Equilibrium of a Spherical Pendulum by Energy Control

In this paper a successful equilibrium method for the the spherical pendulum around it's pivot is studied. In order to reach a balance, it was applied two types of controls, with each applying on the pivot. One of the controls increases or decreases the polar angle variation by moving the pivot in the polar direction, the other type of control alters the azimuthal angle of the pendulum by exercing a radial movement on the pivot based on the difference of energy between the stable position ($\theta = 0$, $\theta' = 0$) and the current iteration. The numerical method applied, fourth order Runge-Kutta, didn't have a negative influence in the final result in the sense that no anomalies were found.

I. INTRODUCTION

A. 1D Control method

Stabilization of a pendulum around it's pivot point in one dimension by energy control was studied by Astrom & Furtura [1] and R. Dilão [3]. Considering θ in polar coordinates, the angle measured from the vertical where the positive sign is clockwise, l its length, and the fact that the pivot of the pendulum is mobile, we can seperate the decription of the problem in two parts, the position of the pendulum from the pivot and a function, $f_x(t)$ the position of the pivot:

$$x(t) = f_x(t) + l \sin\theta(t), \tag{1}$$

$$y(t) = l \cos\theta(t). \tag{2}$$

In cartesian coordinates the Lagrangian may be written as:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy, \tag{3}$$

substituting in with Eqs. (1) and (2). We have the following equation of motion for θ,

$$\ddot{\theta} - \frac{g}{l} \sin\theta = -u(t)\cos\theta, \tag{4}$$

where $u(t) = \ddot{f_x}/l$.

We are interested in controlling only the position of the pendulum and one of the methods is by controlling the energy. This method involves calculating the energy and its derivative in the pivot frame point. The expression for the derivative of the energy can be obtained,

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{g}{l}\cos\theta \tag{5}$$

$$\frac{d\mathcal{H}}{dt} = \dot{\theta}\left(\ddot{\theta} - \frac{g}{l}\sin\theta\right) \tag{6}$$

Substituting Eq. (4),in the derivative we get.

$$\frac{d\mathcal{H}}{dt} = -u(t)\,\dot{\theta}\,\cos\theta \tag{7}$$

Eq. (7) exhibits the strategy of the control method used. Fixing u, in case the coefficient $\dot{\theta}\cos\theta$ is bigger than 0, the energy gets lowered, otherwise is raised. We can invert the effect of this coefficient and only base our control function $u(t)$, so that it depends only on the difference of the energies between the pendulum in an equilibrium around it's pivot, $E_1 = g/l$, (aligned with the vertical axis at $\theta = 0$ and $\dot{\theta} = 0$) and the energy at the current iteration E. This way the control function $u(t)$ can be expressed as,

$$u(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \tag{8}$$

where, $\mu > 0$ is a control parameter.

When the energy E is lower (higher) than E_1, the control function $u(t)$ raises (lowers) the current energy E until $E = E_1$ so that the pendulum remains in an equilibrium position around it's pivot.

B. 2D Control Method

A 2D control method was implemented for the spherical pendulum regarding the information of Sec. (I A). In this case, similarly with Eq. (1) the position of the pendulum with a 2D control function is given by,

$$x(t) = f_x(t) + l, \sin\theta\cos\phi \tag{9}$$

$$y(t) = f_y(t) + l\sin\theta\sin\phi, \tag{10}$$

$$z(t) = l\cos\theta, \tag{11}$$

θ is the azimuthal angle and ϕ is the polar angle. The movement equations become,

$$\ddot{\theta} - \tfrac{g}{l}\sin\theta - \dot{\phi}^2\cos\theta\,\sin\theta = -\cos\theta(\cos\phi\,u_x(t) + \sin\phi\,u_y(t)), \tag{12}$$

$$\ddot{\phi}\sin^2\theta + 2\,\dot{\phi}\,\dot{\theta}\sin\theta\cos\theta = -\sin\theta\sin\phi\,u_x(t) + \sin\theta\cos\phi\,u_y(t), \tag{13}$$

with $u_x(t) = \ddot{f_x}(t)/l$ and $u_y(t) = \ddot{f_y}(t)/l$, the control functions for respectively the x and y direction. The variation of energy can be determined as it was done in Sec. (I A),

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{1}{2}\dot{\phi}^2\sin^2\theta + \frac{g}{l}\cos\theta \tag{14}$$

Substituing Eqs. (12) and (13) in the derivative of the energy in order to time, we have,

$$\begin{aligned}\frac{d\mathcal{H}}{dt} =& -\dot{\phi}\,\sin\theta(-\sin\phi\, u_x(t) + \cos\phi\, u_y(t)) \\ &-\dot{\theta}\,\cos\theta(\cos\phi\, u_x(t) + \sin\phi\, u_y(t))\end{aligned} \tag{15}$$

Eq. (15) displays the geometric relationship between, $u_x(t)$ and $u_y(t)$.

In case,

$$\vec{o}(t) = \begin{cases} u_x(t) = o(t)\cos\phi, \\ u_y(t) = o(t)\sin\phi, \end{cases} \tag{16}$$

this is, if $u_x(t)$ and $u_y(t)$ are aligned with the projection of the radial versor, $\hat{e_r}$, with the xy plane, Eq. (15) reduces to very similar form of Eq. (7) with the exception that $\vec{o}$ is a 2D variable and not 1D. This way, the control function $o(t)$ monitors the energy regarding the azimuthal variation, it's rotation about $\hat{e_\phi}$ versor.

In case,

$$\vec{s}(t) = \begin{cases} u_x(t) = -s(t)\sin\phi, \\ u_y(t) = s(t)\cos\phi, \end{cases} \tag{17}$$

so that $u_x(t)$ and $u_y(t)$ are aligned with the polar versor, $\hat{e_\phi}$, the term, in Eq. (15) proportional to $\dot{\theta}$ goes to 0 leaving the the control function $s(t)$ to modify the energy regarding the polar angle variation ϕ', in other words its rotation about z axis. We can then implement the two types of control by summing vectorially each acceleration component.

$$\vec{u} = \begin{cases} u_x(t) = o(t)\cos\phi - s(t)\sin\phi, \\ u_y(t) = o(t)\sin\phi + s(t)\cos\phi. \end{cases} \tag{18}$$

In order to control the pendulum so that it remains balanced with it's pivot, ($\theta = 0$, $\dot{\theta} = 0$), the strategy was use the control function $s(t)$ to deaccelerate the rotation of the pendulum, until $\dot{\phi} = 0$. This way, $s(t)$ has the following form,

$$s(t) = \nu\,\mathrm{sign}\,(\dot{\phi}\sin\theta), \tag{19}$$

with $\nu > 0$ as an external parameter to the problem. At the same time that the control mechanism in Eq. (8) is applied to raise or lower the energy E to the final state ,$E_1 = g/l$, by modifying the radial position of the pivot. The control function $o(t)$ has then the following form,

$$o(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \tag{20}$$

with, $\mu > 0$ as a external control parameter.

Fig (1) illustrates an examples of the 2D control functions $\vec{o}(t)$ and $\vec{s}(t)$.

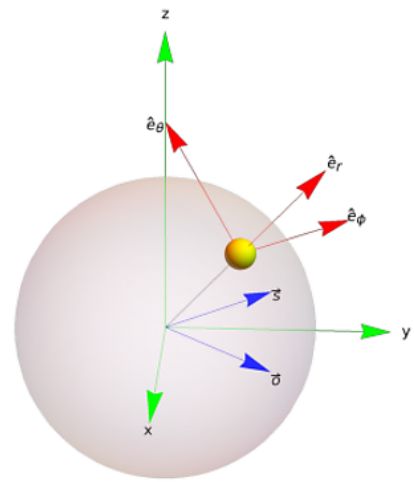

Figure 1. Example of control functions $\vec{o}(t)$ and $\vec{s}(t)$ displayed as blue arrows, to balance the yellow pendulum around it's pivot. The axis x, y, z are displayed in green and the spherical versors, $\hat{e_\phi}$, $\hat{e_\theta}$, $\hat{e_r}$ in red. The control $o(t)$ has the function of raising the Z position to $Z = l$, and the control $s(t)$ is responsible for slowing the rotation around Z axis, $\dot{\phi}$.

II. NUMERICAL METHOD

The numerical method applied in order to solve the system was a fourth order Runge-Kutta method [2], for both $\dot{\phi}$ and $\dot{\theta}$. Having a timestep, Δt and a max number of iterations $Nmax$, this method is applied when we have a generic function x such that,

$$\dot{x} = g(x, t), \tag{21}$$

the next iteration of x by the fourth order Runge-Kutta method is given by,

$$x_{n+1} = x_n + \frac{\Delta t}{6}(k1 + 2\,k2 + 2\,k3 + k4). \tag{22}$$

The k coefficients are given by,

$$k_1 = g(t, x_n), \tag{23}$$

$$k_2 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_1}{2}\right), \tag{24}$$

$$k_3 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_2}{2}\right), \tag{25}$$

$$k_4 = g(t + \Delta t, x_n + k_3). \tag{26}$$

Associating Eq. (12) with $\ddot{\theta} = g_{\dot{\theta}}$ and Eq. (13) with $\ddot{\phi} = g_{\dot{\phi}}$ we can obtain the next iteration of $\dot{\theta}$ and $\dot{\phi}$. The functions to integrate by the Runge-Kutta method can be written in the following way,

$$g_{\dot{\theta}} = \cos\theta_n \left(\sin\theta_n \, \dot{\phi_n}^2 - \cos\phi_n \, u_{x_n} - \sin\phi_n \, u_{y_n}\right) + \frac{g}{l}\sin\theta_n, \tag{27}$$

$$g_{\dot{\phi}} = \frac{2\,\dot{\phi_n}\,\dot{\theta_n}\cos\theta_n - \sin\phi_n \, u_{x_n} + \cos\phi_n \, u_{y_n}}{\sin\theta_n}, \tag{28}$$

$$g_{\theta} = \dot{\theta_n}, \tag{29}$$

$$g_{\phi} = \dot{\phi_n}. \tag{30}$$

We can apply the Runge-Kutta method to Eqs. (18) as done with θ and ϕ. This way, the position of the mobile pivot, the functions f_x, f_y, can be known along side it's derivatives, $\dot{f_x}$ and $\dot{f_y}$. In the end, with the proper set of initial values, and applying the method until the last iteration, $Nmax$, a solution to the problem may be determined so that the pendulum remains in a balanced position.

III. RESULTS

Figs. (2)-(7) display the result from a simulation with S.I. units and the following parameters: $l = 1$ m, $m = 1$ Kg, $g = 1$ m/s^2, μ,ν = 0.3 s^{-2},$\phi_0' = 2$ s^{-1}, $\theta_0 = \frac{3\pi}{4}$, $\Delta t = 0.001$ s, $Nmax = 20000$, and

$$\phi_0, f_{x_0}, f_{y_0}, \dot{f_{x_0}}, \dot{f_{y_0}} = 0. \tag{31}$$

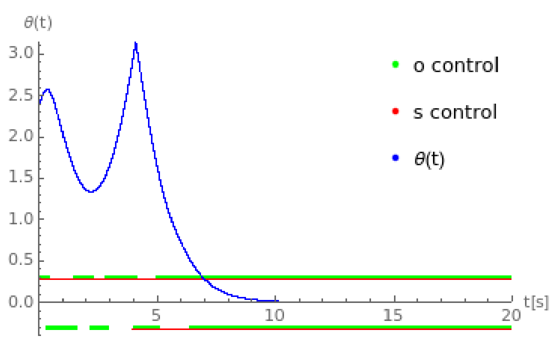

Figure 2. In this figure the variation of θ with time is displayed in blue. After the simulation starts, the pendulum takes approximetely 10 seconds to balance around the pivot. The $o(t)$ and $s(t)$ control functions are displayed respectively in green and red. It can be seen that $s(t)$ initially is assymetric becouse the pendulum is rotating around the Z axis, but with time, this rotation decreases, and it's average equals 0 to correct any deviation from equilibrium, $\phi' = 0$.

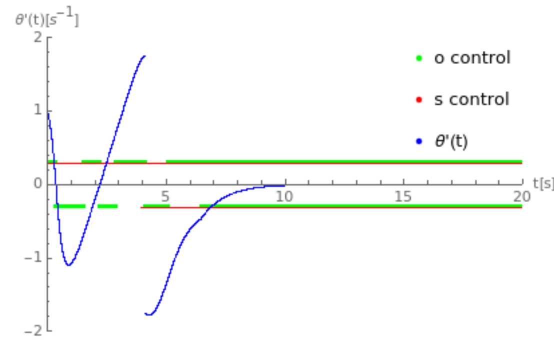

Figure 3. This figure displays the variation of θ' with time in blue. This function has a great oscilation, but it decreases to 0 after $t = 10$ s where the pendulum remains in equilibrium until the end of the simulation.

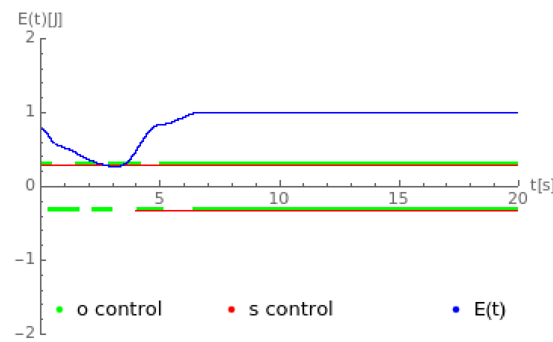

Figure 4. This figure displays the variation of energy marked with blue points. The control $o(t)$ initially oscillates with a certain frequency, so that the pendulum has enough energy to reach the top. After the pendulum reaches equilibrium , $o(t)$ oscilates very rapidly toward 0, similarly to the $s(t)$ control in the balance stage, to correct any deviation from stability.

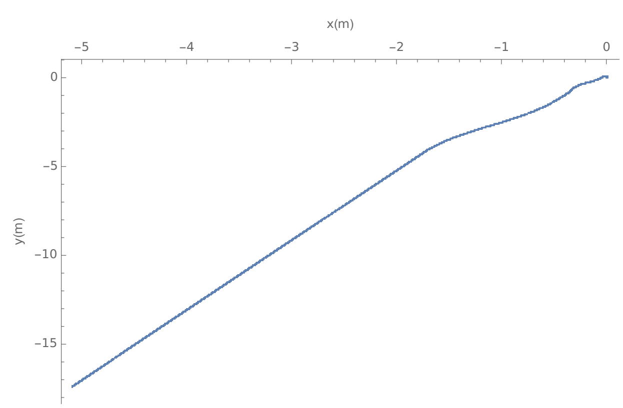

Figure 5. In this figure the trajectory of the pivot, $f_x(t)$ and $f_y(t)$ is displayed in blue. Most of the rotational movement in the beginning of the simulation is transfered to the pivot so that the center of the pendulum remains in a constant drift after it reaches the equilibrium, this is, approximately, the moment when the slope dy/dx is constant.

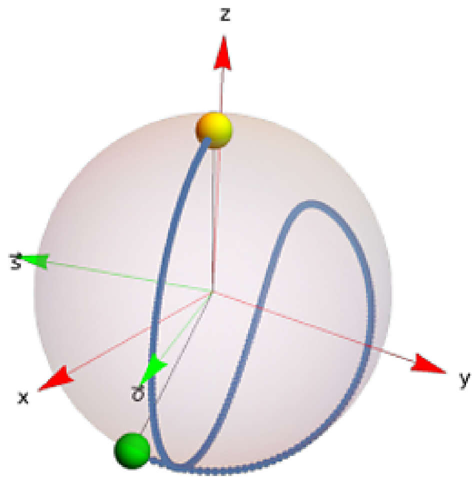

Figure 6. This figure displays the trajectory from the pivot frame. The green pendulum is the initial state and the yellow pendulum is the final state of the simulation. As it can be seen, initially the rotation around the Z axis is reduced and afterwards the pendulum reaches the balanced state where $z = l$. The vectors $\vec{s}(t)$ and $\vec{o}(t)$ correspond to the forces per length that the pivot is subjected.

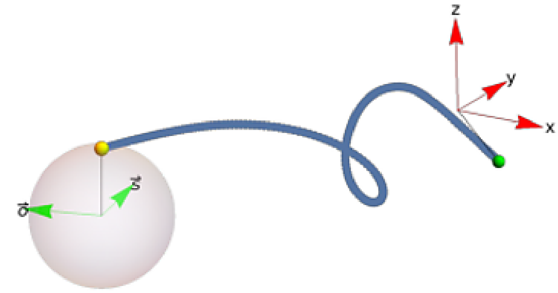

Figure 7. This figure displays the trajectory from the laboratory frame. As in Fig. (6), the initial state corresponds to the initial state and the yellow pendulum corresponds to the moment the pendulum becomes balanced. It is possible to see that when the pendulum arrives its balanced stance, it remains with the same velocity to not break it's stability

IV. CONCLUSIONS

In this work a spherical pendulum control was studied so that after a certain time a vertical balance was reached by by controlling the movement of it's pivot. It was seen in the previous sections that the requirement to obtain such an equilibrium was to make two types of control, one to stop the rotation around the Z axis and another to modify the energy of the pendulum to reach the state where the pendulum is balanced, $\theta = 0$ and $\theta' = 0$.

Figs. (2)-(7) show that such balance was possible along with the exploration of the trajectory of the pivot. The system was successfully represented, with no instabilities, and the numerical method applied, the fourth order Runge-Kutta [2], remained consistent and without significant errors through the simulation.

Eventhough it was possible to achieve a balance, it was not done under the minimal time nor effiency. A detailed stability analysis of the parameters μ and ν can be done to study the effects of each parameter in the dynamics and stability of the system.

In the future, a more realistic model may be studied by introducing air resistance and aerodinamic coefficients to study the different types of cases where this method to achieve balance is possible and in which situations. Besides an energy control method, other methods may also decrease the time to reach a balance or increase the range of stability and therefore maximize the efficiency to balance a pendulum around it's pivot.

[1] Åström, Karl Johan, and Katsuhisa Furuta. "Swinging up a pendulum by energy control." Automatica 36.2 (2000): 287-295.

[2] Wikipedia contributors. "Runge–Kutta methods." Wikipedia, The Free Encyclopedia. 22 Nov. 2018. Web. 23 Nov. 2018.

[3] R. Dilão , "Uma Introdução à Teoria dos Sistemas Dinâmicos e do Caos"(2018):330-335, IST.