UNIFORM BOUNDEDNESS OF PARAMETRIC BILINEAR FRACTIONAL INTEGRALS

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Abstract. We provide weak-type bounds for a family of bilinear fractional integrals that arise in the study of Euler–Riesz systems. These bounds are uniform in the natural parameter that describes the family and are sharp, in the sense that they do not hold for any larger set of indices.

1. Introduction

Let $d \in \mathbb{N}$ and $0 < \alpha < d$. For $\theta \in [0,1]$, we consider a bilinear fractional integral operator I_α^θ, defined for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$I_\alpha^\theta(f,g)(x) = \int_{\mathbb{R}^d} f(x+(\theta-1)y)\, g(x+\theta y)\, |y|^{\alpha-d}\, \mathrm{d}y. \tag{1.1}$$

This operator was introduced in [1] by the authors and Tzavaras, motivated by a reformulation of the Euler–Riesz system that leads to an a priori gain of integrability for the density. The relevant equations model the evolution of a compressible fluid with a nonlocal repulsive potential of Riesz type and read as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma + \rho \nabla K_\alpha * \rho = 0, \end{cases} \tag{1.2}$$

where $\rho\colon [0,\infty)\times\mathbb{R}^d \to [0,\infty)$ denotes the density, $u\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ the linear velocity, and $\gamma > 1$ the adiabatic exponent. We use the notation $u \otimes u$ for the matrix with entries $u_i u_j$. The kernel K_α is given by

$$K_\alpha(x) = \tfrac{1}{d-\alpha}|x|^{\alpha-d}$$

and describes the nonlocal interaction of particles.

A smooth solution (ρ, u) of (1.2) with fast decay at infinity satisfies the conservation of energy and mass identities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \tfrac{1}{2}\rho|u|^2 + \tfrac{1}{\gamma-1}\rho^\gamma + \tfrac{1}{2}\rho(K_\alpha * \rho)\,\mathrm{d}x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho\,\mathrm{d}x = 0, \tag{1.3}$$

which provides an a priori estimate for solutions. This, in particular, implies the following regularity for the density:

$$\rho \in L^\infty\big((0,\infty); L^1 \cap L^\gamma(\mathbb{R}^d)\big). \tag{1.4}$$

Interestingly, by exploiting the structure of the equations, and in particular the symmetry of the interaction kernel K_α, it is possible to improve this a priori regularity of the density for finite-energy solutions. The key idea is to rewrite system (1.2) as a space-time divergence-free condition for a suitable positive-definite symmetric tensor. Then, a gain

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in integrability for the density ρ follows from Serre's theory of *compensated integrability* [16, 17], as established in [1, Theorem 3.2], yielding

$$\rho \in L^{\gamma+\frac{1}{d}}\big((0,\infty)\times\mathbb{R}^d\big). \tag{1.5}$$

Among all terms in the system, the only one that is not in divergence form is the nonlinear nonlocal interaction $\rho\nabla K_\alpha * \rho$. Using the symmetry of K_α, this term can be formally rewritten as the divergence of a positive-definite symmetric tensor:

$$\rho\nabla K_\alpha * \rho = \nabla\cdot S_\alpha(\rho) \tag{1.6}$$

where

$$S_\alpha(\rho) = \tfrac{1}{2}\int_0^1\int_{\mathbb{R}^d} \rho(x+(\theta-1)y)\,\rho(x+\theta y)\,|y|^{\alpha-d-2}\,y\otimes y\,\mathrm{d}y\,\mathrm{d}\theta.$$

By a tensor we mean a matrix-valued function. See [1, Appendix A] for the details of this derivation.

The structural identity (1.6) allows the entire Euler–Riesz system to be cast in divergence form,

$$\nabla_{t,x}\cdot\begin{bmatrix}\rho & (\rho u)^\top\\ \rho u & \rho u\otimes u+\rho^\gamma I_d+S_\alpha(\rho)\end{bmatrix}=0, \tag{1.7}$$

where I_d is the $d\times d$ identity matrix. The tensor in (1.7) is a divergence-free positive symmetric tensor, thus fitting into the framework of compensated integrability.

The operator I_α^θ thus arises naturally in the study of the integrability mapping properties of S_α, a key step in understanding the nonlocal term of the reformulated Euler–Riesz system (1.7). The main result of [1] establishes that I_α^θ maps $L^p(\mathbb{R}^d)\times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ uniformly in θ under some conditions on p,q,r. The result is as follows:

Theorem 1.1 ([1])**.** *Let p,q,r be integrability exponents satisfying $1<p,q<d/\alpha$ and*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{\alpha}{d}. \tag{1.8}$$

Then there is a constant $C=C(\alpha,d,p,q)>0$ independent of θ such that

$$\big\|I_\alpha^\theta(f,g)\big\|_r \le C\,\|f\|_p\,\|g\|_q \tag{1.9}$$

for all $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$.

Theorem 1.1 provides uniform strong-type boundedness for I_α^θ when the pair $(1/p,1/q)$ lies inside the square given by the convex hull of $\{(1,1),(1,\alpha/d),(\alpha/d,1),(\alpha/d,\alpha/d)\}$. In this work, we complement this result in two directions: First, we prove uniform weak-type endpoint estimates along the boundary of this square; interestingly, different edges of the square exhibit distinct behaviors with respect to the uniformity of the estimates in the parameter θ; see Theorems 2.1 and 2.2. Secondly, we establish strong-type boundedness over a larger range of exponents, corresponding to the interior of a pentagon — these estimates are uniform in θ away from 0 and 1; see Theorem 2.3. This completes the mapping properties of I_α^θ on Lebesgue and Lorentz spaces initiated in [1].

Bilinear fractional integral operators have been an important object of study in harmonic analysis over the past three decades. Their significance comes from their singular nature and the challenge of determining whether such operators admit bounds in terms of the norms of the functions on which they act. This question is far from trivial and over the years, a variety of boundedness results have been obtained.

One of the earliest and most studied examples is the operator B_α, defined for nonnegative measurable functions by

$$B_\alpha(f,g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(x+y)\, |y|^{\alpha-d}\, \mathrm{d}y. \tag{1.10}$$

This operator was introduced by the second author in [3] and further investigated by Kenig and Stein [12], as well as by the second author and Kalton [6], where bounds in Lebesgue spaces were established. Extensions of these boundedness results for B_α have been obtained in the rough kernel case [2], on weighted Lebesgue spaces [15, 14, 11, 13], and on Morrey spaces [9, 10].

We note that $I_\alpha^\theta(f,g)$ interpolates between $I_\alpha(f)\, g$ and $f\, I_\alpha(g)$, which correspond to the cases $\theta = 0$ and $\theta = 1$, respectively, where I_α is the classical singular fractional integral (or Riesz potential)

$$I_\alpha(f)(x) = \int_{\mathbb{R}^d} f(x-y)\, |y|^{\alpha-d}\, \mathrm{d}y. \tag{1.11}$$

The mid point of this interpolation corresponds to the operator B_α through the identity

$$I_\alpha^{1/2}(f,g) = 2^\alpha B_\alpha(f,g). \tag{1.12}$$

Thus, the known bounds for the operator B_α are easily recovered from the results we obtain for I_α^θ. We emphasize, however, that the search for uniform bounds of the I_α^θ makes its study more intricate than that of B_α. Moreover, extensions of these uniform estimates to weighted L^p or Morrey spaces would provide a broader context for this family of operators and may be of independent interest.

The manuscript is organized as follows. Section 2 contains the statement of the main results. In Section 3, we consider an auxiliary operator I_j^θ and deduce several boundedness results for it. The proofs of Theorems 2.1 and 2.3 are given in Section 4, and the proof of Theorem 2.2 appears in Section 5. In the final part of this work, Section 6, we investigate the sharpness of Theorem 2.1.

Notation.

We explain the notation used in the statements of the theorems and throughout the text.

For $0 < p \leq \infty$, we denote by $L^p(\mathbb{R}^d)$ the standard Lebesgue space of measurable functions on $\mathbb{R}^d$, equipped with the norm $\|\cdot\|_p$. The weak Lebesgue space is denoted by $L^{p,\infty}(\mathbb{R}^d)$, with quasi-norm $\|\cdot\|_{p,\infty}$, and $L^{p,1}(\mathbb{R}^d)$ stands for the Lorentz space with indices p and 1, whose quasi-norm is denoted by $\|\cdot\|_{p,1}$. For a thorough exposition of the basic properties of these spaces, we refer the reader to [4, Chapter 1].

Given $R > 0$ and $x \in \mathbb{R}^d$, the open ball centered at x with radius R is denoted by $B_x(R)$; for $x = 0$, we simply write $B(R)$.

If $E \subseteq \mathbb{R}^d$ is measurable, its Lebesgue measure is denoted by $|E|$, and its characteristic function by χ_E.

The symbol c denotes (possibly different) positive constants depending at most on α, d, or both. Similarly, C denotes positive constants that may depend on α and d, as well as on the integrability exponents p, q.

2. Main results

Our first main result, stated below, provides the complete range of exponents for which weak-type bounds hold uniformly in θ.

Theorem 2.1. *Let $0 < \delta \leq 1/2$. The following assertions hold:*

(i) If $1 \leq p < d/\alpha$, then there is a constant $C = C(\alpha, d, p) > 0$ independent of θ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{r,\infty} \leq C \left\|f\right\|_p \left\|g\right\|_1 \tag{2.1}$$

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, where $1/r = 1/p + 1 - \alpha/d$.

(ii) If $1 \leq q < d/\alpha$, then there is a constant $C = C(\alpha, d, q) > 0$ independent of θ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{r,\infty} \leq C \left\|f\right\|_1 \left\|g\right\|_q \tag{2.2}$$

for all $f \in L^1(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, where $1/r = 1/q + 1 - \alpha/d$.

(iii) If $0 \leq \theta \leq 1 - \delta$, and $1 \leq p < d/\alpha$, then there is a constant $C = C(\delta, \alpha, d, p) > 0$ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{p,\infty} \leq C \left\|f\right\|_p \left\|g\right\|_{\frac{d}{\alpha}} \tag{2.3}$$

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^{\frac{d}{\alpha}}(\mathbb{R}^d)$.

(iv) If $\delta \leq \theta \leq 1$ and $1 \leq q < d/\alpha$, then there is a constant $C = C(\delta, \alpha, d, q) > 0$ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{q,\infty} \leq C \left\|f\right\|_{\frac{d}{\alpha}} \left\|g\right\|_q \tag{2.4}$$

for all $f \in L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$.

(v) If $\delta \leq \theta \leq 1 - \delta$, then there is a constant $C = C(\delta, \alpha, d) > 0$ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{\frac{d}{\alpha},\infty} \leq C \left\|f\right\|_{\frac{d}{\alpha}} \left\|g\right\|_{\frac{d}{\alpha}} \tag{2.5}$$

for all $f \in L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ and $g \in L^{\frac{d}{\alpha}}(\mathbb{R}^d)$.

In Theorem 2.1, the different edges of the square inside which uniform strong-type bounds hold exhibit varying behavior with respect to the uniformity in θ of the estimates; see Figure 1. The weak-type estimates are uniform in $\theta \in [0,1]$ on the upper and right edges, uniform in θ away from 1 on the lower edge, and uniform in θ away from 0 on the left edge. The southwestern corner is where the behavior is most delicate as uniformity holds *only* away from both critical extremes of the unit interval.

At this point, one may wonder whether full uniform weak-type estimates can hold on the "bad" edges and corners at the cost of restricting the domain of the operator I_α^θ. The answer is yes, as long as the domains are appropriate Lorentz spaces $L^{p,1}$. This as shown in the next theorem.

Theorem 2.2. *The following estimates hold uniformly in θ:*

(i) If $1 \leq p \leq d/\alpha$, then there is a constant $C = C(\alpha, d, p) > 0$ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{p,\infty} \leq C \left\|f\right\|_{p,1} \left\|g\right\|_{\frac{d}{\alpha},1} \tag{2.6}$$

for all $f \in L^{p,1}(\mathbb{R}^d)$ and $g \in L^{\frac{d}{\alpha},1}(\mathbb{R}^d)$.

(ii) If $1 \leq q \leq d/\alpha$, then there is a constant $C = C(\alpha, d, q) > 0$ such that

$$\left\|I_\alpha^\theta(f,g)\right\|_{q,\infty} \leq C \left\|f\right\|_{\frac{d}{\alpha},1} \left\|g\right\|_{q,1} \tag{2.7}$$

for all $f \in L^{p,1}(\mathbb{R}^d)$ and $g \in L^{\frac{d}{\alpha},1}(\mathbb{R}^d)$.

Our last result provides strong-type bounds in the best possible range of exponents (a pentagon), uniformly away from the "bad directions". This is expected, since uniform bounds do not hold in this pentagon.

Theorem 2.3. *Let $1 \le p, q \le \infty$ be integrability exponents such that $(1/p, 1/q)$ lies in the pentagon determined by the interior of the convex hull of $\{(1,0),(0,1),(\alpha/d,0),(0,\alpha/d),(1,1)\}$, and let r be as in* (1.8). *If $0 < \delta \le 1/2$ and $\delta \le \theta \le 1-\delta$, then there is a constant $C = C(\delta,\alpha,d,p,q) > 0$ such that*

$$\left\| I^\theta_\alpha(f,g) \right\|_r \le C \, \|f\|_p \, \|g\|_q \tag{2.8}$$

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$.

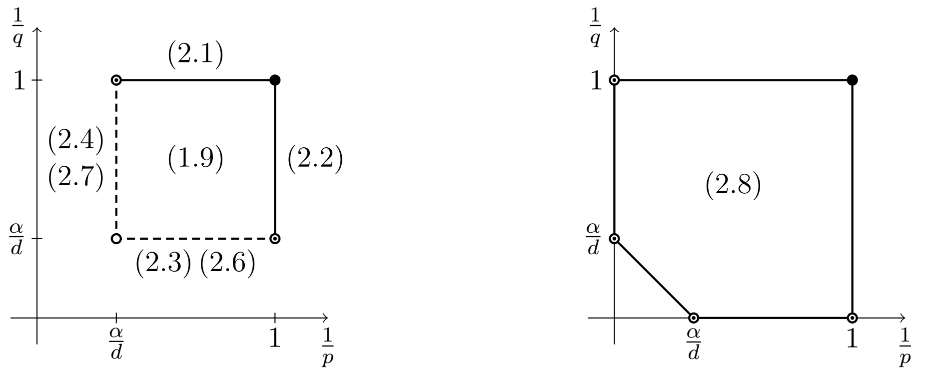

Figure 1. Each numbered estimate above corresponds to boundedness on an open region or a boundary segment. The bounds in Theorems 2.1 and 2.2 are depicted in the square on the left, while the region of boundedness in Theorem 2.3 is shown in the figure on the right.

3. Preliminary estimates

For $j \in \mathbb{Z}$, let I^θ_j be the operator defined for nonnegative measurable functions f and g on $\mathbb{R}^d$ by

$$I^\theta_j(f,g)(x) = \int_{|y| \le 2^j} f(x+(\theta-1)y)\, g(x+\theta y)\, \mathrm{d}y \,. \tag{3.1}$$

The first lemma, proved in [1], yields that the operator I^θ_j maps $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to both $L^1(\mathbb{R}^d)$ and $L^{\frac{1}{2}}(\mathbb{R}^d)$ uniformly in the parameter θ.

Lemma 3.1. *The operator I^θ_j satisfies the following estimates uniformly in θ:*

$$\left\| I^\theta_j(f,g) \right\|_1 \le \|f\|_1 \, \|g\|_1 \,, \tag{3.2}$$

$$\left\| I^\theta_j(f,g) \right\|_{\frac{1}{2}} \le c\, 2^{dj} \, \|f\|_1 \, \|g\|_1 \,. \tag{3.3}$$

Proof. See [1, Lemma 4.2 and Lemma 4.3]. □

Lemma 3.2. *For any $1 \le p \le \infty$, the operator I^θ_j satisfies the following estimates uniformly in θ:*

$$\left\| I^\theta_j(f,g) \right\|_{\frac{p}{p+1}} \le \begin{cases} c\, 2^{dj} \, \|f\|_1 \, \|g\|_p \,, & (3.4) \\ c\, 2^{dj} \, \|f\|_p \, \|g\|_1 \,. & (3.5) \end{cases}$$

Proof. Estimates (3.4) and (3.5) are symmetrical; we provide a proof of the former. Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$ be nonnegative. By Fubini's theorem and the change of variables $z = x + (\theta - 1)y$, we have:

$$\begin{aligned}\left\|I_j^\theta(f,g)\right\|_1 &\le \|g\|_\infty \int_{|y|\le 2^j}\int_{\mathbb{R}^d} f(x+(\theta-1)y)\,\mathrm{d}x\,\mathrm{d}y\\ &\le \|g\|_\infty \int_{|y|\le 2^j}\int_{\mathbb{R}^d} f(z)\,\mathrm{d}z\,\mathrm{d}y\\ &= c\,2^{dj}\|g\|_\infty\|f\|_1.\end{aligned}$$

Estimate (3.4) follows from this bound together with (3.3), by interpolation; see Theorem A.2 in the appendix. ☐

Lemma 3.3. *Let E be a measurable subset of $\mathbb{R}^d$ with $|E| < \infty$. The following estimates are valid for $1 \le p \le \infty$:*

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le \begin{cases} C\,(2^{dj}|E|)^{1-\frac1p}\,\|f\|_p\,\|g\|_1\,\min\{2^{dj},|E|\}^{\frac1p}, & (3.6)\\ C\,(2^{dj}|E|)^{1-\frac1p}\,\|f\|_1\,\|g\|_p\,\min\{2^{dj},|E|\}^{\frac1p}, & (3.7)\\ C\,2^{dj\left(1-\frac1p\right)}\,|E|^{2-\frac{\alpha}{d}-\frac1p}\,\|f\|_p\,\|g\|_{\frac{d}{\alpha}}\,\min\left\{2^{dj},\dfrac{|E|}{(1-\theta)^{d-\alpha}}\right\}^{\frac1p}, & (3.8)\\ C\,2^{dj\left(1-\frac1p\right)}\,|E|^{2-\frac{\alpha}{d}-\frac1p}\,\|f\|_{\frac{d}{\alpha}}\,\|g\|_p\,\min\left\{2^{dj},\dfrac{|E|}{\theta^{d-\alpha}}\right\}^{\frac1p}, & (3.9)\end{cases}$$

for some $C = C(\alpha,d,p) > 0$ independent of θ.

Proof. We start by deducing (3.6). Observe that the change of variables $z = x + \theta y$ gives

$$\int_{\mathbb{R}^d} I_j^\theta(f,g)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(z)\int_{|y|\le 2^j} f(z-y)\,\mathrm{d}y\,\mathrm{d}z \le C\|g\|_1\|f\|_p 2^{dj\left(1-\frac1p\right)}$$

and thus by the Cauchy-Schwarz inequality we have

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le |E|\left\|I_j^\theta(f,g)\right\|_1 \le C|E|\|f\|_p\|g\|_1 2^{dj\left(1-\frac1p\right)}. \tag{3.10}$$

On the other hand, using (3.5) and Hölder's inequality we obtain

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le |E|^{1-\frac1p}\left\|I_j^\theta(f,g)\right\|_{\frac{p}{p+1}} \le C|E|^{1-\frac1p}2^{dj}\|f\|_p\|g\|_1. \tag{3.11}$$

Combining (3.10) and (3.11) gives (3.6). An analogous argument based on (3.4) yields (3.7).

We now proceed to the proof of (3.8). Using Hölder's inequality we have

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le |E|^{2-\frac{\alpha}{d}}\left\|I_j^\theta(f,g)\right\|_{\frac{d}{\alpha}}.$$

By Jensen's inequality, Fubini's theorem, and the change of variables $z = x+\theta y$ we deduce

$$\begin{aligned}\left\|I_j^\theta(f,g)\right\|_{\frac{d}{\alpha}}^{\frac{d}{\alpha}} &= \int_{\mathbb{R}^d}\left(\int_{|y|\le 2^j} f(x+(\theta-1)y)\,g(x+\theta y)\,\mathrm{d}y\right)^{\frac{d}{\alpha}}\mathrm{d}x\\ &\le C2^{dj\left(\frac{d}{\alpha}-1\right)}\|f\|_\infty^{\frac{d}{\alpha}}\int_{|y|\le 2^j}\int_{\mathbb{R}^d} g(x+\theta y)^{\frac{d}{\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\ &= C2^{dj\frac{d}{\alpha}}\|f\|_\infty^{\frac{d}{\alpha}}\|g\|_{\frac{d}{\alpha}}^{\frac{d}{\alpha}}\end{aligned}$$

and hence

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le C2^{dj}|E|^{2-\frac{\alpha}{d}}\|f\|_\infty\|g\|_{\frac{d}{\alpha}}\,. \tag{3.12}$$

Moreover, by Hölder's inequality and (3.4) with $p=d/\alpha$,

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le |E|^{1-\frac{\alpha}{d}}\left\|I_j^\theta(f,g)\right\|_{\frac{d}{d+\alpha}} \le c2^{dj}|E|^{1-\frac{\alpha}{d}}\|f\|_1\|g\|_{\frac{d}{\alpha}}\,. \tag{3.13}$$

Additionally, by the Cauchy-Schwarz inequality and Fubini's theorem,

$$\begin{aligned}\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} &\le |E|\left\|I_j^\theta(f,g)\chi_E\right\|_1\\ &\le |E|\int_{B(2^j)}\int_E f(x+(\theta-1)y)g(x+\theta y)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Using the change of variables $z=x+(\theta-1)y$ we see that the previous expression equals

$$|E|\int_{B(2^j)}\int_{E+(\theta-1)y} f(z)g(z+y)\,\mathrm{d}z\,\mathrm{d}y$$

which, in turn, is bounded by

$$|E|\int_{\mathbb{R}^d} f(z)\int_{\frac{z-E}{\theta-1}} g(z+y)\,\mathrm{d}y\,\mathrm{d}z\,.$$

Applying Hölder's inequality to the inner integral above yields that the whole expression is bounded by

$$|E|\|f\|_1\|g\|_{\frac{d}{\alpha}}\left|\frac{E}{\theta-1}\right|^{1-\frac{\alpha}{d}} = |E|\|f\|_1\|g\|_{\frac{d}{\alpha}}|E|^{1-\frac{\alpha}{d}}|\theta-1|^{\alpha-d}$$

and so we conclude

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le (1-\theta)^{\alpha-d}|E|^{2-\frac{d}{\alpha}}\|f\|_1\|g\|_{\frac{d}{\alpha}}\,. \tag{3.14}$$

Now, for $1\le p\le\infty$, interpolating (3.12) and (3.13) results in

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le C2^{dj}|E|^{2-\frac{\alpha}{d}-\frac1p}\|f\|_p\|g\|_{\frac{d}{\alpha}}\,, \tag{3.15}$$

and similarly, interpolation between (3.12) and (3.14) yields

$$\left\|I_j^\theta(f,g)\chi_E\right\|_{\frac12} \le C(1-\theta)^{\frac{\alpha-d}{p}}2^{dj\left(1-\frac1p\right)}|E|^{2-\frac{\alpha}{d}}\|f\|_p\|g\|_{\frac{d}{\alpha}}\,. \tag{3.16}$$

Estimate (3.8) follows at once from (3.15) and (3.16). Analogously, we deduce (3.9). □

We conclude this section with two useful auxiliary estimates.

Lemma 3.4. *Let $d\in\mathbb{N}$, $0<\alpha<d$, $1\le p\le d/\alpha$, and $R,S>0$. There exists a positive constant $C=C(\alpha,d,p)$ such that:*

(i) If $p<d/\alpha$, then

$$\left(\sum_{j\in\mathbb{Z}}2^{(\alpha-\frac{d}{p})\frac{j}{2}}\min\{2^{dj},R\}^{\frac{1}{2p}}\right)^2 \le C\,R^{\frac{\alpha}{d}}\,. \tag{3.17}$$

(ii) If $1<p$, then

$$\sum_{j\in\mathbb{Z}}2^{\alpha(1-p)j}\min\{2^{\alpha pj}S,R\} \le C\,S\left(\frac{R}{S}\right)^{\frac1p}\,. \tag{3.18}$$

The proof of the inequalities in Lemma 3.4 follows by splitting the indices j into the ranges where $2^j < R^{1/d}$ and the complementary range in (3.17), and where $2^j < (R/S)^{1/(\alpha p)}$ and its complement in (3.18). In both cases, the corresponding series converge and yield the stated bounds; see also [1, Lemma 4.7], which corresponds to the cases $p = 1$ in (3.17) and $p = d/\alpha$ in (3.18). We note that the series on the left-hand side of (3.17) diverges for $p \geq d/\alpha$, and the series on the left-hand side of (3.18) diverges for $p \leq 1$.

4. Proofs of Theorems 2.1 and 2.3

In this section, we provide the proofs of Theorem 2.1 and Theorem 2.3. Recall that if h is a function in $L^{r,\infty}(\mathbb{R}^d)$ for some $0 < r < \infty$, then

$$\|h\|_{r,\infty} \leq \sup_{0<|E|<\infty} |E|^{-\frac{1}{s}+\frac{1}{r}} \|h\chi_E\|_s \tag{4.1}$$

where $0 < s < r$ and the supremum is taken over measurable sets $E \subseteq \mathbb{R}^d$ of finite measure [4].

Expressing $\mathbb{R}^d$ as the union of annuli,

$$\mathbb{R}^d = \bigcup_{j\in\mathbb{Z}} B(2^j) \setminus B(2^{j-1})$$

we deduce the following pointwise estimate for I_α^θ in terms of the auxiliary operator I_j^θ (3.1):

$$I_\alpha^\theta(f,g)(x) \leq c \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j}\, I_j^\theta(f,g)(x)\,. \tag{4.2}$$

Proof of Theorem 2.1 $(i), (ii), (iii), (iv)$.

We start by proving part (i). Let us fix $1 \leq p < d/\alpha$ and r such that $1/r + \alpha/d = 1/p + 1$. Using (4.1) with $s = 1/2$, the pointwise estimate (4.2), and (3.6) yields

$$\begin{aligned}
\left\|I_\alpha^\theta(f,g)\right\|_{r,\infty} &\leq \sup_{0<|E|<\infty} |E|^{-2+\frac{1}{r}} \left\|I_\alpha^\theta(f,g)\chi_E\right\|_{\frac{1}{2}} \\
&\leq c \sup_{0<|E|<\infty} |E|^{\frac{1}{p}-1-\frac{\alpha}{d}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \left\|I_j^\theta(f,g)\chi_E\right\|_{\frac{1}{2}}^{\frac{1}{2}}\right)^2 \\
&\leq C \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \|f\|_p \|g\|_1 \left(\sum_{j\in\mathbb{Z}} 2^{\left(\alpha-\frac{d}{p}\right)\frac{j}{2}} \min\{2^{dj}, |E|\}^{\frac{1}{2p}}\right)^2 \\
&\leq C\, \|f\|_p \|g\|_1,
\end{aligned}$$

where we have used (3.17) with $R = |E|$. The proof of part (ii) is analogous and is based on (3.7).

We now focus on part (*iii*). Assume that $0 \le \theta \le 1-\delta$ and $1 \le p < d/\alpha$. Then, by (4.1) with $s = 1/2$, (4.2), and (3.8),

$$\begin{aligned}
\left\| I_\alpha^\theta(f,g) \right\|_{p,\infty} &\le \sup_{0<|E|<\infty} |E|^{-2+\frac{1}{p}} \left\| I_\alpha^\theta(f,g)\chi_E \right\|_{\frac12} \\
&\le c \sup_{0<|E|<\infty} |E|^{-2+\frac{1}{p}} \left(\sum_{j\in\mathbb{Z}} 2^{\frac{(\alpha-d)j}{2}} \left\| I_j^\theta(f,g)\chi_E \right\|_{\frac12}^{\frac12} \right)^2 \\
&\le C \sup_{0<|E|<\infty} |E|^{-\frac{\alpha}{d}} \|f\|_p \|g\|_1 \left(\sum_{j\in\mathbb{Z}} 2^{\left(\alpha-\frac{d}{p}\right)\frac{j}{2}} \min\left\{ 2^{dj}, \frac{|E|}{(1-\theta)^{d-\alpha}} \right\}^{\frac{1}{2p}} \right)^2 \\
&\le C\delta^{(\alpha-d)\frac{\alpha}{d}} \|f\|_p \|g\|_{\frac{d}{\alpha}},
\end{aligned}$$

where we used (3.17) with $R = (1-\theta)^{\alpha-d}|E|$. Part (*iv*) is deduced similarly, based on (3.9). $\square$

Proof of Theorem 2.3.

It suffices to obtain restricted weak-type estimates for I_α^θ, with bounds depending on positive powers of θ^{-1} and $(1-\theta)^{-1}$, at the following points:

$$(1/p, 1/q) \in \{(1,1), (1,0), (0,1), (\alpha/d, 0), (0, \alpha/d)\}$$

and corresponding r given by (1.8). The result then follows by Marcinkiewicz interpolation, Theorem A.1 in the appendix.

The required estimate at the point $(1,1)$ is an immediate consequence of (2.1) with $p = 1$ (or equivalently, (2.2) with $q = 1$).

Regarding the points $(1,0), (0,1)$, we prove that

$$\left\| I_\alpha^\theta(f,g) \right\|_{\frac{d}{d-\alpha},\infty} \le \begin{cases} c\,(1-\theta)^{-\alpha}\, \|f\|_1\, \|g\|_\infty\,, & (4.3) \\ c\,\theta^{-\alpha}\, \|f\|_\infty\, \|g\|_1\,. & (4.4) \end{cases}$$

To establish (4.3) we note that if $g \in L^\infty(\mathbb{R}^d)$, then

$$I_\alpha^\theta(f,g)(x) \le \|g\|_\infty \int_{\mathbb{R}^d} f(x + (\theta-1)y)|y|^{\alpha-d}\,\mathrm{d}y = |\theta-1|^{-\alpha}\|g\|_\infty \int_{\mathbb{R}^d} f(z)|z-x|^{\alpha-d}\,\mathrm{d}z,$$

where in the second step we used the change of variables $z = x + (\theta-1)y$. Hence (4.3) follows from classical fractional integration. Estimate (4.4) is deduced similarly.

At last, we deduce restricted weak-type bounds for I_α^θ at $(\alpha/d, 0)$ and $(0, \alpha/d)$:

$$\left\| I_\alpha^\theta(\chi_A, \chi_B) \right\|_\infty \le \begin{cases} c\,(1-\theta)^{-\alpha}\, |A|^{\frac{\alpha}{d}}\,, & (4.5) \\ c\,\theta^{-\alpha}\, |B|^{\frac{\alpha}{d}}\,, & (4.6) \end{cases}$$

where A and B are measurable subsets of $\mathbb{R}^d$ of finite measure. As the estimates are analogous, we only provide a proof of the first. Note that

$$I_\alpha^\theta(\chi_A, \chi_B)(x) \le \int_{\mathbb{R}^d} \chi_A(x + (\theta-1)y)|y|^{\alpha-d}\,\mathrm{d}y = \int_{\frac{-x+A}{\theta-1}} |y|^{\alpha-d}\,\mathrm{d}y\,.$$

Since the function $y \mapsto |y|^{\alpha-d}$ blows up at the origin (and it is radially symmetric), we can estimate the integral above by

$$\int_{B(R)} |y|^{\alpha-d}\,\mathrm{d}y \quad \text{with} \quad R = \frac{c|A|^{\frac{1}{d}}}{(1-\theta)}$$

so that $\left|\frac{-x+A}{\theta-1}\right| \leq |B(R)|$. Thus,

$$I_\alpha^\theta(\chi_A,\chi_B)(x) \leq \int_0^R \int_{\partial B(\rho)} |y|^{\alpha-d}\,\mathrm{d}S(y)\,\mathrm{d}\rho = \frac{c}{(1-\theta)^\alpha}\,|A|^{\frac{\alpha}{d}}\,.$$

□

Proof of Theorem 2.1 (v).
Since $(\alpha/d,\alpha/d)$ belongs to the interior of the convex hull of

$$\{(1,0),(0,1),(\alpha/d,0),(0,\alpha/d),(1,1)\}$$

we have, by Theorem 2.3,

$$\left\|I_\alpha^\theta(f,g)\right\|_{\frac{d}{\alpha},\infty} \leq \left\|I_\alpha^\theta(f,g)\right\|_{\frac{d}{\alpha}} \leq \frac{c}{\theta^{\kappa_1}(1-\theta)^{\kappa_2}}\,\|f\|_{\frac{d}{\alpha}}\,\|g\|_{\frac{d}{\alpha}}$$

for some positive constants κ_1,κ_2 depending only on α and d. □

5. Proof of Theorem 2.2

In this section we provide a proof of Theorem 2.2. First, we need a lemma.

Lemma 5.1. *Let $1 < p \leq d/\alpha$ and A,B and E be measurable subsets of $\mathbb{R}^d$ of finite measure. Then, the auxiliary operator I_j^θ defined in* (3.1) *satisfies*

$$\int_E I_j^\theta(\chi_A,\chi_B)\,\mathrm{d}x \leq \begin{cases} C\,2^{dj\left(1-\frac{\alpha p}{d}\right)}\min\left\{2^{\alpha pj}|E|,|A||B|^{\frac{\alpha p}{d}}\right\}, & (5.1)\\ C\,2^{dj\left(1-\frac{\alpha p}{d}\right)}\min\left\{2^{\alpha pj}|E|,|A|^{\frac{\alpha p}{d}}|B|\right\}, & (5.2)\end{cases}$$

for some $C = C(\alpha,d,p) > 0$.

Proof. We first note that if $p = d/\alpha$ then (5.1) and (5.2) coincide; this case is handled in [1, Lemma 4.4]. Therefore we assume for the rest of the proof that $p < d/\alpha$. Due to symmetry we only prove (5.1). Using Fubini's theorem, the change of variables $z = x + (\theta-1)y$, and Hölder's inequality with $d/(\alpha p) > 1$ we deduce

$$\begin{aligned}\int_E I_j^\theta(\chi_A,\chi_B)\,\mathrm{d}x &\leq \int_{\mathbb{R}^d}\chi_A(z)\int_{\mathbb{R}^d}\chi_B(z+y)\chi_{B(2^j)}(y)\,\mathrm{d}y\,\mathrm{d}z\\ &\leq \int_{\mathbb{R}^d}\chi_A(z)\left(\int_{\mathbb{R}^d}\chi_B(z+y)\,\mathrm{d}y\right)^{\frac{\alpha p}{d}}\left(\int_{\mathbb{R}^d}\chi_{B(2^j)}(y)\,\mathrm{d}y\right)^{1-\frac{\alpha p}{d}}\\ &= C\,|A|\,|B|^{\frac{\alpha p}{d}}\,2^{dj\left(1-\frac{\alpha p}{d}\right)}.\end{aligned}$$

Estimate (5.1) follows from the bound above together with the fact that

$$\int_E I_j^\theta(\chi_A,\chi_B)\,\mathrm{d}x \leq c2^{dj}|E| = c\,2^{dj\left(1-\frac{\alpha p}{d}\right)}\,2^{\alpha pj}\,|E|\,.$$

□

Proof of Theorem 2.2.
In view of symmetry we only prove the estimate in (2.6). We first assume that $1 < p \leq d/\alpha$. Note that in this case the weak space $L^{p,\infty}(\mathbb{R}^d)$ is a Banach space. Therefore, when $1 < p \leq d/\alpha$ it suffices to show that (2.6) holds when f and g are characteristic functions

of sets of finite measure; see Theorem A.3 in the appendix. Let A and B be such sets. Using (4.1) with $s = 1$, (4.2) and (5.1) we deduce

$$\begin{aligned}\big\|I_\alpha(\chi_A,\chi_B)\big\|_{p,\infty} &\le c \sup_{0<|E|<\infty} |E|^{-1+\frac{1}{p}} \sum_{j\in\mathbb{Z}} 2^{(\alpha-d)j} \int_E I_j^\theta(f,g)\,\mathrm{d}x \\ &\le C \sup_{0<|E|<\infty} |E|^{-1+\frac{1}{p}} \sum_{j\in\mathbb{Z}} 2^{\alpha(1-p)j} \min\left\{2^{\alpha p j}|E|, |A||B|^{\frac{\alpha p}{d}}\right\} \\ &\le C\,|A|^{\frac{1}{p}}\,|B|^{\frac{\alpha}{d}},\end{aligned}$$

where in the last step we have used (3.18) with $S = |E|$ and $R = |A||B|^{\frac{\alpha p}{d}}$. Analogously, one deduces the estimate in (2.7) with $1 < q \le d/\alpha$, based on (5.2).

We now turn our attention to the case $p = 1$ in (2.6) (the case $q = 1$ in (2.7) is obtained similarly). By appropriate changes of variables we have

$$\begin{aligned}\big\|I_\alpha^\theta(f,g)\big\|_{1,\infty} &\le \big\|I_\alpha^\theta(f,g)\big\|_1 \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+(\theta-1)y)\,g(x+\theta y)|y|^{\alpha-d}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}^d} g(z+y)\,|y|^{\alpha-d}\,\mathrm{d}y\,\mathrm{d}z \\ &\le \|f\|_1\,\|I_\alpha(g)\|_\infty\,.\end{aligned}$$

The desired result follows by duality: the linear fractional integral I_α maps $L^{\frac{d}{d-\alpha},\infty}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, therefore it maps $L^{\frac{d}{\alpha},1}(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. □

6. Sharpness of Theorem 2.1

In this section, we investigate the sharpness of all the estimates in Theorem 2.1. To do so, we make use of two auxiliary functions. Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative radial function with $\mathrm{supp}(\Phi) \subseteq B(1/8)$. Additionally, we consider a function h on $\mathbb{R}^d$ defined by

$$h(x) = |x|^{-\alpha}\left(\log\frac{1}{|x|}\right)^{-\kappa}\chi_{|x|\le 1/e}(x), \tag{6.1}$$

where $\kappa = (d+\alpha)/(2d)$. Note that $\alpha/d < \kappa < 1$ since $\alpha < d$. One can readily see that h belongs to $L^{\frac{d}{\alpha}}(\mathbb{R}^d)$. Moreover, we have the following lower bound for the linear fractional integral of h:

$$\begin{aligned}I_\alpha(h)(x) &= \int_{|y|\le 1/e} |y|^{-\alpha}\left(\log\frac{1}{|y|}\right)^{-\kappa}|x-y|^{\alpha-d}\,\mathrm{d}y \\ &\ge \int_{|x|\le|y|\le 1/e} |y|^{-\alpha}\left(\log\frac{1}{|y|}\right)^{-\kappa}|x-y|^{\alpha-d}\,\mathrm{d}y\,\chi_{|x|\le 1/8}(x) \\ &\ge c\int_{|x|\le|y|\le 1/e} |y|^{-d}\,\mathrm{d}y\left(\log\frac{1}{|x|}\right)^{-\kappa}\chi_{|x|\le 1/8}(x) \\ &= c\left(\log\frac{1}{|x|}-1\right)\left(\log\frac{1}{|x|}\right)^{-\kappa}\chi_{|x|\le 1/8}(x)\,.\end{aligned}$$

Observe that for $|x| \le 1/8 < e^{-2}$ we have

$$\log\frac{1}{|x|} - 1 = \log\frac{1}{|x|}\left(1+\frac{1}{\log|x|}\right) \ge \frac{1}{2}\log\frac{1}{|x|}$$

and so, it follows from the above estimate that

$$I_\alpha(h)(x) \geq c\left(\log\frac{1}{|x|}\right)^{1-\kappa}\chi_{|x|\leq 1/8}(x) \tag{6.2}$$

for $x \in \mathbb{R}^d$.

Case I. We begin with the fact that we cannot take $p = d/\alpha$ in (2.1). Suppose, towards a contradiction, that this were the case. Then, by letting $\theta \to 0$ and using Fatou's lemma (which also holds for weak L^p spaces), we would obtain

$$\|I_\alpha(h)\varphi_t\|_{1,\infty} \leq C\|\varphi_t\|_1 \tag{6.3}$$

for all $t > 0$, where $\varphi_t(x) = t^{-d}\Phi(x/t)$. The right-hand side of (6.3) is constant, equal to $C\|\Phi\|_1$; we prove that the left-hand side tends to infinity as $t \to 0$, leading to a contradiction. Assuming that $0 < t < 1$ and using (6.2) we deduce

$$\begin{aligned} I_\alpha(h)(x)\varphi_t(x) &\geq c\left(\log\frac{1}{|x|}\right)^{1-\kappa} t^{-d}\Phi(x/t) \\ &= c\left(\log\frac{1}{t} + \log\frac{t}{|x|}\right)^{1-\kappa} t^{-d}\Phi(x/t) \\ &\geq c\left(\log\frac{1}{t}\right)^{\frac{1-\kappa}{2}}\left(\log\frac{t}{|x|}\right)^{\frac{1-\kappa}{2}} t^{-d}\Phi(x/t) \end{aligned}$$

and hence

$$\big\|I_\alpha(h)\varphi_t\big\|_{1,\infty} \geq c\left(\log\frac{1}{t}\right)^{\frac{1-\kappa}{2}}\big\|(\log|\cdot|^{-1})^{\frac{1-\kappa}{2}}\Phi\big\|_{1,\infty}$$

where we have used that $L^{1,\infty}(\mathbb{R}^d)$ has the same dilation structure as $L^1(\mathbb{R}^d)$. It is straightforward to verify that $\big\|(\log|\cdot|^{-1})^{\frac{1-\kappa}{2}}\Phi\big\|_{1,\infty}$ is a finite constant. Therefore, letting $t \to 0$ in the last inequality above yields the desired conclusion.

Case II. In a similar fashion (but letting $\theta \to 1$ instead) we obtain the sharpness of (2.2), that is, one cannot take $q = d/\alpha$ in that inequality.

Case III. Next, we obtain the sharpness of (2.3) in terms of δ. If (2.3) were valid for $\delta = 0$, then, by the same argument as in Case I, we would have

$$\|\psi_t I_\alpha(h)\|_{p,\infty} \leq C\|\psi_t\|_p \tag{6.4}$$

for all $t > 0$, where $\psi_t(x) = t^{-d/p}\Phi(x/t)$. The right-hand side of (6.4) equals $C\|\Phi\|_p$ and is therefore finite. We estimate the left-hand side using (6.2) (and assuming $0 < t < 1$):

$$\big\|\psi_t I_\alpha(h)\big\|_{p,\infty} \geq c\left(\log\frac{1}{t}\right)^{\frac{1-\kappa}{2}}\big\|(\log|\cdot|^{-1})^{\frac{(1-\kappa)p}{2}}\Phi^p\big\|_{1,\infty}^{1/p}$$

which blows-up as $t \to 0$, a contradiction.

Case IV. Analogously to Case III, we obtain the sharpness of (2.4), that is, one cannot take $\delta = 0$ in that inequality.

Case V. The sharpness of (2.5) in terms of δ; that is, the fact that we cannot take $\delta = 0$ in that estimate, is achieved as in Case III with $p = d/\alpha$.

Appendix A. Interpolation theorems

In this section we collect the interpolation results used in this work for the reader's convenience. First, we recall that a bilinear operator T acting on measurable functions is said to be of *restricted weak-type* (p, q, r) (with constant $c > 0$) if

$$\|T(\chi_A, \chi_B)\|_{r,\infty} \leq c\, |A|^{\frac{1}{p}}\, |B|^{\frac{1}{q}}$$

for all measurable sets A and B of finite measure. The integrability exponents p, q, r belong to $(0, \infty]$.

The first interpolation result we state is the Marcinkiewicz interpolation theorem. It provides strong-type bounds from a finite set of restricted weak-type estimates and is used in the proofs of Theorem 1.1 in [1] and Theorem 2.3. General multilinear versions of this interpolation theorem are proved in [7] and [5].

Theorem A.1. *Let $0 < p_i, q_i, r_i \leq \infty$ for $i = 1, 2, 3$. Suppose that the points*

$$\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big), \quad \Big(\frac{1}{p_2}, \frac{1}{q_2}\Big), \quad \Big(\frac{1}{p_3}, \frac{1}{q_3}\Big),$$

do not lie on the same line in $\mathbb{R}^2$. For $0 < \theta_1, \theta_2, \theta_3 < 1$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$ consider the points $0 < p, q, r \leq \infty$ such that

$$\Big(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\Big) = \theta_1\Big(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}\Big) + \theta_2\Big(\frac{1}{p_2}, \frac{1}{q_2}, \frac{1}{r_2}\Big) + \theta_3\Big(\frac{1}{p_3}, \frac{1}{q_3}, \frac{1}{r_3}\Big)$$

and

$$\frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\,.$$

Let T be a bilinear operator that is of restricted weak-type (p_i, q_i, r_i) (with constant $c_i > 0$) for $i = 1, 2, 3$. Then there is a constant $C > 0$ depending only on p_i, q_i, r_i, and θ_i ($i = 1, 2, 3$) such that

$$\|T(f, g)\|_{L^r} \leq C\, c_1^{\theta_1} c_2^{\theta_2} c_3^{\theta_3} \|f\|_{L^p} \|g\|_{L^q}$$

for all functions $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$.

The next result concerns interpolation of bilinear operators, in the sense that if an operator satisfies strong-type bounds at two points, then it satisfies strong-type bounds at all points in between. This result follows from general multilinear complex interpolation theorems (see [5, Theorem 7.2.9, Corollary 7.2.11] and [8, Theorem 3.2]), and it is used in the proofs of Lemmas 3.2 and 3.3.

Theorem A.2. *Let $0 < p_i, q_i, r_i \leq \infty$ ($i = 1, 2$) and let T be a bilinear operator acting of measurable functions such that*

$$\|T(f, g)\|_{r_i} \leq c_i\, \|f\|_{p_i}\, \|g\|_{q_i}$$

for all functions $f \in L^{p_i}(\mathbb{R}^d)$ and $g \in L^{q_i}(\mathbb{R}^d)$ and some positive constants c_i ($i = 1, 2$). Then for $0 < \varphi < 1$ and p, q, r such that

$$\frac{1}{p} = \frac{1-\varphi}{p_1} + \frac{\varphi}{p_2}, \qquad \frac{1}{q} = \frac{1-\varphi}{q_1} + \frac{\varphi}{q_2}, \qquad \frac{1}{r} = \frac{1-\varphi}{r_1} + \frac{\varphi}{r_2}$$

we have

$$\|T(f, g)\|_r \leq c_1^{1-\varphi}\, c_2^{\varphi}\, \|f\|_p\, \|g\|_q$$

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$.

At last, we present a result that yields sufficient conditions for a bilinear operator to admit an extension to the product of Lorentz spaces $L^{p,1}(\mathbb{R}^d) \times L^{q,1}(\mathbb{R}^d)$, $0 < p, q < \infty$, being the (Euclidean) bilinear counterpart of the result in [4, Exercise 1.4.7]. We use this result in the proof of Theorem 2.2.

Theorem A.3. *Let Z be a Banach space of real-valued measurable functions on $\mathbb{R}^d$, and let T be a bilinear operator defined on the space of finitely simple functions on $\mathbb{R}^d$ and taking values in Z. Suppose that for $0 < p, q < \infty$ and some constant $C > 0$ the following restricted weak-type estimate*

$$\|T(\chi_A, \chi_B)\|_Z \leq C\,|A|^{\frac{1}{p}}\,|B|^{\frac{1}{q}}$$

holds for all measurable sets A, B of finite measure. Then, T has a bounded extension from $L^{p,1}(\mathbb{R}^d) \times L^{q,1}(\mathbb{R}^d)$ to Z.

We note that the proof of the theorem above is straightforward and follows exactly the idea as in the linear case; the standard proof is omitted.

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References

[1] N. J. Alves, L. Grafakos, and A. E. Tzavaras, A bilinear fractional integral operator for Euler–Riesz systems, arXiv:2409.18309, 2024, to appear in *Anal. PDE.*

[2] Y. Ding and C. C. Lin, Rough bilinear fractional integrals, *Math. Nachr.*, 246-247, 47–52, 2002.

[3] L. Grafakos, On multilinear fractional integrals, *Studia Math.*, 102(1), 49–56, 1992.

[4] L. Grafakos, *Classical Fourier analysis*, Vol. 249 of Graduate Texts in Mathematics, Springer, New York, Third Edition, 2014.

[5] L. Grafakos, *Modern Fourier analysis*, Vol. 250 of Graduate Texts in Mathematics, Springer, New York, Third Edition, 2014.

[6] L. Grafakos and N. Kalton, Some remarks on multilinear maps and interpolation, *Math. Ann.*, 319, 151–180, 2001.

[7] L. Grafakos, L. Liu, S. Lu, and F. Zhao, The multilinear Marcinkiewicz interpolation theorem revisited: The behavior of the constant, *J. Funct. Anal.*, 262(5), 2289–2313, 2012.

[8] L. Grafakos and E. M. Ouhabaz, Interpolation for analytic families of multilinear operators on metric measure spaces. *Studia Math.*, 267(1), 37–57, 2022

[9] N. Hatano and Y. Sawano, A note on the bilinear fractional integral operator acting on Morrey spaces, *Trans. A. Razmadze Math. Inst.*, 173(3), 37–44, 2019.

[10] Q. He and D. Yan, Bilinear fractional integral operators on Morrey spaces, *Positivity*, 25(2), 399–429, 2021.

[11] C. Hoang and K. Moen, Weighted estimates for bilinear fractional integral operators and their commutators, *Indiana Univ. Math. J.*, 67(1), 397–428, 2018.

[12] C. E. Kenig and E. M. Stein, Multilinear estimates and fractional integration, *Math. Res. Lett.*, 6(1), 1–15, 1999.

[13] Y. Komori-Furuya, Weighted estimates for bilinear fractional integral operators: a necessary and sufficient condition for power weights, *Collect. Math.*, 71(1), 25–37, 2020.

[14] K. Li and W. Sun, Two weight norm inequalities for the bilinear fractional integrals, *Manuscripta Math.*, 150(1-2), 159–175, 2016.

[15] K. Moen, New weighted estimates for bilinear fractional integral operators, *Trans. Amer. Math. Soc.*, 366(2), 627–646, 2014.

[16] D. Serre, Divergence-free positive symmetric tensors and fluid dynamics, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(5), 1209–1234, 2018.

[17] D. Serre, Compensated integrability. Applications to the Vlasov–Poisson equation and other models in mathematical physics, *J. Math. Pures Appl. (9)*, 127, 67–88, 2019.

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