ON F-SPACES OF ALMOST-LEBESGUE FUNCTIONS

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ABSTRACT. We consider the space of functions almost in L_p with the topology of asymptotic L_p-convergence. We prove that this space forms a completely metrizable topological vector space, which extends the space of measurable functions with the topology of convergence in measure to infinite measure spaces. We investigate classical properties such as dominated convergence, Vitali convergence theorems, and Marcinkiewicz interpolation. For the case where the underlying measure space is $\mathbb{R}^d$, we establish results on approximation by smooth functions, separability, and the boundedness of maximal functions. Additionally, we explore key topological features, such as local boundedness, local convexity, and duality, which fundamentally distinguish these spaces from the standard L_p spaces.

1. Introduction

Let (X, μ) be a (positive) measure space with a σ-algebra Σ of measurable sets. The space of real-valued measurable functions on X (identified μ-a.e.) is denoted by $L_0(\mathrm{X})$. Such functions are simply called *measurable*. For each $p \in [1, \infty)$, the Lebesgue space of measurable functions f satisfying $\|f\|_p^p = \int_{\mathrm{X}} |f|^p \,\mathrm{d}\mu < \infty$ is denoted by $L_p(\mathrm{X})$. Recall that a sequence of measurable functions (f_n) is said to *converge in measure* to a measurable function f if for each $\delta > 0$, $\mu(|f_n - f| > \delta) \to 0$ as $n \to \infty$. It is well known that if X is of finite measure, then the space $L_0(\mathrm{X})$ with the topology of convergence in measure is a complete metric vector space [5]. However, when X does not have finite measure, the metrizability of $L_0(\mathrm{X})$ is lost.

In this work, we begin by addressing this gap by introducing a complete metric vector space of measurable functions that extends $L_0(\mathrm{X})$ to general measure spaces, including those of infinite measure. For each $p \in [1, \infty)$, we consider a space $\Lambda_p(\mathrm{X})$ defined as

$$\Lambda_p(\mathrm{X}) = \big\{ f \in L_0(\mathrm{X}) \mid \forall\, \delta > 0 \;\exists\, E_\delta \in \Sigma \;:\; \mu(E_\delta) < \delta \text{ and } f\chi_{E_\delta^c} \in L_p(\mathrm{X}) \big\} \tag{1.1}$$

where χ_E denotes the characteristic function of a set $E \subseteq \mathrm{X}$. The space $\Lambda_p(\mathrm{X})$, called *almost L_p-space*, was initially considered in [6, 7] in a context of Banach function spaces. We will endow this space with a map $\|\cdot\|_{\alpha_p}$ defined by

$$\|f\|_{\alpha_p} = \|\min(|f|, 1)\|_p \tag{1.2}$$

for every measurable function f. For $p = 1$, we shorten the notation to $\|\cdot\|_\alpha$.

This investigation initiated in [2] where two modes of convergence for measurable functions were introduced and some of their properties were studied. The main notion of convergence considered there, *asymptotic L_p-convergence*, was motivated by a question in diffusive relaxation, specifically which notion of convergence could be deduced from convergence in relative entropy (a question that has since been settled in [3]). A sequence of measurable functions (f_n) is said to *asymptotically L_p-converge* (in short, *α_p-converge*) to a measurable function f if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that

$$\int_{B_n} |f_n - f|^p \,\mathrm{d}\mu \to 0 \quad \text{as } n \to \infty.$$

2020 *Mathematics Subject Classification.* 28A20, 46A16, 54G05.

Key words and phrases. asymptotic L_p-convergence, almost L_p-spaces, F-spaces.

Secondly, we considered the notion of *convergence almost in* L_p, which is the natural mode of convergence for the space $\Lambda_p(\mathrm{X})$. A sequence (f_n) of measurable functions is said to *converge almost in* L_p to a measurable function f if for each $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f_n - f|^p \, \mathrm{d}\mu \to 0 \quad \text{as } n \to \infty.$$

Convergence in L_p clearly implies both these notions of convergence, which in turn, imply convergence in measure. Additionally, if (f_n) converges to f almost in L_p, then it α_p-converges to f, and reciprocally, if (f_n) α_p-converges to f, then there exists a subsequence that converges to f almost in L_p [2].

One of the results in [2] states that the space of measurable functions, together with asymptotic L_p-convergence, forms a sequential convergence class [8]. It is always the case that metric convergence notions yield sequential convergence classes. This raised the question of whether the topology of asymptotic L_p-convergence can be induced by a metric. A crucial insight in this direction was given in [1], where it was proved that on finite measure spaces, convergence in measure and asymptotic L_p-convergence are equivalent. Moreover, when X has finite measure, it is known that $\mathrm{d}_{\alpha_p}(f,g) \coloneqq \|f-g\|_{\alpha_p}$ defines a metric on $L_0(\mathrm{X})$ that generates the topology of convergence in measure; see [5, Exercise 4.7.60] where the case $p = 1$ is addressed.

For general measure spaces, we prove that the extended metric d_{α_p} generates the topology of asymptotic L_p-convergence; see Section 2.1. The next key step involves identifying a suitable linear space of measurable functions on which d_{α_p} is finite. This then defines a metric linear space of measurable functions that inherits the topology of asymptotic L_p-convergence.

We achieve this by observing that the map $\|\cdot\|_{\alpha_p}$, which is known to define an extended F-norm [20], becomes finite when restricted to $\Lambda_p(\mathrm{X})$, thus providing an F-norm for this space; see Definition A.1 and Proposition A.2 in the appendix. Consequently, d_{α_p} defines a metric on $\Lambda_p(\mathrm{X})$, which is clearly translation-invariant. An *F-space* is a metrizable topological vector space which is complete with respect to a translation-invariant metric [16]. Our first main result is the following:

Theorem 1.1. *For each* $p \in [1,\infty)$*, the space* $\Lambda_p(\mathrm{X})$*, endowed with the* F*-norm* $\|\cdot\|_{\alpha_p}$*, is an* F*-space with the topology of asymptotic* L_p*-convergence. Moreover, if* X *has finite measure, then* $\Lambda_p(\mathrm{X})$ *coincides with the space* $L_0(\mathrm{X})$ *with the topology of convergence in measure.*

This work thus contributes another example to the general theory of F-spaces. Other examples include, for $0 < p < 1$, the Lebesgue spaces $L_p(\mathrm{X})$, the sequence spaces l_p, and the Hardy spaces H^p of analytic functions; see [16, 21] and the many references therein. Furthermore, as $\Lambda_p(\mathrm{X})$ extends the space $L_0(\mathrm{X})$ to general measure spaces, this opens an interesting line of research aimed at understanding which properties of $\Lambda_p(\mathrm{X})$, when $\mu(\mathrm{X}) = \infty$, align or diverge from those of $L_0(\mathrm{X})$, when $\mu(\mathrm{X}) < \infty$. For an extensive literature on the space $L_0(0,1)$ we refer to [12, 15, 17, 19, 13, 14, 9].

A notable feature of the spaces $\Lambda_p(\mathrm{X})$ is that, although they represent all measurable functions when X has finite measure, their almost-L_p structure allows for many results that have classical counterparts in the theory of Lebesgue spaces. We obtain analogs of several classical results, such as the Lebesgue dominated convergence theorem, the Vitali convergence theorem, and a Marcinkiewicz-type interpolation theorem that deals with essentially bounded functions. We also explore the case where the underlying measure space is $\mathbb{R}^d$, establishing results on approximation by smooth functions, separability, and boundedness of maximal functions. Furthermore, we prove that $\Lambda_p(\mathbb{R}^d)$ is neither locally bounded nor locally convex, and that its dual space is trivial, highlighting key differences from Lebesgue spaces.

The paper is organized as follows. In Section 2.1, we establish the equivalence between α_p-convergence and convergence with respect to the F-norm $\|\cdot\|_{\alpha_p}$. In Section 2.2, we identify the condition under which a function in $\Lambda_p(\mathrm{X})$ belongs to $L_p(\mathrm{X})$. Section 2.3 contains the proof of Theorem 1.1, divided into three lemmas. The dominated convergence theorems are presented in Section 3, followed by two Vitali-type convergence theorems in Section 4. Approximation and separability are discussed in Section 5, where we show that $\Lambda_p(\mathbb{R}^d)$ is separable. Interpolation of operators, in the spirit of the Marcinkiewicz interpolation theorem, is explored in Section 6 and applied in Section 7 to deduce the boundedness of maximal functions. Finally, in Section 8, we investigate the topological properties of local boundedness, local convexity, and duality of $\Lambda_p(\mathbb{R}^d)$.

2. Preliminary results

2.1. Metrizability of asymptotic L_p-convergence.

We first characterize the asymptotic L_p-convergence of measurable functions in terms of convergence with respect to the (extended) F-norm $\|\cdot\|_{\alpha_p} = \|\min(|\cdot|, 1)\|_p$.

Proposition 2.1. *Let $(f_n), f$ be measurable functions. Then (f_n) α_p-converges to f if, and only if, $\|f_n - f\|_{\alpha_p} \to 0$ as $n \to \infty$.*

Proof. The result follows from the observation that for every measurable functions f and g one has

$$\begin{aligned}\int_{\{|f-g|\le 1\}} |f-g|^p \,\mathrm{d}\mu + \mu(|f-g|>1) &= \int_{\mathrm{X}} |\min(|f-g|,1)|^p \,\mathrm{d}\mu \\ &\le \int_B |f-g|^p \,\mathrm{d}\mu + \mu(B^c)\end{aligned}$$

for every measurable set B. □

Next, we characterize asymptotically L_p-Cauchy sequences in terms of Cauchy sequences with respect to $\|\cdot\|_{\alpha_p}$. Recall that a sequence (f_n) of measurable functions is said to be *asymptotically L_p-Cauchy* (in short, *α_p-Cauchy*) if there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ as $n \to \infty$ such that

$$\int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu \to 0 \quad \text{as } n, m \to \infty.$$

Proposition 2.2. *Let (f_n) be a sequence of measurable functions. Then (f_n) is α_p-Cauchy if, and only if, $\|f_n - f_m\|_{\alpha_p} \to 0$ as $n, m \to \infty$.*

Proof. First, suppose that (f_n) is α_p-Cauchy and let (B_n) be the sequence of measurable sets associated with this property. Then

$$\|f_n - f_m\|_{\alpha_p}^p \le \int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu + \mu(B_n^c) + \mu(B_m^c) \to 0 \quad \text{as } n, m \to \infty$$

and hence (f_n) is Cauchy with respect to $\|\cdot\|_{\alpha_p}$.

Now, assume that (f_n) is Cauchy with respect to $\|\cdot\|_{\alpha_p}$. Then

$$\int_{\{|f_n - f_m|\le 1\}} |f_n - f_m|^p \,\mathrm{d}\mu \to 0 \quad \text{and} \quad \mu(|f_n - f_m|>1) \to 0 \quad \text{as } n, m \to \infty.$$

We claim that (f_n) is Cauchy in measure. Indeed, for $\delta > 0$ it holds that

$$\begin{aligned}\delta^p \mu(|f_n - f_m| > \delta) &= \delta^p \mu(\{|f_n - f_m| > \delta\} \cap \{|f_n - f_m| \le 1\}) \\ &\quad + \delta^p \mu(\{|f_n - f_m| > \delta\} \cap \{|f_n - f_m| > 1\}) \\ &\le \int_{\{|f_n - f_m|\le 1\}} |f_n - f_m|^p \,\mathrm{d}\mu + \delta^p \mu(|f_n - f_m| > 1) \to 0 \quad \text{as } n, m \to \infty\end{aligned}$$

which proves the claim. Consequently, there exists a measurable function f to which (f_n) converges in measure. For each $n \in \mathbb{N}$, let $B_n = \{|f_n - f| \leq 1/2\}$ and note that $\mu(B_n^c) \to 0$ as $n \to \infty$. Thus, $B_n \cap B_m \subseteq \{|f_n - f_m| \leq 1\}$ and hence

$$\int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu \leq \int_{\{|f_n - f_m| \leq 1\}} |f_n - f_m|^p \,\mathrm{d}\mu \to 0 \quad \text{as } n, m \to \infty$$

which finishes the proof. □

2.2. **Lebesgue vs. almost-Lebesgue.**

The first result of this section provides a necessary and sufficient condition for a function in $\Lambda_p(\mathrm{X})$ to also belong to $L_p(\mathrm{X})$. The condition is as follows: A measurable function f is said to have an *absolutely continuous p-integral* if for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\int_E |f|^p \,\mathrm{d}\mu < \varepsilon^p$ for every measurable set E with $\mu(E) < \delta_\varepsilon$.

Proposition 2.3. *A measurable function f belongs to $L_p(\mathrm{X})$ if, and only if, it belongs to $\Lambda_p(\mathrm{X})$ and has an absolutely continuous p-integral.*

Proof. It is well known that if f belongs to $L_p(\mathrm{X})$, then f has an absolutely continuous p-integral. Moreover, it is clear that $L_p(\mathrm{X}) \subseteq \Lambda_p(\mathrm{X})$. This establishes one direction.

For the other direction, assume that f belongs to $\Lambda_p(\mathrm{X})$ and has an absolutely continuous p-integral. Let $\delta > 0$ be such that $\int_E |f|^p \,\mathrm{d}\mu < 1$ whenever $\mu(E) < \delta$. There exists E_δ with $\mu(E_\delta) < \delta$ such that $\int_{E_\delta^c} |f|^p \,\mathrm{d}\mu < \infty$. Then

$$\int_X |f|^p \,\mathrm{d}\mu = \int_{E_\delta} |f|^p \,\mathrm{d}\mu + \int_{E_\delta^c} |f|^p \,\mathrm{d}\mu < 1 + \int_{E_\delta^c} |f|^p \,\mathrm{d}\mu < \infty$$

which finishes the proof. □

Example 2.4. The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

belongs to $\Lambda_p(\mathbb{R}) \setminus L_p(\mathbb{R})$ for every $p > 1$.

The next result provides a condition under which a measurable function with finite F-norm $\|\cdot\|_{\alpha_p}$ belongs to $\Lambda_p(\mathrm{X})$.

Proposition 2.5. *A measurable function f belongs to $\Lambda_p(\mathrm{X})$ if, and only if, $\|f\|_{\alpha_p} < \infty$ and for each $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that*

$$\int_{E_\delta^c \cap \{|f| > 1\}} |f|^p \,\mathrm{d}\mu < \infty. \tag{2.1}$$

Proof. If $f \in \Lambda_p(\mathrm{X})$, then $\|f\|_{\alpha_p} < \infty$ and given $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c \cap \{|f| > 1\}} |f|^p \,\mathrm{d}\mu \leq \int_{E_\delta^c} |f|^p \,\mathrm{d}\mu < \infty.$$

On the other hand, if $\|f\|_{\alpha_p} < \infty$ and f satisfies (2.1), then given $\delta > 0$ there is E_δ with $\mu(E_\delta) < \delta$ such that

$$\begin{aligned} \int_{E_\delta^c} |f|^p \,\mathrm{d}\mu &= \int_{E_\delta^c \cap \{|f| > 1\}} |f|^p \,\mathrm{d}\mu + \int_{E_\delta^c \cap \{|f| \leq 1\}} |f|^p \,\mathrm{d}\mu \\ &\leq \int_{E_\delta^c \cap \{|f| > 1\}} |f|^p \,\mathrm{d}\mu + \|f\|_{\alpha_p}^p < \infty \end{aligned}$$

which completes the proof. □

In the last result of this section we observe that functions in $\Lambda_p(\mathrm{X})$ have finite F-norm $\|\cdot\|_{\alpha_q}$ for $q \geq p$.

Proposition 2.6. *If $f \in \Lambda_p(\mathrm{X})$ for some $p \geq 1$, then $\|f\|_{\alpha_q} < \infty$ for every $q \geq p$.*

Proof. Let $1 \leq p \leq q$ and assume that $f \in \Lambda_p(\mathrm{X})$. Then $\|f\|_{\alpha_p}$ is finite, and hence $\mu(|f| > 1)$ and $\int_{|f|\leq 1} |f|^p \, \mathrm{d}\mu$ are both finite. Since for $|f| \leq 1$ we have $|f|^q \leq |f|^p$ it follows that

$$\begin{aligned} \|f\|_{\alpha_q}^q &= \mu(|f| > 1) + \int_{|f|\leq 1} |f|^q \, \mathrm{d}\mu \\ &\leq \mu(|f| > 1) + \int_{|f|\leq 1} |f|^p \, \mathrm{d}\mu < \infty \end{aligned}$$

which concludes the proof. □

2.3. **Proof of Theorem 1.1.**

We first evaluate the continuity of the operations of addition, $(f, g) \mapsto f+g$, and scalar multiplication, $(\lambda, f) \mapsto \lambda f$. The continuity of the addition operation follows immediately from the triangle inequality. The continuity of scalar multiplication is established in the next lemma. Consequently, $\Lambda_p(\mathrm{X})$, endowed with the F-norm $\|\cdot\|_{\alpha_p}$, is a metrizable topological vector space.

Lemma 2.7. *Let $(f_n), f$ belong to $\Lambda_p(\mathrm{X})$ and $(\lambda_n), \lambda$ belong to $\mathbb{R}$. If (f_n) α_p-converges to f and (λ_n) converges to λ, then $(\lambda_n f_n)$ α_p-converges to λf.*

Proof. Let $\varepsilon > 0$. Since $f \in \Lambda_p(\mathrm{X})$, there exists a measurable set E_ε with $\mu(E_\varepsilon) < \varepsilon/2^{p+1}$ such that $\int_{E_\varepsilon^c} |f|^p \, \mathrm{d}\mu = C_\varepsilon < \infty$. Moreover, from the α_p-convergence of (f_n) towards f, there exist a sequence of measurable sets (B_n) and $M_\varepsilon \in \mathbb{N}$ so that for all $n \geq M_\varepsilon$,

$$\int_{B_n} |f_n - f|^p \, \mathrm{d}\mu < \frac{\varepsilon}{2^{p+1}(1+|\lambda|)^p} \quad \text{and} \quad \mu(B_n^c) < \frac{\varepsilon}{2^{p+1}}.$$

Additionally, let $N, K_\varepsilon \in \mathbb{N}$ be such that

$$|\lambda_n - \lambda| < \frac{\varepsilon}{2^{p+1}C_\varepsilon} \quad \forall n \geq K_\varepsilon$$

and

$$|\lambda_n - \lambda| < 1 \quad \forall n \geq N.$$

Set $N_\varepsilon = \max\{N, M_\varepsilon, K_\varepsilon\}$ and note that for $n \geq N_\varepsilon$, $|\lambda_n| < 1 + |\lambda|$.

Thus, for $n \geq N_\varepsilon$ we estimate

$$\begin{aligned} \|\lambda_n f_n - \lambda f\|_{\alpha_p}^p &\leq 2^{p-1}\left(\|\lambda_n f_n - \lambda_n f\|_{\alpha_p}^p + \|\lambda_n f - \lambda f\|_{\alpha_p}^p\right) \\ &= 2^{p-1}\left(\int_X |\min(|\lambda_n||f_n - f|, 1)|^p \, \mathrm{d}\mu + \|(\lambda_n - \lambda) f\|_{\alpha_p}^p\right) \\ &\leq 2^{p-1}\left(|\lambda_n|^p \int_{B_n} |f_n - f|^p \, \mathrm{d}\mu + \mu(B_n^c)\right) \\ &\quad + 2^{p-1}\left(|\lambda_n - \lambda|^p \int_{E_\varepsilon^c} |f|^p \, \mathrm{d}\mu + \mu(E_\varepsilon)\right) \\ &< \varepsilon \end{aligned}$$

which concludes the proof. □

The next result establishes the completeness of $\Lambda_p(\mathrm{X})$, meaning that every Cauchy sequence in $\Lambda_p(\mathrm{X})$ converges to some element of $\Lambda_p(\mathrm{X})$. As a consequence, we have that $\Lambda_p(\mathrm{X})$ is an F-space.

Lemma 2.8. *Let (f_n) be a sequence of functions in $\Lambda_p(\mathrm{X})$. If (f_n) is α_p-Cauchy, then there exists $f \in \Lambda_p(\mathrm{X})$ such that $\|f_n - f\|_{\alpha_p} \to 0$ as $n \to \infty$.*

Proof. Since (f_n) is α_p-Cauchy, there exists a measurable function f to which (f_n) α_p-converges [2, Theorem 3.3]. Then, there exists a subsequence $(g_n) \subseteq (f_n)$ converging to f almost in L_p [2, Proposition 2.11]. Let $\delta > 0$. There exists a measurable set E_δ with $\mu(E_\delta) < \delta/2$ such that

$$\int_{E_\delta^c} |g_n - f|^p \, \mathrm{d}\mu \to 0 \quad \text{as } n \to \infty.$$

Pick $N \in \mathbb{N}$ so that

$$\int_{E_\delta^c} |g_N - f|^p \, \mathrm{d}\mu < \frac{1}{2^{p-1}}.$$

Since the function f_N belongs to $\Lambda_p(\mathrm{X})$, there exists a measurable set F_δ with $\mu(F_\delta) < \delta/2$ such that

$$\int_{F_\delta^c} |g_N|^p \, \mathrm{d}\mu < \infty.$$

Set $G_\delta = E_\delta \cup F_\delta$. Then $\mu(G_\delta) < \delta$ and

$$\begin{aligned} \int_{G_\delta^c} |f|^p \, \mathrm{d}\mu &\leq 2^{p-1} \int_{E_\delta^c} |g_N - f|^p \, \mathrm{d}\mu + 2^{p-1} \int_{F_\delta^c} |g_N|^p \, \mathrm{d}\mu \\ &< 1 + 2^{p-1} \int_{F_\delta^c} |g_N|^p \, \mathrm{d}\mu < \infty \end{aligned}$$

from which we conclude that $f \in \Lambda_p(\mathrm{X})$, as desired. □

To conclude the proof of Theorem 1.1, we show that on finite measure spaces, measurable functions are almost in L_p for any $p \geq 1$.

Lemma 2.9. *If $\mu(\mathrm{X}) < \infty$, then $L_0(\mathrm{X}) = \Lambda_p(\mathrm{X})$ for all $p \geq 1$.*

Proof. We show that $L_0(\mathrm{X}) \subseteq \Lambda_p(\mathrm{X})$. Let $f \in L_0(\mathrm{X})$ and note that, by definition, f is finite μ-almost everywhere. For each $n \in \mathbb{N}$, let E_n be the set $E_n = \{x \in X \mid |f(x)|^p \geq n\}$. It is clear that $E_{n+1} \subseteq E_n$ for every natural n. We claim that $\mu(E_n) \to 0$ as $n \to \infty$. Suppose not, then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $\tilde{k}_n \geq n$ so that $\mu(E_{\tilde{k}_n}) \geq \varepsilon$. Set $k_1 = \tilde{k}_1$. If $\tilde{k}_2 \geq k_1$, set $k_2 = \tilde{k}_2$; otherwise $k_2 = k_1$. Similarly, if $\tilde{k}_3 \geq k_2$, set $k_3 = \tilde{k}_3$; otherwise $k_3 = k_2$. Continuing this process, we find a sequence $(k_n) \to \infty$ as $n \to \infty$, such that $k_{n+1} \geq k_n$ and so $E_{k_{n+1}} \subseteq E_{k_n}$. Then

$$\lim_{n\to\infty} \mu(E_{k_n}) = \mu\left(\bigcap_{n\in\mathbb{N}} E_{k_n}\right) \geq \varepsilon$$

which implies that f is infinite on a set of positive measure, which is impossible, thus establishing the claim. Now, let $\delta > 0$ and choose $N \in \mathbb{N}$ so that $\mu(E_N) < \delta$. We have

$$\int_{E_N^c} |f|^p \, \mathrm{d}\mu \leq N\mu(E_N^c) < \infty$$

which finishes the proof. □

3. Dominated convergence results

The first result of this section is a Lebesgue dominated convergence theorem for the space $\Lambda_p(\mathrm{X})$, being a straightforward adaptation of [4, Theorem 5.6].

Proposition 3.1. *Let $(f_n) \subseteq \Lambda_1(\mathrm{X})$ and f be measurable. Assume that (f_n) converges to f a.e. and that there exists $g \in \Lambda_1(\mathrm{X})$ such that $|f_n| \leq g$ a.e. for every $n \in \mathbb{N}$. Then $f \in \Lambda_1(\mathrm{X})$ and for every $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that*

$$\int_{E_\delta^c} f_n \, \mathrm{d}\mu \to \int_{E_\delta^c} f \, \mathrm{d}\mu \qquad \textit{as } n \to \infty.$$

The next result is a consequence of the previous proposition, and the first part of its proof is very similar to that of [4, Theorem 7.2].

Proposition 3.2. *Let $(f_n) \subseteq \Lambda_p(\mathrm{X})$ and f be measurable. Assume that (f_n) converges to f almost everywhere and that there exists $g \in \Lambda_p(\mathrm{X})$ such that $\sup_n |f_n| \leq g$ almost everywhere. Then $f \in \Lambda_p(\mathrm{X})$ and $\|f_n - f\|_{\alpha_p} \to 0$ as $n \to \infty$.*

Proof. Since $f_n \to f$ and $\sup_n |f_n| \leq g$ a.e., it follows that $|f| \leq g$ a.e., and hence $f \in \Lambda_p(\mathrm{X})$. Moreover, we have $|f_n - f|^p \leq 2^p g^p$ a.e., which together with the facts that $\lim |f_n - f|^p = 0$ a.e. and $2^p g^p \in A_1(\mathrm{X})$, implies, by Proposition 3.1, that for every $\delta > 0$ there exists E_δ with $\mu(E_\delta) < \delta$ such that

$$\int_{E_\delta^c} |f_n - f|^p \, \mathrm{d}\mu \to 0$$

as $n \to \infty$, that is, (f_n) converges to f almost in L_p.

Next, we provide a direct proof that convergence almost in L_p implies convergence in the F-norm $\|\cdot\|_{\alpha_p}$, even though this result can also be easily derived from [2, Proposition 2.9] and Proposition 2.1. Given $\varepsilon > 0$, there exist a measurable set E_ε, with $\mu(E_\varepsilon) < \varepsilon/2$, and $N_\varepsilon \in \mathbb{N}$ such that

$$\int_{E_\varepsilon^c} |f_n - f|^p \mathrm{d}\mu < \frac{\varepsilon}{2} \qquad \forall\, n \geq N_\varepsilon.$$

Consequently, for every $n \geq N_\varepsilon$,

$$\begin{aligned}
\|f_n - f\|_{\alpha_p}^p &= \int_X |\min(|f_n - f|, 1)|^p \, \mathrm{d}\mu \\
&= \int_{E_\varepsilon} |\min(|f_n - f|, 1)|^p \, \mathrm{d}\mu + \int_{E_\varepsilon^c} |\min(|f_n - f|, 1)|^p \, \mathrm{d}\mu \\
&\leq \mu(E_\varepsilon) + \int_{E_\varepsilon^c} |f_n - f|^p \, \mathrm{d}\mu < \varepsilon
\end{aligned}$$

which finishes the proof. □

The previous result also holds if convergence almost everywhere is replaced by convergence in measure, serving as a counterpart to [4, Theorem 7.8] in the present setting.

Proposition 3.3. *Let $(f_n) \subseteq \Lambda_p(\mathrm{X})$ and f be measurable. Assume that (f_n) converges to f in measure and that there exists $g \in \Lambda_p(\mathrm{X})$ such that $\sup_n |f_n| \leq g$ almost everywhere. Then $f \in \Lambda_p(\mathrm{X})$ and $\|f_n - f\|_{\alpha_p} \to 0$ as $n \to \infty$.*

4. Vitali convergence theorems

A classical Vitali convergence theorem provides necessary and sufficient conditions for a sequence in $L_p(\mathrm{X})$ that converges in measure to a measurable function f to also converge in $L_p(\mathrm{X})$.

Theorem 4.1 (Vitali convergence theorem [4])**.** *Let $(f_n) \subseteq L_p(\mathrm{X})$ and f be measurable. Then (f_n) converges to f in $L_p(\mathrm{X})$ if, and only if,*

(i) (f_n) converges to f in measure,

(ii) for every $\varepsilon > 0$ there exist $E_\varepsilon \in \Sigma$ with $\mu(E_\varepsilon) < \infty$ and $\delta_\varepsilon > 0$ such that

$$\sup_n \int_{E_\varepsilon^c} |f_n|^p \, \mathrm{d}\mu < \varepsilon^p$$

and, if $F \in \Sigma$ and $\mu(F) < \delta_\varepsilon$, then

$$\sup_n \int_{E_\varepsilon \cap F} |f_n|^p \, \mathrm{d}\mu < \varepsilon^p.$$

Remark 4.2. If $\mu(X) < \infty$, then the first part of condition (ii) of Theorem 4.1 becomes superfluous as one can simply take $E_\varepsilon = X$ for every $\varepsilon > 0$. In this case, condition (ii) is replaced by *uniform p-integrability* of the sequence (f_n), that is, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $F \in \Sigma$ and $\mu(F) < \delta_\varepsilon$, then $\sup_n \int_F |f_n|^p \, \mathrm{d}\mu < \varepsilon^p$.

The next result states that for uniformly p-integrable sequences, α_p-convergence and convergence almost in L_p are equivalent to L_p-convergence. This is analogous to the relationship between L_p-convergence and convergence in measure on finite measure spaces.

Theorem 4.3.
Let $(f_n) \subseteq L_p(\mathrm{X})$ and f be measurable. Then (f_n) converges to f in $L_p(\mathrm{X})$ if, and only if,

(i) (f_n) α_p-converges (or converges almost in L_p) to f,
(ii) (f_n) is uniformly p-integrable.

Proof. First, assume that (f_n) converges to f in L_p. The condition (i) is clear. Regarding condition (ii), it is known that it is satisfied by L_p-converging sequences.

To establish the converse, we show that (f_n) is Cauchy in $L_p(\mathrm{X})$. Since convergence almost in L_p implies α_p-convergence, it suffices to assume the latter to establish the result. Suppose that (f_n) is uniformly p-integrable and let $\varepsilon > 0$ and δ_ε be such that $\sup_n \int_F |f_n|^p \,\mathrm{d}\mu < \varepsilon^p$ whenever $\mu(F) < \delta_\varepsilon$. Assume that (f_n) α_p-converges to f. Then, (f_n) is α_p-Cauchy and hence there exists a sequence of measurable sets (B_n) with $\mu(B_n^c) \to 0$ such that

$$\int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu \to 0 \qquad \text{as } n, m \to \infty.$$

Let $N_1(\varepsilon), N_2(\varepsilon) \in \mathbb{N}$ be such that

$$\forall n \geq N_1(\varepsilon) \qquad \mu(B_n^c) < \frac{\delta_\varepsilon}{2}$$

and

$$\forall n, m \geq N_2(\varepsilon) \qquad \int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu < \varepsilon^p.$$

Then, for $n, m \geq N_\varepsilon = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$

$$\begin{aligned}\int_X |f_n - f_m|^p \,\mathrm{d}\mu &= \int_{B_n \cap B_m} |f_n - f_m|^p \,\mathrm{d}\mu + \int_{B_n^c \cup B_m^c} |f_n - f_m|^p \,\mathrm{d}\mu \\ &< \varepsilon^p + 2^{p-1} \left(\int_{B_n^c \cup B_m^c} |f_n|^p \,\mathrm{d}\mu + \int_{B_n^c \cup B_m^c} |f_m|^p \,\mathrm{d}\mu \right).\end{aligned}$$

Furthermore, for $n, m \geq N_\varepsilon$ we have $\mu(B_n^c \cup B_m^c) < \delta_\varepsilon$, hence

$$\int_{B_n^c \cup B_m^c} |f_n|^p \,\mathrm{d}\mu \leq \sup_k \int_{B_n^c \cup B_m^c} |f_k|^p \,\mathrm{d}\mu < \varepsilon^p$$

and similarly for the remaining term. Consequently, for $n, m \geq N_\varepsilon$

$$\int_X |f_n - f_m|^p \,\mathrm{d}\mu < (1 + 2^p)\varepsilon^p$$

whence

$$\|f_n - f_m\|_p < (1 + 2^p)^{\frac{1}{p}}\, \varepsilon$$

which finishes the proof. ∎

We conclude this section with a result that serves as the counterpart of Theorem 4.1 for the space $\Lambda_p(\mathrm{X})$ endowed with the F-norm $\|\cdot\|_{\alpha_p}$.

Theorem 4.4. *Let $(f_n) \subseteq \Lambda_p(\mathrm{X})$ and f be measurable. Then (f_n) α_p-converges to f if, and only if,*

(i) (f_n) converges to f in measure,
(ii) for every $\varepsilon > 0$ there exists a measurable set E_ε with $\mu(E_\varepsilon) < \infty$ such that

$$\sup_n \|f_n \chi_{E_\varepsilon^c}\|_{\alpha_p} < \varepsilon.$$

Proof. First, assume that (f_n) α_p-converges to f (hence $f \in \Lambda_p(\mathrm{X})$) and let $\varepsilon > 0$. Since α_p-convergence implies convergence in measure [2, 1], we put our attention on the second condition. Let $N = N(\varepsilon) \in \mathbb{N}$ be such that $\|f_n - f\|_{\alpha_p} < \varepsilon/2$ whenever $n > N$. Since $f \in \Lambda_p(\mathrm{X})$, there exists a measurable set F_ε with $\mu(F_\varepsilon) < \varepsilon$ so that $f\chi_{F_\varepsilon^c} \in L_p(\mathrm{X})$. Then, $\|f\chi_{F_\varepsilon^c \cap G_\varepsilon^c}\|_p < \varepsilon/2$ for some subset $G_\varepsilon \subseteq \mathrm{X}$ of finite measure. Therefore, for $n > N$,

$$\begin{aligned}\|f_n\chi_{F_\varepsilon^c\cap G_\varepsilon^c}\|_{\alpha_p} &\leq \|(f_n - f)\chi_{F_\varepsilon^c\cap G_\varepsilon^c}\|_{\alpha_p} + \|f\chi_{F_\varepsilon^c\cap G_\varepsilon^c}\|_{\alpha_p}\\ &\leq \|f_n - f\|_{\alpha_p} + \|f\chi_{F_\varepsilon^c\cap G_\varepsilon^c}\|_p < \varepsilon.\end{aligned}$$

For each $n = 1, \ldots, N$, since $f_n \in \Lambda_p(\mathrm{X})$, there exists $H_{\varepsilon,n}$ with $\mu(H_{\varepsilon,n}) < \varepsilon$ such that $f_n\chi_{H_{\varepsilon,n}} \in L_p(\mathrm{X})$, and so $\|f_n\chi_{H_{\varepsilon,n}^c\cap I_{\varepsilon,n}^c}\|_p < \varepsilon$ for some $I_{\varepsilon,n}$ with finite measure. Define a measurable set E_ε as

$$E_\varepsilon = F_\varepsilon \cup G_\varepsilon \cup \bigcup_{n=1}^{N} H_{\varepsilon,n} \cup \bigcup_{n=1}^{N} I_{\varepsilon,n}$$

and note that $\mu(E_\varepsilon) < \infty$. Thus, for $n > N$ we have

$$\|f_n\chi_{E_\varepsilon^c}\|_{\alpha_p} \leq \|f_n\chi_{F_\varepsilon^c\cap G_\varepsilon^c}\|_{\alpha_p} < \varepsilon$$

and for $n = 1, \ldots, N$ it holds

$$\|f_n\chi_{E_\varepsilon^c}\|_{\alpha_p} \leq \|f_n\chi_{H_{\varepsilon,n}^c\cap I_{\varepsilon,n}^c}\|_{\alpha_p} \leq \|f_n\chi_{H_{\varepsilon,n}^c\cap I_{\varepsilon,n}^c}\|_p < \varepsilon$$

establishing one direction.

Now, assume that (f_n) converges to f in measure and that condition (ii) holds. We prove that (f_n) is α_p-Cauchy. Let $\varepsilon > 0$ and let a measurable set E_ε be such that $\mu(E_\varepsilon) < \infty$ and $\|f_n\chi_{E_\varepsilon^c}\|_{\alpha_p} < \varepsilon/4$ for each $n \in \mathbb{N}$. Note that, for every $n, m \in \mathbb{N}$,

$$\begin{aligned}\|f_n - f_m\|_{\alpha_p} &\leq \|(f_n - f_m)\chi_{E_\varepsilon}\|_{\alpha_p} + \|f_n\chi_{E_\varepsilon^c}\|_{\alpha_p} + \|f_m\chi_{E_\varepsilon^c}\|_{\alpha_p}\\ &< \|(f_n - f_m)\chi_{E_\varepsilon}\|_{\alpha_p} + \varepsilon/2.\end{aligned}$$

Set $\delta = \varepsilon/2\mu(E_\varepsilon)^{1/p}$ and let $N_\varepsilon \in \mathbb{N}$ be such that $\mu(|f_n - f_m| \geq \delta) < \varepsilon/2$ whenever $n, m \geq N_\varepsilon$. Then, for this range of n and m we estimate

$$\begin{aligned}\|(f_n - f_m)\chi_{E_\varepsilon}\|_{\alpha_p}^p &= \int_{E_\varepsilon} |\min(f_n - f_m, 1)|^p \,\mathrm{d}\mu\\ &= \int_{E_\varepsilon\cap\{|f_n - f_m| > \delta\}} |\min(f_n - f_m, 1)|^p \,\mathrm{d}\mu\\ &\quad + \int_{E_\varepsilon\cap\{|f_n - f_m| \leq \delta\}} |\min(f_n - f_m, 1)|^p \,\mathrm{d}\mu\\ &\leq \mu(|f_n - f_m| > \delta) + \delta^p\mu(E_\varepsilon) < \left(\frac{\varepsilon}{2}\right)^p.\end{aligned}$$

Thus, for $n, m \geq N_\varepsilon$,

$$\|f_n - f_m\|_{\alpha_p} < \varepsilon$$

which completes the proof. □

Remark 4.5. If $\mu(\mathrm{X})$ is finite, then condition (ii) of Theorem 4.4 is automatically satisfied by setting $E_\varepsilon = \mathrm{X}$ for every $\varepsilon > 0$. This, together with Lemma 2.9, establishes the equivalence between asymptotic L_p-convergence and convergence in measure on finite measure spaces, a result initially proved in [1].

Remark 4.6. We observe that Theorems 4.3 and 4.4 provide a decomposition of Theorem 4.1. Condition *(ii)* of Theorem 4.3 corresponds to the second part of condition *(ii)* of Theorem 4.1, whereas condition *(ii)* of Theorem 4.4 is the α_p-version of the first part of condition *(ii)* of Theorem 4.1.

5. Approximation and separability

Denote by $\mathcal{S}$ the subspace of $L_0(\mathrm{X})$ that comprises those functions $s : \mathrm{X} \to \mathbb{R}$ that are linear combinations of characteristic functions of measurable sets, that is, $s = \sum_{i=1}^{k} a_i \chi_{F_i}$, for some $a_i \in \mathbb{R}$ and $F_i \in \Sigma$, $i = 1, \ldots, k$. Elements of $\mathcal{S}$ are called *simple functions*.

The first result of this section establishes the density of simple functions in $\Lambda_p(\mathrm{X})$, as a natural extension of [10, Proposition 6.7] to our setting.

Proposition 5.1. *The subspace $\mathcal{S} \cap \Lambda_p(\mathrm{X})$ is dense in $\Lambda_p(\mathrm{X})$, that is, given $f \in \Lambda_p(\mathrm{X})$ there exists a sequence $(s_n) \subseteq \Lambda_p(\mathrm{X})$ of simple functions such that $\|s_n - f\|_{\alpha_p} \to 0$ as $n \to \infty$.*

Note that $s \in \mathcal{S} \cap \Lambda_p(\mathrm{X})$ if and only if for every $\delta > 0$ there exists a measurable set E_δ with $\mu(E_\delta) < \delta$ such that the set $\{s \neq 0\} \cap E_\delta^c$ has finite measure.

The next result establishes the density of $L_p(\mathrm{X})$ in $\Lambda_p(\mathrm{X})$.

Proposition 5.2. *The Lebesgue space $L_p(\mathrm{X})$ is dense in $\Lambda_p(\mathrm{X})$, that is, given $f \in \Lambda_p(\mathrm{X})$ and $\varepsilon > 0$, there exists $g_\varepsilon \in L_p(\mathrm{X})$ such that $\|g_\varepsilon - f\|_{\alpha_p} < \varepsilon$.*

Proof. Let $f \in \Lambda_p(\mathrm{X})$ and $\varepsilon > 0$. There exists a measurable set E_ε with $\mu(E_\varepsilon) < \varepsilon^p$ such that $g_\varepsilon = f\chi_{E_\varepsilon^c}$ belongs to $L_p(\mathrm{X})$. We have

$$\begin{aligned}\|f - g_\varepsilon\|_{\alpha_p}^p &= \int_{\mathrm{X}} |\min(|f - g_\varepsilon|)|^p \,\mathrm{d}\mu \\ &= \int_{E_\varepsilon^c} |\min(|f - f\chi_{E_\varepsilon^c}|, 1)|^p \,\mathrm{d}\mu + \int_{E_\varepsilon} |\min(|f - f\chi_{E_\varepsilon^c}|, 1)|^p \,\mathrm{d}\mu.\end{aligned}$$

Note that the first integral vanishes while the second is controlled from above by $\mu(E_\varepsilon)$. The result follows. $\square$

Now, consider $\mathrm{X} = \mathbb{R}^d$ endowed with the Borel σ-algebra and the Lebesgue measure. We prove that $C_c^\infty(\mathbb{R}^d)$, the space of smooth and compactly supported functions on $\mathbb{R}^d$, is dense in $\Lambda_p(\mathbb{R}^d)$, and that $\Lambda_p(\mathbb{R}^d)$ is separable.

Proposition 5.3. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in $\Lambda_p(\mathbb{R}^d)$, that is, given $f \in \Lambda_p(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ such that $\|f - \varphi_\varepsilon\|_{\alpha_p} < \varepsilon$.*

Proof. Let $f \in \Lambda_p(\mathbb{R}^d)$ and $\varepsilon > 0$. There exists $g_\varepsilon \in L_p(\mathbb{R}^d)$ such that $\|f - g_\varepsilon\|_{\alpha_p} < \varepsilon/2$. Moreover, since $C_c^\infty(\mathbb{R}^d)$ is dense in $L_p(\mathbb{R}^d)$, there exists $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ such that $\|g_\varepsilon - \varphi_\varepsilon\|_{\alpha_p} \leq \|g_\varepsilon - \varphi_\varepsilon\|_p < \varepsilon/2$. The triangle inequality yields the desired conclusion. $\square$

Proposition 5.4. *The space $\Lambda_p(\mathbb{R}^d)$ is separable.*

Proof. Simply note that $L_p(\mathbb{R}^d)$ is a dense separable subspace of $\Lambda_p(\mathbb{R}^d)$. $\square$

6. Marcinkiewicz interpolation theorems

In this section we establish an interpolation result for the spaces $\Lambda_p(\mathrm{X})$ in the spirit of the Marcinkiewicz interpolation theorem. This classical theorem provides strong L_p-bounds for subadditive operators assuming weaker estimates at endpoints q and r such that $1 \leq q < p < r \leq \infty$. We will focus on the case $r = \infty$ as it is the version that can be adapted straightforwardly to our setting. An operator $T : \mathrm{dom}(T) \subseteq L_0(\mathrm{X}) \to L_0(\mathrm{X})$ is called *subadditive* if $|T(f+g)| \leq |T(f)| + |T(g)|$ for all measurable functions f and g in its domain $\mathrm{dom}(T)$.

Theorem 6.1 (Marcinkiewicz interpolation theorem [11])**.** *Let (X, μ) be a σ-finite measure space and $1 \leq q < \infty$. Let $T : L_q(\mathrm{X}) + L_\infty(\mathrm{X}) \to L_0(\mathrm{X})$ be a subadditive operator and assume that there exist $A, B > 0$ such that*

$$\sup_{\lambda > 0} \lambda^q D(T(f), \lambda) \leq A^q \|f\|_q \qquad \forall f \in L_q(\mathrm{X}) \tag{6.1}$$

and

$$\|T(f)\|_\infty \le B\,\|f\|_\infty \qquad \forall f \in L_\infty(\mathrm{X}). \tag{6.2}$$

Then, for every $p > q$,

$$\|T(f)\|_p \le C\,\|f\|_p \qquad \forall f \in L_p(\mathrm{X}) \tag{6.3}$$

where

$$C = 2\left(\frac{p}{p-q}\right)^{\frac{1}{p}} A^{\frac{q}{p}} B^{1-\frac{q}{p}}. \tag{6.4}$$

Here, $L_\infty(\mathrm{X})$ denotes the standard Lebesgue space of essentially bounded measurable functions with norm $\|\cdot\|_\infty$, and $D(f,\cdot) : [0,\infty) \to [0,\infty]$ is the *distribution function* of a measurable function f given by $D(f,\lambda) = \mu(|f| > \lambda)$.

In order to obtain an analogous result for the spaces $\Lambda_p(\mathrm{X})$, one starts by defining for any measurable function f its *restricted distribution function* $\rho(f,\cdot) : [0,1] \to [0,\infty]$ by

$$\rho(f,\lambda) = D(\min(|f|,1),\lambda) = \mu\big(\min(|f|,1) > \lambda\big).$$

If X is σ-finite, then for each measurable function f and $p \ge 1$, Tonelli's theorem yields that

$$p\int_0^1 \lambda^{p-1}\rho(f,\lambda)\,\mathrm{d}\lambda = \|f\|_{\alpha_p}^p. \tag{6.5}$$

Also, for each $\lambda \in (0,1)$ we have

$$\rho(f,\lambda) \le \frac{\|f\|_{\alpha_p}^p}{\lambda^p}. \tag{6.6}$$

Lemma 6.2. *Let f be a measurable function and $\lambda \in (0,1)$. Then f can be decomposed as follows*

$$f = f\chi_{\{\min(|f|,1)>\lambda\}} + f\chi_{\{|f|\le\lambda\}}. \tag{6.7}$$

Proof. Simply note that for $\lambda \in (0,1)$ it holds that

$$\{\min(|f|,1) > \lambda\} = \{|f| > \lambda\} \tag{6.8}$$

for each measurable function f. □

Identity (6.8) yields that the restricted distribution function of a measurable function f equals the distribution function of f restricted to the interval $(0,1)$; that is, $\rho(f,\cdot) = D(f,\cdot)|_{(0,1)}$.

Lemma 6.3. *Let $1 \le q < r < \infty$. Then, for each $p \in (q,r)$ we have*

$$\Lambda_p(\mathrm{X}) \subseteq \Lambda_q(\mathrm{X}) + \Lambda_r(\mathrm{X}) \cap L_\infty(\mathrm{X}).$$

Proof. Let $f \in \Lambda_p(\mathrm{X})$ for some $p \in (q,r)$, and fix $\lambda > 0$. We write $f = g_\lambda + b_\lambda$, where $g_\lambda = f\chi_{\{|f|>\lambda\}}$ and $b_\lambda = f\chi_{\{|f|\le\lambda\}}$. The result follows from noting that $g_\lambda \in \Lambda_q(\mathrm{X})$ and $b_\lambda \in \Lambda_r(\mathrm{X}) \cap L_\infty(\mathrm{X})$. □

The next result provides a sufficient condition for a measurable function belonging to the intersection of two almost-Lebesgue spaces to also belong to Lebesgue spaces.

Proposition 6.4. *Let (X,μ) be a σ-finite measure space and let $f \in \Lambda_q(\mathrm{X}) \cap \Lambda_r(\mathrm{X})$ with $1 \le q < r < \infty$. Suppose that for every $\varepsilon > 0$ there exists δ_ε such that*

$$\sup_{\lambda>0} \lambda \max\Big(\mu(|f|\chi_E > \lambda)^{\frac{1}{q}}, \mu(|f|\chi_E > \lambda)^{\frac{1}{r}}\Big) < \varepsilon \tag{6.9}$$

for each measurable set E with $\mu(E) < \delta_\varepsilon$. Then $f \in L_p(\mathrm{X})$ for every $p \in (q,r)$.

Proof. It is clear that $\Lambda_q(\mathrm{X}) \cap \Lambda_r(\mathrm{X}) \subseteq \Lambda_p(\mathrm{X})$ for every $p \in (q,r)$. Moreover, if f satisfies (6.9) then f is has an absolutely continuous p-integral for every $p \in (q,r)$. Thus, the result follows by Proposition 2.3. □

The previous result shows that the almost-Lebesgue spaces are not very well suited for L_p-interpolation, as one needs the extra condition (6.9) on a function in $\Lambda_q(\mathrm{X})\cap\Lambda_r(\mathrm{X})$ to deduce that it belongs to $L_p(\mathrm{X})$ for $p\in(q,r)$. Nevertheless, a version of Theorem 6.1 for the present setting still holds. Its proof is a simple adaptation of the proof of Theorem 6.1 in [11, Theorem 1.3.3], with minor modifications, and is included here for convenience.

Theorem 6.5. *Let* (X,μ) *be a* σ*-finite measure space and* $1\le q<\infty$. *Let* $T:\Lambda_q(\mathrm{X})+L_\infty(\mathrm{X})\to L_0(\mathrm{X})$ *be a subadditive operator and assume that there exist* $A>0$ *such that*

$$\sup_{0<\lambda<1}\lambda^q\rho(T(f),\lambda)\le A^q\|f\|_{\alpha_q}^q\qquad\forall f\in\Lambda_q(\mathrm{X}),\tag{6.10}$$

and (6.2) holds for some $B\ge 1/2$. *Then, for every* $p>q$,

$$\|T(f)\|_{\alpha_p}\le C\,\|f\|_{\alpha_p}\qquad\forall f\in\Lambda_p(\mathrm{X})\tag{6.11}$$

where C *is as in (6.4).*

Proof. Let $f\in\Lambda_p(\mathrm{X})$ for some $p>q$. We use the decomposition (6.7) with $\lambda_t=t/(2B)$, $t\in(0,1)$. The hypothesis $B\ge1/2$ is imposed to ensure that λ_t belongs to $(0,1)$ for all $t\in(0,1)$. Set

$$g_t=f\chi_{\{\min(|f|,1)>t/(2B)\}}\quad\text{and}\quad b_t=f\chi_{\{|f|\le t/(2B)\}}.$$

It is clear that $\|b_t\|_\infty\le t/(2B)$, and hence, by (6.2) we deduce that

$$|T(b_t)(x)|\le\|T(b_t)\|_\infty\le B\|b_t\|_\infty\le t/2$$

for almost every $x\in X$, and so $\rho(T(b_t),t/2)=\mu(|T(b_t)|>t/2)=0$, where we used identity (6.8). Therefore, from the subadditivity of T it follows that

$$\rho(T(f),t)\le\rho(T(g_t),t/2)+\rho(T(b_t),t/2)=\rho(T(g_t),t/2).$$

Moreover, using (6.10) with $\lambda=t/2$ leads to

$$\rho(T(g_t),t/2)\le 2^qA^qt^{-q}\int_{\{2B\min(|f|,1)>t\}}\min(|f|,1)^q\,\mathrm{d}\mu.$$

Combining the previous expressions with (6.5) and using Tonelli's theorem to interchange the order of integration yields

$$\begin{aligned}\|T(f)\|_{\alpha_p}^p&=p\int_0^1t^{p-1}\rho(T(f),t)\,\mathrm{d}t\\&\le p2^qA^q\int_0^1t^{p-q-1}\int_{\{2B\min(|f|,1)>t\}}\min(|f|,1)^q\,\mathrm{d}\mu\,\mathrm{d}t\\&=p2^qA^q\int_X\int_0^{2B\min(|f|,1)}t^{p-q-1}\min(|f|,1)^q\,\mathrm{d}t\,\mathrm{d}\mu\\&=\frac{p}{p-q}2^pA^qB^{p-q}\|f\|_{\alpha_p}^p\end{aligned}$$

which finishes the proof. ☐

Remark 6.6. Note that the left-hand side of condition (6.10) is obtained from the left-hand side of condition (6.1) by restricting the values of λ considered in the supremum, while enlarging the set of functions that satisfy it. Additionally, even if an operator satisfies condition (6.10), restricting it to functions only in $L_q(\mathrm{X})$ does not necessarily imply condition (6.1), as the supremum in that case is taken over a larger interval.

7. Hardy-Littlewood maximal operators

Consider $\mathrm{X} = \mathbb{R}^d$ with the Borel σ-algebra and the Lebesgue measure. We denote by $|E|$ the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^d$.

Given a measurable function f on $\mathbb{R}^d$, recall its *uncentered Hardy-Littlewood maximal function* $M(f)$ given by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, \mathrm{d}y$$

where the supremum is taken over all open balls that contain x. For each measurable function f on $\mathbb{R}^d$ and every $\lambda > 0$ it holds that [11, Theorem 1.4.6]:

$$\lambda |\{M(f) > \lambda\}| \leq 3^d \int_{\{M(f)>\lambda\}} |f(y)| \, \mathrm{d}y. \tag{7.1}$$

The aim of this section is to provide an application of Theorem 6.5 to establish a boundedness result for a maximal operator suited for the present setting. Given a measurable function f, we define its *bounded maximal function* $M_b(f)$ by

$$M_b(f) = M(\min(|f|, 1)) = \sup_{B \ni x} \frac{1}{|B|} \|f\|_\alpha.$$

Clearly, $\|M_b(f)\|_\infty \leq \min(\|f\|_\infty, 1)$ for $f \in L_\infty(\mathbb{R}^d)$.

Theorem 7.1. *For every $p > 1$ and $f \in \Lambda_p(\mathbb{R}^d)$ we have*

$$\|M_b(f)\|_{\alpha_p} \leq C \, \|f\|_{\alpha_p} \tag{7.2}$$

where

$$C = 2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} 3^{\frac{d}{p}}.$$

Proof. It suffices to show that the operator M_b satisfies (6.10) with $q = 1$ for some $A > 0$. Let $\lambda \in (0, 1)$ and note that by (6.8) we have $\{\min(M_b(f), 1) > \lambda\} = \{M(\min(|f|, 1)) > \lambda\}$. Thus, for $\lambda \in (0, 1)$, using (7.1) we deduce

$$\lambda \rho(M_b(f), \lambda) = \lambda |\{M(\min(|f|, 1) > \lambda)\}| \leq 3^d \|f\|_\alpha.$$

The result follows upon applying Theorem 6.5 with $A = 3^d$ and $B = 1$. □

Remark 7.2. Alternatively, instead of using Theorem 6.5 to prove the previous result, one could apply the maximal function theorem, which states that $\|M(f)\|_p \leq C \|f\|_p$ for every $f \in L_p(\mathrm{X})$; a result derived using (7.1) and Theorem 6.1. In this case, Theorem 7.1 follows by observing that for every $f \in \Lambda_p(\mathrm{X})$, $\min(|f|, 1) \in L_p(\mathrm{X})$.

We conclude this section with an *almost-Lebesgue differentiation theorem*, being the counterpart of [11, Theorem 1.5.1] in the present theory. After a careful inspection of the proof of [11, Theorem 1.5.1], one concludes that it can be adapted to the present case since we have at our disposal the approximation of functions in $\Lambda_p(\mathrm{X})$ by smooth functions, and also a Chebyshev's inequality – identity (6.6).

Theorem 7.3. *Let f be a function in $\Lambda_1(\mathbb{R}^d)$. Then, for almost every $x \in \mathbb{R}^d$ we have*

$$\lim_{\delta \to 0} \frac{1}{|B(x, \delta)|} \|(f - f(x)) \chi_{B(x,\delta)}\|_\alpha = 0. \tag{7.3}$$

Remark 7.4. If $f \in \Lambda_1(\mathbb{R}^d)$, then $\min(|f|, 1) \in L_1(\mathbb{R}^d)$, and hence, by the Lebesgue differentiation theorem,

$$\lim_{\delta \to 0} \frac{1}{|B(x, \delta)|} \|(\min(|f|, 1) - \min(|f(x)|, 1)) \chi_{B(x,\delta)}\|_1 = 0$$

which does not necessarily imply (7.3). However, if $f \in L_1(\mathbb{R}^d)$, then (7.3) follows from the classical theorem since

$$\|(f - f(x)) \chi_{B(x,\delta)}\|_\alpha \leq \|(f - f(x)) \chi_{B(x,\delta)}\|_1.$$

8. Local boundedness, local convexity and duality

Let V be real vector space with an F-norm $\|\cdot\|$. The open balls $B_r = \{f \in V \mid \|f\| < r\}$ centered at zero form a base of neighborhoods at zero. A subset B of V is called *norm bounded* if $\sup\{\|f\| \mid f \in B\} < \infty$, and it is called *topologically bounded* if for each neighborhood U of zero there exists a positive number t such that $tB \subseteq U$. If $\|\cdot\|$ is a norm, then its homogeneity property yields that these two boundedness notions are equivalent. Moreover, a subset B is topologically bounded if, and only if, whenever (f_n) is a sequence in B and (λ_n) is a sequence of scalars converging to zero, then $\|\lambda_n f_n\|$ also converges to zero [18]. The space V is called *locally bounded* if it has a topologically bounded neighborhood of zero, and it is called *locally convex* if it has a convex neighborhood of zero. We denote by V^* the dual space of V, that is, the space of all continuous linear functionals on V.

In the previous sections, we observed that $\Lambda_p(\mathbb{R}^d)$ shares some similarities with $L_p(\mathbb{R}^d)$. Since $L_p(\mathbb{R}^d)$ is a normed space, it is locally bounded and locally convex. Moreover, its dual is nontrivial as it is isomorphic to $L_q(\mathbb{R}^d)$ with $q = p/(p-1)$. In stark contrast to this, we will prove that the balls in $\Lambda_p(\mathbb{R}^d)$ centered at 0 are neither topologically bounded nor convex, and hence $\Lambda_p(\mathbb{R}^d)$ is neither locally bounded nor locally convex. Furthermore, the dual space of $\Lambda_p(\mathbb{R}^d)$ is trivial.

Proposition 8.1. *The space $\Lambda_p(\mathbb{R}^d)$ is not locally bounded.*

Proof. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, consider an open cube $E_n \subseteq \mathbb{R}^d$ given by

$$E_n = \left(n, n + \left(\frac{\varepsilon}{2}\right)^{\frac{p}{d}}\right)^d$$

and note that $|E_n| = (\varepsilon/2)^p$. Let $f_n = n\chi_{E_n}$. Then

$$\|f_n\|_{\alpha_p}^p = \int_{\mathbb{R}^d} |\min(n\chi_{E_n}, 1)|^p \,\mathrm{d}x = \int_{E_n} |\min(n, 1)|^p \,\mathrm{d}x = |E_n| = \left(\frac{\varepsilon}{2}\right)^p$$

and so, the sequence (f_n) belongs to the ball in $\Lambda_p(\mathbb{R}^d)$ centered at 0 with radius ε. Denote this ball by B_ε. If B_ε were (topologically) bounded, then we would have $\|a_n f_n\|_{\alpha_p} \to 0$ as $n \to \infty$ for any sequence of scalars (a_n) converging to 0. However, taking $a_n = 1/n$ we have

$$\|(1/n)f_n\|_{\alpha_p}^p = \int_{\mathbb{R}^d} |\min(\chi_{E_n}, 1)|^p \,\mathrm{d}x = |E_n| = \left(\frac{\varepsilon}{2}\right)^p$$

which does not converge to 0 as $n \to \infty$. This proves that B_ε is not topologically bounded, which concludes the proof. □

Proposition 8.2. *The space $\Lambda_p(\mathbb{R}^d)$ is not locally locally convex.*

Proof. Let $\varepsilon > 0$ and B_ε be the ball in $\Lambda_p(\mathbb{R}^d)$ centered at 0 with radius ε. We prove that B_ε is not convex. For a fixed $K \in \mathbb{N}$, let g_K be given by

$$g_K = \frac{1}{K}\sum_{n=1}^{K} f_n$$

where f_n is as in the proof of the previous result. We have

$$\|g_K\|_{\alpha_p}^p = \sum_{n=1}^{K} \int_{E_n} |\min(n/K, 1)|^p \,\mathrm{d}x = \left(\frac{\varepsilon}{2}\right)^p \frac{1}{K^p} \sum_{n=1}^{K} n^p.$$

Choosing K large enough so that $(1/K^p)\sum_{n=1}^{K} n^p > 2^p$ yields that g_K does not belong B_ε. Since g_K is a convex combination of elements in B_ε, it follows that B_ε is not convex. Therefore, the only nontrivial open convex subset of $\Lambda_p(\mathbb{R}^d)$ is $\Lambda_p(\mathbb{R}^d)$ itself, and hence $\Lambda_p(\mathbb{R}^d)$ is not locally convex. □

Proposition 8.3. *The dual space of $\Lambda_p(\mathbb{R}^d)$ is trivial, that is, $\Lambda_p(\mathbb{R}^d)^* = \{0\}$.*

Proof. Suppose, towards a contradiction, that there exists $\varphi \in \Lambda_p(\mathbb{R}^d)^*$ and $f \in \Lambda_p(\mathbb{R}^d)$ such that $\varphi(f) \neq 0$. By linearity, φ is surjective, and by continuity, the set $\varphi^{-1}(-1,1)$ is open. Moreover, by linearity again, $\varphi^{-1}(-1,1)$ is convex, and so $\varphi^{-1}(-1,1) = \Lambda_p(\mathbb{R}^d)$ by the previous proposition, which contradicts the surjectivity. The result follows. $\square$

Appendix: F-norms

We use the notion of F-norm considered in [16].

Definition A.1. Let V be a real vector space. An F-norm on V is a map $\|\cdot\| : V \to \mathbb{R}$ satisfying the following conditions:

(i) $\|f\| > 0$ for every $f \in V \setminus \{0\}$,

(ii) $\|\lambda f\| \leq \|f\|$ for each $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ and every $f \in V$,

(iii) $\lim_{\lambda \to 0} \|\lambda f\| = 0$ for every $f \in V$,

(iv) $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in V$.

Note that conditions (i) and (iii) imply that $\|f\| = 0$ if and only if $f = 0$. Additionally, an F-norm $\|\cdot\|$ on a real vector space V determines a metric $\mathrm{d_F}$, given by $\mathrm{d_F}(f,g) = \|f - g\|$, which is translation-invariant; that is, $\mathrm{d_F}(f + h, g + h) = \mathrm{d_F}(f,g)$ for every $f, g, h \in V$. Moreover, it is clear that any norm on V is an F-norm.

Proposition A.2. *The map $\|\cdot\|_{\alpha_p} : \Lambda_p(\mathrm{X}) \to [0,\infty)$ is an* F*-norm.*

Proof. First, we prove that $\|\cdot\|_{\alpha_p} = \|\min(|\cdot|, 1)\|_p$ is well defined on $\Lambda_p(\mathrm{X})$, that is, $\|f\|_{\alpha_p} < \infty$ for every $f \in \Lambda_p(\mathrm{X})$. Given $f \in \Lambda_p(\mathrm{X})$, let $E \subseteq \mathrm{X}$ be a measurable set with $\mu(E) < 1$ so that $\int_{E^c} |f|^p \,\mathrm{d}\mu = C < \infty$. Then

$$\begin{aligned}
\|f\|_{\alpha_p} &= \int_X |\min(|f|, 1)|^p \,\mathrm{d}\mu \\
&\leq \int_{E^c} |f|^p \,\mathrm{d}\mu + \mu(E) \\
&< C + 1 < \infty.
\end{aligned}$$

In addition, it is clear that $\|\cdot\|_{\alpha_p}$ is nonnegative. Regarding condition (i) of the definition, if f in $\Lambda_p(\mathrm{X})$ is such that $\|f\|_{\alpha_p} = 0$, then $\min(|f|, 1) = 0$ almost everywhere, which implies that $f = 0$.

We proceed to condition (ii). Let $f \in \Lambda_p(\mathrm{X})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. Given $x \in \mathrm{X}$, if $|f(x)| < 1$, then $|\lambda f(x)| \leq 1$ and so

$$\min(|\lambda f(x)|, 1) = |\lambda f(x)| \leq |f(x)| = \min(|f(x)|, 1).$$

On the other hand, if $|f(x)| > 1$ then either $|\lambda f(x)| > 1$ or $|\lambda f(x)| \leq 1$. Assuming the former we have

$$\min(|\lambda f(x)|, 1) = 1 = \min(|f(x)|, 1)$$

while assuming the latter it holds

$$\min(|\lambda f(x)|, 1) = |\lambda f(x)| \leq 1 = \min(|f(x)|, 1).$$

Applying the L_p norm to both sides of the inequality $\min(|\lambda f|, 1) \leq \min(|f|, 1)$ yields the desired conclusion.

Next, we establish condition (iii). Let $f \in \Lambda_p(\mathrm{X})$ and $\varepsilon > 0$. There exists a measurable set E_ε with $\mu(E_\varepsilon) < \varepsilon^p/2$ such that $\int_{E_\varepsilon^c} |f|^p \,\mathrm{d}\mu = C_\varepsilon < \infty$. Then

$$\begin{aligned}
\|\lambda f\|_{\alpha_p}^p &\leq \int_{E_\varepsilon^c} |\lambda f|^p \,\mathrm{d}\mu + \mu(E_\varepsilon) \\
&< |\lambda|^p C_\varepsilon + \frac{\varepsilon^p}{2}.
\end{aligned}$$

Choosing $\delta_\varepsilon = \varepsilon^p/(2C_\varepsilon)$ we see that for $|\lambda| < \delta_\varepsilon$ one has $\|\lambda f\|_{\alpha_p} < \varepsilon$.

Finally, we provide a proof of the triangle inequality. Let $x \in \mathrm{X}$ and $f, g \in \Lambda_p(\mathrm{X})$. If $\min(|f(x)|, 1) < 1$ and $\min(|g(x)|, 1) < 1$ then

$$\begin{aligned}\min(|f(x)+g(x)|,1) &\leq |f(x)+g(x)| \\ &\leq |f(x)|+|g(x)| \\ &= \min(|f(x)|,1)+\min(|g(x)|,1).\end{aligned}$$

Now, suppose that either $\min(|f(x)|, 1) \geq 1$ or $\min(|g(x)|, 1) \geq 1$. Assume the former (the latter being similar). Then

$$\begin{aligned}\min(|f(x)+g(x)|,1) &\leq 1+\min(|g(x)|,1) \\ &\leq \min(|f(x)|,1)+\min(|g(x)|,1).\end{aligned}$$

Thus, the triangle inequality in $\|\cdot\|_{\alpha_p}$ follows from the triangle inequality in $\|\cdot\|_p$. $\square$

Acknowledgments

The author thanks João Paulos and Olena Karlova for helpful discussions. This work was supported by the Austrian Science Fund (FWF), project number 10.55776/F65.

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