Properties of the Natural Ordering

- ID: $x \leq x$
- Transitive: $x \leq y \wedge y \leq z \to x \leq z$
- Anti-symmetric: $x \leq y \wedge y \leq x \to x = y$
- Guaranteed existence: $x \leq y \vee y \leq x$

Proof $x \leq x$

$$x \leq x \leftrightarrow \exists z \in \mathbb{N}\colon x + z = x \to z = 0$$
$$\square$$

Proof Transitivity

$$
\begin{aligned}
&x \leq y \leftrightarrow \exists w \in \mathbb{N}\colon x + w = y\\
&y \leq z \leftrightarrow \exists r \in \mathbb{N}\colon y + r = z\\
&\qquad x + w + r = (x + w) + r = y + r = z\\
&\qquad \square
\end{aligned}
$$

Proof Anti-symmetry

$$
\begin{aligned}
&\exists z \in \mathbb{N}\colon x + z = y \wedge y + z = x\\
&\qquad x + y = x + y + 2z\\
&\qquad 2z = 0\\
&\qquad z = 0\\
&\qquad \square
\end{aligned}
$$

Proof Guaranteed Existence:

Doubt

If there is no natural number z such that $x + z = y$ then $z = x - y < 0$, so $-z = y - x > 0$, and
$y + (-z) = y + (y - x) = x$

seems ok