# Equilibrium Control of the Spherical Pendulum

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In this work I study the generalization of Ashtrom e futria work (por a ref) to the stabilization of the spherical 3d Pendulum. It was applied two types of controls, one to control the rotational energy around the Z axis and another to control the Z position.

## I. INTRODUCTION

### A. 1D Control method

Stabilization of a pendulum around it's pivot point in one dimension was studied by [Dilão],[Astruma]. Considering $\theta$, the angle measured from the vertical where the positive sign is clockwise and $l$ is its length, and the fact that the pivot of the pendulum is mobile, we can seperate the decription of the problem in two parts, the position of the mass from the pivot and a function, $f_x(t)$ the position of the pivot:

$$x(t) = f_x(t) + l\sin\theta(t), \tag{1}$$

$$y(t) = l\cos\theta(t). \tag{2}$$

The Lagrangian may be written as:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{3}$$

Substituting in with Eqs. (1) and (2). We have the following movement equation for $\theta$.

$$\ddot{\theta} + u\cos\theta - \frac{g}{l}\sin\theta = 0. \tag{4}$$

where $u(t) = \ddot{f}_x/l$ We are interested in controlling only the position of the pendulum, one of the methods is by controlling the energy. This method involves calculating the energy and its derivative when $f_x = 0$. We obtain the following expression for the derivative of the energy, this is,

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{g}{l}\cos\theta \tag{5}$$

$$\frac{d\mathcal{H}}{dt} = \dot{\theta}\left(\ddot{\theta} - \frac{g}{l}\sin\theta\right) \tag{6}$$

Substituting Eq. (4),in the derivative we get.

$$\frac{d\mathcal{H}}{dt} = -u(t)\,\dot{\theta}\,\cos\theta \tag{7}$$

Eq. (7) exhibits the strategy of the control method used. Fixing $u$, in case the coefficient $\dot{\theta}\cos\theta$ is bigger than 0, the energy gets lowered, otherwise is raised. We can invert the effect of this coefficient and only base our control function $u$ so that the energy function $\mathcal{H}$, depends only on the difference of the energies between the pendulum in an equilibrium around it's pivot point,$E_1 = g/l$, (aligned with the vertical axis at $\theta = 0$ and $\dot{\theta} = 0$) and the energy at the current iteration $E$,

$$u(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \tag{8}$$

where, $\mu > 0$. When the energy $E$ is lower (higher) than $E_1$, the control function $u$ raises (lowers) the current energy $E$ until $E = E_1$ so that the pendulum remains in an equilibrium position around it's pivot.

### B. 2D Control Method

A 2D control method was implemented for the spherical pendulum regarding the information of Sec. (I A). In this case, similarly with Eq. (1) the position of the pendulum with a control function is given by,

$$x(t) = f_x(t) + l\sin\theta\cos\phi \tag{9}$$

$$y(t) = f_y(t) + l\sin\theta\sin\phi \tag{10}$$

$$z(t) = l\cos\theta \tag{11}$$

where $\theta$ is the azimuthal angle and $\phi$ is the polar angle. The movement equations become,

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \cos\theta\left(\sin\theta\,\dot{\phi}^2 - \cos\phi\,u_x - \sin\phi\,u_y\right) \tag{12}$$

$$\ddot{\phi} = -\frac{2\,\dot{\phi}\,\dot{\theta}\cos\theta - \sin\phi\,u_x + \cos\phi\,u_y}{\sin\theta} \tag{13}$$

where $u_x = \ddot{f}_x/l$ and $u_y = \ddot{f}_y/l$. With this, we can determine the variation of energy as it was done in Sec. (I A), the energy to control is given by

$$\mathcal{H} = \frac{\dot{\theta}^2}{2} + \frac{1}{2}\dot{\phi}^2\sin^2\theta + \frac{g}{l}\cos\theta \tag{14}$$

Substituing Eqs. (12) and (13) in the derivative of the energy in order to time, we have,

$$\frac{d\mathcal{H}}{dt} = -\dot{\phi}\,\sin\theta(-\sin\phi\,u_x + \cos\phi\,u_y) \\ -\dot{\theta}\,\cos\theta(\cos\phi\,u_x + \sin\phi\,u_y) \quad (15)$$

Eq. (15) displays the geometric relationship between, $u_x$ and $u_y$. In case,

$$\vec{o} = \begin{cases} u_x = o\cos\phi, \\ u_y = o\sin\phi, \end{cases} \quad (16)$$

this is, if $u_x$ and $u_y$ are aligned with the projection of the radial versor, $\hat{e_r}$, with the $xy$ plane, Eq. (15) reduces to very similar form of Eq. (7) with the exception that $o$ is a 2D variable and not 1D. This way, the parameter $o$ controls the energy regarding the azimuthal variation, it's rotation about $\hat{e_\phi}$ versor. In case,

$$\vec{s} = \begin{cases} u_x = -s\sin\phi, \\ u_y = s\cos\phi, \end{cases} \quad (17)$$

so that $u_x$ and $u_y$ are aligned with the polar versor, $\hat{e_\phi}$, the term, in Eq. (15) proportional to $\dot{\theta}$ goes to 0 and the parameter $s$ controls the energy regarding the polar variation, in other words its rotation about $z$ axis. We can then implement the two types of control by summing vectorially each acceleration component.

$$\vec{u} = \begin{cases} u_x = o\cos\phi - s\sin\phi, \\ u_y = o\sin\phi + s\cos\phi. \end{cases} \quad (18)$$

In order to control the pendulum so that it remains balanced with it's center point, ($\theta = 0$, $\phi = 0$, $\dot{\phi} = 0$), the strategy was use the parameter v to desaccelerate the rotation of the pendulum, until $\dot{\phi} = 0$

$$s(t) = \nu\,\mathrm{sign}\,(\dot{\phi}\sin\theta), \quad (19)$$

at the same time that the control mechanism in Eq. (8) is applied to raise or lower the energy $E$ to the final state ,$E_1 = g/l$,

$$r(t) = -\mu\,\mathrm{sign}\,(E_1 - E)\,\mathrm{sign}\,(\dot{\theta}\cos\theta), \quad (20)$$

with, $\mu$, $\nu > 0$. Fig (1) ilustrates an examples of the 2D control functions $\vec{o}$ and $\vec{s}$.

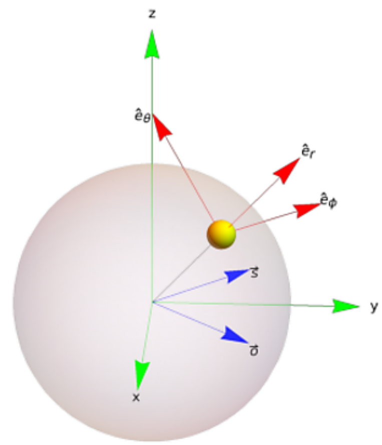


Figure 1: Example of control functions $\vec{o}$ and $\vec{s}$ filled in blue, to balance the yellow pendulum around it's pivot point. The axis $x$, $y$, $z$ are displayed in green and the spherical versors, $\hat{e_\phi}$, $\hat{e_\theta}$, $\hat{e_r}$ in red. The control $o$ has the function of raising the $Z$ position to $Z = l$, and the control $s$ is responsible for slowing the rotation around $Z$ axis, $\dot{\phi}$.

## II. NUMERICAL METHOD

The numerical method applied in order to solve the system was a fourth order Runge-Kutta method for both $\dot{\phi}$ and $\dot{\theta}$. This method is applied when we have a generic function $x$ such that,

$$\dot{x} = g(x,t), \quad (21)$$

the next iteration of $x$ by the fourth order Runge-Kutta method is given by,

$$x_{n+1} = x_n + \frac{\Delta t}{6}(k1 + 2\,k2 + 2\,k3 + k4). \quad (22)$$

The $k$ coefficients are given by,

$$k_1 = g(t, x_n), \quad (23)$$

$$k_2 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_1}{2}\right), \quad (24)$$

$$k_3 = g\left(t + \frac{\Delta t}{2}, x_n + \frac{k_2}{2}\right), \quad (25)$$

$$k_4 = g(t + \Delta t, x_n + k_3). \quad (26)$$

Associating Eq. (12) with $\ddot{\theta} = g_\theta$ and Eq. (13) with $\ddot{\phi} = g_\phi$ we can obtain the next iteration of $\dot{\theta}$ and $\dot{\phi}$. Upon obtaining the derivatives, an invertion of the first order forward finite difference method [ref wikipedia] was used to determine the next iterations of $\phi$ and $\theta$,

$$\theta_{n+1} = \dot{\theta_n} + \theta_n, \tag{27}$$

$$\phi_{n+1} = \dot{\phi_n} + \phi_n. \tag{28}$$

If we integrate numerically Eqs. (18) as done with $\theta$ and $\phi$ we can obtain first, the velocity and then, the position of the mobile point this is, respectively the functions $\dot{f}_x$, $\dot{f}_y$ and $f_x$, $f_y$.

With the proper set of initial values, a solution to the problem may be determined so that the pendulum remains in a balanced position.

## III. RESULTS

A series of results were obtained for different initial conditions

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[1] F. R. Joaquim and A. Rossi, Nucl. Phys. B **765** (2007) 71 [hep-ph/0607298].