What is induction? It refers to two different definitions. One is about the composition of the set of natural numbers $\mathbb{N}$. The other is a method of proof of mathematical statements. Lets start by the first one.

Set of Natural Numbers

> **Definition: Set of Natural Numbers**
>
> The set of Natural Numbers is the smallest set such that,
>
> - $0 \in \mathbb{N}$
> - $n \in \mathbb{N} \rightarrow n + 1 \in \mathbb{N}$

This definition contains 3 of the 5 ideas behind Peano axioms. They are,

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \rightarrow n + 1 \in \mathbb{N}$
- $n + 1 = 0 \rightarrow n \notin \mathbb{N}$
- $n + 1 = m + 1 \rightarrow n = m$
- $(0 \in A \wedge n \in A \rightarrow n + 1 \in A) \rightarrow A \subseteq \mathbb{N}$

Proof by Induction

To prove a family of statements $\forall n \in \mathbb{N}: P(n)$, we can use proof by induction.

> **Definition: Principle of Mathematical Induction**
>
> By proving the *initial case* $P(0)$ and the *induction step* $P(k) \rightarrow P(k+1)$ we conclude $\forall n \in \mathbb{N}: P(n)$. Formally,
>
> $$(P(0) \wedge P(k) \rightarrow P(k+1)) \rightarrow \forall n \in \mathbb{N}: P(n)$$

> **Example 1 Intro to Uni Math Sheet 1: Upper bounded sum and the lower bounded product**
>
> Let $x_1 + x_2 + \ldots + x_n \leq \frac{1}{3}$. Show that $(1 - x_1)(1 - x_2) \ldots (1 - x_n) \geq \frac{2}{3}$
>
> > **Doubt**
> >
> > What is the trick here?

> **Example : Sum of first n natural numbers**
>
> We want to prove the following,
>
> $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
>
> Initial Case: n=1
>
> $$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$$
>
> $\square$

Induction step:

$$\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right) \to \left(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}\right)$$

To prove the induction step,

$$\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i \qquad \text{Using the induction hypothesis we get}$$

$$= (n+1) + \frac{n(n+1)}{2}$$

$$= \frac{2n+2}{2} + \frac{n(n+1)}{2}$$

$$= \frac{2n+2+n^2+n}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$\square$

We can also reduce the scope of the statement to a subset of natural numbers. Instead of the initial case being $P(0)$, we can start with the property $P(k)$ of a given natural number k.

Principle: Advice on proving a statement by Induction

Always try to decompose conclusion into induction hypothesis and other term.

Strong Induction

The induction hypothesis is now,

$$(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \to P(k+1)$$

Example Recurrence

Let a_n be a sequence where $a_1 = 1$ and $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$. We want to prove that,

$$a_n = 3 \cdot 2^{n-1} + 2(-1)^n$$

We prove by induction on n.

Initial case: $n = 3$

$$a_3 = 10$$

$\square$

Inductive Step: $3 \leq n \leq k \to a_n = 3 \cdot 2^{n-1} + 2(-1)^n$

$$a_{n+1} = a_n + 2a_{n-1} \qquad \text{We can use the induction hypothesis twice}$$
$$= 3 \cdot 2^{n-1} + 2(-1)^n + 2\left(3 \cdot 2^{n-2} + 2(-1)^{n-1}\right)$$
$$= 2(3 \cdot 2^{n-1}) + 2(-1)^n + 2^2(-1)^{n-1}$$
$$= 3 \cdot 2^n + 2(-1)^{n-1}(-1+2)$$
$$= 3 \cdot 2^n + 2(-1)^{n-1}$$
$$= 3 \cdot 2^n + 2(-1)^{n+1}$$
$\square$

Principle: Strong Induction

We notice that weak induction is not enough when we need more than a single hypothesis in the induction step.

Forward-Backward Induction

We have the same 2 steps in a regular proof by induction, the initial case and the inductive step, but the inductive step is composed by two different steps.

- Initial case: $P(0)$
- Inductive step:
    - Forward step: $P(k) \to P(f(k))$ where f is an increasing function.
    - Backward step: $P(k) \to P(k-1)$

Example Brilliant.org: AM-GM Inequality

Initial Case: $\frac{a+b}{2} \geq \sqrt{ab}$

$$(a-b)^2 \geq 0 \leftrightarrow$$
$$a^2 - 2ab + b^2 \geq 0 \leftrightarrow$$
$$a^2 + 2ab + b^2 - 4ab \geq 0 \leftrightarrow$$
$$a^2 + 2ab + b^2 \geq 4ab \leftrightarrow$$
$$a^2 + 2ab + b^2 \geq 4ab \leftrightarrow$$
$$(a+b)^2 \geq 4ab \leftrightarrow$$
$$|a+b| \geq 2\sqrt{ab} \leftrightarrow \qquad \text{Since both a and b are positive so is their addition}$$
$$\to a + b \geq 2\sqrt{ab}$$
$$\to \frac{a+b}{2} \geq \sqrt{ab}$$

Inductive Step:

Induction Hypothesis:

$$\frac{\sum\limits_{i=1}^{k} a_i}{k} \geq \sqrt[k]{\prod_{i=1}^{k} a_i}$$

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Forward pass: We want to show,

$$\frac{\sum_{i=1}^{2k} a_i}{2k} \geq \sqrt[2k]{\prod_{i=1}^{2k} a_i}$$

We can start by splitting the summation in the left hand side, [CONTINUE]

Backward pass: We want to show,

$$\frac{\sum_{i=1}^{k-1} a_i}{k-1} \geq \sqrt[k-1]{\prod_{i=1}^{k-1} a_i}$$

We start by using the inductive hypothesis [CONTINUE LATER]

Double Induction

We start by defining the addition of two natural numbers recursively,

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Definition: Recursive Definition of Addition of Two Natural Numbers

Let m and n be natural numbers.

- m+0=m

- m+(n+1)=(m+n)+1

Example Intro to Uni Math ex.2: Commutativity of Addition of Natural Numbers

We want to show that the addition is commutative using its recursive definition,

$$m + n = n + m$$

Let $n = 0$. We want to show that $m + 0 = 0 + m$. We show by induction on m. The base case is skipped. We assume $k + 0 = 0 + k$. We want to prove $0 + (k + 1) = (k + 1) + 0$

$$
\begin{aligned}
0 + (k + 1) &= (0 + k) + 1 && \text{Recursive Part of Definition} \\
&= (k + 0) + 1 && \text{Inductive Hypothesis} \\
&= k + 1 && \text{Base Case for Recursive Definition LR} \\
&= (k + 1) + 0 && \text{Base Case for Recursive Definition RL}
\end{aligned}
$$

Let $n = 1$. We want to show that $m + 1 = 1 + m$. We show by induction on m. Base case $0 + 1 = 1 + 0$ was already proven when $n = 0 \land m = 1$. We assume $k + 1 = 1 + k$. We want to prove $1 + (k + 1) = (k + 1) + 1$

$$
\begin{aligned}
(k + 1) + 1 &= (1 + k) + 1 && \text{Inductive Hypothesis} \\
&= 1 + (k + 1) && \text{Recursive Part of Definition RL} \\
\square
\end{aligned}
$$

We want to show that $m + n = n + m$. We prove by induction on n. The base case was already shown. We assume $m + k = k + m$. We want to prove $m + (k + 1) = (k + 1) + m$

$$
\begin{aligned}
(m + k) + 1 &= (k + m) + 1 && \text{Inductive Hypothesis} \\
&= 1 + (k + m) && \text{Using (ii)} \\
&= (1 + k) + m && \text{Recursive Part of Definition} \\
&= (k + 1) + m && \text{Using (ii)} \\
\square
\end{aligned}
$$

Infinite Descent

Example Wikipedia: Proof by Infinite Descent

The square root of a non-integer is always irrational, formally,

$$k \notin \mathbb{N} \rightarrow \sqrt{k} \notin \mathbb{R} - \mathbb{Q}$$

Let $\sqrt{k}$ be a rational number and q the last integer before $\sqrt{k}$,

$$\sqrt{k} \in \mathbb{Q} \leftrightarrow \sqrt{k} = \frac{m}{n}, \ m, n \in \mathbb{N}$$

$$q \in \mathbb{N} \wedge q < \sqrt{k} \wedge q + 1 > \sqrt{k}$$

We start by describing k,

$$\sqrt{k} = \frac{m}{n}$$
$$= \frac{m(\sqrt{k} - q)}{n(\sqrt{k} - q)}$$
$$= \frac{m\sqrt{k} - mq}{n\sqrt{k} - nq}$$
$$= \frac{n\sqrt{k}\sqrt{k} - mq}{n\frac{m}{n} - nq}$$
$$= \frac{nk - mq}{m - nq}$$

$\square$

Since there is an irreducible fraction for every rational number then the last expression is a contraction.

Structural Induction

The power set of a set A can be defined recursively as follows

- $\emptyset \in \mathbb{P}(A)$
- $B \in \mathbb{P}(A) \wedge x \in A \leftrightarrow B \cup \{x\} \in \mathbb{P}(A)$

Doubt: Recursive Definition of Power Set

Does the previous definition qualify as a recursive one?

Tricks

TODO