What is induction? It refers to two different definitions. One is about the composition of the set of natural numbers $\mathbb{N}$. The other is a method of proof of mathematical statements. Lets start by the first one.

Set of Natural Numbers

Definition: Set of Natural Numbers

The set of Natural Numbers is the smallest set such that,

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \to n+1 \in \mathbb{N}$

This definition contains 3 of the 5 ideas behind Peano axioms. They are,

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \to n+1 \in \mathbb{N}$
- $n+1=0 \to n \notin \mathbb{N}$
- $n+1=m+1 \to n=m$
- $(0 \in A \wedge n \in A \to n+1 \in A) \to A \subseteq \mathbb{N}$

Proof by Induction

To prove a family of statements $\forall n \in \mathbb{N} \colon P(n)$, we can use proof by induction.

Definition: Principle of Mathematical Induction

By proving the *initial case* $P(0)$ and the *induction step* $P(k) \to P(k+1)$ we conclude $\forall n \in \mathbb{N} \colon P(n)$. Formally,

$$(P(0) \wedge P(k) \to P(k+1)) \to \forall n \in \mathbb{N} \colon P(n)$$

Example : Sum of first n natural numbers

We want to prove the following,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Initial Case: n=1

$$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$$

$\square$

Induction step:

$$\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right) \to \left(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}\right)$$

To prove the induction step,

$$\begin{aligned}\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i && \text{Using the induction hypothesis we get}\\ &= (n+1) + \frac{n(n+1)}{2}\\ &= \frac{2n+2}{2} + \frac{n(n+1)}{2}\\ &= \frac{2n+2+n^2+n}{2}\\ &= \frac{n^2+3n+2}{2}\\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

$\square$

We can also reduce the scope of the statement to a subset of natural numbers. Instead of the initial case being $P(0)$, we can start with the property $P(k)$ of a given natural number k.

Principle: Advice on proving a statement by Induction

Always try to decompose conclusion into induction hypothesis and other term.

Strong Induction

The induction hypothesis is now,

$$(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$$

Example Recurrence

Let a_n be a sequence where $a_1 = 1$ and $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$. We want to prove that,

$$a_n = 3 \cdot 2^{n-1} + 2(-1)^n$$

We prove by induction on n.

Initial case: $n = 3$

$$a_3 = 10$$

$\square$

Inductive Step: $3 \leq n \leq k \rightarrow a_n = 3 \cdot 2^{n-1} + 2(-1)^n$

$$\begin{aligned}a_{n+1} &= a_n + 2a_{n-1} && \text{We can use the induction hypothesis twice}\\ &= 3 \cdot 2^{n-1} + 2(-1)^n + 2\left(3 \cdot 2^{n-2} + 2(-1)^{n-1}\right)\\ &= 2(3 \cdot 2^{n-1}) + 2(-1)^n + 2^2(-1)^{n-1}\\ &= 3 \cdot 2^n + 2(-1)^{n-1}(-1+2)\\ &= 3 \cdot 2^n + 2(-1)^{n-1}\\ &= 3 \cdot 2^n + 2(-1)^{n+1}\end{aligned}$$

$\square$

Principle: Strong Induction

We notice that weak induction is not enough when we need more than a single hypothesis in the induction step.