Weak Induction

To prove a family of statements $\forall n \in \mathbb{N} \colon P(n)$, we can use proof by induction.

Definition: Principle of Mathematical Induction

By proving the *initial case* $P(0)$ and the *induction step* $P(k) \to P(k+1)$ we conclude $\forall n \in \mathbb{N} \colon P(n)$. Formally,

$$(P(0) \land P(k) \to P(k+1)) \to \forall n \in \mathbb{N} \colon P(n)$$

Example 1 Intro to Uni Math Sheet 1: Upper bounded sum and the lower bounded product

Let $x_1 + x_2 + \ldots + x_n \leq \frac{1}{3}$. Show that $(1 - x_1)(1 - x_2)\ldots(1 - x_n) \geq \frac{2}{3}$

Doubt

What is the trick here?

Example : Sum of first n natural numbers

We want to prove the following,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Initial Case: n=1

$$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$$

$\square$

Induction step:

$$\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right) \to \left(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}\right)$$

To prove the induction step,

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i && \text{Using the induction hypothesis we get} \\
&= (n+1) + \frac{n(n+1)}{2} \\
&= \frac{2n+2}{2} + \frac{n(n+1)}{2} \\
&= \frac{2n+2+n^2+n}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

$\square$

We can also reduce the scope of the statement to a subset of natural numbers. Instead of the initial case being $P(0)$, we can start with the property $P(k)$ of a given natural number k.

Binomial Theorem

It is a general expression that allows us to decompose the power of the sum of two numbers into sums.

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We can prove by induction on n.

Initial case $n = 0$.

$$(a+b)^0 = 1$$

$$\square$$

The induction hypothesis is,

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

We want to show,

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

We start by showing that,

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i\end{aligned}$$

If $(a+b)$ multiplies the summation, we get,

$$\begin{aligned}(a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i &= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k-(i-1)} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1}\right] a^{k+1-i} b^i\right) + b^{k+1} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i\end{aligned}$$

$$\square$$

Principle: Advice on proving a statement by Induction

Always try to decompose conclusion into induction hypothesis and other term.