ANALYTIC SOLUTIONS FOR VLASOV EQUATIONS WITH NONLINEAR ZERO-MOMENT DEPENDENCE

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ABSTRACT. We consider nonlinear Vlasov-type equations involving powers of the zero-order moment and obtain a local existence and uniqueness result within a framework of analytic functions. The proof employs a Banach fixed point argument, where a contraction mapping is built upon the solutions of a corresponding linearized problem. At a formal level, the considered nonlinear kinetic equations are derived from a generalized Vlasov-Poisson type equation under zero-electron-mass and quasi-neutrality assumptions, and are related to compressible Euler equations through monokinetic distributions.

1. INTRODUCTION

Let $q \in \mathbb{N}_0$ and consider the following kinetic initial-value problem for $t \geq 0$ and $x, v \in \mathbb{R}$:

$$\begin{cases} \partial_t f + v \cdot \partial_x f - \rho^q \partial_x \rho \cdot \partial_v f = 0, \\ \rho = \displaystyle\int f \, \mathrm{d}v, \\ f|_{t=0} = f_0. \end{cases} \tag{1.1}$$

This work aims to establish a local existence and uniqueness result for problem (1.1) within a framework of analytic functions. This extends the existence result for $q = 0$ in [16].

The kinetic equation of problem (1.1) is of Vlasov-type depending on gradients of the first moment. The case $q = 0$ is related to the Benney equations and has interesting connections to the theory of water waves and conformal mappings; see [12]. Such equations are derived in the literature by combining a Vlasov type equation

$$\partial_t f + v \cdot \partial_x f - \partial_x \phi \cdot \partial_v f = 0 \tag{1.2}$$

with a structure equation, where as a result of an asymptotic process the potential satisfies the equation

$$\partial_x \left(\tfrac{1}{\gamma} \rho^\gamma \right) = \rho \partial_x \phi \tag{1.3}$$

where $\gamma \geq 1$ is the adiabatic exponent. The constant $1/\gamma$ is chosen for convenience. For $\gamma = 1$ we recover the usual Boltzmann relation $\rho = \rho_0 e^\phi$, where ρ_0 is some constant; see [14]. By substituting (1.3) into (1.2), we obtain the kinetic equation in (1.1) with $q = \gamma - 2$.

An example of this derivation is provided in [15], where the quasi-neutral limit of a Vlasov-Poisson system, combined with a linearized Maxwell-Boltzmann law, leads to the equation (1.1) with $q = 0$. More general cases can be obtained by starting from a bipolar fluid model and deriving, via asymptotics, the combined zero-electron mass and quasi-neutral limits [2]. Another instance, starting from a generalized Euler-Poisson system, is presented in Section 2.

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Kinetic equations of Vlasov-type have been studied for several decades, with the Vlasov-Poisson equation being among the most prominent due to its significance in collisionless plasmas under their self-consistent electric field. For Vlasov-Poisson systems in three spatial dimensions, global existence of smooth solutions has been obtained in [5] for small initial data, and extended to more general initial data in [17, 22]. In several space dimensions, local existence has been established in [15] for solutions emanating from initial data satisfying a Penrose stability condition. The transport structure of the Vlasov-Poisson kinetic equation has been analyzed in [1] from a Lagrangian viewpoint, which, in particular, yields global existence of weak solutions in any dimension under minimal assumptions on the initial data.

The nonlinearity in (1.1) is more singular than that of the Vlasov-Poisson equation, making its analysis more delicate. When $q = 0$, the equation is called Vlasov-Dirac-Benney equation (VDB). For this case, we refer to [16, 15] for two well-posedness theories. In [15], the solutions to the VDB equation are obtained as quasi-neutral limits of solutions to a Vlasov-Poisson equation. The VDB equation has also been studied extensively in [6, 3, 4].

In this work, we consider a more general case, obtaining local-in-time existence and uniqueness of analytic solutions to problem (1.1) with moderate restrictions on the size of the initial data; see Theorem 4.1 and Remark 4.2. Since we are dealing with analytic solutions, we impose the assumption $q \in \mathbb{N}_0$. This well-posedness result is achieved by applying a fixed point argument to a contraction mapping on a certain metric space of analytic functions; see the space (4.3) with the norm (4.4). Our approach follows the framework developed in [16], where the local existence and uniqueness of analytic solutions for the VDB equation are established. We extend their framework to address a more general equation, while establishing the contraction property using a simpler norm.

This approach has also been used in [8] to examine a Vlasov-type equation that can be interpreted as a Fokker-Planck equation related to a stochastic differential equation. The method is a specific application of the broader analytical techniques developed in the seminal work on nonlinear Landau damping [20]. Analytic methods addressing related topics are discussed in [7, 13, 24], particularly within the context of Gevrey regularity, which lies outside the framework considered here. For a general account of the theory of real analytic functions we refer to [21].

The paper is organized as follows. Section 2 introduces a class of models arising from a formal Hamiltonian structure, which induces a class of (generalized) bipolar Vlasov-Poisson systems. We point out that the well-known bipolar compressible Euler-Poisson system arises from the generalized Vlasov-Poisson system through an hypothesis of monokinetic distributions. Then, we deduce the model (1.1) from the generalized Vlasov-Poisson system via an asymptotic reduction that combines the zero-electron-mass and quasi-neutral limits. Additionally, we show that classical solutions of the one-dimensional Euler equations yield weak monokinetic solutions of the Vlasov equation (1.1); see Proposition 2.1.

The remainder of the manuscript is devoted to the existence theory of (1.1) in a class of analytic functions. Section 3 provides the definition of analytic functions and corresponding norms that will be used in the subsequent analysis. In addition, some basic properties of these objects are deduced. Section 4 presents the statement and respective proof of the main result of this work – Theorem 4.1.

2. A Generalized Vlasov-Poisson System

In this section, we derive a generalized Vlasov-Poisson system induced by an infinite-dimensional Hamiltonian structure. The analysis follows the works of Morrison [18, 19] on the Hamiltonian structure of the Vlasov-Poisson system and uses the Hamiltonian functional (2.1). The emerging (generalized) Vlasov-Poisson system has the property that, for a monokinetic ansatz, it leads to the well-known bipolar Euler-Poisson system. In a second step, we show that in the joint zero-electron-mass and quasi-neutral limits the model asymptotically produces systems of the type (1.1).

2.1. **A bipolar Hamiltonian flow.**

Let $f = f(t,x,v)$ and $g = g(t,x,v)$ be phase-space distribution functions for positively and negatively charged particles standing for ions and electrons, respectively, with opposite charges for simplicity. Assume that $x, v \in \mathbb{R}^d$, with $d \in \mathbb{N}$ being the dimension. We denote by ρ and n the zero velocity moments of f and g, respectively, which represent the ions and electrons mass densities. Next, consider the Hamiltonian $\mathcal{H} = \mathcal{H}(f,g)$ defined as

$$\mathcal{H}(f,g) = \iint \tfrac{1}{2}|v|^2(f+g)\,\mathrm{d}v\,\mathrm{d}x + \mathcal{E}(\rho,n) \tag{2.1}$$

where the functional

$$\mathcal{E}(\rho,n) = \int h_1(\rho) + h_2(n)\,\mathrm{d}x + \int_x\int_y \tfrac{1}{2}(\rho-n)(x)G(x,y)(\rho-n)(y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.2}$$

represents the potential energy. We omit the dependency on time for simplicity. The functions $h_1 = h_1(\rho)$ and $h_2 = h_2(n)$ stand for internal energies of ions and electrons. They are connected to pressure functions $p_1 = p_1(\rho)$ and $p_2 = p(n)$ via $p_1'(\rho) = \rho h_1''(\rho)$ and $p_2'(n) = n h_2''(n)$. The last term in (2.2) represents the electrostatic potential energy induced by the nonhomogeneous Laplace equation

$$-\Delta\phi = \rho - n.$$

The electric potential ϕ induced by the charges is given by

$$\phi(x) = \int_y G(x,y)(\rho-n)(y)\,\mathrm{d}y$$

where $G = G(x,y)$ is the Green function, which is symmetric.

Set $z = (f,g)$ and proceed to express the Hamiltonian system $z_t = [z,H]$ for the functional (2.1). Following [18, 19], the Poisson bracket $[\cdot,\cdot]$ between two functionals $A = A(f)$ and $B = B(f)$ is defined by

$$[A,B] = \int_x\int_v f(x,v)\left\{\frac{\delta A}{\delta f},\frac{\delta B}{\delta f}\right\}\mathrm{d}x\,\mathrm{d}v \tag{2.3}$$

where $\frac{\delta A}{\delta f}$, $\frac{\delta B}{\delta f}$ are the functional derivatives of A, B, and $\{\cdot,\cdot\}$ stands for the canonical Poisson bracket defined by

$$\{F,G\} = \nabla_x F\cdot\nabla_v G - \nabla_v F\cdot\nabla_x G.$$

To express the system $z_t = [z,\mathcal{H}]$ we note that $z = (f,g)$ and use the formulas:

$$\begin{aligned}\partial_t f &= [f,\mathcal{H}] = -\left\{f,\frac{\delta\mathcal{H}}{\delta f}\right\},\\ \partial_t g &= [g,\mathcal{H}] = -\left\{g,\frac{\delta\mathcal{H}}{\delta g}\right\}.\end{aligned} \tag{2.4}$$

The formula $[f,\mathcal{H}] = -\left\{f, \frac{\delta\mathcal{H}}{\delta f}\right\}$ is obtained by noting that, for the functional $A(f) := f(x,v)$, the point evaluation of f at the point (x,v), the functional derivative is $\frac{\delta A}{\delta f} = \delta(x-x')\delta(v-v')$. The result then follows from a formal computation using (2.3) with $A(f) = f(x,v)$ and $B(f) = \mathcal{H}(f,g)$. The second equation in (2.4) is derived in a similar fashion.

The functional derivatives $\delta\mathcal{H}/\delta f$ and $\delta\mathcal{H}/\delta g$ of the Hamiltonian (2.1) – (2.2) are

$$\frac{\delta\mathcal{H}}{\delta f} = \tfrac{1}{2}|v|^2 + \frac{\delta\mathcal{E}}{\delta\rho} = \tfrac{1}{2}|v|^2 + h_1'(\rho) + \phi,$$
$$\frac{\delta\mathcal{H}}{\delta g} = \tfrac{1}{2}|v|^2 + \frac{\delta\mathcal{E}}{\delta n} = \tfrac{1}{2}|v|^2 + h_2'(n) - \phi.$$

Replacing these expressions in (2.4) yields the system

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x(h_1'(\rho)+\phi)\cdot\nabla_v f = 0, \\ \partial_t g + v\cdot\nabla_x g - \nabla_x(h_2'(n)-\phi)\cdot\nabla_v g = 0, \\ -\Delta\phi = \rho - n. \end{cases} \tag{2.5}$$

which we call a (generalized) bipolar Vlasov-Poisson system.

2.2. **Bipolar Euler-Poisson systems.**

In addition to the zero order moments

$$\rho = \int f \,\mathrm{d}v \quad \text{and} \quad n = \int g\,\mathrm{d}v$$

we now introduce the first order moments

$$\rho u = \int vf\,\mathrm{d}v \quad \text{and} \quad nw = \int vg\,\mathrm{d}v.$$

A formal computation of the evolution of these moments using (2.5) leads to

$$\begin{cases} \partial_t\rho + \nabla\cdot\rho u = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\cdot P + \nabla p_1(\rho) = -\rho\nabla\phi, \\ \partial_t n + \nabla\cdot nw = 0, \\ \partial_t(nw) + \nabla\cdot(nw\otimes w) + \nabla\cdot Q + \nabla p_2(n) = n\nabla\phi, \\ -\Delta\phi = \rho - n, \end{cases} \tag{2.6}$$

where

$$P = -\int (v-u)\otimes(v-u)f\,\mathrm{d}v - \int (v-u)\otimes uf\,\mathrm{d}v - \int u\otimes(v-u)f\,\mathrm{d}v$$

and similarly,

$$Q = -\int n(v-w)\otimes(v-w)g\,\mathrm{d}v - \int (v-w)\otimes wg\,\mathrm{d}v - \int w\otimes(v-w)g\,\mathrm{d}v.$$

Observe that if (f,g) is a solution of (2.5) that consists of monokinetic distributions

$$f(t,x,v) = \rho(t,x)\delta\big(v-u(t,x)\big)\,, \quad g(t,x,v) = n(t,x)\delta\big(v-w(t,x)\big) \tag{2.7}$$

then (ρ, u), (n, w) solve the bipolar Euler-Poisson system

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho u = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) = -\rho \nabla \phi, \\ \partial_t n + \nabla \cdot n w = 0, \\ \partial_t(n w) + \nabla \cdot (n w \otimes w) + \nabla p_2(n) = n \nabla \phi, \\ -\Delta \phi = \rho - n. \end{cases} \tag{2.8}$$

2.3. **The zero-electron-mass and quasi-neutral limits.**

We introduce a scaled version of (2.5):

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x (h_1'(\rho) + \phi) \cdot \nabla_v f = 0, \\ \varepsilon(\partial_t g + v \cdot \nabla_x g) - \nabla_x (h_2'(n) - \phi) \cdot \nabla_v g = 0, \\ -\delta \Delta \phi = \rho - n. \end{cases} \tag{2.9}$$

The positive constant ε is the ratio of electron mass to ion mass while $\delta > 0$ is the scaled Debye length squared. We consider the combined asymptotic zero-electron-mass and quasi-neutral limits $(\varepsilon, \delta \to 0)$. In the combined limit, the system (2.9) reduces via the asymptotic relations $n = \rho$ and $\nabla_x h_2'(n) = \nabla_x \phi$. It thus simplifies to the asymptotic system

$$\partial_t f + v \cdot \nabla_x f - \nabla_x (h_1'(\rho) + h_2'(\rho)) \cdot \nabla_v f = 0 \tag{2.10}$$

which is the multi-dimensional version of the kinetic equation in (1.1) when taking

$$h_1(\rho) = h_2(\rho) = \tfrac{1}{2\gamma(\gamma-1)} \rho^\gamma.$$

In this case, taking $\varepsilon = \delta = 0$ in (2.1) and (2.2) yields the energy functional associated with (1.1), which is conserved for a smooth solution f with fast decay at infinity, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\iint \tfrac{1}{2} |v|^2 f \, \mathrm{d}v \, \mathrm{d}x + \int \tfrac{1}{(q+1)(q+2)} \rho^{q+2} \, \mathrm{d}x \right) = 0.$$

2.4. **Monokinetic distributions and compressible Euler equations.**

Consider now the Euler system in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x \left(\rho u^2 + \frac{1}{\gamma} \rho^\gamma \right) = 0. \end{cases} \tag{2.11}$$

These equations model the evolution of a compressible fluid with density ρ and linear velocity u, where the power-law pressure is $\frac{1}{\gamma}\rho^\gamma$ with $\gamma > 1$ being the adiabatic exponent. Classical solutions exist locally in time, which can be deduced from a general existence result for the Cauchy problem of homogeneous systems of conservation laws with a convex entropy, *e.g.* [10, Theorem 5.1.1].

We provide a proof that classical solutions of (2.11) yield weak solutions of monokinetic type to the kinetic equation in (1.1) with $q = \gamma - 2$.

Proposition 2.1. *A classical solution (ρ, u) of (2.11) yields a monokinetic weak solution to the kinetic equation in (1.1) with $q = \gamma - 2$. Precisely, if $f = f(t, x, v)$ is given by*

$$f(t, x, v) = \rho(t, x) \delta(v - u(t, x))$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint f\varphi \,\mathrm{d}x\mathrm{d}v - \iint fv\cdot\partial_x\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x + \iint \rho^q\partial_x\rho\cdot f\partial_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x = 0. \tag{2.12}$$

for every compactly supported test function $\varphi = \varphi(x,v)$.

Proof. Multiplying the continuity equation by the map $(t,x)\mapsto\varphi(x,u(t,x))$, after integrating by parts one obtains

$$\int\partial_t\rho\cdot\varphi(x,u)\,\mathrm{d}x - \int\rho u\cdot\partial_x\varphi(x,u)\,\mathrm{d}x - \int\rho u\partial_x u\cdot\partial_v\varphi(x,u)\,\mathrm{d}x = 0.$$

Now, we evaluate the third term of the expression above using the momentum equation. We have:

$$\int\rho u\partial_x u\cdot\partial_v\varphi(x,u)\,\mathrm{d}x = -\int\rho\partial_t u\cdot\partial_v\varphi(x,u)\,\mathrm{d}x - \int\partial_x\left(\tfrac{1}{\gamma}\rho^\gamma\right)\cdot\partial_v\varphi(x,u)\,\mathrm{d}x.$$

Combining the previous two expressions results in

$$\begin{aligned}\int\partial_t\rho\cdot\varphi(x,u)\,\mathrm{d}x &+ \int\rho\partial_t u\cdot\partial_v\varphi(x,u)\,\mathrm{d}x\\ &- \int\rho u\partial_x u\cdot\partial_v\varphi(x,u)\,\mathrm{d}x + \int\partial_x\left(\tfrac{1}{\gamma}\rho^\gamma\right)\cdot\partial_v\varphi(x,u)\,\mathrm{d}x = 0\end{aligned}$$

which is equivalent to (2.12) by the monokinetic assumption. □

A connection of this sort is also studied in [11] for renormalized solutions of an Euler system, yielding monokinetic solutions to a linear Vlasov equation with a prescribed potential. Moreover, in [9], the existence of monokinetic solutions for a highly singular Vlasov equation is proven as limits of the Wigner transform of solutions to a related logarithmic Schrödinger equation.

Remark 2.2. As previously noted, the highly nonlinear term in the kinetic equation of (1.1) presents significant challenges. For monokinetic distributions $f = \rho\delta(v-u)$ with non-smooth (ρ,u), defining a notion of weak solution requires giving meaning under the integral to terms like $\rho^{q+2}\partial_x\varphi(x,u(t,x))$ and $\partial_v\varphi(x,u(t,x))\rho^{q+2}\partial_x u$, where $\varphi=\varphi(x,v)$ is a smooth, compactly supported test function. Even assuming $u\in BV$, avoiding unnecessary technicalities would still require ρ to be continuous – a level of regularity that is not available nor expected for general weak solutions.

3. Analytic functions and norms

We start by fixing some notation that will be used throughout the rest of the manuscript. The sets of positive and nonnegative integers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. For functions depending on x, v, or both, the corresponding supremum norm, $\|\cdot\|_\infty$, refers to the supremum taken over x, v, or both, respectively. Derivatives with respect to x (resp. v) are denoted by the partial derivative ∂_x (resp. ∂_v). Derivatives with respect to t (resp. λ) are denoted by the total derivative $\mathrm{d}/\mathrm{d}t$ (resp. $\mathrm{d}/\mathrm{d}\lambda$). We use the standard multinomial coefficient notation:

$$\binom{r}{k_1,\cdots,k_q} = \frac{r!}{k_1!\cdots k_q!}$$

where $q\in\mathbb{N}$ and $r,k_1,\ldots,k_q\in\mathbb{N}_0$, with the usual convention $0! = 1$.

3.1. Preliminaries.

Definition 3.1. A function $f \in C^\infty(\mathbb{R}^2)$ is said to be analytic if there exist $C > 0$ and $\Lambda > 0$ such that

$$\frac{\Lambda^{k+l}}{k!l!}\|\partial_x^k \partial_v^l f\|_\infty \leq C \qquad \forall k, l \in \mathbb{N}_0. \tag{3.1}$$

In this case, we say that f is analytic with radius of convergence Λ.

If f only depends on x (resp. v), then it is analytic if (3.1) holds for $l = 0$ (resp. $k = 0$). The constant Λ is called the radius of convergence since, for each $(y, w) \in \mathbb{R}^2$, condition (3.1) implies that

$$\sum_{k,l\geq 0} \frac{\partial_x^k \partial_v^l f(y,w)}{k!l!}(x-y)^k(v-w)^l \leq C \sum_{k\geq 0}\left(\frac{x-y}{\Lambda}\right)^k \sum_{j\geq 0}\left(\frac{v-w}{\Lambda}\right)^l$$

which converges whenever

$$|x-y| < \Lambda \qquad \text{and} \qquad |v-w| < \Lambda.$$

Now let $\lambda > 0$. Within this analytic functions framework, the first norm that we consider is the following:

$$\|f\|_\lambda = \sum_{k,l\geq 0} \frac{\lambda^{k+l}}{k!l!}\|\partial_x^k \partial_v^l f\|_\infty. \tag{3.2}$$

One can readily check that if f is analytic with radius of convergence Λ, then $\|f\|_\lambda < \infty$ for every $\lambda < \Lambda$. Moreover, for each $n \in \mathbb{N}_0$ we consider the n^{th} order derivate of $\|\cdot\|_\lambda$ with respect to λ:

$$|f|_{\lambda,n} = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\|f\|_\lambda = \sum_{k+l\geq n} \frac{(k+l)!}{(k+l-n)!}\frac{\lambda^{k+l-n}}{k!l!}\|\partial_x^k \partial_v^l f\|_\infty. \tag{3.3}$$

Furthermore, we consider a norm $\|\cdot\|_{H,\lambda}$ and a seminorm $|\cdot|_{H,\lambda}$ given by

$$\|f\|_{H,\lambda} = \sum_{n\geq 0} \frac{1}{(n!)^2}|f|_{\lambda,n} \qquad \text{and} \qquad |f|_{H,\lambda} = \sum_{n\geq 1} \frac{n^2}{(n!)^2}|f|_{\lambda,n}. \tag{3.4}$$

Proposition 3.2. *Let f be analytic with radius of convergence Λ. Then for each $0 < \lambda < \Lambda$*

$$\|f\|_{H,\lambda} < \infty \qquad \textit{and} \qquad |f|_{H,\lambda} < \infty.$$

Proof. We first claim that, for each $0 < \lambda < \Lambda$ and $n \in \mathbb{N}_0$,

$$|f|_{\lambda,n} \leq C\Lambda^2 \frac{(n+1)!}{(\Lambda-\lambda)^{n+2}}$$

for some $C > 0$. Indeed, using the definition of $|\cdot|_{\lambda,n}$ together with (3.1) we have:

$$\begin{aligned}
|f|_{\lambda,n} &\leq \frac{C}{\Lambda^n}\sum_{k+l\geq n}\frac{(k+l)!}{(k+l-n)!}\left(\frac{\lambda}{\Lambda}\right)^{k+l-n} \\
&= \frac{C}{\Lambda^n}\frac{\mathrm{d}^n}{\mathrm{d}r^n}\left(\sum_{k,l\geq 0} r^{k+l}\right)\Big|_{r=\lambda/\Lambda} \\
&= \frac{C}{\Lambda^n}\frac{\mathrm{d}^n}{\mathrm{d}r^n}\left(\frac{1}{(1-r)^2}\right)\Big|_{r=\lambda/\Lambda}
\end{aligned}$$

from which the claim follows by noting that

$$\frac{\mathrm{d}^n}{\mathrm{d}r^n}\frac{1}{(1-r)^2} = \frac{(n+1)!}{(1-r)^{n+2}}.$$

Consequently,

$$\|f\|_{H,\lambda} \leq C\Lambda^2 \sum_{n\geq 0} a_n \qquad \text{and} \qquad |f|_{H,\lambda} \leq C\Lambda^2 \sum_{n\geq 1} b_n$$

where

$$a_n = \frac{(n+1)!}{(n!)^2} \frac{1}{(\Lambda-\lambda)^{n+2}} \qquad \text{and} \qquad b_n = n^2 a_n.$$

The result follows from the ratio test since

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{(\Lambda-\lambda)(n+1)^2} \to 0 \quad \text{as} \quad n \to \infty.$$

□

3.2. **A couple of lemmas.**

Lemma 3.3. *Let f be analytic with radius of convergence Λ. Then for $0<\lambda<\Lambda$ and $n \in \mathbb{N}_0$*

$$|\partial_x f|_{\lambda,n} \leq |f|_{\lambda,n+1}. \tag{3.5}$$

Proof. By the definition of $|\cdot|_{\lambda,n}$ we have

$$\begin{aligned}
|\partial_x f|_{\lambda,n} &= \sum_{\substack{k+l\geq n\\ k,l\geq 0}} \frac{(k+l)!}{(k+l-n)!} \frac{\lambda^{k+l-n}}{k!l!} \|\partial_x^{k+1}\partial_v^l f\|_\infty \\
&= \sum_{\substack{k+l\geq n+1\\ k\geq 1\\ l\geq 0}} \frac{(k+l-1)!}{(k+l-n-1)!} \frac{\lambda^{k+l-n-1}}{(k-1)!l!} \|\partial_x^{k}\partial_v^l f\|_\infty.
\end{aligned}$$

Now, using the fact that

$$\frac{(k+l-1)!}{(k-1)!} \leq \frac{(k+l)!}{k!} \qquad \forall k \in \mathbb{N},\ l \in \mathbb{N}_0,$$

we deduce

$$\begin{aligned}
|\partial_x f|_{\lambda,n} &\leq \sum_{\substack{k+l\geq n+1\\ k\geq 1\\ l\geq 0}} \frac{(k+l)!}{(k+l-n-1)!} \frac{\lambda^{k+l-n-1}}{k!l!} \|\partial_x^{k}\partial_v^l f\|_\infty \\
&\leq \sum_{\substack{k+l\geq n+1\\ k,l\geq 0}} \frac{(k+l)!}{(k+l-n-1)!} \frac{\lambda^{k+l-n-1}}{k!l!} \|\partial_x^{k}\partial_v^l f\|_\infty \\
&= |f|_{\lambda,n+1}
\end{aligned}$$

as desired. □

Lemma 3.4. *Let $\rho=\rho(x)$ be analytic with radius of convergence Λ. Then for $0<\lambda<\Lambda$ and $q \in \mathbb{N}$*

$$\|\rho\|_\lambda^q = \sum_{r\geq 0} \sum_{k_1+\cdots+k_q=r} \frac{\lambda^r}{k_1!\cdots k_q!} \prod_{j=1}^q \|\partial_x^{k_j}\rho\|_\infty. \tag{3.6}$$

Proof. We proceed by induction on $q \in \mathbb{N}$. The case $q=1$ follows from the definition of $\|\cdot\|_\lambda$ as

$$\|\rho\|_\lambda = \sum_{r\geq 0} \frac{\lambda^r}{r!} \|\partial_x^r \rho\|_\infty.$$

Now, assume that (3.6) holds for $q-1$ for some $q \in \mathbb{N} \setminus \{1\}$. We have:

$$\begin{aligned}
\sum_{r\geq 0} \sum_{k_1+\cdots+k_q=r} \frac{\lambda^r}{k_1!\cdots k_q!} \prod_{j=1}^q \|\partial_x^{k_j}\rho\|_\infty \\
&= \sum_{r\geq 0} \sum_{k_1=0}^{r} \sum_{k_2+\cdots+k_q=r-k_1} \frac{\lambda^r}{k_1!\cdots k_q!} \prod_{j=1}^q \|\partial_x^{k_j}\rho\|_\infty \\
&= \sum_{k_1\geq 0} \frac{\lambda^{k_1}}{k_1!} \|\partial_x^{k_1}\rho\|_\infty \sum_{r\geq k_1} \sum_{k_2+\cdots+k_q=r-k_1} \frac{\lambda^{r-k_1}}{k_2!\cdots k_q!} \prod_{j=2}^q \|\partial_x^{k_j}\rho\|_\infty \\
&= \|\rho\|_\lambda \sum_{r\geq 0} \sum_{k_2+\cdots+k_q=r} \frac{\lambda^r}{k_2!\cdots k_q!} \prod_{j=2}^q \|\partial_x^{k_j}\rho\|_\infty \\
&= \|\rho\|_\lambda \|\rho\|_\lambda^{q-1}
\end{aligned}$$

as desired. □

3.3. **An auxiliary analytic function.**

Consider a weight function ω given by

$$\omega(v) = \frac{1}{\pi(1+v^2)} \tag{3.7}$$

and note that

$$\int \omega(v)\,\mathrm{d}v = 1.$$

Let $\alpha = \alpha(v)$ be given by

$$\alpha(v) = \frac{\partial_v \omega(v)}{\omega(v)} = -\frac{2v}{1+v^2}. \tag{3.8}$$

The function α is analytic with radius of convergence $\Lambda = 1$, and hence for $0 < \lambda_0 < 1$ we have

$$\alpha_0 \coloneqq \|\alpha\|_{\lambda_0} < \infty. \tag{3.9}$$

4. Main result

This section contains the statement of the main result of this manuscript, as well as its proof in the subsequent subsections.

Consider positive parameters λ_0, K and T satisfying:

$$0 < T < 1, \qquad T < \lambda_0 < 1, \qquad 0 < K < \frac{\lambda_0}{T} - 1. \tag{4.1}$$

Let a function $\lambda : [0,T] \to \mathbb{R}$ be given by

$$\lambda(t) = \lambda_0 - (K+1)t. \tag{4.2}$$

Additionally, for $M > 0$ consider the following set $\mathrm{X}^M_{\lambda_0,K,T}$ of analytic functions:

$$\mathrm{X}^M_{\lambda_0,K,T} = \left\{ f \in C\big([0,T]; C^\infty(\mathbb{R}^2)\big) \;\middle|\; \sup_{t\in[0,T]} \|f(t)\|_{H,\lambda_0} + \int_0^T |f(t)|_{H,\lambda(t)}\,\mathrm{d}t \leq M \right\} \tag{4.3}$$

endowed with the metric induced by a norm $\|\cdot\|_{\mathrm{Z}}$ defined as

$$\|f\|_{\mathrm{Z}} \coloneqq \sup_{t\in[0,T]} \|f(t)\|_{\lambda_0} + \int_0^T |f(t)|_{\lambda(t),1}\,\mathrm{d}t. \tag{4.4}$$

The main result of this work is stated as follows:

Theorem 4.1. *Let $q \in \mathbb{N}_0$, let λ_0, K, T be selected to satisfy* (4.1)*, and assume that M satisfies*

$$(1+qT)(1+\alpha_0)M^{q+1}e^{\alpha_0 M^{q+1}T} < 1 < K - \lambda_0 - 16M^{q+1} \tag{4.5}$$

and $f_0 \in C^\infty(\mathbb{R}^2)$ is such that

$$\|\pi(1+v^2)f_0\|_{H,\lambda_0} \le Me^{-(16+\alpha_0)M^{q+1}} \tag{4.6}$$

where α_0 is as in (3.9).

Then there exists a unique solution f to problem (1.1) that belongs to $\mathrm{X}^M_{\lambda_0,K,T}$.

Remark 4.2. For $T \ll \lambda_0$ we can take K large enough so that M is not required to satisfy $M \ll 1$ for the right-hand side of (4.5) to hold. Thus, M is primarily restricted by the the left-hand side of (4.5), which imposes a moderate restriction compared to $M \ll 1$. Similar conditions on the size of the initial data appear in the analysis of the case $q = 0$ in [16], and also in [8].

The proof of the main theorem relies on a Banach fixed point argument on the space $\mathrm{X}^M_{\lambda_0,K,T}$, which is complete as it is a closed subset of a Banach space $\mathrm{Z}_{\lambda_0,K,T}$ defined as

$$\mathrm{Z}_{\lambda_0,K,T} = \left\{ f \in C\big([0,T]; C^\infty(\mathbb{R}^2)\big) \;\middle|\; \|f\|_{\mathrm{Z}} = \sup_{t\in[0,T]} \|f(t)\|_{\lambda_0} + \int_0^T |f(t)|_{\lambda(t),1}\,\mathrm{d}t < \infty \right\}. \tag{4.7}$$

In Appendix A we provide a proof that $\mathrm{Z}_{\lambda_0,K,T}$ is indeed complete. To check that $\mathrm{X}^M_{\lambda_0,K,T}$ is a closed subset of $\mathrm{Z}_{\lambda_0,K,T}$, we take a sequence $(f_m) \subseteq \mathrm{X}^M_{\lambda_0,K,T}$ such that $\|f_m - f\|_{\mathrm{Z}} \to 0$ as $m \to \infty$ for some $f \in \mathrm{Z}_{\lambda_0,K,T}$. By the definition of $\mathrm{X}^M_{\lambda_0,K,T}$, for every $m \in \mathbb{N}$ we have

$$\sup_{t\in[0,T]} \|f_m(t)\|_{H,\lambda_0} + \int_0^T |f_m(t)|_{H,\lambda(t)}\,\mathrm{d}t \le M. \tag{4.8}$$

Moreover, from the assumed convergence we have for all $t \in [0,T]$ and every $k, l \ge 0$ that

$$\|\partial_x^k \partial_v^l \big(f_m(t) - f(t)\big)\|_\infty \to 0 \quad \text{as } m \to \infty$$

which implies

$$\|\partial_x^k \partial_v^l f_m(t)\|_\infty \to \|\partial_x^k \partial_v^l f(t)\|_\infty \quad \text{as } m \to \infty.$$

By Fatou's lemma we deduce for each $t \in [0,T]$ that

$$\begin{aligned}
\|f(t)\|_{H,\lambda_0} &= \sum_{n\ge 0} \frac{1}{(n!)^2} \sum_{k+l\ge n} \frac{(k+l)!}{(k+l-n)!} \frac{\lambda_0^{k+l-n}}{k!l!} \|\partial_x^k \partial_v^l f(t)\|_\infty \\
&= \sum_{n\ge 0} \frac{1}{(n!)^2} \sum_{k+l\ge n} \frac{(k+l)!}{(k+l-n)!} \frac{\lambda_0^{k+l-n}}{k!l!} \lim_{m\to\infty} \|\partial_x^k \partial_v^l f_m(t)\|_\infty \\
&\le \liminf_{m\to\infty} \|f_m(t)\|_{H,\lambda_0}
\end{aligned}$$

and similarly,

$$\int_0^T |f(t)|_{H,\lambda(t)}\,\mathrm{d}t \le \liminf_{m\to\infty} \int_0^T |f_m(t)|_{H,\lambda(t)}\,\mathrm{d}t \tag{4.9}$$

from which by (4.8) we see

$$\sup_{t\in[0,T]} \|f(t)\|_{H,\lambda_0} + \int_0^T |f(t)|_{H,\lambda(t)}\,\mathrm{d}t \le M \tag{4.10}$$

and so $f \in \mathrm{X}^M_{\lambda_0,K,T}$, as desired.

Next, we define a contraction mapping to which we will apply the Banach fixed point theorem. First, we observe that if f solves (1.1), then $g = f/\omega$ solves the equation

$$\partial_t g + v \cdot \partial_x g - \rho^q \partial_x \rho \cdot (\partial_v g + \alpha g) = 0$$

where ω and α are given by (3.7) and (3.8), respectively. For a general account on the theory of quasilinear partial differential equations we refer to [23].

Define a map $\Psi : \mathrm{X}^M_{\lambda_0,K,T} \to \mathrm{X}^M_{\lambda_0,K,T}$ as follows:

$$\text{Given } h \in \mathrm{X}^M_{\lambda_0,K,T}, \text{ let } \Psi(h) = g \text{ solve the linear initial-value problem}$$
$$\begin{cases} \partial_t g + v \cdot \partial_x g - \sigma^q \partial_x \sigma \cdot (\partial_v g + \alpha g) = 0, \\ \sigma = \displaystyle\int \omega h \,\mathrm{d}v, \\ g|_{t=0} = g_0 = f_0/\omega. \end{cases} \tag{4.11}$$

Clearly, if g is a fixed point of Ψ, then $f = \omega g$ solves problem (1.1).

4.1. **Preliminary estimates.**

Proposition 4.3. *If $g = g(t,x,v)$ is an analytic solution of (4.11), then for each $k, l \in \mathbb{N}_0$*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^k \partial_v^l g\|_\infty \le l \|\partial_x^{k+1} \partial_v^{l-1} g\|_\infty + I_1 + I_2 \tag{4.12}$$

where

$$I_1 = \sum_{i=0}^{k-1} \sum_{r=0}^{k-i} \sum_{k_1+\cdots+k_q=r} \Theta \prod_{s=1}^q \|\partial_x^{k_s} \sigma\|_\infty \cdot \|\partial_x^{k-i-r+1} \sigma\|_\infty \cdot \|\partial_x^i \partial_v^{l+1} g\|_\infty$$
$$I_2 = \sum_{i=0}^{k} \sum_{r=0}^{k-i} \sum_{k_1+\cdots+k_q=r} \Theta \prod_{s=1}^q \|\partial_x^{k_s} \sigma\|_\infty \cdot \|\partial_x^{k-i-r+1} \sigma\|_\infty \sum_{j=0}^{l} \binom{l}{j} \|\partial_v^{l-j} \alpha\|_\infty \cdot \|\partial_x^i \partial_v^j g\|_\infty$$

and

$$\Theta = \binom{k}{i} \binom{k-i}{r} \binom{r}{k_1, \cdots, k_q}. \tag{4.13}$$

Proof. We apply the operator $\partial_x^k \partial_v^l$ to equation (4.11). The first term simply becomes $\partial_t(\partial_x^k \partial_v^l g)$, while the second is computed as follows,

$$\begin{aligned} \partial_x^k \partial_v^l (v \cdot \partial_x g) &= \partial_v^l (v \cdot \partial_x^{k+1} g) \\ &= \sum_{j=0}^{l} \binom{l}{j} \partial_v^j v \cdot \partial_v^{l-j} \partial_x^{k+1} g \\ &= v \cdot \partial_x (\partial_x^k \partial_v^l g) + l \partial_x^{k+1} \partial_v^{l-1} g. \end{aligned}$$

The remaining terms are treated using in particular the general Leibniz rule which we recall here. For a smooth function $\rho = \rho(x)$, for each $r \in \mathbb{N}_0$ and $q \in \mathbb{N}$ it holds:

$$\partial_x^r (\rho^q) = \sum_{k_1+\cdots+k_q=r} \binom{r}{k_1, \cdots, k_q} \prod_{j=1}^q \partial_x^{k_j} \rho. \tag{4.14}$$

We have:

$$\begin{aligned}
\partial_x^k\partial_v^l(\sigma^q\partial_x\sigma\cdot\partial_v g) &= \partial_x^k(\sigma^q\partial_x\sigma\cdot\partial_v^{l+1}g)\\
&= \sigma^q\partial_x\sigma\cdot\partial_v(\partial_x^k\partial_v^l g) + \sum_{i=0}^{k-1}\binom{k}{i}\partial_x^{k-i}(\sigma^q\partial_x\sigma)\cdot\partial_x^i\partial_v^{l+1}g\\
&= \sigma^q\partial_x\sigma\cdot\partial_v(\partial_x^k\partial_v^l g) + \sum_{i=0}^{k-1}\binom{k}{i}\sum_{r=0}^{k-i}\binom{k-i}{r}\partial_x^r\sigma^q\cdot\partial_x^{k-i-r+1}\sigma\cdot\partial_x^i\partial_v^{l+1}g\\
&= \sigma^q\partial_x\sigma\cdot\partial_v(\partial_x^k\partial_v^l g) + \sum_{i=0}^{k-1}\sum_{r=0}^{k-i}\sum_{k_1+\cdots+k_q=r}\Theta\prod_{s=1}^{q}\partial_x^{k_s}\sigma\cdot\partial_x^{k-i-r+1}\sigma\cdot\partial_x^i\partial_v^{l+1}g,
\end{aligned}$$

and similarly,

$$\partial_x^k\partial_v^l(\sigma^q\partial_x\sigma\cdot\alpha g) = \sum_{i=0}^{k}\sum_{r=0}^{k-i}\sum_{k_1+\cdots+k_q=r}\Theta\prod_{s=1}^{q}\partial_x^{k_s}\sigma\cdot\partial_x^{k-i-r+1}\sigma\sum_{j=0}^{l}\binom{l}{j}\partial_v^{l-j}\alpha\cdot\partial_x^i\partial_v^j g.$$

The desired result is obtained upon applying the maximum principle (B.1) to the kinetic equation satisfied by $\partial_x^k\partial_v^l g$, where in this case $a=\sigma^q\partial_x\sigma$ is bounded. □

Proposition 4.4. *If $g=g(t,x,v)$ is an analytic solution of (4.11), then for $\lambda>0$ and each $n\in\mathbb{N}_0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}|g|_{\lambda,n} \le \lambda|g|_{\lambda,n+1} + n|g|_{\lambda,n} + \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}\|\sigma\|_\lambda^q|g|_{\lambda,1} + |\sigma|_{\lambda,1}\|\sigma\|_\lambda^q\|\alpha\|_\lambda\|g\|_\lambda\Big) \tag{4.15}$$

Proof. We first multiply inequality (4.12) by $\dfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda^{k+l}\dfrac{1}{k!l!}$ and then sum over $k,l\in\mathbb{N}_0$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|g|_{\lambda,n} \le \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\sum_{\substack{k\ge0\\ l\ge1}}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_v^{l-1}g\|_\infty + J_1 + J_2$$

where

$$J_1 = \sum_{k,l\ge0}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda^{k+l}\frac{1}{k!l!}I_1, \qquad J_2 = \sum_{k,l\ge0}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda^{k+l}\frac{1}{k!l!}I_2.$$

Now, we analyse each one of the terms on the right-hand-side of the previous inequality. The first term is treated as follows:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\sum_{\substack{k\ge0\\ l\ge1}}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_v^{l-1}g\|_\infty &= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\sum_{k,l\ge0}\frac{\lambda^{k+l+1}}{k!l!}\|\partial_x^{k+1}\partial_v^{l}g\|_\infty\\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda\|\partial_x g\|_\lambda)\\
&= \sum_{m=0}^{n}\binom{n}{m}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\lambda\cdot\frac{\mathrm{d}^{n-m}}{\mathrm{d}\lambda^{n-m}}\|\partial_x\|_\lambda\\
&= \lambda|\partial_x g|_{\lambda,n} + n|\partial_x g|_{\lambda,n-1}\\
&\le \lambda|g|_{\lambda,n+1} + n|g|_{\lambda,n}.
\end{aligned}$$

where in the last step we used inequality (3.5).

The terms J_1, J_2 are more elaborate and the main idea to handle them is to carefully interchange the various sums that appear in their expressions. Let

$$S = \sum_{k_1+\cdots+k_q=r}\binom{r}{k_1,\cdots,k_q}\prod_{s=1}^{q}\|\partial_x^{k_s}\sigma\|_\infty.$$

We have:

$$
\begin{aligned}
J_1 &= \sum_{k,l\geq 0} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda^{k+l} \frac{1}{k!l!} I_1 \\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \sum_{\substack{k\geq 1\\ l\geq 0}} \frac{\lambda^{k+l}}{k!l!} \sum_{i=0}^{k-1} \binom{k}{i} \sum_{r=0}^{k-i} \binom{k-i}{r} S\cdot \|\partial_x^{k-i-r+1}\sigma\|_\infty \cdot \|\partial_x^i \partial_v^{l+1} g\|_\infty \\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \sum_{i,l\geq 0} \frac{\lambda^{i+l}}{i!l!} \|\partial_x^i \partial_v^{l+1} g\|_\infty \sum_{k\geq i+1} \sum_{r=0}^{k-i} \frac{\lambda^{k-i}}{r!(k-i-r!)} S\cdot \|\partial_x^{k-i-r+1}\sigma\|_\infty \\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\|\partial_v g\|_\lambda \sum_{k\geq 1} \sum_{r=0}^{k} \frac{\lambda^k}{r!(k-r)!} S\cdot \|\partial_x^{k-r+1}\sigma\|_\infty \right) \\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\|\partial_v g\|_\lambda \left(\sum_{k\geq 0} \sum_{r=0}^{k} \frac{\lambda^k}{r!(k-r)!} S\cdot \|\partial_x^{k-r+1}\sigma\|_\infty - \|\sigma\|_\infty^q \|\partial_x \sigma\|_\infty \right) \right).
\end{aligned}
$$

Using (3.6) we deduce:

$$
\begin{aligned}
&\sum_{k\geq 0} \sum_{r=0}^{k} \frac{\lambda^k}{r!(k-r)!} S\cdot \|\partial_x^{k-r+1}\sigma\|_\infty \\
&\qquad = \sum_{r\geq 0} \sum_{k\geq r} \frac{\lambda^k}{r!(k-r)!} S\cdot \|\partial_x^{k-r+1}\sigma\|_\infty \\
&\qquad = \sum_{r\geq 0} \sum_{k\geq 0} \frac{\lambda^{k+r}}{r!k!} S\cdot \|\partial_x^{k+1}\sigma\|_\infty \\
&\qquad = \sum_{k\geq 0} \frac{\lambda^k}{k!} \|\partial_x^{k+1}\sigma\|_\infty \sum_{r\geq 0} \sum_{k_1+\cdots+k_q=r} \frac{\lambda^r}{k_1!\cdots k_q!} \prod_{s=1}^{q} \|\partial_x^{k_s}\sigma\|_\infty \\
&\qquad = \|\partial_x \sigma\|_\lambda \|\sigma\|_\lambda^q.
\end{aligned}
$$

Combining the two previous identities and using (3.5) yields:

$$
\begin{aligned}
J_1 &= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\|\partial_v g\|_\lambda \left(\|\partial_x\sigma\|_\lambda \|\sigma\|_\lambda^q - \|\sigma\|_\infty^q \|\partial_x\sigma\|_\infty \right) \Big) \\
&= \sum_{m=0}^{n} \binom{n}{m} \frac{\mathrm{d}^{n-m}}{\mathrm{d}\lambda^{n-m}} \|\partial_v g\|_\lambda \cdot \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\|\partial_x\sigma\|_\lambda \|\sigma\|_\lambda^q - \|\sigma\|_\infty^q \|\partial_x\sigma\|_\infty \Big) \\
&= |\partial_v g|_{\lambda,n} \Big(\|\partial_x\sigma\|_\lambda \|\sigma\|_\lambda^q - \|\sigma\|_\infty^q \|\partial_x\sigma\|_\infty \Big) + \sum_{m=1}^{n} \binom{n}{m} \frac{\mathrm{d}^{n-m}}{\mathrm{d}\lambda^{n-m}} \|\partial_v g\|_\lambda \cdot \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(\|\partial_x\sigma\|_\lambda \|\sigma\|_\lambda^q \Big) \\
&\leq \sum_{m=0}^{n} \binom{n}{m} |\partial_v g|_{\lambda,n-m} \sum_{r=0}^{m} \binom{m}{r} |\partial_x \sigma|_{\lambda,m-r} \frac{\mathrm{d}^r}{\mathrm{d}\lambda^r} \|\sigma\|_\lambda^q \\
&\leq \sum_{m=0}^{n} \binom{n}{m} |g|_{\lambda,n-m+1} \sum_{r=0}^{m} \binom{m}{r} |\sigma|_{\lambda,m-r+1} \frac{\mathrm{d}^r}{\mathrm{d}\lambda^r} \|\sigma\|_\lambda^q \\
&= \sum_{m=0}^{n} \binom{n}{m} |g|_{\lambda,n-m+1} \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} \Big(|\sigma|_{\lambda,1} \|\sigma\|_\lambda^q \Big) \\
&= \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(|\sigma|_{\lambda,1} \|\sigma\|_\lambda^q |g|_{\lambda,1} \Big).
\end{aligned}
$$

Similarly, one deduces that

$$
J_2 \leq \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(|\sigma|_{\lambda,1} \|\sigma\|_\lambda^q \|\alpha\|_\lambda \|g\|_\lambda \Big).
$$

Putting all the previous estimates together yields the desired result. □

4.2. **A time-dependent estimate.**

Recall that $\lambda(t) = \lambda_0 - (K+1)t$ for $t \in [0,T]$ with $\lambda_0 > (1+K)T$.

Proposition 4.5. *If $g = g(t,x,v)$ is an analytic solution of (4.11), then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|_{H,\lambda(t)} \leq{}& (\lambda_0 - K)|g(t)|_{H,\lambda(t)}\\ &+\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma(t)|_{\lambda(t),1}\|\sigma(t)\|^q_{\lambda(t)}|g(t)|_{\lambda(t),1}\Big)\\ &+\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma(t)|_{\lambda(t),1}\|\sigma(t)\|^q_{\lambda(t)}\|\alpha\|_{\lambda(t)}\|g(t)\|_{\lambda(t)}\Big).\end{aligned}\tag{4.16}$$

Proof. First, note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|_{H,\lambda(t)} = \sum_{n\geq 0}\frac{\lambda'(t)}{(n!)^2}|g(t)|_{\lambda(t),n+1} + \sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}}{\mathrm{d}t}|g(t)|_{\lambda,n}\Big|_{\lambda=\lambda(t)}.$$

Now, using (4.15) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|_{H,\lambda(t)} \leq{}& \sum_{n\geq 0}\frac{\lambda'(t)+\lambda(t)}{(n!)^2}|g(t)|_{\lambda(t),n+1} + \sum_{n\geq 0}\frac{n}{(n!)^2}|g(t)|_{\lambda(t),n}\\ &+\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma(t)|_{\lambda(t),1}\|\sigma(t)\|^q_{\lambda(t)}|g(t)|_{\lambda(t),1}\Big)\\ &+\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma(t)|_{\lambda(t),1}\|\sigma(t)\|^q_{\lambda(t)}\|\alpha\|_{\lambda(t)}\|g(t)\|_{\lambda(t)}\Big).\end{aligned}$$

The desired inequality is obtained from the previous expression together with the following facts:

$$\begin{aligned}\sum_{n\geq 0}\frac{\lambda'(t)+\lambda(t)}{(n!)^2}|g(t)|_{\lambda(t),n+1} &\leq (\lambda_0 - 1 - K)|g(t)|_{H,\lambda(t)},\\ \sum_{n\geq 0}\frac{n}{(n!)^2}|g(t)|_{\lambda(t),n} &\leq |g(t)|_{H,\lambda(t)}.\end{aligned}$$

□

4.3. **Supplementary lemmas.**

Lemma 4.6. *Let g and σ be analytic functions. Then for each $q \in \mathbb{N}_0$*

$$\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}\|\sigma\|^q_\lambda|g|_{\lambda,1}\Big) \leq 16\|\sigma\|^{q+1}_{H,\lambda}|g|_{H,\lambda} + 16|\sigma|_{H,\lambda}\|\sigma\|^q_{H,\lambda}\|g\|_{H,\lambda}.\tag{4.17}$$

Proof. We proceed by induction on $q \geq 0$. The case $q = 0$ is handled in [16, 8], nevertheless we provide a proof here for convenience. We have:

$$\begin{aligned}\sum_{n\geq 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}|g|_{\lambda,1}\Big) &= \sum_{n\geq 0}\frac{1}{(n!)^2}\sum_{m=0}^{n}\binom{n}{m}|\sigma|_{\lambda,n-m+1}|g|_{\lambda,m+1}\\ &= \sum_{m\geq 0}|g|_{\lambda,m+1}\sum_{n\geq m}\binom{n}{m}\frac{1}{(n!)^2}|\sigma|_{\lambda,n-m+1}\\ &= \sum_{m\geq 0}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}\sum_{n\geq 0}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}A(n,m)\end{aligned}$$

where

$$A(n,m) = \binom{n+m}{m}\frac{((n+1)!)^2((m+1)!)^2}{((n+m)!)^2}.$$

A straightforward computation yields that $A(n,m) \le 24$ for $n \ge 2$ and $m \ge 2$. Hence,

$$\begin{aligned}
\sum_{n\ge 0} \frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}|g|_{\lambda,1}\Big)& \\
&= |\sigma|_{\lambda,1}\sum_{m\ge 0}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}A(0,m) + |\sigma|_{\lambda,2}\sum_{m\ge 0}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}A(1,m) \\
&\quad + |g|_{\lambda,1}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}A(n,0) + |g|_{\lambda,2}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}A(n,1) \\
&\quad + \sum_{m\ge 2}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}A(n,m) \\
&\le |\sigma|_{\lambda,1}\sum_{m\ge 0}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}(m+1)^2 + 4|\sigma|_{\lambda,2}\sum_{m\ge 0}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}(m+1) \\
&\quad + |g|_{\lambda,1}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}(n+1)^2 + 4|g|_{\lambda,2}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2}(n+1) \\
&\quad + 24\sum_{m\ge 2}\frac{|g|_{\lambda,m+1}}{((m+1)!)^2}\sum_{n\ge 2}\frac{|\sigma|_{\lambda,n+1}}{((n+1)!)^2} \\
&\le (|\sigma|_{\lambda,1} + 4|\sigma|_{\lambda,2})|g|_{H,\lambda} + (|g|_{\lambda,1} + 4|g|_{\lambda,2})|\sigma|_{H,\lambda} \\
&\quad + 24\sum_{m\ge 3}\frac{|g|_{\lambda,m}}{(m!)^2}\sum_{n\ge 3}\frac{|\sigma|_{\lambda,n}}{(n!)^2}.
\end{aligned}$$

Now, we observe that

$$24\sum_{m\ge 3}\frac{|g|_{\lambda,m}}{(m!)^2}\sum_{n\ge 3}\frac{|\sigma|_{\lambda,n}}{(n!)^2} \le 12|g|_{H,\lambda}\sum_{n\ge 3}\frac{|\sigma|_{\lambda,n}}{(n!)^2} + 12|\sigma|_{H,\lambda}\sum_{n\ge 3}\frac{|g|_{\lambda,m}}{(m!)^2}$$

whence

$$\begin{aligned}
\sum_{n\ge 0} \frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}|g|_{\lambda,1}\Big) &\le \Big(|\sigma|_{\lambda,1} + 4|\sigma|_{\lambda,2} + 12\sum_{n\ge 3}\frac{|\sigma|_{\lambda,n}}{(n!)^2}\Big)|g|_{H,\lambda} \\
&\quad + \Big(|g|_{\lambda,1} + 4|g|_{\lambda,2} + 12\sum_{m\ge 3}\frac{|g|_{\lambda,m}}{(m!)^2}\Big)|\sigma|_{H,\lambda} \\
&\le 16\|\sigma\|_{H,\lambda}|g|_{H,\lambda} + 16\|g\|_{H,\lambda}|\sigma|_{H,\lambda}
\end{aligned}$$

as desired.

Regarding the general case, assuming that (4.17) holds for $q-1$, for some $q \in \mathbb{N}$, we have:

$$\begin{aligned}
\sum_{n\ge 0} \frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}\|\sigma\|_\lambda^q|g|_{\lambda,1}\Big)& \\
&= \sum_{n\ge 0}\frac{1}{(n!)^2}\sum_{m=0}^{n}\binom{n}{m}|\sigma|_{\lambda,m}\frac{\mathrm{d}^{n-m}}{\mathrm{d}\lambda^{n-m}}\Big(|\sigma|_{\lambda,1}\|\sigma\|_\lambda^{q-1}|g|_{\lambda,1}\Big) \\
&= \sum_{m\ge 0}\frac{1}{(m!)^2}|\sigma|_{\lambda,m}\sum_{n\ge 0}\frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}\|\sigma\|_\lambda^{q-1}|g|_{\lambda,1}\Big)B(n,m)
\end{aligned}$$

where

$$B(n,m) = \frac{(n!)^2(m!)^2}{((n+m)!)^2}\binom{n+m}{m} = \frac{n!m!}{(n+m)!} \le 1$$

and so,

$$\sum_{n\ge 0} \frac{1}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(|\sigma|_{\lambda,1}\|\sigma\|_\lambda^q|g|_{\lambda,1}\Big) \le \|\sigma\|_{H,\lambda}\big(16\|\sigma\|_{H,\lambda}^q|g|_{H,\lambda} + 16|\sigma|_{H,\lambda}\|\sigma\|_{H,\lambda}^{q-1}\|g\|_{H,\lambda}\big)$$

which concludes the proof. □

Next, we present another estimate. Its proof is omitted as it is analogous to the one of the lemma above. It follows by an induction argument in $q \in \mathbb{N}_0$ with the initial step detailed in [8].

Lemma 4.7. *Let g, σ and α be analytic functions. Then for each $q \in \mathbb{N}_0$*

$$\sum_{n\geq 0} \frac{1}{(n!)^2} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Big(|\sigma|_{\lambda,1} \|\sigma\|_\lambda^q \|\alpha\|_\lambda \|g\|_\lambda \Big) \leq |\sigma|_{H,\lambda} \|\sigma\|_{H,\lambda}^q \|\alpha\|_{H,\lambda} \|g\|_{H,\lambda}. \tag{4.18}$$

4.4. **Contraction mapping.**

Given $h \in \mathrm{X}^M_{\lambda_0,K,T}$, we prove that under some restrictions on the positive parameters λ_0, K, T, M and on the initial data, the function $g = \Psi(h)$ belongs to $\mathrm{X}^M_{\lambda_0,K,T}$ and Ψ is a contraction for the metric induced by the norm $\|\cdot\|_{\mathrm{Z}}$. This establishes the existence and uniqueness result stated in Theorem 4.1.

Proposition 4.8. *If $\lambda_0 > (K+1)T$, $K - \lambda_0 - 16M^{q+1} > 1$ and $\|g_0\|_{H,\lambda_0} \leq Me^{-(16+\alpha_0)M^{q+1}}$ then*

$$\Psi(\mathrm{X}^M_{\lambda_0,K,T}) \subseteq \mathrm{X}^M_{\lambda_0,K,T}. \tag{4.19}$$

Proof. First, note that if $h \in \mathrm{X}^M_{\lambda_0,K,T}$ then

$$\sup_{t\in[0,T]} \|\sigma(t)\|_{H,\lambda_0} + \int_0^T |\sigma(t)|_{H,\lambda(t)} \,\mathrm{d}t \leq M.$$

Combining (4.16) with (4.17) and (4.18) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|_{H,\lambda(t)} \leq (\lambda_0 - K + 16M^{q+1})|g(t)|_{H,\lambda(t)} + (16+\alpha_0)M^q|\sigma(t)|_{H,\lambda(t)}\|g(t)\|_{H,\lambda(t)}$$

and hence, by Gronwall's lemma, for each $t \in [0,T]$ we have

$$\|g(t)\|_{H,\lambda(t)} \leq e^{(16+\alpha_0)M^{q+1}} \Big(\|g_0\|_{H,\lambda_0} + (\lambda_0 - K + 16M^{q+1}) \int_0^T |g(\tau)|_{H,\lambda(\tau)} \,\mathrm{d}\tau \Big).$$

Consequently,

$$\begin{aligned}
\|g(t)\|_{H,\lambda(t)} &+ \int_0^T |g(\tau)|_{H,\lambda(\tau)} \,\mathrm{d}\tau \\
&\leq e^{(16+\alpha_0)M^{q+1}} \|g_0\|_{H,\lambda_0} \\
&\quad + e^{(16+\alpha_0)M^{q+1}} \big(1 - (K - \lambda_0 - 16M^{q+1})\big) \int_0^T |g(\tau)|_{H,\lambda(\tau)} \,\mathrm{d}\tau \\
&\leq e^{(16+\alpha_0)M^{q+1}} \|g_0\|_{H,\lambda_0} \\
&\leq M
\end{aligned}$$

where we used the hypothesis $1 - (K - \lambda_0 - 16M^{q+1}) < 0$. Taking the supremum over $t \in [0,T]$ yields the desired result. □

Proposition 4.9. *In addition to the hypotheses of Proposition 4.8, assume that*

$$(1+qT)(1+\alpha_0)M^{q+1}e^{\alpha_0 M^{q+1} T} < 1.$$

Then $\Psi : \mathrm{X}^M_{\lambda_0,K,T} \to \mathrm{X}^M_{\lambda_0,K,T}$ is a contraction under the metric induced by $\|\cdot\|_{\mathrm{Z}}$.

Proof. The proof is divided into several steps, following the reasoning of the previous sections.

Let $g = \Psi(h),\ \bar{g} = \Psi(\bar{h})$ and $\sigma = \int \omega h \,\mathrm{d}v,\ \bar{\sigma} = \int \omega \bar{h} \,\mathrm{d}v$.

First step.

The difference $g - \bar{g}$ satisfies the kinetic equation

$$\begin{aligned}\partial_t(g-\bar{g}) &+ v\cdot\partial_x(g-\bar{g}) - \sigma^q\partial_x\sigma\cdot\partial_v(g-\bar{g})\\ &= \sigma^q\partial_x\sigma\cdot\alpha(g-\bar{g}) + (\sigma^q\partial_x\sigma - \bar{\sigma}^q\partial_x\bar{\sigma})\cdot(\partial_v\bar{g}+\alpha\bar{g}).\end{aligned}$$

A straightforward computation yields that

$$\begin{aligned}\sigma^q\partial_x\sigma - \bar{\sigma}^q\partial_x\bar{\sigma} &= \frac{1}{q+1}\partial_x(\sigma^{q+1}-\bar{\sigma}^{q+1})\\ &= \frac{1}{q+1}\sum_{p=0}^{q}\sigma^{q-p}\bar{\sigma}^p\cdot\partial_x(\sigma-\bar{\sigma})\\ &\quad + \frac{1}{q+1}(\sigma-\bar{\sigma})\sum_{p=0}^{q-1}(q-p)\sigma^{q-p-1}\bar{\sigma}^p\cdot\partial_x\sigma\\ &\quad + \frac{1}{q+1}(\sigma-\bar{\sigma})\sum_{p=1}^{q}p\sigma^{q-p}\bar{\sigma}^{p-1}\cdot\partial_x\bar{\sigma}.\end{aligned}$$

Second step.

Applying $\partial_x^k\partial_v^l$ to the kinetic equation satisfied by the difference $g-\bar{g}$ results in

$$\begin{aligned}\partial_t\partial_x^k\partial_v^l(g-\bar{g}) &+ v\cdot\partial_x\partial_x^k\partial_v^l(g-\bar{g}) - \sigma^q\partial_x\sigma\cdot\partial_v\partial_x^k\partial_v^l(g-\bar{g})\\ &= -l\partial_x^{k+1}\partial_v^{l-1}(g-\bar{g}) + G_1 + G_2 + G_3 + G_4 + G_5\end{aligned}$$

where

$$\begin{aligned}G_1 &= \sum_{i=0}^{k-1}\sum_{r=0}^{k-i}\sum_{k_1+\cdots+k_q=r}\Theta\prod_{s=1}^{q}\partial_x^{k_s}\sigma\cdot\partial_x^{k-i-r+1}\sigma\cdot\partial_x^i\partial_v^{l+1}(g-\bar{g})\\ G_2 &= \sum_{i=0}^{k}\sum_{r=0}^{k-i}\sum_{k_1+\cdots+k_q=r}\Theta\prod_{s=1}^{q}\partial_x^{k_s}\sigma\cdot\partial_x^{k-i-r+1}\sigma\sum_{j=0}^{l}\binom{l}{j}\partial_v^{l-j}\alpha\cdot\partial_x^i\partial_v^j(g-\bar{g})\end{aligned}$$

and Θ is given by (4.13). Moreover, the remaining terms are as follows:

$$\begin{cases} G_3 &= \dfrac{1}{q+1}\displaystyle\sum_{i=0}^{k}\binom{k}{i}\sum_{r=0}^{k-i}\binom{k-i}{r}\sum_{p=0}^{q}\Gamma_3\cdot\left(\partial_x^i\partial_v^{l+1}\bar{g} + \sum_{j=0}^{l}\binom{l}{j}\partial_v^{l-j}\alpha\cdot\partial_x^i\partial_v^j\bar{g}\right),\\ \Gamma_3 &= \displaystyle\sum_{s=0}^{r}\sum_{k_1+\cdots+k_{q-p}=s}\prod_{m=1}^{q-p}\sum_{\bar{k}_1+\cdots+\bar{k}_p=r-s}\Theta_3\prod_{n=1}^{p}\partial_x^{k_m}\sigma\cdot\partial_x^{\bar{k}_n}\bar{\sigma}\cdot\partial_x^{k-i-r+1}(\sigma-\bar{\sigma}),\\ \Theta_3 &= \dbinom{r}{s}\dbinom{s}{k_1,\cdots,k_{q-p}}\dbinom{r-s}{\bar{k}_1,\cdots,\bar{k}_p},\end{cases}$$

$$\begin{cases} G_4 &= \dfrac{1}{q+1}\displaystyle\sum_{i=0}^{k}\binom{k}{i}\sum_{r=0}^{k-i}\binom{k-i}{r}\partial_x^{k-i-r}(\sigma-\bar\sigma)\cdot\Gamma_4\cdot\left(\partial_x^i\partial_v^{l+1}\bar g+\sum_{j=0}^{l}\binom{l}{j}\partial_v^{l-j}\alpha\cdot\partial_x^i\partial_v^j\bar g\right), \\ \Gamma_4 &= \displaystyle\sum_{p=0}^{q-1}(q-p)\sum_{s=0}^{r}\sum_{a=0}^{s}\sum_{k_1+\cdots+k_{q-p-1}=a}\prod_{m=1}^{q-p-1}\sum_{\bar k_1+\cdots+\bar k_p=s-a}\Theta_4\prod_{n=1}^{p}\partial_x^{k_m}\sigma\cdot\partial_x^{\bar k_n}\bar\sigma\cdot\partial_x^{r-s+1}\sigma, \\ \Theta_4 &= \dbinom{r}{s}\dbinom{s}{a}\dbinom{a}{k_1,\cdots,k_{q-p-1}}\dbinom{s-a}{\bar k_1,\cdots,\bar k_p}, \end{cases}$$

and

$$\begin{cases} G_5 &= \dfrac{1}{q+1}\displaystyle\sum_{i=0}^{k}\binom{k}{i}\sum_{r=0}^{k-i}\binom{k-i}{r}\partial_x^{k-i-r}(\sigma-\bar\sigma)\cdot\Gamma_5\cdot\left(\partial_x^i\partial_v^{l+1}\bar g+\sum_{j=0}^{l}\binom{l}{j}\partial_v^{l-j}\alpha\cdot\partial_x^i\partial_v^j\bar g\right), \\ \Gamma_5 &= \displaystyle\sum_{p=1}^{q}p\sum_{s=0}^{r}\sum_{a=0}^{s}\sum_{k_1+\cdots+k_{q-p}=a}\prod_{m=1}^{q-p}\sum_{\bar k_1+\cdots+\bar k_{p-1}=s-a}\Theta_5\prod_{n=1}^{p-1}\partial_x^{k_m}\sigma\cdot\partial_x^{\bar k_n}\bar\sigma\cdot\partial_x^{r-s+1}\bar\sigma, \\ \Theta_5 &= \dbinom{r}{s}\dbinom{s}{a}\dbinom{a}{k_1,\cdots,k_{q-p}}\dbinom{s-a}{\bar k_1,\cdots,\bar k_{p-1}}. \end{cases}$$

Third step.

The maximum principle (B.1) is applied to the kinetic equation satisfied by $\partial_x^k\partial_v^l(g-\bar g)$ yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^k\partial_v^l(g-\bar g)\|_\infty \le l\|\partial_x^{k+1}\partial_v^{l-1}(g-\bar g)\|_\infty+\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3+\mathcal{G}_4+\mathcal{G}_5$$

where each $\mathcal{G}_i$ is obtained from G_i by taking the norm $\|\cdot\|_\infty$ in each one of its terms. Next, we multiply the previous inequality by $\lambda^{k+l}/k!l!$ and sum over $k,l\in\mathbb{N}_0$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g-\bar g\|_\lambda \le \sum_{k,l\ge 0} l\frac{\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_v^{l-1}(g-\bar g)\|_\infty+\mathbb{G}_1+\mathbb{G}_2+\mathbb{G}_3+\mathbb{G}_4+\mathbb{G}_5$$

where $\mathbb{G}_i=\sum_{k,l\ge0}\frac{\lambda^{k+l}}{k!l!}\mathcal{G}_i$.

Proceeding as in the proof of Proposition 4.4 we obtain the bounds

$$\sum_{k,l\ge 0} l\frac{\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_v^{l-1}(g-\bar g)\|_\infty \le \lambda|g-\bar g|_{\lambda,1}$$

and

$$\mathbb{G}_1+\mathbb{G}_2\le|\sigma|_{\lambda,1}\|\sigma\|_\lambda^q|g-\bar g|_{\lambda,1}+|\sigma|_{\lambda,1}\|\sigma\|_\lambda^q\|\alpha\|_\lambda\|g-\bar g\|_\lambda.$$

Furthermore, concerning the remaining terms, following a meticulous computation involving the careful interchange of sums within the expressions and employing (3.5) and (3.6), we arrive at:

$$\mathbb{G}_3\le\frac{1}{q+1}|\sigma-\bar\sigma|_{\lambda,1}\Big(\sum_{p=0}^{q}\|\sigma\|_\lambda^{q-p}\|\bar\sigma\|_\lambda^p\Big)\big(|\bar g|_{\lambda,1}+\|\alpha\|_\lambda\|\bar g\|_\lambda\big),$$

$$\mathbb{G}_4\le\frac{1}{q+1}\|\sigma-\bar\sigma\|_\lambda|\sigma|_{\lambda,1}\Big(\sum_{p=0}^{q-1}(q-p)\|\sigma\|_\lambda^{q-p-1}\|\bar\sigma\|_\lambda^p\Big)\big(|\bar g|_{\lambda,1}+\|\alpha\|_\lambda\|\bar g\|_\lambda\big),$$

$$\mathbb{G}_5\le\frac{1}{q+1}\|\sigma-\bar\sigma\|_\lambda|\bar\sigma|_{\lambda,1}\Big(\sum_{p=1}^{q}p\|\sigma\|_\lambda^{q-p}\|\bar\sigma\|_\lambda^{p-1}\Big)\big(|\bar g|_{\lambda,1}+\|\alpha\|_\lambda\|\bar g\|_\lambda\big).$$

Fourth step.
Take $\lambda = \lambda(t) = \lambda_0 - (1+K)t \leq \lambda_0$. Since $h, \bar{h} \in \mathrm{X}^M_{\lambda_0,K,T}$, then

$$\|\sigma(t)\|_{\lambda(t)} \leq \|h(t)\|_{\lambda(t)} \leq \sup_{t\in[0,T]} \|h(t)\|_{H,\lambda(t)} \leq M,$$

$$|\sigma(t)|_{\lambda(t),1} \leq |h(t)|_{\lambda(t),1} \leq \sup_{t\in[0,T]} \|h(t)\|_{H,\lambda(t)} \leq M,$$

and similarly for $\bar{\sigma}$. Thus, the estimates of the previous step imply that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)-\bar{g}(t)\|_{\lambda(t)} &= \lambda'(t)|g(t)-\bar{g}(t)|_{\lambda(t),1} + \frac{\mathrm{d}}{\mathrm{d}t}\|g(t)-\bar{g}(t)\|_{\lambda}\Big|_{\lambda=\lambda(t)} \\ &\leq (\lambda_0 - K + M^{q+1})|g(t)-\bar{g}(t)|_{\lambda(t),1} \\ &\quad + \alpha_0 M^{q+1}\|g(t)-\bar{g}(t)\|_{\lambda(t)} \\ &\quad + (1+\alpha_0)M^{q+1}|\sigma(t)-\bar{\sigma}(t)|_{\lambda(t),1} \\ &\quad + q(1+\alpha_0)M^{q+1}\|\sigma(t)-\bar{\sigma}(t)\|_{\lambda(t)}\end{aligned}$$

which, by Gronwall's lemma, results in

$$\begin{aligned}\|g(t)-\bar{g}(t)\|_{\lambda(t)} &\leq (\lambda_0 - K + M^{q+1})e^{\alpha_0 M^{q+1}T}\int_0^T |g(\tau)-\bar{g}(\tau)|_{\lambda(\tau),1}\,\mathrm{d}\tau \\ &\quad + (1+\alpha_0)M^{q+1}e^{\alpha_0 M^{q+1}T}\int_0^T |\sigma(\tau)-\bar{\sigma}(\tau)|_{\lambda(\tau),1}\,\mathrm{d}\tau \\ &\quad + q(1+\alpha_0)M^{q+1}Te^{\alpha_0 M^{q+1}T}\sup_{\tau\in[0,T]}\|\sigma(\tau)-\bar{\sigma}(\tau)\|_{\lambda_0}\end{aligned} \tag{4.20}$$

for every $t \in [0,T]$.

Now, we use the hypothesis

$$K > 1 + \lambda_0 + 16M^{q+1} > 1 + \lambda_0 + M^{q+1}$$

to infer from (4.20) that

$$\begin{aligned}\|g(t)-\bar{g}(t)\|_{\lambda(t)} &+ \int_0^T |g(\tau)-\bar{g}(\tau)|_{\lambda(\tau),1}\,\mathrm{d}\tau \\ &\leq (1+qT)(1+\alpha_0)M^{q+1}e^{\alpha_0 M^{q+1}T}\|\sigma-\bar{\sigma}\|_{\mathrm{Z}} \\ &\leq \kappa\|h-\bar{h}\|_{\mathrm{Z}}\end{aligned}$$

where $\kappa = (1+qT)(1+\alpha_0)M^{q+1}e^{\alpha_0 M^{q+1}T} < 1$ by assumption. Taking the supremum over $t \in [0,T]$ yields the desired result.

□

Appendix A. Banach spaces of analytic functions

Lemma A.1. *Let $\lambda > 0$ and consider the space W_λ of those functions $f \in C^\infty(\mathbb{R})$ satisfying*

$$\|f\|_\lambda = \sum_{k\geq 0}\frac{\lambda^k}{k!}\|\partial_x^k f\|_\infty < \infty.$$

Then W_λ is a Banach space under the norm $\|\cdot\|_\lambda$.

Proof. Let $(f_n) \subseteq W_\lambda$ be a Cauchy sequence. Then, for each $k \in \mathbb{N}_0$, $(\partial_x^k f_n)$ is a Cauchy sequence of the Banach space $C_b(\mathbb{R})$ of bounded continuous functions, and so, there exists $g_k \in C_b(\mathbb{R})$ such that $\|\partial_x^k f_n - g_k\|_\infty \to 0$ as $n \to \infty$. Set $f = g_0$. We claim that for each $k \in \mathbb{N}_0$, $g_k = \partial_x^k f$. Let $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} \partial_x f_n \cdot \varphi \, \mathrm{d}x = -\int_{\mathbb{R}} f_n \cdot \partial_x \varphi \, \mathrm{d}x$$

which, by letting $n \to \infty$ yields

$$\int_{\mathbb{R}} g_1 \cdot \varphi \, \mathrm{d}x = -\int_{\mathbb{R}} f \cdot \partial_x \varphi \, \mathrm{d}x.$$

Hence, g_1 is the weak derivative of f, and since g_1 is continuous, we have that $f \in C^1(\mathbb{R})$ and $g_1 = \partial_x f$. The claim follows by a standard inductive argument.

Moreover, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $\|f_n - f_m\|_\lambda < \varepsilon$. Letting $m \to \infty$ combined with Fatou's lemma results in $\|f_n - g\|_\lambda < \varepsilon$ for $n \geq N$. Thus, $\|g\|_\lambda \leq \|f_N - g\|_\lambda + \|f_N\|_\lambda < \infty$. This proves that (f_n) has a limit in W_λ, establishing the result. $\square$

Lemma A.2. *Let $\lambda, T > 0$ and consider the space $Y_{\lambda,T}$ of those functions $f \in C\big([0,T]; C^\infty(\mathbb{R})\big)$ satisfying*

$$\sup_{t\in[0,T]} \|f(t)\|_\lambda = \sup_{t\in[0,T]} \sum_{k\geq 0} \frac{\lambda^k}{k!} \|\partial_x^k f(t)\|_\infty < \infty.$$

Then $Y_{\lambda,T}$ is a Banach space under the norm $\sup_{t\in[0,T]} \|\cdot\|_\lambda$.

Proof. Let $(f_n) \subseteq Y_{\lambda,T}$ be a Cauchy sequence. Then, for each $t \in [0,T]$, $(f_n(t))$ is Cauchy in the space W_λ defined in the previous lemma. Therefore, there exists $g_t \in W_\lambda$ such that $\|f_n(t) - g_t\|_\lambda \to 0$ as $n \to \infty$. Define $f : [0,T] \to C^\infty(\mathbb{R})$ by $f(t) = g_t$. Note that, for every $n, m \in \mathbb{N}$ and any $t \in [0,T]$, there holds

$$\big|\|f(t) - f_n(t)\|_\lambda - \|f_m(t) - f_n(t)\|_\lambda\big| \leq \|f(t) - f_m(t)\|_\lambda$$

and so

$$\|f(t) - f_n(t)\|_\lambda = \lim_{m\to\infty} \|f_m(t) - f_n(t)\|_\lambda.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$ and all $t \in [0,T]$ we have $\|f_n(t) - f_m(t)\| \leq \varepsilon$. Letting $m \to \infty$ yields $\|f_n(t) - f(t)\|_\lambda < \varepsilon$ for all $n \geq N$ and $t \in [0,T]$. A standard argument then implies that f is continuous in time, which concludes the proof. $\square$

Remark A.3. The previous two lemmas also hold if $C^\infty(\mathbb{R})$ is replaced by $C^\infty(\mathbb{R}^2)$, in which the norm $\|\cdot\|_\lambda$ becomes

$$\|f\|_\lambda = \sum_{k,l\geq 0} \frac{\lambda^{k+l}}{k!l!} \|\partial_x^k \partial_v^l f\|_\infty.$$

Proposition A.4. *Let $\lambda_0, K, T > 0$ be such that $\lambda_0 > (K+1)T$ and consider the function $\lambda(t) = \lambda_0 - (K+1)t$ for $t \in [0,T]$. The space $\mathrm{Z}_{\lambda_0,K,T}$ defined in (4.7) is a Banach space under the norm $\|\cdot\|_{\mathrm{Z}}$ given by* (4.4).

Proof. Let $(f_n) \subseteq \mathrm{Z}_{\lambda_0,K,T}$ be a Cauchy sequence. Then, (f_n) is Cauchy in the $\mathbb{R}^2$ version of the space $Y_{\lambda_0,T}$ of the previous lemma, and hence there exists $f \in C\big([0,T]; C^\infty(\mathbb{R}^2)\big)$ such that

$$\sup_{t\in[0,T]} \|f_n(t) - f(t)\|_{\lambda_0} = \sup_{t\in[0,T]} \sum_{k,l\geq 0} \frac{\lambda_0^{k+l}}{k!l!} \|\partial_x^k \partial_v^l \big(f_n(t) - f(t)\big)\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{A.1}$$

In particular, for all $t \in [0,T]$ and each $k, l \geq 0$ we have

$$\|\partial_x^k \partial_v^l \big(f_n(t) - f(t)\big)\|_\infty \to 0 \quad \text{as } n \to \infty$$

which implies

$$\|\partial_x^k \partial_v^l \big(f_n(t) - f_m(t)\big)\|_\infty \to \|\partial_x^k \partial_v^l \big(f_n(t) - f(t)\big)\|_\infty \quad \text{as } m \to \infty.$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that, for all $n, m \geq N$,

$$\int_0^T |f_n(t) - f_m(t)|_{\lambda(t),1} \, \mathrm{d}t < \varepsilon.$$

Using Fatou's lemma we deduce for $n \geq N$ that

$$\begin{aligned}
&\int_0^T |f_n(t) - f(t)|_{\lambda(t),1} \, \mathrm{d}t \\
&\quad = \int_0^T \sum_{k+l \geq 1} \frac{(k+l)!}{(k+l-1)!} \frac{\lambda(t)^{k+l-1}}{k!l!} \|\partial_x^k \partial_v^l \big(f_n(t) - f(t)\big)\|_\infty \, \mathrm{d}t \\
&\quad = \int_0^T \sum_{k+l \geq 1} \frac{(k+l)!}{(k+l-1)!} \frac{\lambda(t)^{k+l-1}}{k!l!} \lim_{m\to\infty} \|\partial_x^k \partial_v^l \big(f_n(t) - f_m(t)\big)\|_\infty \, \mathrm{d}t \\
&\quad \leq \liminf_{m\to\infty} \int_0^T |f_n(t) - f_m(t)|_{\lambda(t),1} \, \mathrm{d}t < \varepsilon
\end{aligned}$$

and so

$$\int_0^T |f_n(t) - f(t)|_{\lambda(t),1} \, \mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{A.2}$$

Combining (A.1) and (A.2), the definition (4.4) yields

$$\|f_n - f\|_{\mathrm{Z}} \to 0 \quad \text{as } n \to \infty$$

and concludes the proof. $\square$

Appendix B. Maximum principle

Suppose that $g = g(t,x,v)$ is a smooth solution of the equation

$$\partial_t g + v \cdot \partial_x g + a(t,x) \cdot \partial_v g = G(t,x,v)$$

where a is smooth in time and globally Lipschitz in space. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g(t)\|_\infty \leq \|G(t)\|_\infty. \tag{B.1}$$

Indeed, using characteristics we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t, x(t), v(t)) = G(t, x(t), v(t))$$

where $(x(t), v(t))$ solves

$$\begin{cases}
\dot{x}(t) = v(t), \\
\dot{v}(t) = a(t, x(t)), \\
x(0) = x_0, \ v(0) = v_0.
\end{cases}$$

Integrating over $[t, t+h]$, for some $0 < h < t$, yields

$$\begin{aligned} g(t+h, x(t+h), v(t+h)) &= g(t, x(t), v(t)) + \int_t^{t+h} G(s, x(\tau), v(\tau))\,\mathrm{d}\tau \\ &\leq \|g(t)\|_\infty + \int_t^{t+h} \|G(\tau)\|_\infty\,\mathrm{d}\tau. \end{aligned}$$

Since the initial values $x_0, v_0 \in \mathbb{R}$ are arbitrary, it follows that

$$\|g(t+h)\|_\infty - \|g(t)\|_\infty \leq \int_t^{t+h} \|G(\tau)\|_\infty\,\mathrm{d}\tau.$$

Similarly,

$$\|g(t)\|_\infty - \|g(t-h)\|_\infty \leq \int_{t-h}^{t} \|G(\tau)\|_\infty\,\mathrm{d}\tau.$$

The inequality (B.1) is obtained by dividing both previous expressions by h and then letting $h \to 0^+$.

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