

Equilibrium of a Spherical Pendulum by Energy Control

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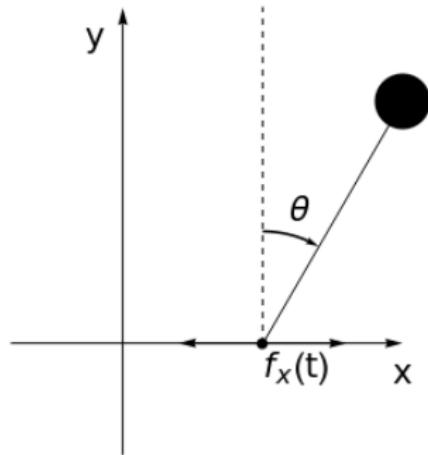
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Equation of Motion

$$x(t) = f_x(t) + l \sin \theta(t) \quad (1)$$

$$y(t) = l \cos \theta(t) \quad (2)$$



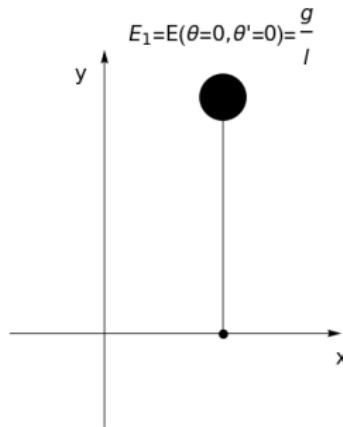
$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \quad (3)$$

$$\ddot{\theta} - \frac{g}{l} \sin \theta = -u(t) \cos \theta \quad (4)$$

with $u(t) = \ddot{f}_x/l$, the control function applied to the pivot

Figure 1: Equilibrium of the pendulum in 1D

Energy Control



$$E_1 = E(\theta=0, \dot{\theta}=0) = \frac{g}{l}$$

In the pivot reference frame

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{g}{l} \cos \theta \quad (5)$$

$$\frac{d\mathcal{H}}{dt} = \dot{\theta} \left(\ddot{\theta} - \frac{g}{l} \sin \theta \right) \quad (6)$$

$$\frac{d\mathcal{H}}{dt} = -u(t) \dot{\theta} \cos \theta \quad (7)$$

Figure 2: Energy control E_1

Control Function

$$u(t) = -\mu \operatorname{sign}(E_1 - E) \operatorname{sign}(\dot{\theta} \cos \theta) \quad (8)$$

Equations of Motion

$$x(t) = f_x(t) + l \sin \theta \cos \phi \quad (9)$$

$$y(t) = f_y(t) + l \sin \theta \sin \phi \quad (10)$$

$$z(t) = l \cos \theta \quad (11)$$

$$\ddot{\theta} - \frac{g}{l} \sin \theta - \dot{\phi}^2 \cos \theta \sin \theta = -\cos \theta (\cos \phi u_x(t) + \sin \phi u_y(t)) \quad (12)$$

$$\ddot{\phi} \sin^2 \theta + 2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta = -\sin \theta (\sin \phi u_x(t) - \cos \phi u_y(t)) \quad (13)$$

with $u_x(t) = f_x(t)/l$ and $u_y(t) = f_y(t)/l$

Energy Control

$$\begin{aligned}\frac{d\mathcal{H}}{dt} = & -\dot{\phi} \sin \theta (-\sin \phi u_x(t) + \cos \phi u_y(t)) \\ & -\dot{\theta} \cos \theta (\cos \phi u_x(t) + \sin \phi u_y(t))\end{aligned}\quad (14)$$

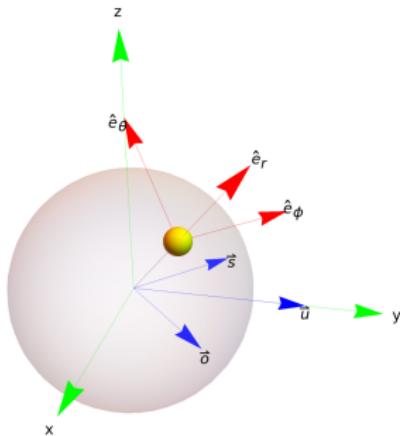
In case $u_x, u_y \parallel (\hat{e}_r)_{cylindrical}$

$$\vec{o}(t) = \begin{cases} u_x(t) = o(t) \cos \phi \\ u_y(t) = o(t) \sin \phi \end{cases} \rightarrow \frac{d\mathcal{H}}{dt} = -o(t) \dot{\theta} \cos \theta$$

In case $u_x, u_y \parallel (\hat{e}_\phi)$

$$\vec{s}(t) = \begin{cases} u_x(t) = -s(t) \sin \phi \\ u_y(t) = s(t) \cos \phi \end{cases} \rightarrow \frac{d\mathcal{H}}{dt} = -s(t) \dot{\phi} \sin \theta$$

2D Control Functions



2D Control Functions

$$o(t) = -\mu \operatorname{sign}(E_1 - E) \operatorname{sign}(\dot{\theta} \cos \theta) \quad (15)$$

$$s(t) = \nu \operatorname{sign}(\dot{\phi} \sin \theta) \quad (16)$$

Net control:

Figure 3: Control of the spherical pendulum in 2D

$$\vec{u}(t) = \vec{o}(t) + \vec{s}(t) \quad (17)$$

$\approx 1D$ Control (Lab). $\mu = 0.4s^{-2}$, $\nu = 0.02s^{-2}$, $\theta_0 = 5\pi/6$

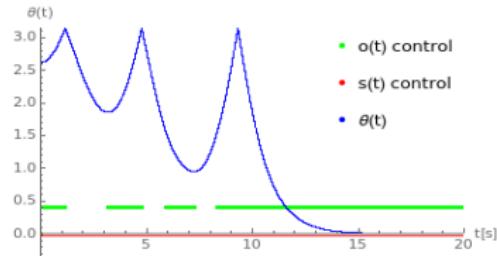


Figure 4: θ with time

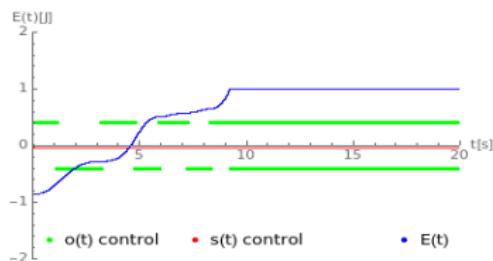


Figure 5: Energy with time

\approx 1D Control (Pivot). $\mu = 0.4\text{s}^{-2}$, $\nu = 0.02\text{s}^{-2}$, $\theta_0 = 5\pi/6$

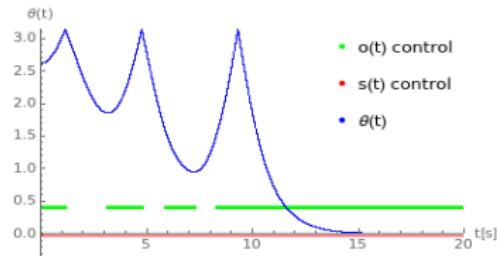


Figure 6: θ with time

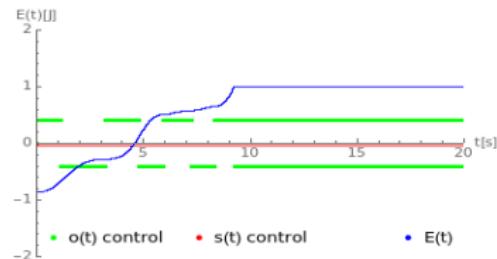


Figure 7: Energy with time

2D Control (Lab). $\nu, \mu = 0.3\text{s}^{-2}$, $\phi'_0 = 2\text{s}^{-1}$, $\theta'_0 = 1\text{s}^{-1}$, $\theta_0 = 3\pi/4$

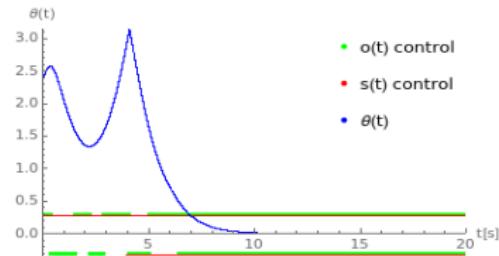


Figure 8: θ with time



Figure 9: Energy with time

2D Control (Pivot). $\nu, \mu = 0.3\text{s}^{-2}$, $\phi'_0 = 2\text{s}^{-1}$, $\theta'_0 = 1\text{s}^{-1}$, $\theta_0 = 3\pi/4$

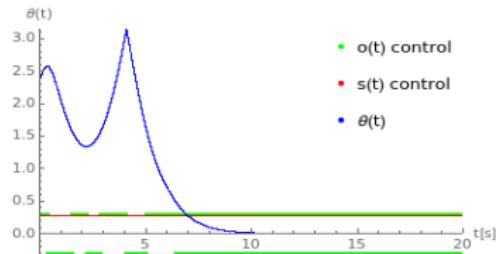


Figure 10: θ with time

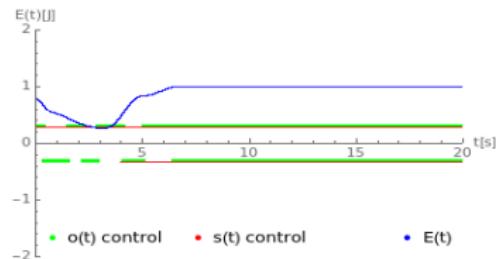


Figure 11: Energy with time

Conclusions

Equilibrium of the spherical pendulum was attainable with success.

Possible Future work:

- Analysis on the stability of the μ and ν parameters
- Produce a more realistic model contemplating air friction

References

- [1] Åström, Karl Johan, and Katsuhisa Furuta. "Swinging up a pendulum by energy control." *Automatica* 36.2 (2000): 287-295
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- [3] R. Dilão, "Uma Introdução à Teoria dos Sistemas Dinâmicos e do Caos",IST 31.2 (2018): 330-335