What is a Natural Number? An element of a special set called the set of natural numbers :O. This set has a special notation $\mathbb{N}$. Informally it has the following composition,

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

Definition: Peano Axioms

- Zero is a Natural Number, $0 \in \mathbb{N}$
- Every successor of a Natural Number is a Natural Number, $n \in \mathbb{N} \to n + 1 \in \mathbb{N}$
- Zero is not the successor of any Natural Number, $n + 1 = 0 \to n \notin \mathbb{N}$
- Every successor is unique, $n + 1 = m + 1 \to n = m$
- Principle of Mathematical Induction, $\forall n \in \mathbb{N} \colon P(n) \leftrightarrow P(0) \wedge \forall k \in \mathbb{N} \colon P(k) \to P(k+1)$

Definition: Binary Operation

Given a set A, we say there is a Binary Operation $\times$ if there is an element $x \times y \in A$ for each ordered pair $(x, y) \in A \times A$.

Definition: Natural Ordering

We say that $x \leq y$ if there is a natural number z such that $x + z = y$.

$$x \leq y \leftrightarrow \exists z \in \mathbb{N} \colon x + z = y$$

Properties of the Natural Ordering

- ID: $x \leq x$
- Transitive: $x \leq y \wedge y \leq z \to x \leq z$
- Anti-symmetric: $x \leq y \wedge y \leq x \to x = y$
- Guaranteed existence: $x \leq y \vee y \leq x$

Proof $x \leq x$

$$x \leq x \leftrightarrow \exists z \in \mathbb{N} \colon x + z = x \to z = 0$$
$$\square$$

Proof Transitivity

$$\begin{aligned} x \leq y &\leftrightarrow \exists w \in \mathbb{N} \colon x + w = y \\ y \leq z &\leftrightarrow \exists r \in \mathbb{N} \colon y + r = z \\ &x + w + r = (x + w) + r = y + r = z \\ &\square \end{aligned}$$

Proof Anti-symmetry

$$\begin{aligned} \exists z \in \mathbb{N} \colon x + z = y \wedge y + z = x \\ x + y = x + y + 2z \\ 2z = 0 \\ z = 0 \\ \square \end{aligned}$$

Proof Guaranteed Existence:

Doubt

If there is no natural number z such that $x+z=y$ then $z=x-y<0$, so $-z=y-x>0$, and
$y+(-z)=y+(y-x)=x$

seems ok

Definition: Set of Natural Numbers $\mathbb{N}$

The set of natural numbers $\mathbb{N}$ is the smallest set such that:

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \rightarrow n+1 \in \mathbb{N}$

These last two points also fit the set of Integers, but since $\mathbb{N}$ is the smallest set that satisfies these properties then it is uniquely defined.