RELATION BETWEEN ASYMPTOTIC L_p-CONVERGENCE AND SOME CLASSICAL MODES OF CONVERGENCE

NUNO J. ALVES AND GIORGI G. ONIANI

Abstract. Asymptotic L_p-convergence, which resembles convergence in L_p, was introduced to address a question in diffusive relaxation. This note aims to compare asymptotic L_p-convergence with convergence in measure and in weak L_p spaces. One of the results characterizes convergence in measure on finite measure spaces in terms of asymptotic L_p-convergence.

Let $(X, \mathbb{X}, \mu)$ be a measure space. By a measurable function, we understand a real-valued μ-measurable function defined on X. The set of all such measurable functions (identified μ-a.e.) is denoted by $M(X)$. We use the notation A^c for the complement of a set $A \subseteq X$ with respect to X. It will be assumed that p is a number belonging to $[1, \infty)$.

The main purpose of this note is to compare the convergences in measure and in weak L_p spaces with asymptotic L_p-convergence. The latter concept was motivated by a question on convergence in diffusive relaxation and introduced in [1]. Specifically, a sequence (f_n) of measurable functions is said to *asymptotically L_p-converge* (in short, *α_p-converge*) to a measurable function f if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that $\int_{B_n} |f_n - f|^p \, \mathrm{d}\mu \to 0$ as $n \to \infty$.

1. Asymptotic L_p-convergence vs. convergence in measure

In [1], it was shown that α_p-convergence implies convergence in measure, and an example was given of a sequence of functions that converges in measure but does not α_p-converge. This example, however, is in an infinite measure space. For finite measure spaces, these two notions of convergence are, in fact, equivalent. Recall that a sequence (f_n) is said to *converge in measure* to a measurable function f if for every $\delta > 0$, $\mu\big(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}\big) \to 0$ as $n \to \infty$. We prove the following:

Theorem 1.1. *Let f_n ($n \in \mathbb{N}$) and f be measurable functions. If (f_n) α_p-converges to f, then (f_n) converges to f in measure. On the other hand, if $\mu(X) < \infty$ and (f_n) converges to f in measure, then (f_n) α_p-converges to f.*

The first part of this theorem was established in [1], but here we provide a simpler proof.

Proof. First, assume that (f_n) α_p-converges to f. For each $\delta > 0$ let the set $E_n(\delta)$ be given by

$$E_n(\delta) = \{x \in X \mid |f_n(x) - f(x)| \geq \delta\}.$$

2020 *Mathematics Subject Classification.* 28A20.

Key words and phrases. convergence in measure, weak L_p spaces, asymptotic L_p-convergence.

Then

$$\begin{aligned}\delta^p\mu(E_n(\delta)) &= \delta^p\mu(E_n(\delta)\cap B_n)+\delta^p\mu(E_n(\delta)\cap B_n^c)\\ &\le \int_{E_n(\delta)\cap B_n}\delta^p\,\mathrm{d}\mu+\delta^p\mu(B_n^c)\\ &\le \int_{B_n}|f_n-f|^p\,\mathrm{d}\mu+\delta^p\mu(B_n^c)\end{aligned}$$

where (B_n) is a sequence of measurable sets associated with the α_p-convergence of (f_n) towards f. Letting $n\to\infty$ yields the desired conclusion.

Now, assume that X has finite measure and that (f_n) converges to f in measure. Then, for each $n\in\mathbb{N}$, there exists $N_n\in\mathbb{N}$ such that $\mu(E_k(1/n))<1/n$ whenever $k\ge N_n$. Assume, without loss of generality, that $N_{n+1}>N_n$ for every natural n. Let (λ_k) be a sequence of positive numbers such that for every $n\in\mathbb{N}$, $\lambda_k=1/n$ whenever $k\in[N_n,N_{n+1})$, and set $B_k=E_k(\lambda_k)^c$ $(k\in\mathbb{N})$. Note that $\mu(B_k^c)\to 0$ as $k\to\infty$ and

$$\int_{B_k}|f_k-f|^p\,\mathrm{d}\mu\le\int_{B_k}\lambda_k^p\,\mathrm{d}\mu\le\lambda_k^p\mu(X)\to 0\quad\text{as } k\to\infty$$

which finishes the proof. □

Furthermore, there are notions of Cauchy sequences related to α_p-convergence and convergence in measure. We say that a sequence (f_n) of measurable functions is *asymptotically L_p-Cauchy* (in short, *α_p-Cauchy*) if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c)\to 0$ as $n\to\infty$ such that

$$\int_{B_n\cap B_m}|f_n-f_m|^p\,\mathrm{d}\mu\to 0\quad\text{as } n,m\to\infty,$$

and it is *Cauchy in measure* if

$$\forall\delta>0\quad \mu\big(\{x\in X\mid |f_n(x)-f_m(x)|\ge\delta\}\big)\to 0\quad\text{as } n,m\to\infty.$$

It is known that α_p-Cauchy sequences are Cauchy in measure, and that α_p-Cauchy sequences (respectively, Cauchy in measure) α_p-converge (respectively, converge in measure) to a measurable function f (see [1, 2]). This, together with Theorem 1.1, yields that in a finite measure space these two notions of Cauchy sequences are equivalent.

Theorem 1.2. *Let (f_n) be a sequence of measurable functions. If (f_n) is α_p-Cauchy, then (f_n) is Cauchy in measure. On the other hand, if $\mu(X)<\infty$ and (f_n) is Cauchy in measure, then (f_n) is α_p-Cauchy.*

Proof. The first part can be deduced similarly to the proof of the first part of Theorem 1.1, but with $E_n(\delta)$ replaced by $E_{n,m}(\delta)=\{x\in X\mid |f_n(x)-f_m(x)|\ge\delta\}$ and B_n replaced by $B_n\cap B_m$.

Regarding the second part, assume that $\mu(X)<\infty$ and that (f_n) is Cauchy in measure. Then, there exists a measurable function f such that (f_n) converges to f in measure. By Theorem 1.1 it follows that (f_n) α_p-converges to f, and hence (f_n) is α_p-Cauchy, as desired. □

2. Asymptotic L_p-convergence vs. convergence in weak L_p spaces

The *weak L_p space*, denoted by $L_{p,\infty}$, consists of all measurable functions f such that

$$\sup_{\delta>0}\delta^p\mu(\{x\in X\mid |f(x)|\ge\delta\})<\infty.$$

Let f_n $(n \in \mathbb{N})$ and f be functions belonging to $L_{p,\infty}$. The sequence (f_n) is said to *converge in* $L_{p,\infty}$ to f (see, e.g., [5]) if

$$\sup_{\delta>0} \delta^p \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}) \to 0 \quad \text{as } n \to \infty.$$

It is clear that this convergence implies convergence in measure.

The next example shows that asymptotic L_p-convergence does not imply convergence in weak L_p spaces even for spaces with finite measure.

Example 2.1. Let $X = [0, 1]$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[0, 1]$, and μ be Lebesgue measure on $\mathbb{X}$. Consider the sequence of functions (f_n) defined by $f_n = n^{1/p}\chi_{[0,1/n]}$. Then (f_n) α_p-converges to 0 (zero function), but it does not converge to 0 in $L_{p,\infty}$. To check the latter, for each $\delta > 0$ let $F_n(\delta) = \delta^p \mu(\{x \in [0, 1] \mid |f_n(x)| \geq \delta\})$ and note that

$$F_n(\delta) = \begin{cases} \dfrac{\delta^p}{n} & \text{if } 0 < \delta \leq n^{1/p}, \\ 0 & \text{if } \delta > n^{1/p}. \end{cases}$$

Then $\sup_{\delta>0} F_n(\delta) = 1$ for every natural n and hence (f_n) does not converge to 0 in $L_{p,\infty}$.

The next example shows that the inverse implication is also false, i.e., convergence in weak L_p spaces does not imply asymptotic L_p-convergence.

Example 2.2. Let $X = [1, \infty)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[1, \infty)$, and μ be Lebesgue measure on $\mathbb{X}$. Let f_n be given by

$$f_n(x) = \frac{1}{(nx)^{\frac{1}{p}}} \quad (x \in [1, \infty)).$$

Then

$$\begin{aligned} F_n(\delta) &= \delta^p \mu(\{x \in [1, \infty) \mid |f_n(x)| \geq \delta\}) \\ &= \begin{cases} \dfrac{1 - n\delta^p}{n} & \text{if } 0 < \delta \leq \dfrac{1}{n^{1/p}}, \\ 0 & \text{if } \delta > \dfrac{1}{n^{1/p}}, \end{cases} \end{aligned}$$

and so $\sup_{\delta>0} F_n(\delta) = 1/n \to 0$ as $n \to \infty$, i.e., (f_n) converges to 0 in $L_{p,\infty}$.

Next, we show that (f_n) does not α_p-converge to 0. Let (B_n) be any sequence of measurable subsets of $[1, \infty)$ such that $\mu(B_n^c) \to 0$ as $n \to \infty$. Let $N \in \mathbb{N}$ be such that $\mu(B_n^c) < 1$ for all $n \geq N$. Then, for $n \geq N$,

$$\frac{1}{n} \int_{B_n^c} \frac{1}{x} \, \mathrm{d}x \leq \frac{1}{n} \mu(B_n^c) < \frac{1}{n} < \infty.$$

Note that

$$\frac{1}{n} \int_{B_n} \frac{1}{x} \, \mathrm{d}x = \frac{1}{n} \int_1^\infty \frac{1}{x} \, \mathrm{d}x - \frac{1}{n} \int_{B_n^c} \frac{1}{x} \, \mathrm{d}x.$$

Since the first integral on the right-hand side is infinite and for $n \geq N$ the second integral on the right-hand side is finite, it follows that for $n \geq N$ the integral on the left-hand side is infinite, i.e., (f_n) does not α_p-converge to 0.

Note that by Theorem 1.1, for spaces with finite measure, convergence in weak L_p spaces implies asymptotic L_p-convergence.

3. Almost L_p spaces vs. weak L_p spaces

Denote by A_p the space of measurable functions that are *almost in* L_p, i.e.,

$$A_p = \left\{ f \in M(X) \mid \forall \delta > 0 \ \exists E_\delta \text{ with } \mu(E_\delta) < \delta \text{ such that } \int_{E_\delta^c} |f|^p \, \mathrm{d}\mu < \infty \right\}.$$

These spaces are named as *almost L_p spaces* and were introduced in [3, 4]. It is clear that $L_p \subseteq A_p$, and it is not hard to see that there are functions belonging to A_p but not to L_p.

Below, we give examples that show that in general measure spaces, $A_p \setminus L_{p,\infty}$ and $L_{p,\infty} \setminus A_p$ are non-empty. Then, at last, we prove that $L_{p,\infty}$ is contained in A_p when $\mu(X) < \infty$.

Example 3.1. Let $X = (0,1)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $(0,1)$, and μ be Lebesgue measure on $\mathbb{X}$. Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x^2} \quad (x \in (0,1)).$$

This function does not belong to $L_{p,\infty}$. Indeed, for every $\delta \geq 1$ we have that

$$\delta^p \mu(\{x \in (0,1) \mid |f(x)| \geq \delta\}) = \delta^{p-1/2}.$$

Letting $\delta \to \infty$ yields the desired conclusion.

On the other hand, since f is decreasing, it is easy to see that f belongs to A_p.

Example 3.2. Let $X = [1,\infty)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[1,\infty)$, and μ be Lebesgue measure on $\mathbb{X}$. Let f be defined by

$$f(x) = \frac{1}{x^{\frac{1}{p}}} \quad (x \in [1,\infty)).$$

Using a similar reasoning as in Example 2.2 one can conclude that f does not belong to A_p but it belongs to $L_{p,\infty}$.

Proposition 3.3. *If $\mu(X) < \infty$, then $L_{p,\infty} \subseteq A_p$.*

Proof. By hypothesis we have

$$\sup_{\delta > 0} \delta^p \mu(\{x \in X \mid |f(x)| \geq \delta\}) = C < \infty.$$

In particular, for each $k \in \mathbb{N}$ it holds that

$$\mu(\{x \in X \mid |f(x)| \geq k\}) \leq \frac{C}{k^p}.$$

Given $\delta > 0$, let $K \in \mathbb{N}$ be such that

$$\mu(\{x \in X \mid |f(x)| \geq K\}) \leq \frac{C}{K^p} < \delta$$

and set $E_\delta = \{x \in X \mid |f(x)| \geq K\}$. Then

$$\int_{E_\delta^c} |f|^p \, \mathrm{d}\mu \leq K\mu(X) < \infty$$

which concludes the proof. □

Acknowledgments

Some of the questions addressed in this note were proposed in the problem session of the 46th Summer Symposium in Real Analysis, which took place at the University of Łódź, Poland, in June 2024. This research was partially funded by the Austrian Science Fund (FWF) project 10.55776/F65.

References

[1] N. J. Alves, and J. Paulos, A mode of convergence arising in diffusive relaxation, *The Quarterly Journal of Mathematics* 75(1), 143–159, 2024.

[2] R. G. Bartle, *Elements of Integration and Lebesgue Measure*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1995.

[3] O. G. Bravo, and E. A. S. Pérez, Optimal range theorems for operators with p-th power factorable adjoints. *Banach Journal of Mathematical Analysis*, 6(1), 61-73, 2012.

[4] J. M. Calabuig, O. G. Bravo, M. A. Juan, and E. A. S. Pérez, Representation and factorization theorems for almost-L_p-spaces. *Indagationes Mathematicae*, 30(5), 930-942, 2019.

[5] R. E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces.* Springer, Cham, 2016.

(Nuno J. Alves) University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria.

Email address: nuno.januario.alves@univie.ac.at

(Giorgi G. Oniani) Kutaisi International University, School of Computer Science and Mathematics, Kutaisi 4600, Georgia.

Email address: giorgi.oniani@kiu.edu.ge