) \end{aligned}$$

Example

$$S = \{(x, y, f(x, y)) : (x, y) \in U \subset \mathbb{R}^2\}$$

$$z = f(x, y) \qquad Df = \nabla f = z - z_0 = \nabla f\big|_{(x_0, y_0)} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$z = z_0 + \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0, y_0)}(y - y_0)$$

13.1 Theorem

If $f(z) = 0$ describes a manifold M and if $DF(c)$ is onto for every $c \in M$, then the tangent space T_cM is the set of solutions of the linear system $DF(c) \cdot (x - c) = 0$, i.e. the tangent space is the kernel of the differential $DF(c)$ at the point c.

$$T_cM = \ker DF(c)$$

Example

$$DF(c)\cdot\begin{bmatrix} x-c_1 \\ x_2-c_2 \\ \vdots \\ x_{n+m}-c_{n+m}\end{bmatrix}=$$

$$DF,\quad \text{if } F:\mathbb{R}^{n+m}\to\mathbb{R}^m,\quad DF=\begin{bmatrix} D_1F & D_2F & \cdots & D_{n+m}F \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}\begin{bmatrix} x_1-c_1 \\ x_2-c_2 \\ \vdots \\ x_{n+m}-c_{n+m}\end{bmatrix}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

Example

The curve is a locus defined at $(1,1)$ by the equation $x^9+2x^3+y+y^5=5$.

$$F:\mathbb{R}^2\to\mathbb{R},\quad DF=\begin{pmatrix}9x^8+6x^2 & 1+5y^4\end{pmatrix}$$

$$\text{As } 1+5y^4\neq 0 \implies y(x)$$

$$\text{The tangent space is the set of solutions of } DF\big|(1,1)\cdot\begin{pmatrix}x-1\\y-1\end{pmatrix}=0$$

$$DF\big|(1,1)=\begin{pmatrix}15 & 6\end{pmatrix}\implies 15(x-1)+6(y-1)=0\implies y=-\frac{5}{2}(x-1)+1$$

Example

The paraboloid $z=x^2+y^2$

$$F:\mathbb{R}^3\to\mathbb{R},\quad F(x,y,z)=z-x^2-y^2$$

$$F(x_0,y_0,z_0)=0\quad\text{and } DF=\begin{pmatrix}-2x & -2y & 1\end{pmatrix}$$

$$DF\big|_{(x_0,y_0,z_0)}=\begin{pmatrix}-2x_0 & -2y_0 & 1\end{pmatrix}\quad z_0=x_0^2+y_0^2$$

$$-2x_0(x-x_0)-2y_0(y-y_0)+(z-z_0)=0$$

14 Taylor Polynomial

Let $U\subset\mathbb{R}^n$ be an open subset and $f:U\to\mathbb{R}$ a continuously differentiable function. The Taylor polynomial of degree 2 at the point (x_0,y_0) is given by:

$$P(x_0+h_1,y_0+h_2)=f(x_0,y_0)+\left\langle \nabla f\big|(x_0,y_0),\begin{pmatrix}h_1\\h_2\end{pmatrix}\right\rangle+\frac{1}{2}\left\langle H\big|(x_0,y_0)\begin{pmatrix}h_1\\h_2\end{pmatrix},\begin{pmatrix}h_1\\h_2\end{pmatrix}\right\rangle$$

where H is the Hessian matrix of f at the point (x_0,y_0), and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$.

$$H=\begin{pmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2}\end{pmatrix}\quad\text{is the Hessian matrix of } f$$

$$P(x_0+h_1,y_0+h_2)=f(x_0,y_0)+\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}h_1+\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}h_2+$$

$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)}h_1^2+\frac{\partial^2 f}{\partial x\partial y}\Big|_{(x_0,y_0)}h_1h_2+\frac{\partial^2 f}{\partial y\partial x}\Big|_{(x_0,y_0)}h_1h_2+\frac{\partial^2 f}{\partial y^2}\Big|_{(x_0,y_0)}h_2^2\right)$$

14.1 Theorem

Let $U \subset \mathbb{R}^2$, $(x_0, y_0) \in U$, and $f : U \to \mathbb{R}$ be a continuously differentiable function. Then:

1. The Taylor polynomial is the unique polynomial of degree 2 with the same partial derivatives up to order 2 at the point (x_0, y_0) of the function f.

2. The Taylor polynomial is the best approximation of the function f at the point (x_0, y_0) in the sense that:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(x_0+h_1, y_0+h_2) - P(x_0+h_1, y_0+h_2)|}{h_1^2 + h_2^2} = 0$$