The Universe´s Stability for Models with more than one Higgs Doublet

September 2022

1 2HDM-Two Higgs Doublet Model:

Considering a Model with Two Doiublets we have:

$$\Phi_1 = \begin{bmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{bmatrix}, \Phi_2 = \begin{bmatrix} \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{bmatrix} \tag{1}$$

Applying a Gauge Symmetry, we can transform the doublets into a more simplified version:

$$\Phi_1 = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 \\ \varphi_3 + i\varphi_4 \end{bmatrix} \tag{2}$$

Using the parameterization:

$$\begin{cases} \varphi_1 = \cos\theta \\ \varphi_2 = \sin\theta\cos\alpha \\ \varphi_3 = \sin\theta\sin\alpha\cos\beta \\ \varphi_4 = \sin\theta\sin\alpha\sin\beta\cos\gamma \end{cases}$$

we build the Higgs Potential for the 2HDM:

$$\begin{aligned} V_H = m_{11}^2\Phi_1^\dagger\Phi_1 + m_{22}^2\Phi_2^\dagger\Phi_2 - (m_{12}^2\Phi_1^\dagger\Phi_2 + H.c) + \frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) \\ +\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + [\frac{\lambda_5}{2}(\Phi_1^\dagger\Phi_2)^2 + \lambda_6(\Phi_1^\dagger\Phi_1)(\Phi_1^\dagger\Phi_2) + \lambda_7(\Phi_2^\dagger\Phi_2)(\Phi_1^\dagger\Phi_2) + h.c] \end{aligned} \tag{3}$$

with $\lambda_5, \lambda_6, \lambda_7 \in \mathbb{C}$

2 3HDM-Three Higgs Doublet Model:

Considering:

$$\Phi_1 = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 \\ \varphi_3 + i\varphi_4 \end{bmatrix} and \Phi_3 = \begin{bmatrix} \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{bmatrix} \tag{4}$$

and the parameterization:

$$\begin{cases} \varphi_1 = \cos\theta \\ \varphi_2 = \sin\theta\cos\alpha \\ \varphi_3 = \sin\theta\sin\alpha\cos\beta \\ \varphi_4 = \sin\theta\sin\alpha\sin\beta\cos\gamma \\ \varphi_5 = \sin\theta\sin\alpha\sin\beta\sin\gamma\cos\delta \\ \varphi_6 = \sin\theta\sin\alpha\sin\beta\sin\gamma\sin\delta\cos\epsilon \\ \varphi_7 = \sin\theta\sin\alpha\sin\beta\sin\gamma\sin\delta\sin\epsilon\cos\eta \\ \varphi_8 = \sin\theta\sin\alpha\sin\beta\sin\gamma\sin\delta\sin\epsilon\sin\eta \end{cases}$$

we can build the Higgs Potential for the 3HDM:

$$V_4 = \frac{\lambda_1}{2}|\Phi_1|^4 + \frac{\lambda_2}{2}|\Phi_2|^4 + \frac{\lambda_3}{2}|\Phi_3|^4 + \lambda_4|\Phi_1|^2|\Phi_2|^2 + \lambda_5|\Phi_1|^2|\Phi_3|^2 + \lambda_6|\Phi_2|^2|\Phi_3|^2 + \lambda_7|\Phi_1^\dagger\Phi_2|^2$$
$$+\lambda_8|\Phi_1^\dagger\Phi_3|^2 + \lambda_9|\Phi_2^\dagger\Phi_3|^2 + [\frac{\lambda_{10}}{2}(\Phi_1^\dagger\Phi_2)^2 + \frac{\lambda_{11}}{2}(\Phi_1^\dagger\Phi_3)^2 + \frac{\lambda_{12}}{2}(\Phi_2^\dagger\Phi_3)^2 + h.c] + [\lambda_{13}(\Phi_1^\dagger\Phi_2)|\Phi_1|^2$$
$$+\lambda_{14}(\Phi_1^\dagger\Phi_2)|\Phi_2|^2 + \lambda_{15}(\Phi_1^\dagger\Phi_2)|\Phi_3|^2 + \lambda_{16}(\Phi_1^\dagger\Phi_3)|\Phi_1|^2 + \lambda_{17}(\Phi_1^\dagger\Phi_3)|\Phi_2|^2 + \lambda_{18}(\Phi_1^\dagger\Phi_3)|\Phi_3|^2$$
$$+\lambda_{19}(\Phi_2^\dagger\Phi_3)|\Phi_1|^2 + \lambda_{20}(\Phi_2^\dagger\Phi_3)|\Phi_2|^2 + \lambda_{21}(\Phi_2^\dagger\Phi_3)|\Phi_3|^2 + h.c] + [\lambda_{22}(\Phi_1^\dagger\Phi_2)(\Phi_1^\dagger\Phi_3) + \lambda_{23}(\Phi_2^\dagger\Phi_3)(\Phi_2^\dagger\Phi_1)$$
$$+\lambda_{24}(\Phi_3^\dagger\Phi_1)(\Phi_3^\dagger\Phi_2) + \lambda_{25}(\Phi_1^\dagger\Phi_2)(\Phi_3^\dagger\Phi_1) + \lambda_{26}(\Phi_2^\dagger\Phi_3)(\Phi_1^\dagger\Phi_2) + \lambda_{27}(\Phi_3^\dagger\Phi_1)(\Phi_2^\dagger\Phi_3) + h.c].$$

If we truncate this potential, such that it turns into a quadratic form, we can derive necessary and sufficient analytical conditions for BFB:

$$V_4 = \frac{\lambda_1}{2}|\Phi_1|^4 + \frac{\lambda_2}{2}|\Phi_2|^4 + \frac{\lambda_3}{2}|\Phi_3|^4 + \lambda_4|\Phi_1|^2|\Phi_2|^2 + \lambda_5|\Phi_1|^2|\Phi_3|^2 + \lambda_6|\Phi_2|^2|\Phi_3|^2 + \lambda_7|\Phi_1^\dagger\Phi_2|^2$$
$$+ \lambda_8|\Phi_1^\dagger\Phi_3|^2 + \lambda_9|\Phi_2^\dagger\Phi_3|^2 + [\frac{\lambda_{10}}{2}(\Phi_1^\dagger\Phi_2)^2 + \frac{\lambda_{11}}{2}(\Phi_1^\dagger\Phi_3)^2 + \frac{\lambda_{12}}{2}(\Phi_2^\dagger\Phi_3)^2 + h.c]$$

Following Kannike, we parameterize the above potential with:

$$\begin{cases} |\Phi_i|^2 = h_i^2 \\ \Phi_i^\dagger\Phi_j = h_i h_j \rho_{ij} e^{i\phi_{ij}} \end{cases} \qquad \text{where } \rho_{ij}\epsilon[0,1] \text{ and } \phi_{ij}\epsilon[0,2\pi]$$

So we have:

$$\begin{aligned} V_4 = \frac{\lambda_1}{2}h_1^4 + \frac{\lambda_2}{2}h_2^4 + \frac{\lambda_3}{2}h_3^4 + \lambda_4 h_1^2h_2^2 + \lambda_5 h_1^2h_3^2 + \lambda_6 h_2^2h_3^2 + \lambda_7\rho_{12}h_1^2h_2^2 \\ + \lambda_8\rho_{13}h_1^2h_3^2 + \lambda_9\rho_{23}h_2^2h_3^2 + |\lambda_{10}|cos(2\phi_{12}+\phi_{10})\rho_{12}h_1^2h_2^2 \\ + |\lambda_{11}|cos(2\phi_{13}+\phi_{11})\rho_{13}h_1^2h_3^2 + |\lambda_{12}|cos(2\phi_{23}+\phi_{12})\rho_{23}h_2^2h_3^2 \end{aligned}$$

Defining:

$\lambda_{ijk} = \lambda_i + \rho_{jk}^2[\lambda_j + |\lambda_k| cos(2\phi_{jk} + \phi_l)]$
where ϕ_l is the phase of the complex coupling constant
and choosing the base:

$$b = \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{bmatrix} \tag{5}$$

We can write the potential in the following quadratic form:

$$V_4 = b^T \frac{1}{2} \Lambda b, \tag{6}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_{4710} & \lambda_{5811} \\ \lambda_{4710} & \lambda_2 & \lambda_{6912} \\ \lambda_{5811} & \lambda_{6912} & \lambda_3 \end{bmatrix} \tag{7}$$

For the potential to be BFB, the Λ matrix must be copositive, so we require that each of all the principal minors of Λ be copositive and that $|\Lambda| > 0$.

The principal minors are:

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & \lambda_{4710} \\ \lambda_{4710} & \lambda_2 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} \lambda_1 & \lambda_{5811} \\ \lambda_{5811} & \lambda_3 \end{bmatrix}, \Lambda_3 = \begin{bmatrix} \lambda_2 & \lambda_{6912} \\ \lambda_{6912} & \lambda_3 \end{bmatrix} \tag{8}$$

Trivially, we need to have $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$.

The other conditions can be derived, by ensuring that: $|\Lambda_1| > 0$, $|\Lambda_2| > 0$ and $|\Lambda_3| > 0$, for all possible values of λ_{ijk}.

If $\lambda_{ijk} > 0$, the matrix is trivially copositive; so we only consider the case $\lambda_{ijk} < 0$.

For a matrix of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \tag{9}$$

$$|A| = a_{11}a_{22} - a_{12}^2, \tag{10}$$

So, to ensure that $|\Lambda_{1,2,3} > 0|$ we need to find the minimum value of λ_{ijk}, which implies that $cos(2\phi_{jk} + \phi_l) = -1$.

So we have: $\lambda_{ijk} = \lambda_i + \rho_{jk}^2(\lambda_j - |\lambda_k|)$, for all $\rho_{jk} \epsilon [0, 1]$.

Clearly we have 2 possible cases for the minimum value: $\begin{cases} \lambda_j - |\lambda_k| >= 0 \implies \lambda_{ijk_{min}} = \lambda_i, \\ \lambda_j - |\lambda_k| < 0 \implies \lambda_{ijk_{min}} = \lambda_i + \lambda_j - \lambda_k \end{cases}$

Knowing this, we define: $\lambda_{4710m} = min(\lambda_4, \lambda_4 + \lambda_7 - |\lambda_{10}|)$, $\lambda_{5811m} = min(\lambda_5, \lambda_5 + \lambda_8 - |\lambda_{11}|)$, $\lambda_{6912m} = min(\lambda_6, \lambda_6 + \lambda_9 - |\lambda_{12}|)$.

So we can derive the following BFB conditions that are analogous to those of the 2HDM model:
$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0,$

$\lambda_{4710m} + \sqrt{\lambda_1\lambda_2} > 0$,
$\lambda_{5811m} + \sqrt{\lambda_1\lambda_3} > 0$,
$\lambda_{6912m} + \sqrt{\lambda_2\lambda_3} > 0$.

We also need an additional condition to ensure that $|\Lambda| > 0$, however, according to Kannike, for a general matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{11}$$

The condition $|A| > 0$ is equivalent to:
$\sqrt{a_{11}a_{22}a_{33}} + a_{12}\sqrt{a_{33}} + a_{13}\sqrt{a_{22}} + a_{23}\sqrt{a_{11}} + \sqrt{2(a_{12} + \sqrt{a_{11}a_{22}})(a_{13} + \sqrt{a_{11}a_{33}})(a_{23} + \sqrt{a_{22}a_{33}})} > 0$.

So we can use this result to write: $\sqrt{\lambda_1\lambda_2\lambda_3} + \lambda_{4710m}\sqrt{\lambda_3} + \lambda_{5811m}\sqrt{\lambda_2} + \lambda_{6912m}\sqrt{\lambda_1} + \sqrt{2(\lambda_{4710m} + \sqrt{\lambda_1\lambda_2})(\lambda_{5811m} + \sqrt{\lambda_1\lambda_3})(\lambda_{6912m} + \sqrt{\lambda_2\lambda_3})} > 0$

The bounded from bellow conditions are then:
$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$;
$\lambda_{4710m} + \sqrt{\lambda_1\lambda_2} > 0$;
$\lambda_{5811m} + \sqrt{\lambda_1\lambda_3} > 0$;
$\lambda_{6912m} + \sqrt{\lambda_2\lambda_3} > 0$ and
$\sqrt{\lambda_1\lambda_2\lambda_3} + \lambda_{4710m}\sqrt{\lambda_3} + \lambda_{5811m}\sqrt{\lambda_2} + \lambda_{6912m}\sqrt{\lambda_1} + \sqrt{2(\lambda_{4710m} + \sqrt{\lambda_1\lambda_2})(\lambda_{5811m} + \sqrt{\lambda_1\lambda_3})(\lambda_{6912m} + \sqrt{\lambda_2\lambda_3})} > 0$.

These are sufficient BFB conditions for this model.

2.1 $U(1) \times U(1)$

We know that $V = V_2 + V_N + V_{CB}$ where:

$V_2 = m_{11}^2(\Phi_1^\dagger\Phi_1) + m_{22}^2(\Phi_2^\dagger\Phi_2) + m_{33}^2(\Phi_3^\dagger\Phi_3)$,

$V_N = \frac{\lambda_{11}}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_{22}}{2}(\Phi_2^\dagger\Phi_2)^2 + \frac{\lambda_{33}}{2}(\Phi_3^\dagger\Phi_3)^2 + \lambda_{12}(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_{13}(\Phi_1^\dagger\Phi_1)(\Phi_3^\dagger\Phi_3) + \lambda_{23}(\Phi_2^\dagger\Phi_2)(\Phi_3^\dagger\Phi_3)$,

$V_{CB} = \lambda_{12}^{'}z_{12} + \lambda_{13}^{'}z_{13} + \lambda_{23}^{'}z_{23}$

2.2 $U(1) \times Z_2$

We know that $V_4 = V_N + V_{CB} + V_{U(1)\times Z_2}$ where:

$V_{U(1)\times Z_2} = \frac{1}{2}[\bar{\lambda}_{12}(\Phi_1^\dagger\Phi_2)^2 + h.c.]$

2.3 $Z_2 \times Z_2$

We know that $V = V_2 + V_N + V_{CB} + V_{Z_2 \times Z_2}$ where:

$V_{Z_2 \times Z_2} = \frac{1}{2}[\bar{\lambda}_{12}(\Phi_1^\dagger \Phi_2)^2 + \bar{\lambda}_{13}(\Phi_1^\dagger \Phi_3)^2 + \bar{\lambda}_{23}(\Phi_2^\dagger \Phi_3)^2 + h.c]$

$\bar{\lambda}_{ij}$ can be either real or complex