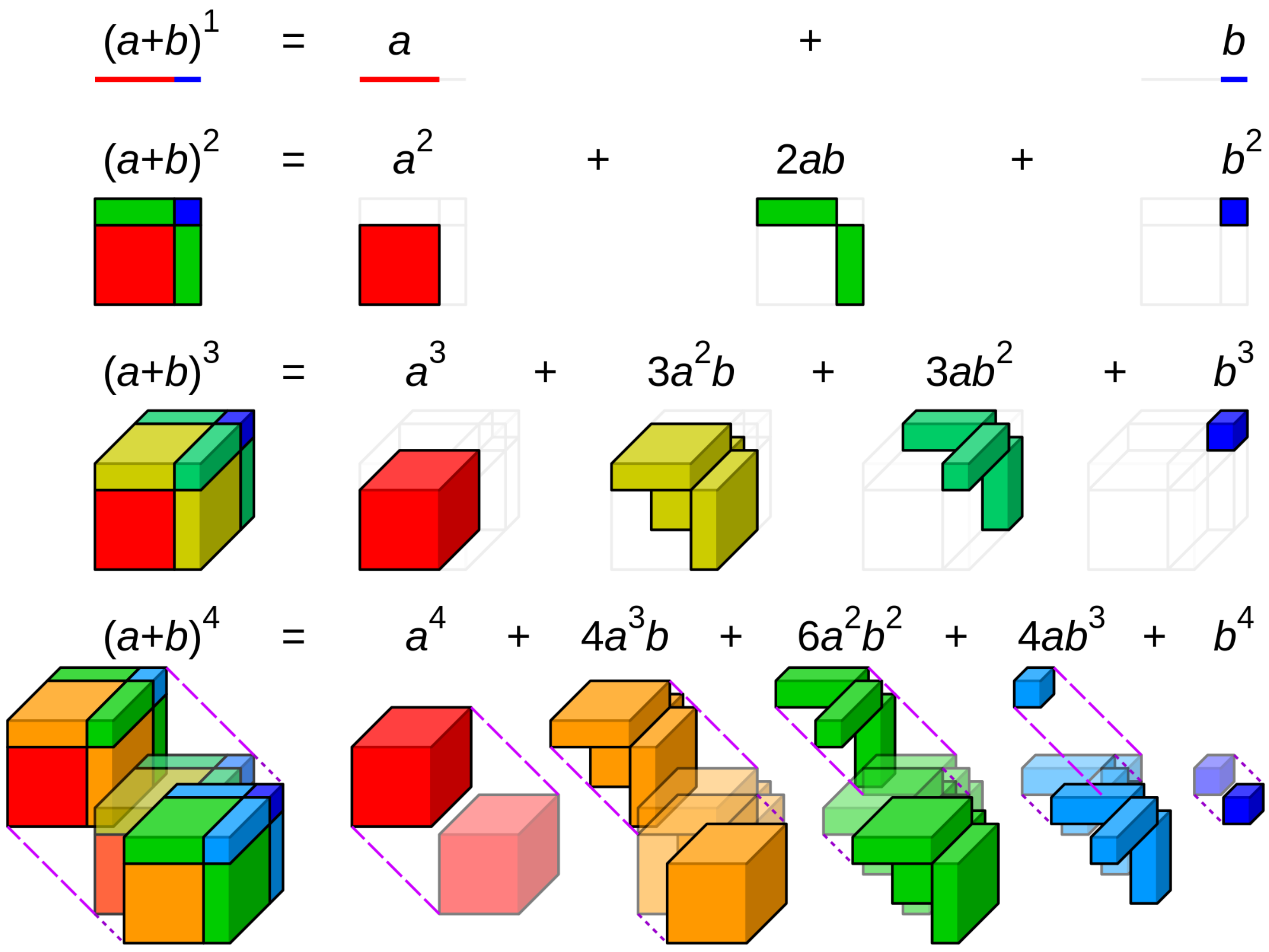
Theory



Doubt: Geometric Interpretation

What is the geometric interpretation of the binomial theorem?

From the sum of two quantities a and b, we can build a line segment of length $a + b$. We can also expand this line segment into a square in two dimensions, a cube in three, or an hypercube in more than three.

The Binomial Theorem shows how we can decompose any hypercube of n dimensions into smaller shapes, cuboids, whose sides can be expressed in terms of a and b.

Doubt

How is it related with $x^n - y^n$?

Binomial Theorem

It is a general expression that allows us to decompose the power of the sum of two numbers into sums.

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We can prove by induction on n.

Initial case $n = 0$.

$$(a+b)^0 = 1$$

$\square$

The induction hypothesis is,

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

We want to show,

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

We start by showing that,

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i\end{aligned}$$

If $(a+b)$ multiplies the summation, we get,

$$\begin{aligned}(a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i &= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k-(i-1)} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1}\right] a^{k+1-i} b^i\right) + b^{k+1} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i\end{aligned}$$

$\square$

Example Mattuck Page 6: Polinomials of degree n

The specification that the proof uses is,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

The proof is on pages 6 and 7 of Mattuck.

Example : Powers of two can be expressed as a sums of combinations

As potências de dois pode ser representadas por uma soma de combinações.

$$(1+1)^n = 2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Example : Binomial Theorem Application

$$\left(1+\frac{1}{x}\right)^x = \sum_{i=0}^{x} \binom{x}{i} \frac{1}{x^i}$$

We can see that the binomial theorem gives us the opportunity to expand any expression of the form $(a+b)^n$. This is great :O.

Quests

Quest

Find out an application of representing a power of two as sums of combinations.