Weak Induction

To prove a family of statements $\forall n \in \mathbb{N} \colon P(n)$, we can use proof by induction.

Definition: Principle of Mathematical Induction

By proving the *initial case* $P(0)$ and the *induction step* $P(k) \to P(k+1)$ we conclude $\forall n \in \mathbb{N} \colon P(n)$. Formally,

$$(P(0) \land P(k) \to P(k+1)) \to \forall n \in \mathbb{N} \colon P(n)$$

Example 1 Intro to Uni Math Sheet 1: Upper bounded sum and the lower bounded product

Let $x_1 + x_2 + \ldots + x_n \leq \frac{1}{3}$. Show that $(1 - x_1)(1 - x_2) \ldots (1 - x_n) \geq \frac{2}{3}$

Doubt

What is the trick here?

Example : Sum of first n natural numbers

We want to prove the following,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Initial Case: n=1

$$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$$

$\square$

Induction step:

$$\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right) \to \left(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \right)$$

To prove the induction step,

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i && \text{Using the induction hypothesis we get} \\
&= (n+1) + \frac{n(n+1)}{2} \\
&= \frac{2n+2}{2} + \frac{n(n+1)}{2} \\
&= \frac{2n+2+n^2+n}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

$\square$

We can also reduce the scope of the statement to a subset of natural numbers. Instead of the initial case being $P(0)$, we can start with the property $P(k)$ of a given natural number k.

Principle: Advice on proving a statement by Induction

Always try to decompose conclusion into induction hypothesis and other term.