

KOLMOGOROV–RIESZ COMPACTNESS IN ASYMPTOTIC L_p SPACES

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ABSTRACT. We extend the classical Kolmogorov–Riesz compactness theorem to the setting of asymptotic L_p spaces on \mathbb{R}^n . These are nonlocally convex F-spaces that contain the standard L_p spaces as dense subspaces and include all measurable functions supported on sets of finite measure. As an application of our main result, we deduce a well-known characterization of relatively compact families of measurable functions in terms of almost equiboundedness and almost equicontinuity. We conclude with illustrative examples.

1. INTRODUCTION

The classical Kolmogorov–Riesz compactness theorem provides necessary and sufficient conditions for a family of functions in $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$, to be totally bounded, and hence relatively compact due to completeness. This result plays a key role in establishing existence results for partial differential equations.

We recall the statement of the theorem using the standard norm $\|\cdot\|_p$ on $L^p(\mathbb{R}^n)$:

Theorem 1.1 (Kolmogorov–Riesz compactness theorem in $L^p(\mathbb{R}^n)$ [11, 12]). *A subset $\mathcal{F} \subseteq L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$, is totally bounded with respect to $\|\cdot\|_p$ if and only if the following two conditions hold:*

(i) *For each $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} |f|^p dx < \varepsilon^p$$

for all $f \in \mathcal{F}$.

(ii) *For each $\varepsilon > 0$, there exists $r > 0$ such that*

$$\int_{\mathbb{R}^n} |\tau_y f - f|^p dx < \varepsilon^p$$

for every $y \in \mathbb{R}^n$ with $|y| < r$, and all $f \in \mathcal{F}$, where $\tau_y f(x) = f(x + y)$.

The classical version of this theorem included a third condition: boundedness in $L^p(\mathbb{R}^n)$ of the family \mathcal{F} . However, this assumption has been shown to be redundant, as it follows from conditions (i) and (ii); see [12]. A related improvement in the setting of bounded metric measure spaces can be found in [9]. For a historical account and a proof of Theorem 1.1 based on a general compactness lemma in metric spaces, we refer to [11].

In this note, we establish an analogous compactness criterion in the nonlocally convex setting of *asymptotic L_p spaces* on \mathbb{R}^n ; see Theorem 3.1. These spaces, denoted by $\Lambda^p(\mathbb{R}^n)$, were introduced in [1] and consist of real-valued measurable functions that are *almost in L^p* , in the sense that they belong to $L^p(\mathbb{R}^n)$ outside sets of arbitrarily small measure. The topology

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is given by *asymptotic L_p -convergence* (abbreviated as α_p -convergence), which endows the space with a complete metric structure. More precisely,

$$\Lambda^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable} : \forall \delta > 0 \exists E_\delta \text{ with } |E_\delta| < \delta \text{ and } f\chi_{E_\delta^c} \in L^p(\mathbb{R}^n) \right\}$$

where χ_E denotes the characteristic function of the set E , and $E^c = \mathbb{R}^n \setminus E$. The topology is generated by the F-norm

$$\|f\|_{\alpha_p} := \|\min(|f|, 1)\|_p.$$

If $p = 1$, we write $\|\cdot\|_\alpha$ for short. This F-norm generates the topology of α_p -convergence and makes $\Lambda^p(\mathbb{R}^n)$ into a complete metric space; see [1] for details. Section 2 recalls the main properties of these spaces.

Given the apparent similarity between $\Lambda^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, a natural first step toward establishing a compactness criterion for $\Lambda^p(\mathbb{R}^n)$ is to try adapting the proof of Theorem 1.1. However, the lack of homogeneity of the F-norm prevents a direct adaptation and leads to the need for an additional condition: *almost equiboundedness*, which controls the measure of the regions where functions take large values. We show that this condition is also necessary; see condition (iii) in Theorem 3.1 and Lemma 4.3.

Furthermore, if $E \subseteq \mathbb{R}^n$ has finite measure, then $\Lambda^p(E)$ coincides with the space of all real-valued measurable functions on E , equipped with the topology of convergence in measure; see [1, Theorem 1.1]. As a consequence of our main theorem, we recover a classical compactness result for families of measurable functions on bounded sets; see Corollary 3.2. This illustrates how our theorem connects compactness in $L^p(\mathbb{R}^n)$ with compactness in the space of measurable functions.

To the best of the author's knowledge, Theorem 3.1 provides the first extension of the Kolmogorov–Riesz compactness theorem to a nonlocally convex F-space on an unbounded domain, with topology generated by a nonhomogeneous F-norm. An F-space is a completely metrizable topological vector space with a translation-invariant metric; standard examples include, for $0 < p < 1$, the spaces L_p , ℓ_p , and the Hardy spaces H^p of analytic functions [14].

Over the past two decades, extensions of Theorem 1.1 have been developed in a wide range of functional settings. These include locally compact Abelian groups [6]; variable exponent Lebesgue spaces [16, 7, 4]; variable exponent Morrey spaces [5]; grand Lebesgue and grand variable exponent spaces [17, 8]; and, more generally, quasi-Banach function spaces [10].

The manuscript is organized as follows. In Section 2, we recall the main properties of the asymptotic L_p spaces. Section 3 contains the statement of the main result and its corollary. Theorem 3.1 is proved in Sections 4 and 5, while Corollary 3.2 is established in Section 6. Finally, in Section 7, we present examples that illustrate our result.

2. ASYMPTOTIC L_p SPACES

In this section, we give an overview of the main properties of the asymptotic L_p spaces $\Lambda^p(\mathbb{R}^n)$.

This line of research was initiated in [3] with the introduction of asymptotic L_p -convergence, motivated by a question related to convergence in relative entropy. A sequence of measurable functions $\{f_k\}_{k \in \mathbb{N}}$ is said to α_p -converge to a function f if there exists a sequence of measurable sets $\{B_k\}_{k \in \mathbb{N}}$ such that

$$\int_{B_k} |f_k - f|^p dx \rightarrow 0, \quad |B_k^c| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Basic properties of this mode of convergence were studied in [3], and it was shown in [2] that, on finite measure spaces, it is equivalent to convergence in measure.

It is easy to see that α_p -convergence is generated by the F-norm $\|\cdot\|_{\alpha_p}$ defined above. An F-norm is a functional similar to a norm, except that homogeneity is replaced by the following two conditions:

$$\|\lambda f\|_{\alpha_p} \leq \|f\|_{\alpha_p} \quad \text{for all } |\lambda| \leq 1, \text{ and all } f \in \Lambda^p(\mathbb{R}^n),$$

and

$$\lim_{\lambda \rightarrow 0} \|\lambda f\|_{\alpha_p} = 0 \quad \text{for all } f \in \Lambda^p(\mathbb{R}^n);$$

see [1, Proposition A.2].

Interestingly, the lack of homogeneity has deep consequences: the space $\Lambda^p(\mathbb{R}^n)$ is neither locally bounded nor locally convex [1, Propositions 7.1 and 7.2], and its dual consists only of the zero functional [1, Proposition 7.3]. This highlights how fundamentally different $\Lambda^p(\mathbb{R}^n)$ is from the standard $L^p(\mathbb{R}^n)$. Nevertheless, many classical results have analogs in this setting. In [1], versions of the dominated convergence and Vitali convergence theorems were established for $\Lambda^p(\mathbb{R}^n)$.

Moreover, it follows from the definitions that if $f \in \Lambda^p(\mathbb{R}^n)$, then there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subseteq L^p(\mathbb{R}^n)$ that α_p -converges to f . Hence, $L^p(\mathbb{R}^n)$ is dense in $\Lambda^p(\mathbb{R}^n)$, and since $L^p(\mathbb{R}^n)$ is separable, so is $\Lambda^p(\mathbb{R}^n)$.

As mentioned in the introduction, when the underlying measure space has finite measure, for instance a bounded set $E \subseteq \mathbb{R}^n$, then $\Lambda^p(E)$ coincides with the space of all real-valued measurable functions on E . Thus, $\Lambda^p(\mathbb{R}^n)$ extends the space of measurable functions, equipped with the topology of convergence in measure, to the unbounded domain \mathbb{R}^n . In this sense, the asymptotic L_p spaces retain features from both the standard $L^p(\mathbb{R}^n)$ and the F-space of measurable functions.

3. MAIN RESULT

In this section, we state the main result of the paper — a characterization of the relatively compact subsets of $\Lambda^p(\mathbb{R}^n)$, given in Theorem 3.1 below. As a consequence, we obtain a classical characterization of relatively compact families of measurable functions defined on a bounded subset of \mathbb{R}^n , stated in Corollary 3.2.

Theorem 3.1 (Kolmogorov–Riesz compactness theorem in $\Lambda^p(\mathbb{R}^n)$). *A subset $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is totally bounded with respect to the F-norm $\|\cdot\|_{\alpha_p}$ if and only if the following three conditions hold:*

(i) *For each $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x| > R} \min(|f|, 1)^p dx < \varepsilon^p$$

for all $f \in \mathcal{F}$.

(ii) *For each $\varepsilon > 0$, there exists $r > 0$ such that*

$$\int_{\mathbb{R}^n} \min(|\tau_y f - f|, 1)^p dx < \varepsilon^p$$

for every $y \in \mathbb{R}^n$ with $|y| < r$ and all $f \in \mathcal{F}$, where $\tau_y f(x) = f(x + y)$.

(iii) *For each $\varepsilon > 0$, there exists $M > 0$ such that*

$$|\{f \in \mathcal{F} : |f| > M\}| < \varepsilon$$

for all $f \in \mathcal{F}$.

We note that the conditions in Theorem 3.1 are sharp, in the sense that none of them can be deduced from the others. The first three examples in Section 7 illustrate this.

The next result, which first appeared in [13] for the case $n = 1$ (see also [15]), is derived here as a consequence of Theorem 3.1.

Corollary 3.2. *Let E be a bounded subset of \mathbb{R}^n . A family \mathcal{F} of real-valued measurable functions on E is totally bounded with respect to the topology of convergence in measure if and only if the following two conditions hold:*

- (i) *Almost equiboundedness: For each $\varepsilon > 0$, there exists $M > 0$ such that for every $f \in \mathcal{F}$ there is a set $S_f \subseteq E$ with $|S_f| < \varepsilon$ and*

$$|f| \leq M \quad \text{on } E \setminus S_f.$$

- (ii) *Almost equicontinuity: For each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in \mathcal{F}$ there is a set $B_f \subseteq E$ with $|B_f| < \varepsilon$ such that*

$$|f(x_1) - f(x_2)| < \varepsilon$$

for all $x_1, x_2 \in E \setminus B_f$ satisfying $|x_1 - x_2| < \delta$.

We observe that condition (iii) of Theorem 3.1 is precisely the notion of almost equiboundedness, which is seen by taking $S_f = \{|f| > M\}$.

4. PROOF OF THEOREM 3.1: NECESSITY

In this section, we prove that a totally bounded family in $\Lambda^p(\mathbb{R}^n)$ satisfies the three conditions of Theorem 3.1. The proof is divided into three lemmas, each corresponding to one of the conditions.

Lemma 4.1. *If a subset $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$ is totally bounded (with respect to $\|\cdot\|_{\alpha_p}$) then for each $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} \min(|f|, 1)^p dx < \varepsilon^p$$

for all $f \in \mathcal{F}$.

Proof. Let $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$ be totally bounded and $\varepsilon > 0$. There exist $f_1, \dots, f_m \in \Lambda^p(\mathbb{R}^n)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^m B_{\alpha_p}(f_i, \varepsilon/2^{1+1/p}).$$

Since $f_i \in \Lambda^p(\mathbb{R}^n)$, there exists a measurable set E_i with $|E_i| < \varepsilon^p/(4m)$ so that $f_i \chi_{E_i^c} \in L^p(\mathbb{R}^n)$. This implies that there is $R_i > 0$ such that

$$\int_{E_i^c \cap \{|x|>R_i\}} |f_i|^p dx < \frac{\varepsilon^p}{2^{p+1}}.$$

Set $E = \bigcup_{i=1}^m E_i$ and $R = \max\{R_i : i = 1, \dots, m\}$. Clearly $|E| < \varepsilon^p/4$.

Let $f \in \mathcal{F}$. Then, for some f_i we have $\|f_i - f\|_{\alpha_p} < \varepsilon/2^{1+1/p}$; in particular

$$\int_{|f-f_i| \leq 1} |f_i - f|^p dx < \frac{\varepsilon^p}{2^{p+1}}, \quad |\{|f_i - f| > 1\}| < \frac{\varepsilon^p}{2^{p+1}} \leq \frac{\varepsilon^p}{4}.$$

Set $G = E \cup \{|f_i - f| > 1\}$. Then $|G| < \varepsilon^p/2$ and

$$\begin{aligned} \left(\int_{G^c \cap \{|x| > R\}} |f|^p dx \right)^{\frac{1}{p}} &\leq \left(\int_{G^c \cap \{|x| > R\}} |f_i - f|^p dx \right)^{\frac{1}{p}} + \left(\int_{G^c \cap \{|x| > R\}} |f_i|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{|f_i - f| \leq 1} |f_i - f|^p dx \right)^{\frac{1}{p}} + \left(\int_{E_i^c \cap \{|x| > R_i\}} |f_i|^p dx \right)^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2^{1/p}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{|x| > R} \min(|f|, 1)^p dx &= \int_{G \cap \{|x| > R\}} \min(|f|, 1)^p dx + \int_{G^c \cap \{|x| > R\}} \min(|f|, 1)^p dx \\ &\leq |G| + \int_{G^c \cap \{|x| > R\}} |f|^p dx \\ &< \varepsilon^p \end{aligned}$$

which finishes the proof. \square

Lemma 4.2. *If a subset $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$ is totally bounded (with respect to $\|\cdot\|_{\alpha_p}$) then for each $\varepsilon > 0$, there exists $r > 0$ such that*

$$\int_{\mathbb{R}^n} \min(|\tau_y f - f|, 1)^p dx < \varepsilon^p$$

for every $y \in \mathbb{R}^n$ with $|y| < r$ and all $f \in \mathcal{F}$.

Proof. Assume that $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^N)$ is totally bounded and let $\varepsilon > 0$. There exist $f_1, \dots, f_m \in \Lambda^p(\mathbb{R}^n)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^m B_{\alpha_p}(f_i, \varepsilon/3).$$

For each $i = 1, \dots, m$ there exists $\varphi_i \in C_c^\infty(\mathbb{R}^n)$ such that $\|\varphi_i - f_i\|_{\alpha_p} < \varepsilon/3$ (see [1, Proposition 6.3]). Hence

$$\|\tau_y f_i - f_i\|_{\alpha_p} \leq \|\tau_y f_i - \tau_y \varphi_i\|_{\alpha_p} + \|\tau_y \varphi_i - \varphi_i\|_{\alpha_p} + \|\varphi_i - f_i\|_{\alpha_p}.$$

The first term on the right-hand side equals the last one, and both are bounded by $\varepsilon/3$. The middle term is bounded by $\|\tau_y \varphi_i - \varphi_i\|_p$. The smoothness of φ_i guarantees the existence of a constant $r_i > 0$ such that $\|\tau_y \varphi_i - \varphi_i\|_p < \varepsilon/3$ whenever $|y| < r_i$. It follows that for $|y| < r_i$ we have

$$\|\tau_y f_i - f_i\|_{\alpha_p} < \varepsilon.$$

Set $r = \min\{r_i : i = 1, \dots, m\}$. Then, for $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < r$ we have

$$\|\tau_y f - f\|_{\alpha_p} \leq \|\tau_y f - \tau_y f_i\|_{\alpha_p} + \|\tau_y f_i - f_i\|_{\alpha_p} + \|f_i - f\|_{\alpha_p}$$

where f_i is such that $f \in B_{\alpha_p}(f_i, \varepsilon/3)$. Each one of the terms on the right-hand side is bounded from above by $\varepsilon/3$, yielding the desired conclusion. \square

Lemma 4.3. *If a subset $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$ is totally bounded (with respect to $\|\cdot\|_{\alpha_p}$) then for each $\varepsilon > 0$, there exists $M > 0$ such that*

$$|\{|f| > M\}| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Suppose, towards a contradiction, that there exists $\varepsilon > 0$ such that for every $M > 0$ one can find $f_M \in \mathcal{F}$ satisfying

$$|\{ |f_M| > M \}| \geq \varepsilon.$$

Since \mathcal{F} is totally bounded, there are $f_1, \dots, f_m \in \Lambda^p(\mathbb{R}^n)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^m B_{\alpha_p}(f_i, (\varepsilon/4)^{1/p}).$$

For each $i = 1, \dots, m$, there exists a measurable set E_i with $|E_i| < \varepsilon/4$ such that $f_i \chi_{E_i^c} \in L^p(\mathbb{R}^n)$. Therefore

$$\begin{aligned} |E_i^c \cap \{ |f_i| > M \}| &= |\{ |f_i \chi_{E_i^c}| > M \}| \\ &\leq \frac{1}{M^p} \int_{E_i^c} |f_i|^p dx \rightarrow 0 \quad \text{as } M \rightarrow \infty. \end{aligned}$$

Choose $M_i > 0$ so that

$$|E_i^c \cap \{ |f_i| > M_i \}| < \frac{\varepsilon}{4}.$$

We thus have for every $i = 1, \dots, m$ that

$$|\{ |f_i| > M_i \}| \leq |E_i| + |E_i^c \cap \{ |f_i| > M_i \}| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Set $M_0 = \max\{M_i : i = 1, \dots, m\}$. There exists $f \in \mathcal{F}$ with $|\{ |f| > M_0 + 1 \}| \geq \varepsilon$. Moreover, $\|f_j - f\|_{\alpha_p}^p < \varepsilon/4$ for some $j \in \{1, \dots, m\}$. Define the sets

$$G = \{ |f| > M_0 + 1 \} \quad \text{and} \quad H = \{ |f_j - f| > 1 \}$$

and note that

$$|H| \leq \|f_j - f\|_{\alpha_p}^p < \frac{\varepsilon}{4}.$$

Now,

$$\varepsilon \leq |G| \leq |H| + |G \cap H^c| < \frac{\varepsilon}{4} + |G \cap H^c|$$

whence

$$|G \cap H^c| > \frac{3\varepsilon}{4}.$$

On $G \cap H^c$ we have

$$M_0 + 1 < |f| \leq |f_j - f| + |f_j| \leq 1 + |f_j|$$

and hence

$$G \cap H^c \subseteq \{ |f_j| > M_0 \}$$

which implies

$$\frac{3\varepsilon}{4} < |G \cap H^c| \leq |\{ |f_j| > M_0 \}|.$$

However, since $M_0 > M_j$,

$$|\{ |f_j| > M_0 \}| \leq |\{ |f_j| > M_j \}| < \frac{\varepsilon}{2} < \frac{3\varepsilon}{4}$$

which is a contradiction. The result follows. \square

5. PROOF OF THEOREM 3.1: SUFFICIENCY

Assume that $\mathcal{F} \subseteq \Lambda^p(\mathbb{R}^n)$ satisfies conditions (i), (ii) and (iii) of Theorem 3.1 and let $\eta > 0$ be given. According to condition (iii) we can choose $M > 1$ such that

$$|\{ |f| > M \}| < \left(\frac{\eta}{2}\right)^p$$

for all $f \in \mathcal{F}$. Let T_M be the truncation function defined for $t \in \mathbb{R}$ by $T_M(t) = \max\{-M, \min\{t, M\}\}$, and define for each $f \in \mathcal{F}$ its truncated version

$$f_M(x) = T_M(f(x)) = \begin{cases} M, & \text{if } f(x) > M, \\ f(x), & \text{if } |f(x)| \leq M, \\ -M, & \text{if } f(x) < -M. \end{cases}$$

Since $f_M \in \Lambda^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ it follows that $f_M \in L^p(\mathbb{R}^n)$.

Now, given $f \in \mathcal{F}$ let $G = \{|f| > M\}$. Note that

$$|f(x) - f_M(x)| = \begin{cases} |f(x)| - M, & \text{if } x \in G, \\ 0, & \text{if } x \in G^c, \end{cases}$$

Therefore

$$\int_{\mathbb{R}^n} \min(|f - f_M|, 1)^p dx = \int_G \min(|f - f_M|, 1)^p dx \leq |G| < \left(\frac{\eta}{2}\right)^p$$

and hence, for every $f \in \mathcal{F}$,

$$\|f - f_M\|_{\alpha_p} < \frac{\eta}{2}.$$

We proceed to show that the family $\mathcal{F}_M = \{f_M : f \in \mathcal{F}\}$ satisfies conditions (i) and (ii) of Theorem 3.1. Regarding the first condition we note that since $M > 1$ then

$$\min(|f_M|, 1) = \min(|f|, 1)$$

being clear that \mathcal{F}_M satisfies condition (i). Regarding condition (ii) we note that the truncation map T_M is Lipschitz continuous with constant 1, which implies that

$$|f_M(x+y) - f_M(x)| = |T_M(f(x+y)) - T_M(f(x))| \leq |f(x+y) - f(x)|.$$

Therefore

$$\min(|f_M(x+y) - f_M(x)|, 1) \leq \min(|f(x+y) - f(x)|, 1)$$

and so

$$\int_{\mathbb{R}^n} \min(|f_M(x+y) - f_M(x)|, 1) dx \leq \int_{\mathbb{R}^n} \min(|f(x+y) - f(x)|, 1) dx.$$

The next step is to prove that \mathcal{F}_M is totally bounded in $L^p(\mathbb{R}^n)$ (with respect to $\|\cdot\|_p$). From condition (i) of Theorem 3.1 we have

$$\int_{|x|>R} \min(|f_M|, 1)^p dx < \left(\frac{\varepsilon}{M}\right)^p$$

for some $R > 0$ and all $f_M \in \mathcal{F}_M$. Then

$$\int_{|x|>R} |f_M|^p dx \leq M^p \int_{|x|>R} \min(|f_M|, 1)^p dx < \varepsilon^p$$

and thus \mathcal{F}_M satisfies condition (i) of Theorem 1.1.

Moreover, given $\varepsilon > 0$, choose $r > 0$ from condition (ii) of Theorem 3.1 so that if $|y| < r$ then

$$\int_{\mathbb{R}^n} \min(|f_M(x+y) - f_M(x)|, 1)^p dx < \frac{\varepsilon^p}{1 + (2M)^p}.$$

We have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_M(x+y) - f_M(x)|^p dx &\leq \int_{|\tau_y f_M - f_M| \leq 1} |f_M(x+y) - f_M(x)|^p dx \\
&\quad + (2M)^p \int_{|\tau_y f_M - f_M| > 1} 1 dx \\
&\leq (1 + (2M)^p) \int_{\mathbb{R}^n} \min(|f_M(x+y) - f_M(x)|, 1)^p dx \\
&< \varepsilon^p.
\end{aligned}$$

It follows by Theorem 1.1 that \mathcal{F}_M is totally bounded in $L^p(\mathbb{R}^n)$. Let $h_1, \dots, h_m \in L^p(\mathbb{R}^n)$ be such that

$$\mathcal{F}_M \subseteq \bigcup_{i=1}^m B_p(h_i, \eta/2).$$

Then, given $f \in \mathcal{F}$, there exists $i \in \{1, \dots, m\}$ so that

$$\begin{aligned}
\|f - h_i\|_{\alpha_p} &\leq \|f - f_M\|_{\alpha_p} + \|f_M - h_i\|_{\alpha_p} \\
&\leq \eta/2 + \|f_M - h_i\|_p \\
&< \eta
\end{aligned}$$

which completes the proof.

6. PROOF OF COROLLARY 3.2

Let E be a bounded subset of \mathbb{R}^n , and recall that $\Lambda^p(E)$, for $1 \leq p < \infty$, consists of all real-valued measurable functions on E , equipped with the topology of convergence in measure.

Since condition (iii) of Theorem 3.1 corresponds to almost equiboundedness, and condition (i) is automatically satisfied for families in $\Lambda^p(E)$, it remains to prove that for almost equibounded families in $\Lambda^p(E)$, condition (ii) of Theorem 3.1 is equivalent to almost equicontinuity. This will complete the proof of Corollary 3.2, and is established in the next two lemmas. For simplicity, we restrict to the case $p = 1$.

Lemma 6.1. *Let \mathcal{F} be a family of real-valued measurable functions defined on a bounded set $E \subseteq \mathbb{R}^n$ (and extended by zero to all of \mathbb{R}^n). If \mathcal{F} is almost equicontinuous, then for each $\varepsilon > 0$, there exists $r > 0$ such that*

$$\int_{\mathbb{R}^n} \min(|\tau_y f - f|, 1) dx < \varepsilon$$

for every $y \in \mathbb{R}^n$ with $|y| < r$ and all $f \in \mathcal{F}$.

Proof. Let $\varepsilon > 0$ and choose $\tilde{\varepsilon} > 0$ so that $(3 + |E|)\tilde{\varepsilon} < \varepsilon$. Since \mathcal{F} is almost equicontinuous, there exists $\delta > 0$ such that for every $f \in \mathcal{F}$, there is a set $B_f \subseteq E$ with $|B_f| < \tilde{\varepsilon}$ such that $|f(x_1) - f(x_2)| < \tilde{\varepsilon}$ whenever $x_1, x_2 \in E \setminus B_f$ satisfy $|x_1 - x_2| < \delta$. Moreover, we note that for all $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$, $|f(x+y) - f(x)| = 0$ for $x \in (E \cup (E - y))^c$.

Now, we write $E \cup (E - y) = (E \cap (E - y)) \cup (E \Delta (E - y))$, where Δ denotes the symmetric difference of sets, and note that

$$|E \Delta (E - y)| \rightarrow 0 \quad \text{as } |y| \rightarrow 0$$

given that E is measurable and bounded. Let $r_0 > 0$ be such that $|E \Delta (E - y)| < \tilde{\varepsilon}$ whenever $|y| < r_0$, and set $r = \min\{\delta, r_0\}$. For fixed $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < r$ we estimate:

$$\begin{aligned}
\int_{\mathbb{R}^n} \min(|\tau_y f - f|, 1) \, dx &= \int_{E \cup (E-y)} \min(|\tau_y f - f|, 1) \, dx \\
&= \int_{E \cap (E-y)} \min(|\tau_y f - f|, 1) \, dx \\
&\quad + \int_{E \Delta (E-y)} \min(|\tau_y f - f|, 1) \, dx \\
&\leq \int_{E \cap (E-y)} \min(|\tau_y f - f|, 1) \, dx + |E \Delta (E-y)| \\
&\leq \int_{E \cap (E-y)} \min(|\tau_y f - f|, 1) \, dx + \tilde{\varepsilon}.
\end{aligned}$$

We proceed by estimating the last integral above. Let $\tilde{B} = B_f \cup (B_f - y) \subseteq E \cup (E - y)$, which satisfies $|\tilde{B}| \leq 2|B_f| < 2\tilde{\varepsilon}$. We have:

$$\begin{aligned}
\int_{E \cap (E-y)} \min(|\tau_y f - f|, 1) \, dx &= \int_{(E \cap (E-y)) \cap \tilde{B}} \min(|\tau_y f - f|, 1) \, dx \\
&\quad + \int_{(E \cap (E-y)) \cap \tilde{B}^c} \min(|\tau_y f - f|, 1) \, dx \\
&< 2\tilde{\varepsilon} + \int_{(E \setminus B_f) \cap ((E-y) \setminus (B_f - y))} \min(|\tau_y f - f|, 1) \, dx.
\end{aligned}$$

For $x \in (E \setminus B_f) \cap ((E-y) \setminus (B_f - y))$ we have $x, x+y \in E \setminus B_f$, and hence for such x it holds, by almost equicontinuity (since $|x+y-x| = |y| < r \leq \delta$), that

$$\min(|f(x+y) - f(x)|, 1) \leq |f(x+y) - f(x)| < \tilde{\varepsilon}$$

and thus the last integral above is controlled by $|E|\tilde{\varepsilon}$. Putting the previous estimates all together yields that

$$\int_{\mathbb{R}^n} \min(|\tau_y f - f|, 1) \, dx < (3 + |E|)\tilde{\varepsilon} < \varepsilon$$

which finishes the proof. \square

For the other implication, we apply Theorem 3.1 to conclude that if $\mathcal{F} \subseteq \Lambda^1(E)$ is almost equibounded and satisfies condition (ii) of the theorem, then it is totally bounded with respect to the F-norm $\|\cdot\|_\alpha$ restricted to E . The next lemma shows that under these assumptions, a totally bounded family is almost equicontinuous.

Lemma 6.2. *Let \mathcal{F} be a family of real-valued measurable functions defined on a bounded set $E \subseteq \mathbb{R}^n$ (and extended by zero to all of \mathbb{R}^n). If \mathcal{F} is totally bounded with respect to the F-seminorm*

$$f \mapsto \int_E \min(|f|, 1) \, dx$$

then \mathcal{F} is almost equicontinuous.

Proof. Let $\varepsilon > 0$ and assume without loss of generality that $\varepsilon < 1$. By hypothesis, there exist real-valued measurable functions on E , f_1, \dots, f_m , such that for every $f \in \mathcal{F}$,

$$\int_E \min(|f - f_i|, 1) \, dx < \frac{\varepsilon^2}{12}$$

for some $i \in \{1, \dots, m\}$. Using Lusin's theorem, for each $i = 1, \dots, m$ there exists a closed set $F_i \subseteq E$ with $|E \setminus F_i| < \varepsilon^2/(12m)$ such that $f_i|_{F_i}$ is continuous. Let $F = \bigcap_{i=1}^m F_i$ and

set $c_i = f_i|_F$. Then F is compact, $|E \setminus F| < \varepsilon^2/12$ and the family $\{c_i : i = 1, \dots, m\}$ is equicontinuous. Therefore, there exists $\delta > 0$ such that for all $i = 1, \dots, m$, $|c_i(x_1) - c_i(x_2)| < \varepsilon/3$ whenever $x_1, x_2 \in F$ satisfy $|x_1 - x_2| < \delta$.

Now, fix $f \in \mathcal{F}$ and let $i \in \{1, \dots, m\}$ be such that f belongs to the ball centered at f_i . We have:

$$\begin{aligned} \int_E \min(|f - c_i|, 1) \, dx &\leq \int_E \min(|f - f_i|, 1) \, dx + \int_E \min(|f_i - c_i|, 1) \, dx \\ &< \frac{\varepsilon^2}{12} + \int_{E \setminus F} \min(|f_i - c_i|, 1) \, dx \\ &\leq \frac{\varepsilon^2}{12} + |E \setminus F| \\ &< \frac{\varepsilon^2}{6}. \end{aligned}$$

Thus, since $0 < \varepsilon/3 < 1$,

$$\begin{aligned} |\{|f - c_i| > \varepsilon/3\}| &= |\{\min(|f - c_i|, 1) > \varepsilon/3\}| \\ &\leq \frac{3}{\varepsilon} \int_E \min(|f - c_i|, 1) \, dx \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Set $B_f = \{|f - c_i| > \varepsilon/3\} \cup (E \setminus F)$. Then $|B_f| < \varepsilon/2 + \varepsilon^2/12 < \varepsilon$ and for $x_1, x_2 \in E \setminus B_f$ satisfying $|x_1 - x_2| < \delta$ it holds

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1) - c_i(x_1)| + |c_i(x_1) - c_i(x_2)| + |c_i(x_2) - f(x_2)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

which completes the proof. \square

7. EXAMPLES

The first three examples of this section concern sequences of functions that satisfy only two of the conditions of Theorem 3.1 but violate a third. This shows that Theorem 3.1 is sharp in the sense that none of the conditions is redundant.

Example 7.1. Let $f_k(x) = k^{1/p} \chi_{[0,1]}(x)$. The sequence $\{f_k\}_{k \in \mathbb{N}}$ satisfies conditions (i) and (ii) but violates condition (iii).

The first condition is clear since for $R > 1$ we have

$$\int_{|x| > R} \min(|f_k|, 1)^p \, dx = 0$$

for all $k \in \mathbb{N}$.

Regarding the second condition, given $\varepsilon > 0$ we choose $r = \min\{1, \varepsilon^p/2\}$ so that for $|y| < r$ and $k \in \mathbb{N}$ it holds

$$\int_{\mathbb{R}} \min(|f_k(x+y) - f_k(x)|, 1)^p \, dx = 2|y| < 2r \leq \varepsilon^p.$$

Now, we note that given $M > 0$, for $k > M^p$ we have

$$|\{|f_k| > M\}| = 1$$

and hence the third condition is not satisfied.

Therefore, this sequence is neither totally bounded in $\Lambda^p(\mathbb{R})$ nor in $L^p(\mathbb{R})$ (in fact, it is not even bounded in $L^p(\mathbb{R})$).

Example 7.2. Let $g_k(x) = \chi_{[k, k+1]}(x)$. The sequence $\{g_k\}_{k \in \mathbb{N}}$ satisfies conditions (ii) and (iii) but violates condition (i).

We start with condition (iii). Simply note that $|g_k| \leq 1$ and hence for $M > 1$ we have

$$|\{ |g_k| > M \}| = 0$$

for all $k \in \mathbb{N}$.

Condition (ii) is analogous to the previous example: given $\varepsilon > 0$, take $r = \min\{1, \varepsilon^p/2\}$ so that for $|y| < r$ and all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \min(|g_k(x+y) - g_k(x)|, 1)^p dx = 2|y| < 2r \leq \varepsilon^p.$$

Regarding the failure of condition (i), we observe that given $R > 0$, choosing $k > R + 1$ yields

$$\int_{|x| > R} \min(|g_k|, 1)^p dx = 1.$$

It follows that $\{g_k\}_{k \in \mathbb{N}}$ is not totally bounded in $\Lambda^p(\mathbb{R})$ (nor in $L^p(\mathbb{R})$).

Example 7.3. Let $h_k(x) = r_k(x)\chi_{[0,1]}(x)$ where r_k is the k^{th} Rademacher function on $[0, 1]$ (and extended by zero elsewhere), that is, $r_k(x) = \text{sign}(\sin(2^k \pi x))$. The sequence $\{h_k\}_{k \in \mathbb{N}}$ satisfies conditions (i) and (iii) but violates condition (ii).

The first and third conditions are clear (take $R > 1$ for (i) and $M > 1$ for (iii)), so we focus on the failure of condition (ii). Let $r > 0$ and choose $k \in \mathbb{N}$ so that $y = 2^{-k} < r$. Note that for $x \in [0, 1 - 2^{-k}]$ we have $r_k(x+y) = -r_k(x)$. Thus

$$\begin{aligned} \int_{\mathbb{R}} \min(|h_k(x+y) - h_k(x)|, 1)^p dx &\geq \int_0^{1-2^{-k}} \min(|2r_k(x)|, 1)^p dx \\ &= 1 - 2^{-k} \\ &\geq \frac{1}{2} \end{aligned}$$

which proves that $\{h_k\}_{k \in \mathbb{N}}$ does not satisfy condition (ii).

The last two examples concern sequences that are totally bounded in $\Lambda^p(\mathbb{R}^n)$ but not in $L^p(\mathbb{R}^n)$.

Example 7.4. Let $\varphi \in L^p(\mathbb{R})$ be a fixed nonnegative function in $L^p(\mathbb{R})$, and consider for each $k \in \mathbb{N}$, $u_k(x) = \varphi(x) + k^{1/p}\chi_{[k, k+1/k]}(x)$. The sequence $\{u_k\}_{k \in \mathbb{N}}$ asymptotically L_p -converges to φ , and hence it is totally bounded in $\Lambda^p(\mathbb{R})$. However, it is not totally bounded in $L^p(\mathbb{R})$. Indeed, we have

$$\int_{\mathbb{R}} \min(|u_k(x) - \varphi(x)|, 1)^p dx = \int_k^{k+1/k} \min(k^{1/p}, 1)^p dx \leq \frac{1}{k} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which proves that $\{u_k(t, \cdot)\}_{k \in \mathbb{N}}$ α_p -converges to φ_t , but given $R > 0$, for $k > R$ it holds

$$\begin{aligned} \int_{|x| > R} |u_k(x)|^p dx &= \int_R^\infty |\varphi(x) + k^{1/p}\chi_{[k, k+1/k]}(x)|^p dx \\ &\geq \int_R^\infty k\chi_{[k, k+1/k]}(x) dx \\ &= 1 \end{aligned}$$

therefore $\{u_k\}_{k \in \mathbb{N}}$ does not satisfy the first condition of Theorem 1.1, and hence it is not totally bounded in $L^p(\mathbb{R})$.

Example 7.5. Let $1 < p < \infty$ and consider for each $k \in \mathbb{N}$, $v_k(x) = x^{-1}\chi_{[1/k, \infty)}$. Then, $\{v_k\}_{k \in \mathbb{N}}$ is not bounded in $L^p(\mathbb{R})$, and hence it is not totally bounded in $L^p(\mathbb{R})$, yet it is totally bounded in $\Lambda^p(\mathbb{R})$ since it α_p -converges to $v \in \Lambda^p(\mathbb{R})$ given by $v(x) = x^{-1}\chi_{(0, \infty)}(x)$.

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