Equilibrium of a Spherical Pendulum by Energy Control

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In this paper a successful equilibrium method for the spherical pendulum at the vertical position is studied. We have applied two types of controls to the pivot position of the pendulum. One of the controls increases or decreases the polar angle variation by moving the pivot in the polar direction. The other type of control alters the azimuthal angle of the pendulum by forcing a radial movement on the pivot based on the difference of energy between the unstable vertical position ($\theta = 0$, $\theta' = 0$) and the actual position. Numerical integration as been done with the fourth order Runge-Kutta method.

I. INTRODUCTION

A. 1D Control Method

Stabilization of a pendulum around it's vertical position in one dimension by energy control was studied by Astrom & Furuta [1] and R. Dilão [3].

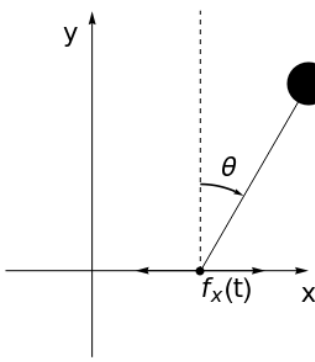

Figure 1. Vertical position of the pendulum in one dimension. The function $f_x(t)$ controls the horizontal movement of the pivot.

Considering the setup in Fig. (1), θ is the polar angle, the angle measured from the vertical where the positive sign is clockwise, l is the length of the pendulum. Since the pivot of the pendulum is mobile, we can seperate the description of the problem in two parts, the position of the pivot, $f_x(t)$, and the position of the mass of the pendulum relative to it's pivot,

$$x(t) = f_x(t) + l \sin\theta(t), \tag{1}$$

$$y(t) = l\cos\theta(t). \tag{2}$$

In cartesian coordinates the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy. \tag{3}$$

Introducing Eqs. (1) and (2) into (3), we obtain the following equation of motion for θ,

$$\ddot{\theta} - \frac{g}{l}\sin\theta = -u(t)\cos\theta, \tag{4}$$

where $u(t) = \ddot{f}_x/l$, is the control function applied to the pivot.

We are interested in controlling the position of the pendulum in the pivot frame. One of the methods to balance the pendulum is by controlling the energy. The expression for the derivative of the energy can thus be obtained,

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{g}{l}\cos\theta \tag{5}$$

$$\frac{d\mathcal{H}}{dt} = \dot{\theta}\left(\ddot{\theta} - \frac{g}{l}\sin\theta\right) \tag{6}$$

Substituting Eq. (4),in the derivative we get.

$$\frac{d\mathcal{H}}{dt} = -u(t)\,\dot{\theta}\,\cos\theta \tag{7}$$

Equation (7) exhibits the strategy of the control method used. Fixing $u(t)$, in case the coefficient $\dot{\theta}\cos\theta$ is bigger than 0, the energy gets lowered, otherwise is raised. We can thus, cancel the effect of the product $\dot{\theta}\cos\theta$. By doing this, the control function $u(t)$, may alter the energy of the system so that the only important variable is the difference of the energies of the pendulum between the vertical position with its energy given by $E_1 = +g/l$, and the energy at the actual time given in Eq. (5). The control function $u(t)$ can be expressed as,

$$u(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \tag{8}$$

where, $\mu > 0$ is an external control parameter. When the energy E is lower (higher) than E_1, the control function $u(t)$ raises (lowers) the current energy E until $E = E_1$ so that the pendulum remains in an equilibrium position in it's unstable vertical position.

B. 2D Control Method

A 2D control method was implemented for the spherical pendulum regarding the information of Sec. (I A).

In this case, similarly with Eq. (1) the position of the pendulum with a 2D control function is given by,

$$x(t) = f_x(t) + l \sin\theta \cos\phi \tag{9}$$

$$y(t) = f_y(t) + l \sin\theta \sin\phi, \tag{10}$$

$$z(t) = l \cos\theta, \tag{11}$$

θ is the azimuthal angle and ϕ is the polar angle. The equations of motion become,

$$\ddot{\theta} - \tfrac{g}{l}\sin\theta - \dot{\phi}^2 \cos\theta \sin\theta = -\cos\theta(\cos\phi\, u_x(t) + \sin\phi\, u_y(t)), \tag{12}$$

$$\ddot{\phi}\sin^2\theta + 2\,\dot{\phi}\,\dot{\theta}\sin\theta\cos\theta = -\sin\theta(\sin\phi\, u_x(t) - \cos\phi\, u_y(t)), \tag{13}$$

with $u_x(t) = \ddot{f_x}(t)/l$ and $u_y(t) = \ddot{f_y}(t)/l$, the control functions for respectively the x and y direction. The variation of energy can be determined as it has been done in Sec. (I A),

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{1}{2}\dot{\phi}^2 \sin^2\theta + \frac{g}{l}\cos\theta. \tag{14}$$

Substituing Eqs. (12) and (13) in the derivative of the energy in order to time, we obtain,

$$\begin{aligned} \frac{d\mathcal{H}}{dt} =& -\dot{\phi}\,\sin\theta(-\sin\phi\, u_x(t) + \cos\phi\, u_y(t)) \\ &- \dot{\theta}\,\cos\theta(\cos\phi\, u_x(t) + \sin\phi\, u_y(t)). \end{aligned} \tag{15}$$

Equation (15) displays the geometric relationship between, $u_x(t)$ and $u_y(t)$.

In case,

$$\vec{o}(t) = \begin{cases} u_x(t) = o(t)\cos\phi, \\ u_y(t) = o(t)\sin\phi, \end{cases} \tag{16}$$

this is, if $u_x(t)$ and $u_y(t)$ are aligned with the projection of the radial versor, $\hat{e_r}$, into the xy plane, Eq. (15) reduces to very similar form of Eq. (7) with the exception that $\vec{o}$ is a 2D variable and not 1D. This way, the control function $o(t)$ monitors the energy regarding the azimuthal variation, it's rotation around the $\hat{e_\phi}$ versor.

In case,

$$\vec{s}(t) = \begin{cases} u_x(t) = -s(t)\sin\phi, \\ u_y(t) = s(t)\cos\phi, \end{cases} \tag{17}$$

so that $u_x(t)$ and $u_y(t)$ are aligned with the polar versor, $\hat{e_\phi}$, the term, in Eq. (15) proportional to $\dot{\theta}$ goes to 0 leaving the the control function $s(t)$ to modify the energy regarding the polar angle variation ϕ', in other words its rotation about z axis. We can then implement the two types of control by summing vectorially each acceleration component, the net control $\vec{u}(t)$ is given by,

$$\vec{u}(t) = \begin{cases} u_x(t) = o(t)\cos\phi - s(t)\sin\phi, \\ u_y(t) = o(t)\sin\phi + s(t)\cos\phi. \end{cases} \tag{18}$$

In order to control the pendulum so that it remains balanced in it's vertical position with $\theta = 0$, $\dot{\theta} = 0$, the strategy was use the control function $s(t)$ to deaccelerate the rotation of the pendulum, until $\dot{\phi} = 0$. This way, $s(t)$ has the following form,

$$s(t) = \nu\,\mathrm{sign}\,(\dot{\phi}\sin\theta), \tag{19}$$

with $\nu > 0$ defined as an external control parameter.

Simultaneously, the control mechanism in Eq. (8) is applied in a radial direction to the pivot in order to raise or lower the energy E to the final state, $E_1 = +g/l$. The control function $o(t)$ has then the following form,

$$o(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \tag{20}$$

with, $\mu > 0$ defined as an external control parameter.

Fig (2) illustrates an examples of the 2D control functions $\vec{o}(t)$ and $\vec{s}(t)$.

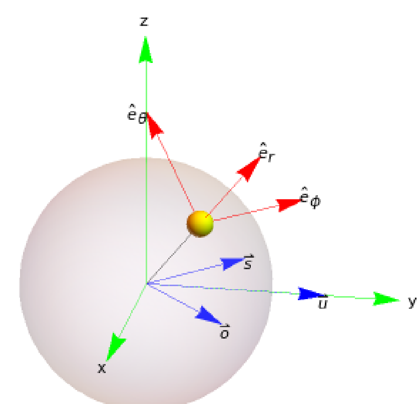

Figure 2. Example of control functions $\vec{o}(t)$, $\vec{s}(t)$ and net control $\vec{u}(t)$ with blue arrows, to balance the yellow pendulum around it's unstable vertical position. The axis x, y, z are displayed in green and the spherical versors, $\hat{e_\phi}$, $\hat{e_\theta}$, $\hat{e_r}$ in red. The control $s(t)$ is responsible for slowing the rotation around z axis, $\dot{\phi}$. The control function $o(t)$ has the purpuse of raising the z position of the pendulum to it's unstable vertical position until $\theta = 0$ and $\theta' = 0$.

II. NUMERICAL METHOD

The numerical method applied in order to solve the system was a fourth order Runge-Kutta method [2], for both $\dot{\phi}$ and $\dot{\theta}$. Having a timestep, Δt and a max number of iterations $Nmax$, this method is applied when we have a generic function x such that,

$$\dot{x} = g(x, t), \tag{21}$$

the next iteration of x by the fourth order Runge-Kutta method is given by,

$$x_{n+1} = x_n + \frac{\Delta t}{6}(k1 + 2\,k2 + 2\,k3 + k4). \tag{22}$$

The k coefficients are given by,

$$k_1 = g(t, x_n), \tag{23}$$

$$k_2 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_1}{2}\right), \tag{24}$$

$$k_3 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_2}{2}\right), \tag{25}$$

$$k_4 = g(t + \Delta t, x_n + k_3). \tag{26}$$

Associating Eq. (12) with $\ddot{\theta} = g_{\dot{\theta}}$ and Eq. (13) with $\ddot{\phi} = g_{\dot{\phi}}$ we can obtain the next iteration of $\dot{\theta}$ and $\dot{\phi}$. The functions to integrate by the Runge-Kutta method can be written in the following way,

$$g_{\dot{\theta}} = \cos\theta_n\,(\sin\theta_n\,\dot{\phi_n}^2 - \cos\phi_n\,u_{x_n} - \sin\phi_n\,u_{y_n}) + \frac{g}{l}\sin\theta_n, \tag{27}$$

$$g_{\dot{\phi}} = \frac{2\,\dot{\phi_n}\,\dot{\theta_n}\cos\theta_n - \sin\phi_n\,u_{x_n} + \cos\phi_n\,u_{y_n}}{\sin\theta_n}, \tag{28}$$

$$g_\theta = \dot{\theta_n}, \tag{29}$$

$$g_\phi = \dot{\phi_n}. \tag{30}$$

We can apply the Runge-Kutta method to Eqs. (18) as done with θ and ϕ. This way, the position of the mobile pivot, the functions f_x, f_y, can be determined along side it's derivatives, $\dot{f_x}$ and $\dot{f_y}$. In the end, with the proper set of initial values, and applying the method until the last iteration, $Nmax$, a solution to the problem may be determined so that the pendulum remains balanced in it's unstable vertical position.

III. RESULTS

Figs. (3)-(8) display the result from a simulation with S.I. units and the following parameters: $l = 1$ m, $m = 1$ Kg, $g = 1$ m/s^2, μ,ν = 0.3 s^{-2},$\phi'_0 = 2$ s^{-1}, $\theta_0 = \frac{3\pi}{4}$, $\Delta t = 0.001$ s, $Nmax = 20000$, and

$$\phi_0, f_{x_0}, f_{y_0}, \dot{f}_{x_0}, \dot{f}_{y_0} = 0. \tag{31}$$

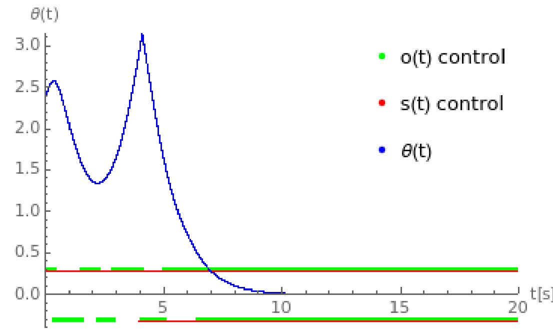

Figure 3. Variation of θ with time in blue. After the simulation starts, the pendulum takes approximetely 10 seconds to reach $\theta = 0$ and to balance in the unstable vertical position. The $o(t)$ and $s(t)$ control functions are displayed respectively in green and red. It can be seen that $s(t)$ initially is assymetric becouse the pendulum is rotating around the z axis, but with time, this rotation decreases, and it's average equals 0 to correct any deviation from equilibrium, $\phi' = 0$.

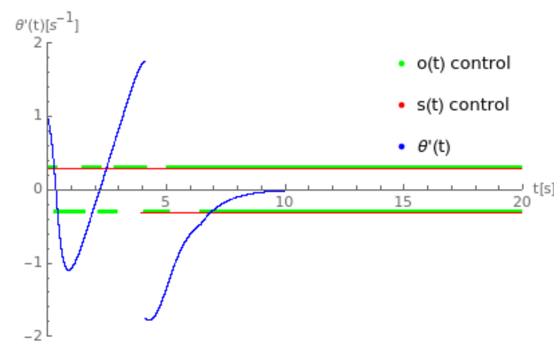

Figure 4. Variation of θ' with time in blue. This function has a great oscilation, but it decreases to 0 after $t = 10$ s where the pendulum remains in equilibrium until the end of the simulation.

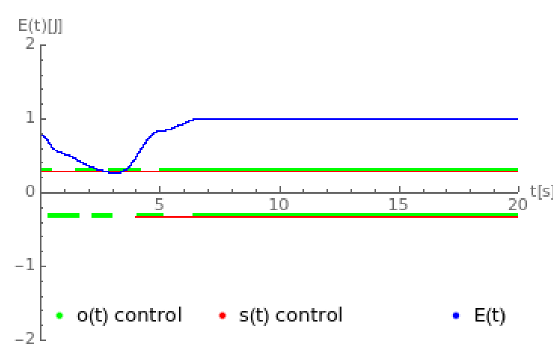

Figure 5. Variation of energy with time in blue. The control $o(t)$ initially oscillates with a certain frequency, so that the pendulum has enough energy to reach the top. After the pendulum is balanced in it's unstable vertical position, $o(t)$ oscilates very rapidly toward 0, similarly to the $s(t)$ control when the pendulum is in equilibrium, in order to correct any deviation from stability.

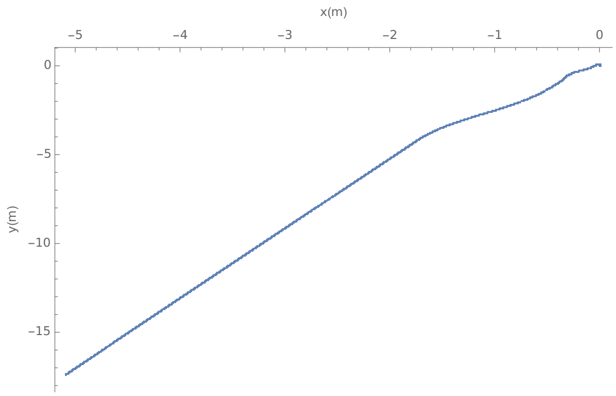

Figure 6. Parametric function with $x = f_x(t)$ and $y = f_y(t)$. This function corresponds to the trajectory of the pivot. Most of the rotational movement in the beginning of the simulation is transfered to the pivot so that the center of the pendulum remains in a constant drift after it reaches the equilibrium. The moment the pendulum reaches equilibrium is, approximately, when the slope dy/dx is constant.

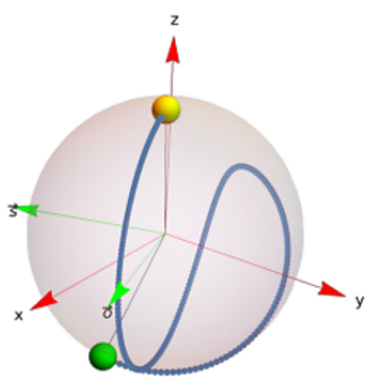

Figure 7. Trajectory of the pendulum from the pivot frame. The green pendulum is the initial state and the yellow pendulum is the final state of the simulation. As it can be seen, initially the rotation around the z axis is reduced because of the effect of the control function $\vec{s}(t)$, and afterwards with the control function $\vec{o}(t)$ the pendulum reaches the balanced state where $\theta = 0$ and $\theta' = 0$.

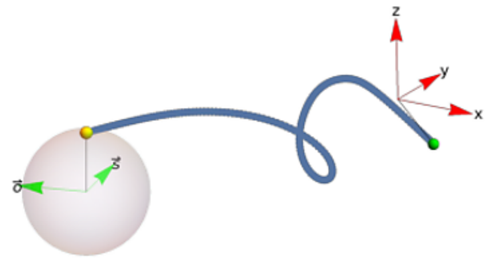

Figure 8. Trajectory of the pendulum from the laboratory frame. As in Fig. (7), the green pendulum corresponds to the initial state and the yellow pendulum corresponds to the moment the pendulum becomes balanced. It is possible to see that when the pendulum arrives its balanced stance, it remains with the same velocity to not break it's stability.

IV. CONCLUSIONS

In this work a spherical pendulum control was studied so that after a certain time a vertical balance was reached by controlling the movement of it's pivot. The requirement to obtain such an equilibrium was to make two types of control, one to stop the rotation around the z axis and another to modify the energy of the pendulum to reach the state where the pendulum is balanced, $\theta = 0$ and $\theta' = 0$.

Figs. (3)-(8) show that such balance was possible along with the determination of the trajectory of the pivot. The system was successfully represented, with no instabilities, and the numerical method applied, the fourth order Runge-Kutta [2], remained consistent and without significant errors throught the simulation.

Eventhough it was possible to achieve a balance, it was not done under the minimal time nor effiency. A detailed stability analysis of the parameters μ and ν can be done to study the effects of each parameter in the dynamics and stability of the system.

In the future, a more realistic model may be contemplated by introducing air resistance, aerodynamic coefficients and moment of inertia, to study the different types of cases where this method to achieve balance is possible and in which situations. Besides an energy control method, it is possible that other methods may also decrease the time to reach a balance or increase the range of stability and therefore maximize the efficiency to equilibrate a pendulum in it's unstable vertical position.

[1] Åström, Karl Johan, and Katsuhisa Furuta. "Swinging up a pendulum by energy control." Automatica 36.2 (2000): 287-295

[2] Süli, Endre; Mayers, David, An Introduction to Numerical Analysis, Cambridge University Press,(2003): ISBN 0-521-00794-1

[3] R. Dilão, "Uma Introdução à Teoria dos Sistemas Dinâmicos e do Caos",IST 31.2 (2018): 330-335