Binomial Theorem

It is a general expression that allows us to decompose the power of the sum of two numbers into sums.

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We can prove by induction on n.

Initial case $n = 0$.

$$(a+b)^0 = 1$$

$\square$

The induction hypothesis is,

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

We want to show,

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

We start by showing that,

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i\end{aligned}$$

If $(a+b)$ multiplies the summation, we get,

$$\begin{aligned}(a+b)\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i &= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k-(i-1)} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i\right) + \left(\sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^{i}\right) + b^{k+1} \\ &= a^{k+1} + \left(\sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1}\right] a^{k+1-i} b^i\right) + b^{k+1} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i\end{aligned}$$

$\square$