

# Random Variables

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## 1 Expected Value

The expected value of a random variable is the mean of its instantiated values. It can be thought of as the average of all possible values of the variable, weighed by the probability of observing those values. For a discrete random variable, it is given by:

$$E[X] = \sum_{i=1}^n x_i P(X = x_i)$$

Where  $x_i$  are the possible values of the random variable  $X$  and  $P(X = x_i)$  is the probability of observing that value.

For a continuous random variable, it is given by:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Where  $f(x)$  is the probability density function of the random variable  $X$ .

### 1.1 Properties

Expected value is a linear operator, meaning that it satisfies the following properties:

- $E[aX + b] = aE[X] + b$  for any constants  $a$  and  $b$ .
- $E[X + Y] = E[X] + E[Y]$  for any two random variables  $X$  and  $Y$ .
- $E[XY] = E[X]E[Y]$  if  $X$  and  $Y$  are independent random variables.

## 2 Variance

The variance of a random variable is the mean squared distance of its instantiated values from its mean. It is given by:

$$Var[X] = E[(X - E[X])^2]$$

It is also equivalent to:

$$\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= E[X^2] - 2E[X]E[X] + E[X]^2 \\
&= E[X^2] - E[X]^2
\end{aligned}$$

Therefore, the variance of a random variable is equal to the difference in the expected values of the random variable squared and the variable itself.

## 2.1 Case where $E[X] = 0$

In the special case where  $E[X] = 0$ , the variance simplifies to:

$$Var[X] = E[X^2]$$

## 2.2 Standard deviation

The standard deviation is the square root of the variance:

$$\sigma = \sqrt{Var[X]}$$

It can be interpreted in the same units as the random variable itself.

## 2.3 Properties

$$Var[aX + b] = a^2 Var[X]$$

Where  $a$  and  $b$  are constants. This can be derived by:

$$\begin{aligned}
Var[aX + b] &= E[(aX + b)^2] - E[aX + b]^2 \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
&= a^2E[X^2] - a^2E[X]^2 \\
&= a^2(E[X^2] - E[X]^2) \\
&= a^2Var[X]
\end{aligned}$$

## 3 Covariance

The covariance of two random variables is a measure of how much the two variables vary together. It is given by:

$$Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$$

It can also be expressed as:

$$\begin{aligned}
Cov[X, Y] &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - E[X]Y - E[Y]X + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

The covariance of a variable with itself is equivalent to its variance:

$$\begin{aligned}
Cov[X, X] &= E[(X - E[X])(X - E[X])] \\
&= E[(X - E[X])^2] \\
&= Var[X]
\end{aligned}$$

### 3.1 Case where $E[X] = E[Y] = 0$

In the special case where  $E[X] = 0$  and  $E[Y] = 0$ , the covariance simplifies to:

$$Cov[X, Y] = E[XY]$$

## 4 Correlation

Correlation is a standardized measure describing the association between two random variables. There are multiple measures of correlation, although the most commonly used is the Pearson correlation coefficient.

### 4.1 Pearson correlation coefficient

The Pearson correlation coefficient, denoted by  $\rho$ , standardizes the covariance of two random variables X and Y by dividing it by the product of their standard deviations. That is:

$$\rho_{X,Y} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y} = \frac{E[(X - E[X])(Y - E[Y])]}{\sqrt{E[(X - E[X])^2]} \sqrt{E[(Y - E[Y])^2]}}$$

### 4.2 Case where X and Y are standardized

In the special case where  $E[X] = 0$  and  $E[Y] = 0$ , the Pearson correlation coefficient simplifies to:

$$\rho_{X,Y} = \frac{E[XY]}{\sqrt{E[X^2]} \sqrt{E[Y^2]}}$$

If  $\text{Var}[X] = \text{Var}[Y] = 1$ , then the correlation coefficient further simplifies to:

$$\rho_{X,Y} = E[XY]$$

Which is equivalent to the covariance between  $X$  and  $Y$  in this case.