RELATION BETWEEN ASYMPTOTIC L_p-CONVERGENCE AND SOME CLASSICAL MODES OF CONVERGENCE

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Abstract. Asymptotic L_p-convergence, which resembles convergence in L_p, was introduced to address a question in diffusive relaxation. This note aims to compare asymptotic L_p-convergence with convergence in measure and in weak L_p spaces. One of the results characterizes convergence in measure on finite measure spaces in terms of asymptotic L_p-convergence.

Let $(X, \mathbb{X}, \mu)$ be a measure space. By a measurable function, we understand a real-valued μ-measurable function defined on X. The set of all such measurable functions (identified μ-a.e.) is denoted by $M(X)$. We use the notation A^c for the complement of a set $A \subseteq X$ with respect to X. It will be assumed that p is a number belonging to $[1, \infty)$.

The main purpose of this note is to compare the convergences in measure and in weak L_p spaces with asymptotic L_p-convergence. The latter concept was motivated by a question on convergence in diffusive relaxation and introduced in [1]. Specifically, a sequence (f_n) of measurable functions is said to *asymptotically L_p-converge* (in short, *α_p-converge*) to a measurable function f if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that $\int_{B_n} |f_n - f|^p \,\mathrm{d}\mu \to 0$ as $n \to \infty$.

1. Asymptotic L_p-convergence vs. convergence in measure

In [1], it was shown that α_p-convergence implies convergence in measure, and an example was given of a sequence of functions that converges in measure but does not α_p-converge. This example, however, is in an infinite measure space. For finite measure spaces, these two notions of convergence are, in fact, equivalent. Recall that a sequence (f_n) is said to *converge in measure* to a measurable function f if for every $\delta > 0$, $\mu\big(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}\big) \to 0$ as $n \to \infty$. We prove the following:

Theorem 1.1. *Let f_n $(n \in \mathbb{N})$ and f be measurable functions. If (f_n) α_p-converges to f, then (f_n) converges to f in measure. On the other hand, if $\mu(X) < \infty$ and (f_n) converges to f in measure, then (f_n) α_p-converges to f.*

The first part of this theorem was established in [1], but here we provide a simpler proof.

Proof. First, assume that (f_n) α_p-converges to f. For each $\delta > 0$ let the set $E_n(\delta)$ be given by

$$E_n(\delta) = \{x \in X \mid |f_n(x) - f(x)| \geq \delta\}.$$

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Then

$$\begin{aligned}\delta^p\mu(E_n(\delta)) &= \delta^p\mu(E_n(\delta)\cap B_n)+\delta^p\mu(E_n(\delta)\cap B_n^c)\\ &\le \int_{E_n(\delta)\cap B_n}\delta^p\,\mathrm{d}\mu+\delta^p\mu(B_n^c)\\ &\le \int_{B_n}|f_n-f|^p\,\mathrm{d}\mu+\delta^p\mu(B_n^c)\end{aligned}$$

where (B_n) is a sequence of measurable sets associated with the α_p-convergence of (f_n) towards f. Letting $n\to\infty$ yields the desired conclusion.

Now, assume that X has finite measure and that (f_n) converges to f in measure. Then, for each $n\in\mathbb{N}$, there exists $N_n\in\mathbb{N}$ such that $\mu(E_k(1/n))<1/n$ whenever $k\ge N_n$. Assume, without loss of generality, that $N_{n+1}>N_n$ for every natural n. Let (λ_k) be a sequence of positive numbers such that for every $n\in\mathbb{N}$, $\lambda_k=1/n$ whenever $k\in[N_n,N_{n+1})$, and set $B_k=E_k(\lambda_k)^c$ $(k\in\mathbb{N})$. Note that $\mu(B_k^c)\to 0$ as $k\to\infty$ and

$$\int_{B_k}|f_k-f|^p\,\mathrm{d}\mu\le\int_{B_k}\lambda_k^p\,\mathrm{d}\mu\le\lambda_k^p\mu(X)\to 0\quad\text{as }k\to\infty$$

which finishes the proof. □

Furthermore, there are notions of Cauchy sequences related to α_p-convergence and convergence in measure. We say that a sequence (f_n) of measurable functions is *asymptotically L_p-Cauchy* (in short, *α_p-Cauchy*) if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c)\to 0$ as $n\to\infty$ such that

$$\int_{B_n\cap B_m}|f_n-f_m|^p\,\mathrm{d}\mu\to 0\quad\text{as }n,m\to\infty,$$

and it is *Cauchy in measure* if

$$\forall\delta>0\quad\mu\big(\{x\in X\mid|f_n(x)-f_m(x)|\ge\delta\}\big)\to 0\quad\text{as }n,m\to\infty.$$

It is known that α_p-Cauchy sequences are Cauchy in measure, and that α_p-Cauchy sequences (respectively, Cauchy in measure) α_p-converge (respectively, converge in measure) to a measurable function f (see [1, 2]). This, together with Theorem 1.1, yields that in a finite measure space these two notions of Cauchy sequences are equivalent.

Theorem 1.2. *Let (f_n) be a sequence of measurable functions. If (f_n) is α_p-Cauchy, then (f_n) is Cauchy in measure. On the other hand, if $\mu(X)<\infty$ and (f_n) is Cauchy in measure, then (f_n) is α_p-Cauchy.*

Proof. The first part can be deduced similarly to the proof of the first part of Theorem 1.1, but with $E_n(\delta)$ replaced by $E_{n,m}(\delta)=\{x\in X\mid|f_n(x)-f_m(x)|\ge\delta\}$ and B_n replaced by $B_n\cap B_m$.

Regarding the second part, assume that $\mu(X)<\infty$ and that (f_n) is Cauchy in measure. Then, there exists a measurable function f such that (f_n) converges to f in measure. By Theorem 1.1 it follows that (f_n) α_p-converges to f, and hence (f_n) is α_p-Cauchy, as desired. □

2. Asymptotic L_p-convergence vs. convergence in weak L_p spaces

The *weak L_p space*, denoted by $L_{p,\infty}$, consists of all measurable functions f such that

$$\sup_{\delta>0}\delta^p\mu(\{x\in X\mid|f(x)|\ge\delta\})<\infty.$$

Let f_n ($n \in \mathbb{N}$) and f be functions belonging to $L_{p,\infty}$. The sequence (f_n) is said to *converge in* $L_{p,\infty}$ to f (see, e.g., [5]) if

$$\sup_{\delta>0} \delta^p \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}) \to 0 \quad \text{as } n \to \infty.$$

It is clear that this convergence implies convergence in measure.

The next example shows that asymptotic L_p-convergence does not imply convergence in weak L_p spaces even for spaces with finite measure.

Example 2.1. Let $X = [0,1]$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[0,1]$, and μ be Lebesgue measure on $\mathbb{X}$. Consider the sequence of functions (f_n) defined by $f_n = n^{1/p}\chi_{[0,1/n]}$. Then (f_n) α_p-converges to 0 (zero function), but it does not converge to 0 in $L_{p,\infty}$. To check the latter, for each $\delta > 0$ let $F_n(\delta) = \delta^p \mu(\{x \in [0,1] \mid |f_n(x)| \geq \delta\})$ and note that

$$F_n(\delta) = \begin{cases} \dfrac{\delta^p}{n} & \text{if } 0 < \delta \leq n^{1/p}, \\ 0 & \text{if } \delta > n^{1/p}. \end{cases}$$

Then $\sup_{\delta>0} F_n(\delta) = 1$ for every natural n and hence (f_n) does not converge to 0 in $L_{p,\infty}$.

The next example shows that the inverse implication is also false, i.e., convergence in weak L_p spaces does not imply asymptotic L_p-convergence.

Example 2.2. Let $X = [1,\infty)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[1,\infty)$, and μ be Lebesgue measure on $\mathbb{X}$. Let f_n be given by

$$f_n(x) = \frac{1}{(nx)^{\frac{1}{p}}} \quad (x \in [1,\infty)).$$

Then

$$\begin{aligned} F_n(\delta) &= \delta^p \mu(\{x \in [1,\infty) \mid |f_n(x)| \geq \delta\}) \\ &= \begin{cases} \dfrac{1 - n\delta^p}{n} & \text{if } 0 < \delta \leq \dfrac{1}{n^{1/p}}, \\ 0 & \text{if } \delta > \dfrac{1}{n^{1/p}}, \end{cases} \end{aligned}$$

and so $\sup_{\delta>0} F_n(\delta) = 1/n \to 0$ as $n \to \infty$, i.e., (f_n) converges to 0 in $L_{p,\infty}$.

Next, we show that (f_n) does not α_p-converge to 0. Let (B_n) be any sequence of measurable subsets of $[1,\infty)$ such that $\mu(B_n^c) \to 0$ as $n \to \infty$. Let $N \in \mathbb{N}$ be such that $\mu(B_n^c) < 1$ for all $n \geq N$. Then, for $n \geq N$,

$$\frac{1}{n}\int_{B_n^c} \frac{1}{x}\,\mathrm{d}x \leq \frac{1}{n}\mu(B_n^c) < \frac{1}{n} < \infty.$$

Note that

$$\frac{1}{n}\int_{B_n} \frac{1}{x}\,\mathrm{d}x = \frac{1}{n}\int_1^\infty \frac{1}{x}\,\mathrm{d}x - \frac{1}{n}\int_{B_n^c} \frac{1}{x}\,\mathrm{d}x.$$

Since the first integral on the right-hand side is infinite and for $n \geq N$ the second integral on the right-hand side is finite, it follows that for $n \geq N$ the integral on the left-hand side is infinite, i.e., (f_n) does not α_p-converge to 0.

Note that by Theorem 1.1, for spaces with finite measure, convergence in weak L_p spaces implies asymptotic L_p-convergence.

3. Almost L_p spaces vs. weak L_p spaces

Denote by A_p the space of measurable functions that are *almost in* L_p, i.e.,

$$A_p = \left\{ f \in M(X) \mid \forall \delta > 0 \ \exists E_\delta \text{ with } \mu(E_\delta) < \delta \text{ such that } \int_{E_\delta^c} |f|^p \, \mathrm{d}\mu < \infty \right\}.$$

These spaces are named as *almost L_p spaces* and were introduced in [3, 4]. It is clear that $L_p \subseteq A_p$, and it is not hard to see that there are functions belonging to A_p but not to L_p.

Below, we give examples that show that in general measure spaces, $A_p \setminus L_{p,\infty}$ and $L_{p,\infty} \setminus A_p$ are non-empty. Then, at last, we prove that $L_{p,\infty}$ is contained in A_p when $\mu(X) < \infty$.

Example 3.1. Let $X = (0,1)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $(0,1)$, and μ be Lebesgue measure on $\mathbb{X}$. Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x^2} \quad (x \in (0,1)).$$

This function does not belong to $L_{p,\infty}$. Indeed, for every $\delta \geq 1$ we have that

$$\delta^p \mu(\{x \in (0,1) \mid |f(x)| \geq \delta\}) = \delta^{p-1/2}.$$

Letting $\delta \to \infty$ yields the desired conclusion.

On the other hand, since f is decreasing, it is easy to see that f belongs to A_p.

Example 3.2. Let $X = [1,\infty)$, $\mathbb{X}$ be the collection of all Lebesgue measurable subsets of $[1,\infty)$, and μ be Lebesgue measure on $\mathbb{X}$. Let f be defined by

$$f(x) = \frac{1}{x^{\frac{1}{p}}} \quad (x \in [1,\infty)).$$

Using a similar reasoning as in Example 2.2 one can conclude that f does not belong to A_p but it belongs to $L_{p,\infty}$.

Proposition 3.3. *If $\mu(X) < \infty$, then $L_{p,\infty} \subseteq A_p$.*

Proof. By hypothesis we have

$$\sup_{\delta > 0} \delta^p \mu(\{x \in X \mid |f(x)| \geq \delta\}) = C < \infty.$$

In particular, for each $k \in \mathbb{N}$ it holds that

$$\mu(\{x \in X \mid |f(x)| \geq k\}) \leq \frac{C}{k^p}.$$

Given $\delta > 0$, let $K \in \mathbb{N}$ be such that

$$\mu(\{x \in X \mid |f(x)| \geq K\}) \leq \frac{C}{K^p} < \delta$$

and set $E_\delta = \{x \in X \mid |f(x)| \geq K\}$. Then

$$\int_{E_\delta^c} |f|^p \, \mathrm{d}\mu \leq K\mu(X) < \infty$$

which concludes the proof. □

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References

[1] N. J. Alves and J. Paulos, A mode of convergence arising in diffusive relaxation, *The Quarterly Journal of Mathematics* 75(1), 143–159, 2024.

[2] R. G. Bartle, *Elements of Integration and Lebesgue Measure*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1995.

[3] O. G. Bravo and E. A. S. Pérez, Optimal range theorems for operators with p-th power factorable adjoints. *Banach Journal of Mathematical Analysis*, 6(1), 61-73, 2012.

[4] J. M. Calabuig, O. G. Bravo, M. A. Juan, and E. A. S. Pérez, Representation and factorization theorems for almost-L_p-spaces. *Indagationes Mathematicae*, 30(5), 930-942, 2019.

[5] R. E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces*. Springer, Cham, 2016.