

What is a Natural Number? An element of a special set called the set of natural numbers :O. This set has a special notation \mathbb{N} . Informally it has the following composition,

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

We can define a type of operation with these numbers,

Definition: Binary Operation

Given a set A , we say there is a Binary Operation \times if there is an element $x \times y \in A$ for each ordered pair $(x, y) \in A \times A$.

Definition: Natural Ordering

We say that $x \leq y$ if there is a natural number z such that $x + z = y$.

$$x \leq y \leftrightarrow \exists z \in \mathbb{N}: x + z = y$$

Properties of the Natural Ordering

- ID: $x \leq x$
- Transitive: $x \leq y \wedge y \leq z \rightarrow x \leq z$
- Anti-symmetric: $x \leq y \wedge y \leq x \rightarrow x = y$
- Guaranteed existence: $x \leq y \vee y \leq x$

Proof $x \leq x$

$$x \leq x \leftrightarrow \exists z \in \mathbb{N}: x + z = x \rightarrow z = 0$$

□

Proof Transitivity

$$x \leq y \leftrightarrow \exists w \in \mathbb{N}: x + w = y$$

$$y \leq z \leftrightarrow \exists r \in \mathbb{N}: y + r = z$$

$$x + w + r = (x + w) + r = y + r = z$$

□

Proof Anti-symmetry

$$\exists z \in \mathbb{N}: x + z = y \wedge y + z = x$$

$$x + y = x + y + 2z$$

$$2z = 0$$

$$z = 0$$

□

Proof Guaranteed Existence:

Doubt

If there is no natural number z such that $x + z = y$ then $z = x - y < 0$, so $-z = y - x > 0$, and
 $y + (-z) = y + (y - x) = x$

seems ok

Definition: Set of Natural Numbers \mathbb{N}

The set of natural numbers \mathbb{N} is the smallest set such that:

- $0 \in \mathbb{N}$
- $n \in \mathbb{N} \rightarrow n + 1 \in \mathbb{N}$

These last two points also fit the set of Integers, but since \mathbb{N} is the smallest set that satisfies these properties then it is uniquely defined.