Inequality

Inequalities are of the form,

$$a < b \lor a > b$$

But what are they?

Definition: Inequality

The relation $<$ between two numbers a and b is defined as,

$$a < b \leftrightarrow b - a \in P,$$

, where P is the set of positive numbers.

This definition is rooted in the axioms of the rational numbers.

There seems to be a connection between the concepts of inequality and absolute value.

Definition: Absolute Value

The absolute value of a number a can be defined in at least two ways, through the concept of square root or by cases.

$$|a| = \sqrt{a^2} = \begin{cases} a, & \text{se } a \geq 0 \\ -a, & \text{se } a < 0 \end{cases}$$

The first theorem that appears on Spivak's Calculus is the Triangle Inequality. In this theorem we see the first relation between the concepts of absolute value and inequality.

Theorem: Triangle Inequality

We want to show,

$$|a + b| \leq |a| + |b|$$

The first attempt should be to decompose the square of one side to see if one of the terms is the other side we are trying to prove, or a function of it.

$$(a + b)^2 = a^2 + 2ab + b^2$$

Já temos parte dela though,

$$\begin{aligned} (a + b)^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \end{aligned}$$

Since,

$$2ab \leq 2|a||b|$$

We get,

$$\begin{aligned} (a + b)^2 &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

If we compute the square root on both sides,

$$\begin{aligned} |a+b| &\leq ||a|+|b|| && \text{The argument is always non-negative} \\ &= |a|+|b| \\ &\square \end{aligned}$$

Done.

Theorem: The square root of a square is the absolute value of its argument

$$\begin{aligned} \sqrt{a^2} = y &\rightarrow \\ a^2 = y^2 &\rightarrow && \text{Square both sides} \\ a^2 - y^2 = 0 &\rightarrow \\ (a+y)(a-y) = 0 &\rightarrow \\ a+y=0 \vee a-y=0 &\rightarrow \\ y=-a \vee y=a &\leftrightarrow \end{aligned}$$

Now we still have to figure out when $\sqrt{a^2}$ é a ou $-a$. We need to know how the square root function is defined,

$$\begin{aligned} \sqrt{}\colon [0,\infty[&\rightarrow [0,\infty[\\ x &\mapsto \sqrt{x} \end{aligned}$$

In order to be consistent with the image of the square root function we need to prevent the number from begin negative. For that,

$$\sqrt{a^2} = \begin{cases} a & \text{se } a \geq 0 \\ -a & \text{se } a < 0 \end{cases}$$

This coincides perfectly with the definition of absolute value!

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$

$\square$

Its funny to think the notion of absolute value only transform into a function when we set a rule to decide when $y = a \vee y = -a$.

Building Inequality Properties

The most foundational property is,

Theorem: Adding a constant to a inequality

We want to show that,

$$a < b \rightarrow a + c < b + c$$

Using the definition we get,

$$\begin{aligned} a < b &\leftrightarrow b - a \in P \\ &\leftrightarrow b - a + (c - c) \in P \\ &\leftrightarrow b + c - a - c \in P \\ &\leftrightarrow (b + c) - (a + c) \in P \\ &\leftrightarrow a + c < b + c \end{aligned}$$

Another foundation property we need to prove is the inequality transitivity.

Property: Inequality Transitivity

We want to show,

$$a < b \wedge b < c \rightarrow a < c$$

Using the definition,

$$\begin{aligned} b - a \in P \wedge c - b \in P \rightarrow& \\ (b - a) + (c - b) \in P \rightarrow& \qquad \text{Applied the axiom of the closure under addtion} \\ b - b + c - a \in P \rightarrow& \\ c - a \in P \leftrightarrow& \\ a < c& \\ \square& \end{aligned}$$

Now we go to Spivak's Chapter 1 exercises, to build the rest of the properties.

Exercise Spivak 1/5i): Sum of Inequalities

We want to show,

$$a < b \wedge c < d \rightarrow a + c < d + b$$

Let $a < b$ and $c < d$,

$$\begin{aligned} a + c < b + c \wedge c + b < d + b& \\ a + c < b + c \wedge b + c < d + b& \\ a + c < d + b& \qquad \text{By the Inequality Transitivity} \end{aligned}$$

Exercise Spivak 1/5ii: Negative Equivalence of an Inequality

$$\begin{aligned} a < b &\leftrightarrow a - b < 0 \\ &\leftrightarrow -b < -a \\ &\leftrightarrow -a > -b \end{aligned}$$

Exercise Spivak 1/5iii: Quasi-Subtraction of Inequalities

Let $a < b \wedge c > d$,

$$\begin{aligned} c < d &\leftrightarrow -c < -d \\ &\rightarrow a - c < b - d \qquad \text{By the sum of inequalities} \end{aligned}$$

Exercise Spivak 1/5iv: Multiplying an inequality by a positive number

Queremos mostrar,

$$a < b \wedge c > 0 \rightarrow ac < bc$$

We start by representing the inequality through its definition,

$$a < b \leftrightarrow b - a \in P$$

Since $c > 0 \leftrightarrow c \in P$, by the axiom of closure under addition,

$$\begin{aligned} c(b-a) \in P &\leftrightarrow cb - ca \in P \\ &\leftrightarrow bc - ac \in P \\ &\leftrightarrow ac < bc \end{aligned}$$

Exercise Spivak 1/5v: Multiplying an inequality by a negative number

Queremos mostrar,

$$a < b \wedge c < 0 \rightarrow bc < ac$$

Basta representar $c < 0$ como $-c > 0$,

$$\begin{aligned} b - a \in P \wedge -c \in P &\rightarrow -c(b-a) \in P \\ &\leftrightarrow -bc + ac \in P \\ &\leftrightarrow ac - bc \in P \\ &\leftrightarrow bc < ac \end{aligned}$$

Exercise Spivak 1/5vi: Number greater than one is less than its squares

We want to show that,

$$a > 1 \rightarrow a^2 > a$$

It suffices to multiply the hypothesis by a.

Exercise Spivak 1/5vii: Number between zero and one is greater than its square

We want to show,

$$0 < a < 1 \rightarrow a^2 < a$$

It suffices to multiply the hypothesis $a < 1$ by a.

Exercise Spivak 1/5viii: Multiplying Positive Inequalities

Queremos mostrar,

$$0 \leq a < b \wedge 0 \leq c < d \rightarrow ac < bd$$

Lets represent the inequalities through their definition,

$$\begin{gathered} a < b \leftrightarrow b - a \in P \wedge \\ c < d \leftrightarrow d - c \in P \end{gathered}$$

If we multiply the first inequality by c and the second by b we get,

$$ac < bc \wedge bc < bd$$

By the transitive property of the inequality we get,

$$ac < bd$$

Exercise Spivak 1/5ix: Squaring a Positive Inequality

Queremos mostrar,

$$0 \leq a < b \to a^2 < b^2$$

It suffices to multiply the inequality by itself.

Exercise Spivak 1/5x: Square Root of a Positive Inequality

We want to show,

$$a \geq 0 \wedge b > 0 \wedge a^2 < b^2 \to a < b$$

We start by representing $a^2 < b^2$ by its definition,

$$\begin{aligned} a^2 < b^2 &\leftrightarrow b^2 - a^2 \in P \\ &\leftrightarrow (b+a)(b-a) \in P \end{aligned}$$

For this last expression to be positive, it means that the two terms are both positive or negative. Since we know by the hypothesis that $b + a$ is positive,

$$\begin{aligned} b + a > 0 \wedge b - a > 0 &\to b > -a \wedge b > a \\ &\to a < b \end{aligned}$$

$$\square$$

Exercise Spivak 1/6a: Powering a Positive Inequality

We want to show,

$$0 \leq a < b \to a^n < b^n$$

We prove by induction on n. We already showed for $n = 2$. The induction hypothesis is,

$$0 \leq a < b \to a^k < b^k$$

We want to show,

$$0 \leq a < b \to a^{k+1} < b^{k+1}$$

This is a direct result since we can multiply positive inequalities.

$$\begin{aligned} a^k < b^k &\to aa^k < bb^k \\ &\to a^{k+1} < b^{k+1} \end{aligned}$$

Exercise Spivak 12i: Product of absolute values is the absolute value of the product

We want to show,

$$|xy| = |x||y|$$

We start by squaring $|xy|$,

$$\begin{aligned} |xy|^2 &= (xy)^2 \\ &= x^2y^2 \end{aligned}$$

If we apply the square root on both sides, we show the equality.

Exercise Spivak 1/12iv

We want to show,

$$|a-b| \leq |a| + |b|$$

By the triangle inequality,

$$\begin{aligned} |a+(-b)| &\leq |a| + |-b| \\ &= |a| + |b| \end{aligned}$$

Exercise 1/12v: Sums of absolute values are an opportunity to use the triangle inequality

We want to show,

$$|x| - |y| \leq |x-y|$$

Consegui resolver o exercício ao usar o mesmo método que o spivak usou para provar a desigualdade triangular. No entanto, existe uma prova muito mais bonita através da mesma desigualdade. We start by representing x in the following manner,

$$x = (x-y) + y \rightarrow$$
$$|x| = |(x-y) + y|$$

By the triangle inequality we get,

$$|x| \leq |x-y| + |y| \leftrightarrow$$
$$|x-y| + |y| \geq |(x-y) + y| = |x|$$

This is still not what we want. But we saw earlier that we can also deduce the absolute value of the subtraction,

$$|x-y| + |y| \geq |(x-y) - y| = |x|$$

Reorganizing we get,

$$|x| - |y| \leq |x-y|$$

The takeaway messafe of this exercise is: every sum of absolute values, even with zero, are a possible opportunity to use the triangle inequality.

$\square$

By trying to solve this exercise, i tried to use two theorems (neither was actually used to solve the exercise).

Theorem: Supplementary Subtraction of an Inequality

We show,

$$c < a - b \wedge b > 0 \rightarrow a > c$$

By the definition we get,

$$a - b - c \in P \wedge b \in P$$

By the axiom of closure under addition,

$$\begin{aligned} a - b - c \in P \land b \in P &\to a - b - c + b \in P \\ &\leftrightarrow a - c \in P \\ &\leftrightarrow c < a \\ &\leftrightarrow a > c \end{aligned}$$

This theorem has a brother,

Theorem: Supplementary Addition of an Inequality

We show that,

$$a + b < c \land b > 0 \to a < c$$

We start by using the definition,

$$a + b < c \leftrightarrow c - b - a \in P$$

By the closure under addition we get,

$$\begin{aligned} c - b - a \in P \land b \in P &\to c - b - a + b \\ &\leftrightarrow c - a \in P \\ &\leftrightarrow a < c \end{aligned}$$