What is induction? It refers to two different definitions. One is about the composition of the set of natural numbers $\mathbb{N}$. The other is a method of proof of mathematical statements. Lets start by the first one.

## Set of Natural Numbers

**Definition: Set of Natural Numbers**

The set of Natural Numbers is the smallest set such that,

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \to n+1 \in \mathbb{N}$

This definition contains 3 of the 5 ideas behind Peano axioms. They are,

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \to n+1 \in \mathbb{N}$
- $n+1 = 0 \to n \notin \mathbb{N}$
- $n+1 = m+1 \to n = m$
- $(0 \in A \wedge n \in A \to n+1 \in A) \to A \subseteq \mathbb{N}$

## Proof by Induction

To prove a family of statements $\forall n \in \mathbb{N} \colon P(n)$, we can use proof by induction.

**Definition: Principle of Mathematical Induction**

By proving the *initial case* $P(0)$ and the *induction step* $P(k) \to P(k+1)$ we conclude $\forall n \in \mathbb{N} \colon P(n)$. Formally,

$$(P(0) \wedge P(k) \to P(k+1)) \to \forall n \in \mathbb{N} \colon P(n)$$

**Example 1 Intro to Uni Math Sheet 1: Upper bounded sum and the lower bounded product**

Let $x_1 + x_2 + \ldots + x_n \leq \frac{1}{3}$. Show that $(1-x_1)(1-x_2)\ldots(1-x_n) \geq \frac{2}{3}$

**Doubt**

What is the trick here?

**Example : Sum of first n natural numbers**

We want to prove the following,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Initial Case: n=1

$$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$$

□

Induction step:

$$\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right) \rightarrow \left(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}\right)$$

To prove the induction step,

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i && \text{Using the induction hypothesis we get} \\
&= (n+1) + \frac{n(n+1)}{2} \\
&= \frac{2n+2}{2} + \frac{n(n+1)}{2} \\
&= \frac{2n+2+n^2+n}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

□

We can also reduce the scope of the statement to a subset of natural numbers. Instead of the initial case being $P(0)$, we can start with the property $P(k)$ of a given natural number $k$.

**Principle: Advice on proving a statement by Induction**

Always try to decompose conclusion into induction hypothesis and other term.

## Strong Induction

The induction hypothesis is now,

$$(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$$

**Example University of Illinois: Proof Recurrence Relation by Strong Induction**

Let $a_n$ be a sequence where $a_1 = 1$ and $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$. We want to prove that,

$$a_n = 3 \cdot 2^{n-1} + 2(-1)^n$$

We prove by induction on $n$. Initial case: $n = 3$

$$a_3 = 10$$

□

Inductive Step: $3 \leq n \leq k \to a_n = 3 \cdot 2^{n-1} + 2(-1)^n$

$$
\begin{aligned}
a_{n+1} &= a_n + 2a_{n-1} && \text{We can use the induction hypothesis twice} \\
&= 3 \cdot 2^{n-1} + 2(-1)^n + 2\left(3 \cdot 2^{n-2} + 2(-1)^{n-1}\right) \\
&= 2(3 \cdot 2^{n-1}) + 2(-1)^n + 2^2(-1)^{n-1} \\
&= 3 \cdot 2^n + 2(-1)^{n-1}(-1+2) \\
&= 3 \cdot 2^n + 2(-1)^{n-1} \\
&= 3 \cdot 2^n + 2(-1)^{n+1} \\
&\square
\end{aligned}
$$

**Principle: Strong Induction**

We notice that weak induction is not enough when we need more than a single hypothesis in the induction step.

## Forward-Backward Induction

We have the same 2 steps in a regular proof by induction, the initial case and the inductive step, but the inductive step is composed by two different steps.

- Initial case: $P(0)$
- Inductive step:
  - Forward step: $P(k) \to P(f(k))$ where $f$ is an increasing function.
  - Backward step: $P(k) \to P(k-1)$

**Quest: More patterns in forward backward induction**

Find an application of a proof that requires more than one forward and/or backward step. Example:

- $f_1(k) = 2^k$
- $f_2(k) = 2^k - 1$
- $b(k) = k - 2$

**Example brilliant.org: AM-GM Inequality**

Initial Case: $\frac{a+b}{2} \geq \sqrt{ab}$

$$
\begin{aligned}
(a-b)^2 \geq 0 \leftrightarrow \\
& a^2 - 2ab + b^2 \geq 0 \leftrightarrow \\
& a^2 + 2ab + b^2 - 4ab \geq 0 \leftrightarrow \\
& a^2 + 2ab + b^2 \geq 4ab \leftrightarrow \\
& a^2 + 2ab + b^2 \geq 4ab \leftrightarrow \\
& (a+b)^2 \geq 4ab \leftrightarrow \\
& |a+b| \geq 2\sqrt{ab} \leftrightarrow && \text{Since both a and b are positive so is their addition} \\
& \to a + b \geq 2\sqrt{ab} \\
& \to \frac{a+b}{2} \geq \sqrt{ab}
\end{aligned}
$$

Inductive Step:

Induction Hypothesis:

$$\frac{\sum_{i=1}^{k} a_i}{k} \geq \sqrt[k]{\prod_{i=1}^{k} a_i}$$

Forward pass: We want to show,

$$\frac{\sum_{i=1}^{2k} a_i}{2k} \geq \sqrt[2k]{\prod_{i=1}^{2k} a_i}$$

We can start by splitting the summation in the left hand side,

$$\begin{aligned}
a_1 + a_2 + \cdots + a_{2k} &= \frac{\frac{a_1+a_2+\cdots+a_k}{k} + \frac{a_{k+1}+a_{k+2}+\cdots+a_{2k}}{k}}{2} \\
&\geq \frac{\sqrt[k]{a_1 a_2 \cdots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}}{2} \\
&\geq \sqrt{\sqrt[k]{a_1 a_2 \cdots a_k}\,\sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}} \\
&= \sqrt[2k]{a_1 a_2 \cdots a_{2k}}
\end{aligned}$$

$\square$

This completes the proof for the forward pass.

Backward pass: We want to show,

$$\frac{\sum_{i=1}^{k-1} a_i}{k-1} \geq \sqrt[k-1]{\prod_{i=1}^{k-1} a_i}$$

We start by using the inductive hypothesis,

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_k}{k} &\geq \sqrt[k]{a_1 a_2 \cdots a_k} \\
\frac{a_1 + a_2 + \cdots + a_{k-1} + \frac{a_1+a_2+\cdots+a_{k-1}}{k-1}}{k} &\geq \sqrt[k]{a_1 a_2 \cdots a_{k-1} \cdot \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}} \qquad (*1) \\
\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} &\geq \sqrt[k]{a_1 a_2 \cdot \cdot a_{k-1} \cdot \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}} \\
\left(\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}\right)^k &\geq a_1 a_2 \cdots a_{k-1} \cdot \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \\
\left(\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}\right)^{k-1} &\geq a_1 a_2 \cdots a_{k-1} \\
\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} &\geq \sqrt[k-1]{a_1 a_2 \cdots a_{k-1}}
\end{aligned}$$

$\square$

This completes the backward pass and the proof of the AM-GM Inequality.

(*1) The used property is: the arithmetic mean of $k$ numbers in which one of them is the mean of the other $k-1$ numbers is actually the mean of the same $k-1$ numbers.

## Double Induction

We start by defining the addition of two natural numbers recursively,

**Definition: Recursive Definition of Addition of Two Natural Numbers**

Let $m$ and $n$ be natural numbers.

- m+0=m
- m+(n+1)=(m+n)+1

**Example Intro to Uni Math ex.2: Commutativity of Addition of Natural Numbers**

We want to show that the addition is commutative using its recursive definition,

$$m + n = n + m$$

Let $n = 0$. We want to show that $m + 0 = 0 + m$. We show by induction on $m$. The base case is skipped. We assume $k + 0 = 0 + k$. We want to prove $0 + (k + 1) = (k + 1) + 0$

$$\begin{aligned} 0 + (k+1) &= (0+k)+1 && \text{Recursive Part of Definition} \\ &= (k+0)+1 && \text{Inductive Hypothesis} \\ &= k+1 && \text{Base Case for Recursive Definition LR} \\ &= (k+1)+0 && \text{Base Case for Recursive Definition RL} \end{aligned}$$

Let $n = 1$. We want to show that $m + 1 = 1 + m$. We show by induction on $m$. Base case $0 + 1 = 1 + 0$ was already proven when $n = 0 \wedge m = 1$. We assume $k + 1 = 1 + k$. We want to prove $1 + (k + 1) = (k + 1) + 1$

$$\begin{aligned} (k+1)+1 &= (1+k)+1 && \text{Inductive Hypothesis} \\ &= 1+(k+1) && \text{Recursive Part of Definition RL} \end{aligned}$$

$\square$

We want to show that $m + n = n + m$. We prove by induction on $n$. The base case was already shown. We assume $m + k = k + m$. We want to prove $m + (k + 1) = (k + 1) + m$

$$\begin{aligned} (m+k)+1 &= (k+m)+1 && \text{Inductive Hypothesis} \\ &= 1+(k+m) && \text{Using (ii)} \\ &= (1+k)+m && \text{Recursive Part of Definition} \\ &= (k+1)+m && \text{Using (ii)} \end{aligned}$$

$\square$

## Infinite Descent

**Example Wikipedia: Proof by Infinite Descent**

The square root of a non-integer is always irrational, formally,

$$k \notin \mathbb{N} \rightarrow \sqrt{k} \notin \mathbb{R} - \mathbb{Q}$$

Let $\sqrt{k}$ be a rational number and $q$ the last integer before $\sqrt{k}$,

$$\sqrt{k} \in \mathbb{Q} \leftrightarrow \sqrt{k} = \frac{m}{n},\ m, n \in \mathbb{N}$$
$$q \in \mathbb{N} \wedge q < \sqrt{k} \wedge q + 1 > \sqrt{k}$$

We start by describing $k$,

$$\begin{aligned}\sqrt{k} &= \frac{m}{n} \\ &= \frac{m(\sqrt{k}-q)}{n(\sqrt{k}-q)} \\ &= \frac{m\sqrt{k}-mq}{n\sqrt{k}-nq} \qquad (*2) \\ &= \frac{n\sqrt{k}\sqrt{k}-mq}{n\frac{m}{n}-nq} \\ &= \frac{nk-mq}{m-nq}\end{aligned}$$

$\square$

Since there is an irreducible fraction for every rational number then the last expression is a contraction.

(*2) We want to get rid of $\sqrt{k}$ so we have to try to replace either the square itself or the term multiplying by the square root.

### Structural Induction

The power set of a set $A$, $\mathbb{P}(A)$, can be defined recursively.

**Definition: Power Set**

Let $A$ be a set. The smallest set that satisfies the properties below is the power set of $A$. An equivalent definition is that the power set is composed by all the subsets of $A$.

- $\emptyset \in \mathbb{P}(A)$
- $B \in \mathbb{P}(A) \land x \in A \leftrightarrow B \cup \{x\} \in \mathbb{P}(A)$

**Doubt**

NEED HELP! Is the definition OK? Should probably prove the equivalence. Does the previous definition qualify as a recursive one?

### Tricks

TODO