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Equilibrium of a Spherical Pendulum by Energy Control

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Equation of Motion

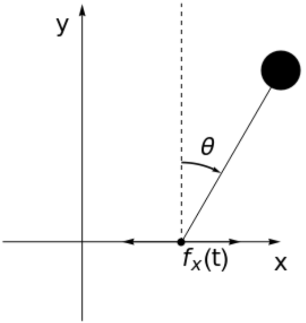


Figure 1: Equilibrium of the pendulum in 1D

$$x(t) = f_x(t) + l \sin\theta(t) \qquad (1)$$

$$y(t) = l \cos\theta(t) \qquad (2)$$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy \qquad (3)$$

$$\ddot{\theta} - \frac{g}{l} \sin\theta = -u(t)\cos\theta \qquad (4)$$

with $u(t) = \ddot{f}_x / l$, the control function applied to the pivot

Energy Control

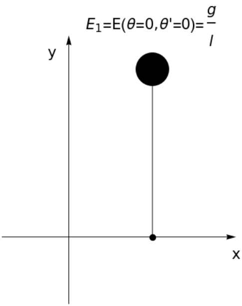


Figure 2: Energy control E_1

In the pivot reference frame

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{g}{l}\cos\theta \qquad (5)$$

$$\frac{d\mathcal{H}}{dt} = \dot{\theta}\left(\ddot{\theta} - \frac{g}{l}\sin\theta\right) \qquad (6)$$

$$\frac{d\mathcal{H}}{dt} = -u(t)\,\dot{\theta}\,\cos\theta \qquad (7)$$

Control Function

$$u(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}(\dot{\theta}\cos\theta) \qquad (8)$$

Equations of Motion

$$x(t) = f_x(t) + l \sin\theta \cos\phi \tag{9}$$

$$y(t) = f_y(t) + l \sin\theta \sin\phi \tag{10}$$

$$z(t) = l \cos\theta \tag{11}$$

$$\ddot{\theta} - \frac{g}{l} \sin\theta - \dot{\phi}^2 \cos\theta \, \sin\theta = -\cos\theta(\cos\phi \, u_x(t) + \sin\phi \, u_y(t)) \tag{12}$$

$$\ddot{\phi} \sin^2\theta + 2\,\dot{\phi}\,\dot{\theta} \sin\theta \cos\theta = -\sin\theta(\sin\phi \, u_x(t) - \cos\phi \, u_y(t)) \tag{13}$$

with $u_x(t) = \ddot{f_x}(t)/l$ and $u_y(t) = \ddot{f_y}(t)/l$

Energy Control

$$\frac{d\mathcal{H}}{dt} = -\dot{\phi}\sin\theta(-\sin\phi\, u_x(t) + \cos\phi\, u_y(t)) - \dot{\theta}\cos\theta(\cos\phi\, u_x(t) + \sin\phi\, u_y(t)) \tag{14}$$

In case $u_x, u_y \parallel (\hat{e_r})_{cylindrical}$

$$\vec{o}(t) = \begin{cases} u_x(t) = o(t)\cos\phi \\ u_y(t) = o(t)\sin\phi \end{cases} \rightarrow \frac{d\mathcal{H}}{dt} = -o(t)\,\dot{\theta}\cos\theta$$

In case $u_x,\ u_y \parallel (\hat{e_\phi})$

$$\vec{s}(t) = \begin{cases} u_x(t) = -s(t)\sin\phi \\ u_y(t) = s(t)\cos\phi \end{cases} \rightarrow \frac{d\mathcal{H}}{dt} = -s(t)\,\dot{\phi}\sin\theta$$

2D Control Functions

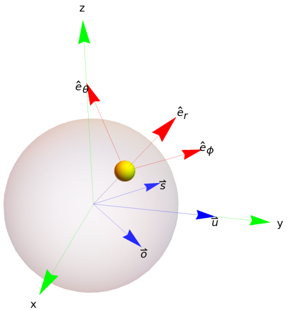


Figure 3: Control of the spherical pendulum in 2D

2D Control Functions

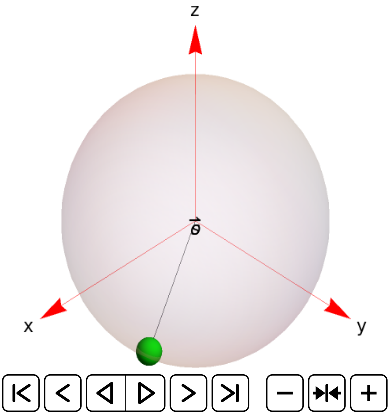
$$o(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta) \quad (15)$$

$$s(t) = \nu\,\mathrm{sign}\,(\dot{\phi}\sin\theta) \quad (16)$$

Net control:

$$\vec{u}(t) = \vec{o}(t) + \vec{s}(t) \quad (17)$$

$\approx$ 1D Control (Lab). $\mu = 0.4\text{s}^{-2}$, $\nu = 0.02\text{s}^{-2}$, $\theta_0 = 5\pi/6$



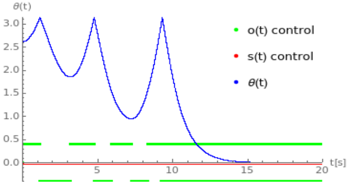


Figure 4: θ with time

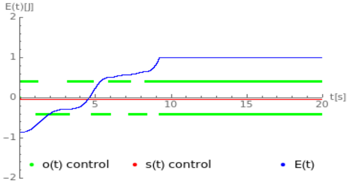
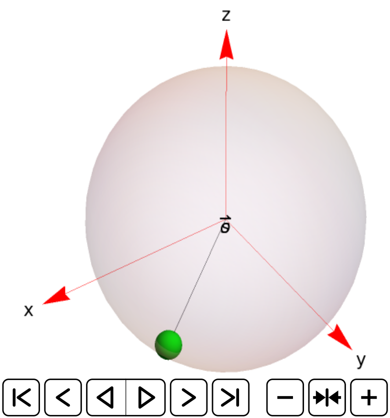


Figure 5: Energy with time

$\approx$ 1D Control (Pivot). $\mu = 0.4s^{-2}$, $\nu = 0.02s^{-2}$, $\theta_0 = 5\pi/6$



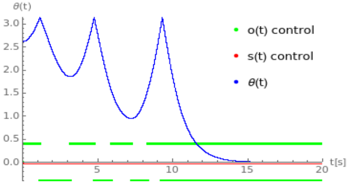


Figure 6: θ with time

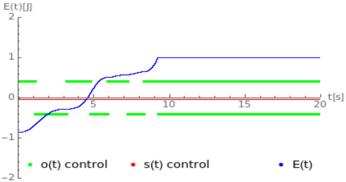
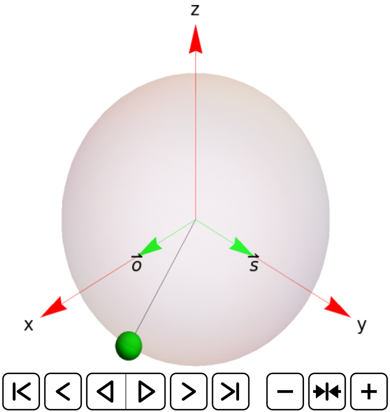


Figure 7: Energy with time

2D Control (Lab). $\nu,\ \mu = 0.3\text{s}^{-2},\ \phi_0' = 2\text{s}^{-1},\ \theta_0' = 1\text{s}^{-1},\ \theta_0 = 3\pi/4$



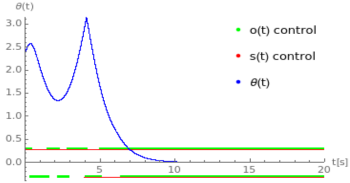


Figure 8: θ with time

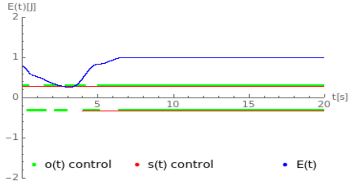
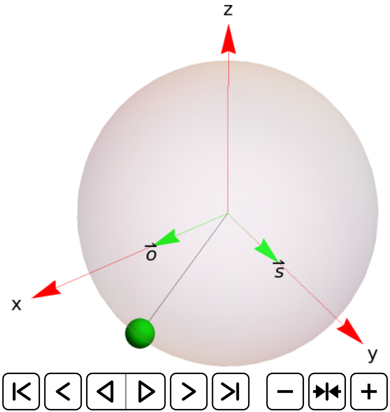


Figure 9: Energy with time

2D Control (Pivot). ν, $\mu = 0.3s^{-2}$, $\phi_0' = 2s^{-1}$, $\theta_0' = 1s^{-1}$, $\theta_0 = 3\pi/4$



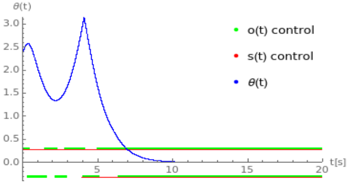


Figure 10: θ with time

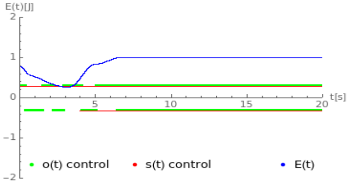


Figure 11: Energy with time

Conclusions

Equilibrium of the spherical pendulum was attainable with success.

Possible Future work:

- Analysis on the stability of the μ and ν parameters
- Compare with other equilibrium methods
- Produce a more realistic model contemplating air friction

References

[1] Åström, Karl Johan, and Katsuhisa Furuta. "Swinging up a pendulum by energy control." Automatica 36.2 (2000): 287-295

[2] Süli, Endre; Mayers, David, An Introduction to Numerical Analysis, Cambridge University Press,(2003): ISBN 0-521-00794-1

[3] R. Dilão, "Uma Introdução à Teoria dos Sistemas Dinâmicos e do Caos",IST 31.2 (2018): 330-335