

The Universe's Stability for Models with more than one Higgs Doublet

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1 2HDM-Two Higgs Doublet Model:

Considering a Model with Two Doublets we have:

$$\Phi_1 = \begin{bmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{bmatrix}, \Phi_2 = \begin{bmatrix} \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{bmatrix} \quad (1)$$

Applying a Gauge Symmetry, we can transform the doublets into a more simplified version:

$$\Phi_1 = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 \\ \varphi_3 + i\varphi_4 \end{bmatrix} \quad (2)$$

Using the parameterization:

$$\begin{cases} \varphi_1 = \cos \theta \\ \varphi_2 = \sin \theta \cos \alpha \\ \varphi_3 = \sin \theta \sin \alpha \cos \beta \\ \varphi_4 = \sin \theta \sin \alpha \sin \beta \cos \gamma \end{cases}$$

we build the Higgs Potential for the 2HDM:

$$\begin{aligned} V_H = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - (m_{12}^2 \Phi_1^\dagger \Phi_2 + H.c.) + \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) \\ + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + [\frac{\lambda_5}{2} (\Phi_1^\dagger \Phi_2)^2 + \lambda_6 (\Phi_1^\dagger \Phi_1) (\Phi_1^\dagger \Phi_2) + \lambda_7 (\Phi_2^\dagger \Phi_2) (\Phi_1^\dagger \Phi_2) + h.c.] \end{aligned} \quad (3)$$

with $\lambda_5, \lambda_6, \lambda_7 \in \mathbb{C}$

2 3HDM-Three Higgs Doublet Model:

Considering:

$$\Phi_1 = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 \\ \varphi_3 + i\varphi_4 \end{bmatrix} \text{ and } \Phi_3 = \begin{bmatrix} \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{bmatrix} \quad (4)$$

and the parameterization:

$$\begin{cases} \varphi_1 = \cos \theta \\ \varphi_2 = \sin \theta \cos \alpha \\ \varphi_3 = \sin \theta \sin \alpha \cos \beta \\ \varphi_4 = \sin \theta \sin \alpha \sin \beta \cos \gamma \\ \varphi_5 = \sin \theta \sin \alpha \sin \beta \sin \gamma \cos \delta \\ \varphi_6 = \sin \theta \sin \alpha \sin \beta \sin \gamma \sin \delta \cos \epsilon \\ \varphi_7 = \sin \theta \sin \alpha \sin \beta \sin \gamma \sin \delta \sin \epsilon \cos \eta \\ \varphi_8 = \sin \theta \sin \alpha \sin \beta \sin \gamma \sin \delta \sin \epsilon \sin \eta \end{cases}$$

we can build the Higgs Potential for the 3HDM:

$$\begin{aligned} V_4 = & \frac{\lambda_1}{2} |\Phi_1|^4 + \frac{\lambda_2}{2} |\Phi_2|^4 + \frac{\lambda_3}{2} |\Phi_3|^4 + \lambda_4 |\Phi_1|^2 |\Phi_2|^2 + \lambda_5 |\Phi_1|^2 |\Phi_3|^2 + \lambda_6 |\Phi_2|^2 |\Phi_3|^2 + \lambda_7 |\Phi_1^\dagger \Phi_2|^2 \\ & + \lambda_8 |\Phi_1^\dagger \Phi_3|^2 + \lambda_9 |\Phi_2^\dagger \Phi_3|^2 + [\frac{\lambda_{10}}{2} (\Phi_1^\dagger \Phi_2)^2 + \frac{\lambda_{11}}{2} (\Phi_1^\dagger \Phi_3)^2 + \frac{\lambda_{12}}{2} (\Phi_2^\dagger \Phi_3)^2 + h.c.] + [\lambda_{13} (\Phi_1^\dagger \Phi_2) |\Phi_1|^2 \\ & + \lambda_{14} (\Phi_1^\dagger \Phi_2) |\Phi_2|^2 + \lambda_{15} (\Phi_1^\dagger \Phi_2) |\Phi_3|^2 + \lambda_{16} (\Phi_1^\dagger \Phi_3) |\Phi_1|^2 + \lambda_{17} (\Phi_1^\dagger \Phi_3) |\Phi_2|^2 + \lambda_{18} (\Phi_1^\dagger \Phi_3) |\Phi_3|^2 \\ & + \lambda_{19} (\Phi_2^\dagger \Phi_3) |\Phi_1|^2 + \lambda_{20} (\Phi_2^\dagger \Phi_3) |\Phi_2|^2 + \lambda_{21} (\Phi_2^\dagger \Phi_3) |\Phi_3|^2 + h.c.] + [\lambda_{22} (\Phi_1^\dagger \Phi_2) (\Phi_1^\dagger \Phi_3) + \lambda_{23} (\Phi_2^\dagger \Phi_3) (\Phi_2^\dagger \Phi_1) \\ & + \lambda_{24} (\Phi_3^\dagger \Phi_1) (\Phi_3^\dagger \Phi_2) + \lambda_{25} (\Phi_1^\dagger \Phi_2) (\Phi_3^\dagger \Phi_1) + \lambda_{26} (\Phi_2^\dagger \Phi_3) (\Phi_1^\dagger \Phi_2) + \lambda_{27} (\Phi_3^\dagger \Phi_1) (\Phi_2^\dagger \Phi_3) + h.c.]. \end{aligned}$$

If we truncate this potential, such that it turns into a quadratic form, we can derive necessary and sufficient analytical conditions for BFB:

$$\begin{aligned} V_4 = & \frac{\lambda_1}{2} |\Phi_1|^4 + \frac{\lambda_2}{2} |\Phi_2|^4 + \frac{\lambda_3}{2} |\Phi_3|^4 + \lambda_4 |\Phi_1|^2 |\Phi_2|^2 + \lambda_5 |\Phi_1|^2 |\Phi_3|^2 + \lambda_6 |\Phi_2|^2 |\Phi_3|^2 + \lambda_7 |\Phi_1^\dagger \Phi_2|^2 \\ & + \lambda_8 |\Phi_1^\dagger \Phi_3|^2 + \lambda_9 |\Phi_2^\dagger \Phi_3|^2 + [\frac{\lambda_{10}}{2} (\Phi_1^\dagger \Phi_2)^2 + \frac{\lambda_{11}}{2} (\Phi_1^\dagger \Phi_3)^2 + \frac{\lambda_{12}}{2} (\Phi_2^\dagger \Phi_3)^2 + h.c.] \end{aligned}$$

Following Kannike, we parameterize the above potential with:

$$\begin{cases} |\Phi_i|^2 = h_i^2 \\ \Phi_i^\dagger \Phi_j = h_i h_j \rho_{ij} e^{i\phi_{ij}} \end{cases} \quad \text{where } \rho_{ij} \in [0, 1] \text{ and } \phi_{ij} \in [0, 2\pi]$$

So we have:

$$\begin{aligned} V_4 = & \frac{\lambda_1}{2} h_1^4 + \frac{\lambda_2}{2} h_2^4 + \frac{\lambda_3}{2} h_3^4 + \lambda_4 h_1^2 h_2^2 + \lambda_5 h_1^2 h_3^2 + \lambda_6 h_2^2 h_3^2 + \lambda_7 \rho_{12} h_1^2 h_2^2 \\ & + \lambda_8 \rho_{13} h_1^2 h_3^2 + \lambda_9 \rho_{23} h_2^2 h_3^2 + |\lambda_{10}| \cos(2\phi_{12} + \phi_{10}) \rho_{12} h_1^2 h_2^2 \\ & + |\lambda_{11}| \cos(2\phi_{13} + \phi_{11}) \rho_{13} h_1^2 h_3^2 + |\lambda_{12}| \cos(2\phi_{23} + \phi_{12}) \rho_{23} h_2^2 h_3^2 \end{aligned}$$

Defining:

$\lambda_{ijk} = \lambda_i + \rho_{jk}^2 [\lambda_j + |\lambda_k| \cos(2\phi_{jk} + \phi_l)]$
where ϕ_l is the phase of the complex coupling constant
and choosing the base:

$$b = \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{bmatrix} \quad (5)$$

We can write the potential in the following quadratic form:

$$V_4 = b^T \frac{1}{2} \Lambda b, \quad (6)$$

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_{4710} & \lambda_{5811} \\ \lambda_{4710} & \lambda_2 & \lambda_{6912} \\ \lambda_{5811} & \lambda_{6912} & \lambda_3 \end{bmatrix} \quad (7)$$

For the potential to be BFB, the Λ matrix must be copositive, so we require that each of all the principal minors of Λ be copositive and that $|\Lambda| > 0$.

The principal minors are:

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & \lambda_{4710} \\ \lambda_{4710} & \lambda_2 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} \lambda_1 & \lambda_{5811} \\ \lambda_{5811} & \lambda_3 \end{bmatrix}, \Lambda_3 = \begin{bmatrix} \lambda_2 & \lambda_{6912} \\ \lambda_{6912} & \lambda_3 \end{bmatrix} \quad (8)$$

Trivially, we need to have $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$.

The other conditions can be derived, by ensuring that: $|\Lambda_1| > 0$, $|\Lambda_2| > 0$ and $|\Lambda_3| > 0$, for all possible values of λ_{ijk} .

If $\lambda_{ijk} > 0$, the matrix is trivially copositive; so we only consider the case $\lambda_{ijk} < 0$.

For a matrix of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad (9)$$

$$|A| = a_{11}a_{22} - a_{12}^2, \quad (10)$$

So, to ensure that $|\Lambda_{1,2,3}| > 0$ we need to find the minimum value of λ_{ijk} , which implies that $\cos(2\phi_{jk} + \phi_l) = -1$.

So we have: $\lambda_{ijk} = \lambda_i + \rho_{jk}^2(\lambda_j - |\lambda_k|)$, for all $\rho_{jk} \in [0, 1]$.

Clearly we have 2 possible cases for the minimum value: $\begin{cases} \lambda_j - |\lambda_k| > 0 \implies \lambda_{ijk_{min}} = \lambda_i, \\ \lambda_j - |\lambda_k| < 0 \implies \lambda_{ijk_{min}} = \lambda_i + \lambda_j - \lambda_k \end{cases}$

Knowing this, we define: $\lambda_{4710m} = \min(\lambda_4, \lambda_4 + \lambda_7 - |\lambda_{10}|)$, $\lambda_{5811m} = \min(\lambda_5, \lambda_5 + \lambda_8 - |\lambda_{11}|)$, $\lambda_{6912m} = \min(\lambda_6, \lambda_6 + \lambda_9 - |\lambda_{12}|)$.

So we can derive the following BFB conditions that are analogous to those of the 2HDM model:

$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0,$

$$\begin{aligned}\lambda_{4710m} + \sqrt{\lambda_1 \lambda_2} &> 0, \\ \lambda_{5811m} + \sqrt{\lambda_1 \lambda_3} &> 0, \\ \lambda_{6912m} + \sqrt{\lambda_2 \lambda_3} &> 0.\end{aligned}$$

We also need an additional condition to ensure that $|\Lambda| > 0$, however, according to Kannike, for a general matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (11)$$

The condition $|A| > 0$ is equivalent to:

$$\sqrt{a_{11}a_{22}a_{33}} + a_{12}\sqrt{a_{33}} + a_{13}\sqrt{a_{22}} + a_{23}\sqrt{a_{11}} + \sqrt{2(a_{12} + \sqrt{a_{11}a_{22}})(a_{13} + \sqrt{a_{11}a_{33}})(a_{23} + \sqrt{a_{22}a_{33}})} > 0.$$

$$\text{So we can use this result to write: } \sqrt{\lambda_1 \lambda_2 \lambda_3} + \lambda_{4710m} \sqrt{\lambda_3} + \lambda_{5811m} \sqrt{\lambda_2} + \lambda_{6912m} \sqrt{\lambda_1} + \sqrt{2(\lambda_{4710m} + \sqrt{\lambda_1 \lambda_2})(\lambda_{5811m} + \sqrt{\lambda_1 \lambda_3})(\lambda_{6912m} + \sqrt{\lambda_2 \lambda_3})} > 0$$

The bounded from below conditions are then:

$$\begin{aligned}\lambda_1 &> 0, \lambda_2 > 0, \lambda_3 > 0; \\ \lambda_{4710m} + \sqrt{\lambda_1 \lambda_2} &> 0; \\ \lambda_{5811m} + \sqrt{\lambda_1 \lambda_3} &> 0; \\ \lambda_{6912m} + \sqrt{\lambda_2 \lambda_3} &> 0 \text{ and} \\ \sqrt{\lambda_1 \lambda_2 \lambda_3} + \lambda_{4710m} \sqrt{\lambda_3} + \lambda_{5811m} \sqrt{\lambda_2} + \lambda_{6912m} \sqrt{\lambda_1} + \\ \sqrt{2(\lambda_{4710m} + \sqrt{\lambda_1 \lambda_2})(\lambda_{5811m} + \sqrt{\lambda_1 \lambda_3})(\lambda_{6912m} + \sqrt{\lambda_2 \lambda_3})} &> 0.\end{aligned}$$

These are sufficient BFB conditions for this model.

2.1 $U(1) \times U(1)$

We know that $V = V_2 + V_N + V_{CB}$ where:

$$V_2 = m_{11}^2(\Phi_1^\dagger \Phi_1) + m_{22}^2(\Phi_2^\dagger \Phi_2) + m_{33}^2(\Phi_3^\dagger \Phi_3),$$

$$V_N = \frac{\lambda_{11}}{2}(\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_{22}}{2}(\Phi_2^\dagger \Phi_2)^2 + \frac{\lambda_{33}}{2}(\Phi_3^\dagger \Phi_3)^2 + \lambda_{12}(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_{13}(\Phi_1^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3) + \lambda_{23}(\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3),$$

$$V_{CB} = \lambda'_{12} z_{12} + \lambda'_{13} z_{13} + \lambda'_{23} z_{23}$$

2.2 $U(1) \times Z_2$

We know that $V_4 = V_N + V_{CB} + V_{U(1) \times Z_2}$ where:

$$V_{U(1) \times Z_2} = \frac{1}{2}[\bar{\lambda}_{12}(\Phi_1^\dagger \Phi_2)^2 + h.c.]$$

2.3 $Z_2 \times Z_2$

We know that $V = V_2 + V_N + V_{CB} + V_{Z_2 \times Z_2}$ where:

$$V_{Z_2 \times Z_2} = \frac{1}{2}[\bar{\lambda}_{12}(\Phi_1^\dagger \Phi_2)^2 + \bar{\lambda}_{13}(\Phi_1^\dagger \Phi_3)^2 + \bar{\lambda}_{23}(\Phi_2^\dagger \Phi_3)^2 + h.c.]$$

$\bar{\lambda}_{ij}$ can be either real or complex